# Odd Symplectic Supermanifolds

**Motivation:** Quantization of gauge theories

$G$ : Lie group (gauge group)
$\circlearrowleft$
$X$ : manifold (space of fields)

$$Z = \int_X \mu_X \, e^{\frac{i}{\hbar} S}$$

Problem: stationary phase approximation will fail as critical points come in $G$-orbits, i.e. are highly degenerate.

Fadeev-Popov $\Rightarrow$ Introduce (fermionic) ghost fields

$$X \rightsquigarrow \mathcal{F} : \text{ supermanifold}$$

$$Z = \int_{\mathcal{F}} \mu_{BRST} \, e^{\frac{i}{\hbar}(S + Q(\Psi))}$$

cohomological vector field, i.e. $Q \in \mathfrak{X}(\mathcal{F})$, $Q^2 = 0$.

gauge fixing fermion $\Psi \in C^\infty(\mathcal{F})_{-1}$

$\overset{BV}{\Rightarrow}$ $\mathcal{F}_{BV} = T^*[-1]\mathcal{F}$ (has odd symplectic structure)

$$Z = \int_{\mathcal{L} \subset \mathcal{F}_{BV}} \sqrt{\mu_{BV}} \, e^{\frac{i}{\hbar} S_{BV}}$$

$\hookrightarrow$ Lagrangian submanifold

gauge fixing means choosing a Lagrangian!

**Differential forms:**

$M$: smooth manifold $\quad \Omega^\bullet(M) = \Gamma(\Lambda^\bullet T^* M) \equiv C^\infty(\Pi TM)$

$\mathcal{M}$: smooth supermfd $\quad \Omega^\bullet(\mathcal{M}) := C^\infty(\Pi T\mathcal{M})$

[Batchelor]
$\Rightarrow$ $\mathcal{M} \cong \Pi E$ for $E \to M$ a vector bundle

$$\Rightarrow \Omega^p(\mathcal{M}) \ni f = \sum_{|I|+|J|=p} \underbrace{f_{IJ}(x,\theta)}_{C^\infty(U)\otimes\Lambda^\bullet E} \, \underbrace{dx^I}_{\Lambda^{|I|}T^*M} \, \underbrace{d\theta^J}_{\mathrm{Sym}^{|J|}E}$$

$$\Rightarrow \Omega^p(\mathcal{M}) = \Gamma\Big(\bigoplus_{i=1}^{p} \Lambda^\bullet E \,\hat{\otimes}\, \Lambda^i T^*M \,\hat{\otimes}\, \mathrm{Sym}^{p-i} E\Big)$$

**Exterior derivative:** $d : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$

$$d = dx^i \frac{\partial}{\partial x^i} + d\theta^i \frac{\partial}{\partial \theta^i}$$

## Odd symplectic supermanifolds

$(\mathcal{M}, \omega)$: $\mathcal{M}$ smooth supermanifold,
$\omega \in \Omega^2(\mathcal{M})$ odd, closed & non-degenerate
$\equiv \sum_{ij} \omega_{ij}(x, \theta)\, dx^i d\theta^j$ (locally)
where $\omega_{ij}(x,\theta)$ is invertible.

<u>Remark</u> non-degeneracy implies $\dim \mathcal{M} = (n|n)$

<u>Example</u> $M$: smooth manifold
$\Rightarrow (\Pi T^* M, dx^i d\theta_i)$ is odd-symplectic
globally defined as $dx^i$'s transform inverse to $d\theta_i$'s.

**Thm [Schwarz]**

symplectomorphism

$$\exists\, \Phi : (\mathcal{M}, \omega) \xrightarrow{\sim} (\Pi T M, \omega_0 := \sum_i dx^i d\theta^i)$$

<u>Rem.</u> This can be seen as global version of Darboux theorem in ordinary symplectic geometry.

<u>proof $\mathcal{M} \cong \Pi T^*M$:</u>

Batchelor $\Rightarrow \mathcal{M} = \Pi E$, $E \to M$ rank $n$ vector bundle

$\omega$ non-deg. $\Rightarrow$ $\omega_{ij}(x,0)$ invertible

$$\Rightarrow \quad \begin{aligned} E_x &\longrightarrow T^*_x M \\ v^i \partial_{\theta^i} &\longmapsto v^i \omega_{ij}(x,0)\, dx^j \end{aligned}$$

$$\Rightarrow \quad E \cong T^*M$$

Remark: Observe that under this identification the symplectic form $\omega$ agrees with $\omega_0 = dx^i d\theta_i$ on the zero-section, since

$$dx^i d(\omega_{ij}(x,0)\theta^j) \underset{\omega \text{ closed}}{=} \omega_{ij}(x,0)\, dx^i d\theta^j$$

<u>proof $(\Pi T^*M, \omega) \cong (\Pi T^*M, \omega_0)$:</u>

Remark $\Rightarrow$ $\omega_t = t\omega_0 + (1-t)\omega$ non-degenerate & $[\omega_t] = 0 \in H^2(\mathcal{M})$

(since all odd 2-forms are exact)

<u>Moser trick:</u>

$\varphi_t : \mathcal{M} \to \mathcal{M}$ isotopy

$$0 = \frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* ( \underbrace{L_{X_t} \omega_t}_{= d\iota_{X_t}\omega_t \text{ as } \omega \text{ closed}} + \dot{\omega}_t )$$

exact as $[\omega_t]$ constant

$X_t$: time-dependent vector field generated by $\varphi_t$.

$$\Leftrightarrow \quad d\iota_{X_t}\omega_t = d\sigma_t \quad \Leftarrow \quad \iota_{X_t}\omega_t = \sigma_t$$

can be solved by non-deg. of $\omega_t$.

