Super Riemann Surfaces and Perturbative Superstring Theory

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1 Introduction

In perturbative string theory one calculates string amplitudes by summing over all possible world sheet topologies with all possible choices of conformal structures. The world sheet topologies that can arise in closed oriented

bosonic string theory are precisely Riemann surfaces and the summation over conformal structure can be made precise by an integration over the moduli spaces $\mathcal{M}_g$ of Riemann surfaces of genus g.
If we now consider super string theory, we need to introduce an odd variable on the world sheet, i.e. we need replace the Riemann surfaces by super Riemann surfaces.
In order to perform the integration over all possible super Riemann surfaces we need to study the moduli space (or rather stack) $\mathfrak{M}_g$ of super Riemann surfaces of genus g. For genus 2 the supermoduli space is projected and hence the calculation of the superstring amplitude can be broken to an integration over the moduli space$\mathcal{M}_{spin,2}$ of spin curves after integrating over the fibers. This calculation was performed by D'Hoker and Phong [3]. However in [4] Donagi and Witten showed that $\mathfrak{M}_g$ is non-projected for $g \geq 5$. The question is still unanswered for $g = 3, 4$. Besides the integration over all super Riemann surfaces, the path integral also includes so called vertex operators, that are inserted on the super Riemann surfaces. Unlike in bosonic string theory where it is clear how to insert the operators, in super string theory we need to distinguish between the Neveu-Schwarz sector which leads to spacetime bosons and the Ramond sector which leads to spacetime fermions. We rediscover this distinction in the theory of super Riemann surfaces, where it leads to different kinds of divisors along which the operators are inserted. A detailed overview of the theory of super Riemann surface can be found in [7]. For applications to superstring theory see [6].

2 Preliminary

2.1 Supermanifolds over a basis

In supergeometry, as in algebraic geometry, it is beneficial to consider families of supermanifolds. For a detailed overview of relative supergeometry and applications to super Riemann surfaces we refer to [5].

Definition 2.1. Let $U \subset \mathbb{R}^{m|n}$, $V \subset R^{m|n}$ and $B \subset \mathbb{R}^{p|q}$ be superdomains with coordinates $X^A = (x^a, \eta^\alpha), Y^B = (y^b, \theta^\beta)$ and $L^C = (l^c, \iota^\gamma)$ respectively. For a map of superdomains $f : U \times B \to V \times B$ over B and $u \in |U \times B|$ a point. We call the map f a submersion at u if the following holds:
There exists an open neighbourhood $V' \subset V \times B$ of $f_{red}(u)$ and a superdomain $U' \subset \mathbb{R}^{k|l}$ and an neighbourhood of u which is isomorphic to $V' \times U'$ such

that $f|_{V'\times U'} : V' \times U' \to V'$ is given by projection on the first factor. Equivalently, the matrix $\frac{\partial f^{\#} y^b}{\partial x^a}$ has rank k and the matrix $\frac{\partial f^{\#}\theta^\beta}{\partial \eta^\alpha}$ has rank l. If f is a submersion for all $u \in |U \times B|$, it is called a submersion.

In algebraic geometry the analog of a submersion is a smooth morphism. A morphism $f : X \to B$ of schemes is called smooth if

i) f is locally of finite presentation

ii) f is flat

iii) for every geometric point $\bar{b} \to B$, the fiber $X \times_B \bar{b}$ is regular.

The last point says, that each fiber is non-singular. Equivalently, one can show that for a submersion each fiber is a smooth supermanifold (the same is true for ordinary manifolds). Just as smooth morphisms, submersions are stable under base change. All this motivates the following definition.

Definition 2.2. (Family of Supermanifolds) A supermanifold M with a submersion $M \to B$ is called a family over the base B or shorter a supermanifold over B.

By the definition of a submersion, for every point $p \in |M|$ there is a neighbourhood $U \subseteq M$ such that the submersion coincides with the projection $U_1 \times U_2 \to U_2$ for $U_1 \subseteq \mathbb{R}^{m|n}$ and $U_2 \subseteq B$. In this case we call M a supermanifold of relative dimension $m|n$ over B. Coordinates on U_1 are called relative coordinates for M. A morphism of supermanifolds over B is a smooth map $f : M \to M'$, such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & M' \arrow[dl] \\ & B & \end{tikzcd}$$

Remark 2.2.1. Any supermanifold is a supermanifold over a point $\mathbb{R}^{0|0}$.
This observation let us establish the following principle. Any supermanifold M is to be understood implicitly as a family $b_M : M \to B$ of supermanifolds. Any map of supermanifolds is to be understood as a map of supermanifolds over a basis B.
One should only consider properties of supermanifolds and maps that are invariant under base change. That is, if a property holds for a supermanifold

M or a map $f : M \to N$ this property should also hold for b^*M and b^*f, where $b : B' \to B$ is any map of supermanifolds. Properties that are invariant under base change are called "*geometric*".

2.2 Divisors

In projective space a codimension 1 subvariety is defined by the vanishing of one homogeneous polynomial. In contrast subvarieties of higher dimension are much more difficult to understand.

Example 2.2.1. *(Twisted Cubic)* We construct construct a projective scheme as follows. Let

$$\begin{aligned} f_1 &= x_0x_3 - x_1x_2 \\ f_2 &= x_1^2 - x_0x_2 \\ f_3 &= x_2^2 - x_1x_3 \end{aligned}$$

The twisted cubic is the projective scheme

$$\mathrm{Proj}\left(\frac{\mathbb{C}[x_0, x_1, x_2, x_3]}{(f_1, f_2, f_3)}\right) \tag{1}$$

The embedding $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ is given by

$$[s : t] \to [s^3 : s^2t : st^2 : t^3] \tag{2}$$

One easily checks that this embedding respects f_1, f_2 and f_3.

Motivated by the observation we study subvarieties of codimension 1 in varieties (or also supermanifolds). In a general variety the condition to be a subvariety of codimension 1 is generalized by the definition of a Weil divisor. A subvariety cut out by 1 equation leads to the definition of a Cartier divisor.

Definition 2.3. Let X be integral, locally notherian scheme. A prime Weil divisor is an integral, closed subscheme Z of codimension 1 in X.
A Weil divisor in the free abelian group generated on the set of prime divisors. An general Weil divisor is a locally finite sum

$$D = \sum_Z n_Z Z. \tag{3}$$

The group of all Weil divisors is denoted $Div(X)$. A Weil divisor D is effective if all the coefficients are non-negative. One writes $D \geq D'$ if the difference $D - D'$ is effective. If X is defined over an algebraic closed field, we define the degree $\deg(D) = \sum_Z n_Z$.
If $f \in \mathcal{O}_{X,Z}$ where $\mathcal{O}_{X,Z}$ is the stalk at the generic point of Z. We define the order of vanishing of f along Z

$$\mathrm{ord}_Z(f) := \mathrm{length}\,(\mathcal{O}_{X,Z}/(f)) \tag{4}$$

This length is finite and $\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$. For a non-zero rational function $f \in k(X)^\times$, the principal Weil divisor associated to f is defined to be the Weil divisor

$$\mathrm{div}\ f = \sum_Z \mathrm{ord}_Z(f) Z. \tag{5}$$

Let X be a normal integral noetherian scheme. Every Weil divisor D determines a coherent sheaf $\mathcal{O}_X(D)$ on X by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X) \,|\, f = 0 \text{ or } \mathrm{div}(f) + D \geq 0 \text{ on U}\} \tag{6}$$

The idea behind this definition is that for a prime divisor Z the sheaf $\mathcal{O}_X(-Z)$ is the sheaf of functions vanishing on Z. For a general effective divisor D we get a corresponding subscheme of X, and a exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i^*\mathcal{O}_D \to 0. \tag{7}$$

We already saw that there is a strong connection between line bundles and divisors. This observation motivates the definition of a Cartier divisor

Definition 2.4. Let X be an integral noetherian scheme. Let $\mathcal{M}_X$ be the sheaf of rational functions. There is a exact sequence:

$$0 \to \mathcal{O}_X^\times \to \mathcal{M}_X^\times \to \mathcal{M}_X^\times/\mathcal{O}_X^\times \to 0 \tag{8}$$

A Cartier divisor on X is a global section of $\mathcal{M}_X^\times/\mathcal{O}_X^\times$. An equivalent description is that a Cartier divisor is a collection $\{(U_i, f_i)\}$, where $f_i = f_j$ on $U_i \cap U_j$ up to multiplication by a section of $\mathcal{O}_X^\times$. The idea is that on the patch U_i we cut out a subscheme by a single equation f_i.
By the exact sequence above, there is an exact sequence of sheaf cohomology groups:

$$H^0(X, \mathcal{M}_X^\times) \to H^0(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) \to 0 \tag{9}$$