<u>Rem:</u> In ordinary sympl. geometry one needs to assume the sympl. mfd. to be compact to assure integrability for $t \in [0,1]$. Here $X_t$ generates the zero vector field

on the base and we can drop the assumption.

**Example** Take $\mathcal{L}$ a $(k, n-k)$ supermanifold

$$\Rightarrow (\Pi T^*\mathcal{L}, dq^i dp_i) \text{ is symplectic}$$

**Observation:** $\mathcal{L} = \Pi E$ for $E \to L$ rank $n-k$ v.b.

$$\Rightarrow (\Pi T^*\mathcal{L}, dq^i dp_i) \xrightarrow{\sim} (\Pi T^* E, dx^i d\theta_i)$$

$$q^i \longmapsto \begin{cases} x^i & i = 1, \dots, k \\ \theta^i & i = k+1, \dots, n \end{cases}$$

$$p^i \longmapsto \begin{cases} \theta^i & i = 1, \dots, k \\ x^i & i = k+1, \dots, n \end{cases}$$

## Lagrangian Submanifolds

$\mathcal{L} \subset \mathcal{M}$: $\omega|_{\mathcal{L}} = 0$ & $\dim \mathcal{L} = (n-k, k)$ for $0 \leq k \leq n$.

**Examples:**

1. Conormal Lagrangian: $(\mathcal{M} = \Pi T^* N, \omega = dx\, d\theta)$
   - $K \subset N$ submanifold
   - $N^*K$ conormal bundle

   $\Rightarrow \Pi N^* K \subset \Pi T^* N$ Lagrangian

Rem. The same works in classical symplectic geometry, i.e. $N^*K \subset (T^*N, dp \wedge dx)$ is Lagrange.

# Classification of Lagrangians

Fix a splitting $\mathcal{M} \cong \Pi T^*M$, take a Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$

$\Rightarrow$ $|\mathcal{L}| =: L \subset M$ & $\mathcal{L} = \Pi E$, $E \to L$ v.b.

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & T^*M \arrow[d] \\
L \arrow[r, hook] \arrow[u, dashed, bend left, "0_E"] & M \arrow[u, dashed, bend right, "s_L \in \Gamma(N^*L)"']
\end{tikzcd}$$

$\Rightarrow$ $\boxed{\text{Every Lagrangian of } \Pi T^*M \text{ is a graph over the conormal Lagrangian of its base.}}$

**Remark** $s_L \in \Gamma(N^*L)$ determines an odd function $\psi_L \in C^\infty(\mathcal{L}_L)_{odd}$, i.e.

$$\psi_L(x, \Theta) = s_L(x)_i \Theta^i$$

This corresponds to the gauge-fixing fermion in BRST.

**Remark:** If we don't fix the splitting $\mathcal{M} \cong \Pi T^*M$ we can find a splitting such that

$$\mathcal{L} \subset \mathcal{M} \cong \Pi T^*M$$

is the conormal Lagrangian over its base.

# Appendix: Alternative proof of Schwarz ?!

**Lemma:** Closed odd two-forms are exact.

*proof:* $\omega = \omega_{ij}(x,\Theta)\, dx^i d\Theta^j$

$$0 = d\omega = \partial_{x_k} \omega_{ij}(x,\Theta)\, dx^k dx^i d\Theta^j + \partial_{\Theta_k} \omega_{ij}(x,\Theta)\, dx^i d\Theta^j d\Theta^k$$

$$\Rightarrow \partial_{x_k} \omega_{ij}(x,\Theta) = \partial_{x_i} \omega_{kj}(x,\Theta) \quad (*)$$

$$\partial_{\Theta_k} \omega_{ij}(x,\Theta) = -\partial_{\Theta_j} \omega_{ik}(x,\Theta) \Rightarrow \Theta^j \partial_{\Theta_k} \omega_{ij} = -\omega_{ik} \quad (**)$$

$\omega_{ij}\Theta^j \neq 0$ ?

Set $v = \frac{1}{2}\Theta^j \omega_{ij}\, dx^i$ globally defined ???

$$\Rightarrow dv = \frac{1}{2}\Big(\underbrace{\omega_{ij}\, d\Theta^j dx^i}_{=\omega} + \underbrace{\Theta^j \partial_{x_k}\omega_{ij}\, dx^k dx^i}_{=0 \text{ by } (*)} - \underbrace{\Theta^j \partial_{\Theta_k}\omega_{ij}}_{=-\omega_{ik} \text{ by } (**)} d\Theta^k dx^i\Big)$$

$$= \omega$$

*proof of Schwarz:*

Since $\omega = d\left(\frac{1}{2}\Theta^j \omega_{ij}\right) dx^i$ we see that

$$(x,\Theta) \longrightarrow \left(x, \frac{1}{2}\Theta^j \omega_{ij}\right)$$

intertwines $\omega$ and $d\Theta_i dx^i$.