A Cartier divisor is called principal if it is in the image of $H^0(X, \mathcal{M}_X^\times) \to H^0(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times)$. We see that

$$\mathrm{Pic}(X) \cong \{\text{Cartier divisors}\}/\{\text{principal Cartier divisors}\} \tag{10}$$

Let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles and $[\mathcal{L}_1], [\mathcal{L}_2]$ the corresponding equivalence classes of divisors. We have a group structure on $H^0(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times)$ and also on $\mathrm{Pic}(X)$, which are compatible, i.e. $[\mathcal{L}_1] + [\mathcal{L}_2] = [\mathcal{L}_1 \otimes \mathcal{L}_2]$ and $-[\mathcal{L}_1] = [\mathcal{L}_1^*]$.

We now restrict ourselves to the for us must interesting case of an proper algebraic curve X over $\mathbb{C}$, i.e. a compact Riemann surface of some genus g. On a curve the canonical bundle is isomorphic to the sheaf of Kähler differentials Ω_X^1. The corresponding divisor in called the canonical divisor K.

Theorem 2.5. *(Riemann-Roch) For a divisor D on X*

$$l(D) - l(K - D) = \deg(D) - g + 1 \tag{11}$$

with $l(D) = \dim H^0(X, L)$ where L is the line bundle corresponding to D.

The second important theorem on divisors by Serre (actually the theorem holds for all vector bundles) can seen as a algebraic analog of Poincaré duality.

Theorem 2.6. *(Serre duality) For every line bundle $\mathcal{L}$ on a Riemann surface X there is an isomorphism $H^i(X, \mathcal{L}) \to H^{1-i}(X, K \otimes \mathcal{L})^*$.*

2.3 Spin structures

Definition 2.7. A spin structure on an orientable Riemannian manifold (M, g) is an equivariant lift of the oriented Riemannian manifold (M, g) is an equivariant lift of the oriented orthonormal frame bundle

$$\pi : F_{SO}(M) \to M \tag{12}$$

with respect to the double covering

$$\rho : Spin(n) \to SO(n) \tag{13}$$

I.e. a pair (P, F_p) is a spin structure on the principal bundle $\pi : F_{SO}(M) \to M$, where

a) $\pi_P : P \to M$ is a principal $Spin(n)$-bundle over M.

b) $F_p : P \to F_{SO}(M)$ is an ρ-equivariant 2-fold covering map such that $\pi \circ F_p = \pi_P$ and $F_p(p \cdot q) = F_p(p) \cdot \rho(q)$ for all $p \in P$ and all $q \in Spin(n)$.

The Obstruction to the existence of a spin structure is the a certain cohomology class $w_2(M) \in H^2(M, \mathbb{Z}/2)$ of M.
There is a spin structure if and only if $w_2(M)$ vanishes. If $E \to M$ is spin, the spin structures are in bijection with $H^1(M, \mathbb{Z}/2)$.

Spin structures on curves

Suppose that X is an almost complex manifold so that the structure group of its principal $SO(2n)$-bundle P reduces to $U(n)$. The Serre spectral sequence gives an exact sequence:

$$0 \longrightarrow H^1(P_E, \mathbb{Z}/2) \longrightarrow H^1(SO(n), \mathbb{Z}/2) \cong \mathbb{Z}/2 \xrightarrow{\delta} H^2(M, \mathbb{Z}/2)$$

There is an isomorphism $H^1(SO(n), \mathbb{Z}/2) \cong \mathbb{Z}$ and we have $\delta(1) = w_2(E)$. The maps $U(n) \to SO(2n)$ and $\det : U(n) \to U(1)$ induce isomorphisms in $H^1(-, \mathbb{Z}/2)$. One can show that spin structures over X correspond bijective to those double coverings of the $U(1)$-bundle $\det(P)$ which restrict to the squaring map $U(1) \to U(1)$, i.e. $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle and α is an isomorphism

$$\alpha : \mathcal{L}^2 \to K \tag{14}$$

In the language of divisors this meadns that a spin structure corresponds to a divisor D such that $2\theta = K$ where K is the canonical divisor. For a Riemann surface over $\mathbb{C}$ it is $\dim(H^1(M, \mathbb{Z}/2) = 2g$. Hence there are 2^{2g} different spin structures on a surface of genus g. The divisor associated to a spin structure is also called a theta characteristic. A spin structure (or more precisely the associated divisor) θ can be even if $\dim(H^0(C, \theta)) = 0 \bmod 2$ and odd if $\dim(H^0(C, \theta)) = 1 \bmod 2$. There are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g + 1)$ odd spin structures. For a complete discussion of the correspondence between theta characteristics and spin structures we refer to [1].

3 Super Riemann Surfaces

We now are in the position to define super Riemann surfaces (over some base supermanifold B. For the mathematically minded reader we recommend to consult [5] and for the physicists [7] for more details.

Definition 3.1. A super Riemann surface Σ is a (smooth family of) complex supermanifold of relative complex dimension $1|1$ together with a holomorphic subbundle $\mathcal{D} \subset TM$ of complex rank $0|1$ such that the Lie bracket induces an isomorphism

$$\frac{1}{2}[\cdot,\cdot] : \mathcal{D} \otimes_{\mathbb{C}} \mathcal{D} \to TM/\mathcal{D} \tag{15}$$

The condition (15) is called the complete non-integrability of $\mathcal{D}$. This condition is crucial for the results about super Riemann surfaces presented here, hence the results most probably do not generalize to other definitions of super Riemann surfaces (for instance dropping the distribution $\mathcal{D}$).

Example 3.1.1*.* Let (z, θ) be the standard complex coordinates on $\mathbb{C}^{1|1}$. Note that $\mathcal{D}$ is a line bundle and hence locally generated by one section D of $\mathcal{D}$ over $\mathbb{C}^{1|1}$. A typical example is $D_\theta = \partial_\theta + \theta\partial_z$. Then we have $D_\theta^2 = \partial_z$ and hence $\mathcal{D}_\theta$ and $\mathcal{D}_\theta^2$ is everywhere a basis of $T(\mathbb{C}^{1|1})$.

The example 3.1.1 is typical, because every super Riemann surface is locally equivalent to the standard super Riemann surface $\mathbb{C}^{1|1}$. To see this let $D = a\partial_\eta + b\partial_u$ be a local section of $\mathcal{D}$ where (u, η) are relative local coordinates on M and $a = a(u, \eta)$ is some even holomorphic function and $b = b(u, \eta)$ an odd holomorphic function. Since D and D^2 locally generate TM the function a has to be invertible and we way assume that $a = 1$. Due the complete non-integrability condition, the remaining coefficient of $\frac{1}{2}[D, D] = (\partial_\eta b)\partial_u$ has to be invertible. Consider the coordinate change $z = f(u) + \eta\zeta(u), \theta = \eta$. We obtain:

$$D = \partial_\theta + \left(b\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \eta}\right)\partial_z \tag{16}$$

We decompose b as $b = b_0 + \eta b_1$ (note that b_1 is invertible). The equation $b\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \eta} = \eta$ decomposes to

$$\begin{aligned} b_0 \cdot f' + \zeta(u) &= 0 \\ b_0\zeta' + b_1 f' &= 0 \end{aligned}$$

The set of differential equations is solved by

$$\begin{aligned} \zeta &= -\frac{b_0}{b_1} \\ f' &= (b_1 - b_0 b_0')\, b_1^{-2}. \end{aligned}$$

A morphism of super Riemann surfaces is a holomorphic map $M \to M'$ over B that preserves the distribution $\mathcal{D}$. We call such a map superconformal.
We call coordinates $(z|\theta)$ on a super Riemann surface superconformal coordinates if the distribution $\mathcal{D}$ is locally generated by $\partial_\theta + \theta\partial_z$.
We want to determine the most general form of superconformal coordinate changes, since a super Riemann surface is completely determined by an atlas of superconformal coordinates such that all coordinate changes preserve the line bundle $\mathcal{D}$. The must general holomorphic coordinate change is given by

$$\begin{aligned}\tilde{z} &= f(z) + \theta\zeta(z)\\ \tilde{\theta} &= \xi(z) + \theta g(z)\end{aligned}$$

where f, g are even holomorphic functions and ζ, ξ are odd holomorphic functions. In order to preserve D we want that $D_\theta = FD_{\tilde{\theta}}$ for a non-zero function F (i.e. such that $1/F$ is defined). Acting on $\tilde{\theta}$ determines F: $D_\theta = (D_\theta\tilde{\theta})D_{\tilde{\theta}}$. Using the chain rule we compute

$$D_\theta = (D_\theta\tilde{\theta})D_{\tilde{\theta}} + (D_\theta\tilde{z} - \tilde{\theta}D_\theta\tilde{\theta})\partial_{\tilde{z}} \tag{17}$$

Plugging in the explicit for of the coordinate changes we obtain

$$\begin{aligned}\zeta &= g\xi\\ f' &= g^2 - \xi\xi'\end{aligned}$$

Let's now consider the case that $\xi = 0$. It follows immediately that $\zeta = 0$. Then we get the the transition functions

$$\begin{aligned}z_\alpha &= f(z_\beta)\\ \theta_\alpha &= (f'(z_\beta))^{1/2}\theta_\beta\end{aligned}$$

The transition of the bosonic coordinates are independent of the θ's. This shows that we can consistently forget the θ's which amounts to a projection $\Sigma \to \Sigma_{red}$, so we see explicitly that ever Riemann surface over a point is projected Σ. We already know that ever complex supermanifold with dimension $n|1$ is split so every super Riemann surface over a bosonic base is split. A split supermanifold is determined by the underlying ordinary manifold and a vector bundle on the base. The vector defining the vector bundle on a super Riemann surface is generated by θ.
On a Riemann surface the canonical bundle K is generated by dz_α, which

transform like $dz_\alpha = f'(z_\beta)dz_\beta$, so we see that the θ_α transform as $(dz_\alpha)^{1/2}$. Hence the bundle generated by θ is a square root of the canonical bundle (i.e. a Spin structure) with odd fibers. We conclude the following:

Theorem 3.2. *There exist a bijection between the set of super Riemann surfaces over the point $\mathbb{R}^{0|0}$ and the set of pairs (M, S), where M is an ordinary Riemann surface over a point and S a Spin structure on M.*

Example 3.2.1. From the theory of Riemann surfaces we know, we know, that on a Riemann surface of genus g there are 2^{2g} non-equivalent spin structures. So the spin structure on the sphere $P^1_{\mathbb{C}}$ is unique. Cover $P^{1|1}_{\mathbb{C}}$ by two patch given by $(z_1|\theta_1)$ and $(z_2|\theta_2)$ and the transition function are given by $z_2 = 1/z_1$ and $\theta_2 = \frac{\theta_1}{z_1}$. We fix $D_1 = \partial_{\theta_1} + \theta_1 \partial_{z_1}$ and get

$$\begin{aligned} D_1 &= \frac{\partial \theta_2}{\partial \theta_1}\partial_{\theta_2} + \frac{\partial z_2}{\partial \theta_1}\partial_{z_2} + \theta_2 z_1 \left(\frac{\partial \theta_2}{\partial z_1}\partial_{\theta_2} + \frac{\partial z_2}{\partial z_1}\partial_{\theta_2} \right) \\ &= \frac{1}{z_1}\partial_{\theta_2} - \frac{\theta}{z_1}\partial_{z_2} \end{aligned}$$

Since $D_{\theta_1}\theta_2 = \frac{1}{z_1}$ we see that we can write $D_2 = \theta_{\theta_2} - \theta_2 \partial_{z_2}$ and superconformal coordinates are given by $(z_1|\theta)$ and $(-z_2|\theta_2)$.

3.1 Superconformal vector fields

Infinitesimally, superconformal coordinate transformations are generated by vector fields that preserve D_θ, i.e. $[W, D_\theta] = F D_\theta$. One check that these vector fields are generated given by

$$\begin{aligned} \nu_f &= f(z)\left(\partial_\theta - \theta \partial_z\right) \\ V_g &= g(z)\partial_z + \frac{\partial_z g(z)}{2}\theta\partial_\theta \end{aligned}$$

The vector field ν_f is odd, whereas V_g is even. We call these superconformal vector fields. A short calculation shows

$$\begin{aligned} \{\nu_f, D_\theta\} &= (\theta f')D_\theta \\ [V_g, D_\theta] &= -(\partial_z g/2)D_\theta \end{aligned}$$

We want to understand the sheaf of superconformal vector fields on Σ. Since $\mathcal{D}^2 \cong T\Sigma/\mathcal{D}$, the natural exact sequence $0 \to \mathcal{D} \to T\Sigma \to T\Sigma/\mathcal{D} \to 0$ becomes

$$0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2 \to 0. \tag{18}$$

Given a global super conformal transformation defined by f and g, (i.e. with the odd parameter turned off). We want to combine f and g to a super field. In terms of a superfield $\mathcal{V}(z|\theta) = g(z) + 2\theta f(x)$, a general superconformal vector field is

$$\mathcal{W} = \mathcal{V}(z|\theta)\partial_z + \frac{1}{2}D_\theta \mathcal{V} D_\theta \tag{19}$$

Expanding the right hand side shows

$$\mathcal{W} = V_g + \nu_f \tag{20}$$

A superconformal vector field $\mathcal{W}$ is in particular a vector field, and thus a section of $T\Sigma$. We can project $\mathcal{W}$ from $T\Sigma$ to $T\Sigma/\mathcal{D}$ by dropping the $\mathcal{D}_\theta$ term. In other words $\mathcal{W} \equiv \mathcal{V}(z,\theta)\partial_z \bmod D_\theta$. Since $\mathcal{V}(z,\theta)$ determines $\mathcal{W}$ we conclude that the map from super conformal vector fields to sections of $T\Sigma/\mathcal{D} \cong D^2$ is one-to-one. Hence the sheaf of superconformal vector fields is isomorphic to the sheaf $\mathcal{D}^2$.

3.2 The Berezinian

We want to use super Riemann surface to construct supersymmetric Lagrangian densities. Since we want to integrate this density to obtain an action, the Lagrangian density must be a section of the Berezinian bundle of a super Riemann surface.

Proposition 3.3. *For a super Riemann surface Σ, the Berezinian bundle $Ber(\Sigma)$ is naturally isomorphic to $\mathcal{D}^{-1}$. This means that $Ber(\Sigma) \otimes \mathcal{D}$ should be the trivial bundle and hence, $[dz|d\theta]D_\theta]$ should be invariant under coordinate change.*

Proof. We consider transformed coordinates $(\tilde{z}(z,\theta)|\tilde{\theta}(z,\theta))$. By definition the Berezinian transforms like

$$[d\tilde{z}|d\tilde{\theta}] = [dz|d\vartheta]Ber(M) \tag{21}$$

where

$$M = \begin{pmatrix} \partial_z \tilde{z} & \partial_z \tilde{\theta} \\ \partial_\theta \tilde{z} & \partial_\theta \tilde{\theta} \end{pmatrix} \tag{22}$$

One calculates that

$$Ber(M) = \frac{\partial_z \tilde{z} + \tilde{\theta}\partial_z \tilde{grt}}{D_\theta \tilde{\theta}} = D_\theta. \tag{23}$$

We get that

$$[d\tilde{z}|d\tilde{\theta}]D_{\tilde{\theta}} = [dz|d\vartheta]D_\theta\tilde{\theta}D_{\tilde{\theta}} = [dz|d\vartheta]D_\theta. \tag{24}$$

□

4 The moduli space of super Riemann surfaces

In this section we want to study the moduli space of super Riemann surfaces. The precise definition of a moduli space is quite technical but for our purpose it's enough to characterize the moduli space $\mathfrak{M}$ as the following functor of points: For any super manifold B we set

$$\mathfrak{M}(B) = \{\, \Sigma \to B \,|\, \Sigma \text{ super Riemann surface over } B \,\}$$

We have already seen that the Riemann surfaces over a bosonic base are just given by families of Spin curves. The underlying manifold is just given by reducing the moduli space $\mathfrak{M}$ to bosonic manifolds B_{red}. Then the the moduli space reduces to the moduli space of spin curves $\mathcal{M}_{spin}$. The moduli space of spin curves decomposes into two parts $\mathcal{M}_{spin,\pm}$. We call a spin structure $K^{1/2}$ even if the dimension of $H^0(\Sigma_{red}, K^{1/2})$ is even if it is odd.
If we allow odd parameters they are nilpotent, so they does not change the topology of the situation and the moduli space $\mathfrak{M}$ has two connected components $\mathfrak{M}_\pm$ with reduced space $\mathcal{M}_{spin,\pm}$.
We want to investigate the odd degrees of freedom of the moduli space $\mathfrak{M}$. Recall that Σ is build out of small open sets U_α that are glued together on intersections $U_\alpha \cap U_\beta$. So a first-order deformation of the gluing data is given by an infinitesimal superconformal coordinate transformation $\phi_{\alpha\beta}$ defined on each intersection $U_\alpha \cap U_\beta$. The idea is that before gluing U_β to U_α, we transform by $1 + w\phi_{\alpha\beta}$, with an infinitesimal parameter w. The $\phi_{\alpha\beta}$ must obey the cocycle condition

$$\phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\gamma\alpha} = 0$$

and transformations coming from superconformal vector fields $\phi_\alpha - \phi_\beta$ restricted to the intersection should be considered as trivial since it comes from a global automorphism. All in all the first order deformations of a super Riemann surface are determined by an element of $H^1(\Sigma, \mathcal{S})$ where $\mathcal{S}$ is

the space of superconformal vector fields. What we just explained is that $T\mathfrak{M}|_\Sigma = H^1(\Sigma, \mathcal{S})$. For a super Riemann surface Σ, only the sheaf cohomology groups $H^k(\Sigma, \mathcal{S})$ for $k = 0, 1$ are non-zero.
For the time being, we consider Riemann surfaces without punctures. In this case there are no inner automorphisms for $g \geq 2$, i.e. $H^0(\Sigma, \mathcal{S}) = 0$. We claim without proof that the dimension of $T\mathfrak{M}|_\Sigma$ does not depend on the odd moduli. With this we can choose Σ to be a split super Riemann surface. In this case, we have a decomposition $\mathcal{S} = S_+ \otimes S_-$ where S_+ consist of vector fields V_g and S_- of vector fields ν_f with f odd. In the split, there is also a natural decomposition $T\mathfrak{M}|_\Sigma = T_+\mathfrak{M}|_\Sigma \oplus T_-\mathfrak{M}|_\Sigma$, and $T_\pm\mathfrak{M}|_\Sigma = H^1(\Sigma, S_\pm)$. V_g is determined by $g(z)\partial_z$ which is a section of the tangent bundle $T\Sigma_{red}$, so S_+ is the sheaf of sections of $T\Sigma_{red}$ and

$$T_+\mathfrak{M}|_\Sigma = H^1(\Sigma_{red}, T\Sigma_{red}). \tag{25}$$

Recall the Riemann-Roch theorem: For any line bundle $\mathcal{L}$ of degree n, this theorems asserts that

$$\dim\, H^0(\Sigma_{red}, \mathcal{L}) - \dim\, H^1(\Sigma_{red}, \mathcal{L}) = 1 - g + n. \tag{26}$$

If $n < 0$ then $H^0(\Sigma_{red}, \mathcal{L}) = 0$. For the tangent bundle $\mathcal{L} = T\Sigma_{red} = K^{-1}$ we have that $n = 2 - 2g$ which is negative for $g \geq 2$ and so we get $\dim\, H^1(\Sigma_{red}, T\Sigma_{red)} = 3g - 3$.
To ν_f we associate the object $f(z)\partial_\theta$ which we view as an odd vector field along the fibers of $\Sigma \to \Sigma_{red}$, or in other words as a section of $K^{-1/2} = T\Sigma_{red}^{1/2}$. So $\mathcal{S}_-$ is the sheaf of sections of $K^{-1/2}$ and

$$T_-\Sigma_{red}^{1/2} = \Pi H^1(\Sigma_{red}, T\Sigma_{red}^{1/2}). \tag{27}$$

The degree of $deg(K^{-1/2}) = -deg(K)/2 = 1 - g$, hence for $g \geq 2$ we have $H^0(\Sigma_{red}, K^{-1/2}) = 0$ and the Riemann-Roch formula gives

$$\dim\, H^1(\Sigma_{red}, T\Sigma_{red}^{1/2}) = 2g - 2. \tag{28}$$

Thus for $g \geq 2$, the dimension of the moduli space of super Riemann surfaces of genus g with no punctures is

$$\dim\, \mathfrak{M}_g = 3g - 3|2g - 2 \tag{29}$$

Remark 4.0.1. The moduli space $\mathcal{M}_g$ of ordinary Riemann surfaces Σ_0 of genus g is not a manifold but an orbifold, since Σ_0 may have automorphisms. Similarly a super Riemann surface can have automorphisms, indeed every split Riemann surface admits an automorphism $\tau : z_\alpha|\theta_\alpha \to z_\alpha| - \theta_\alpha$. Since every super Riemann surface is a infinitesimal deformation of a split Riemann surface the locus of enhanced symmetry is dense in $\mathcal{M}_g$. So it's more correct to refer to $\mathfrak{M}_g$ as the moduli "stack" of super Riemann surfaces rather than the moduli "space".

4.1 Non-Projectedness

In [4] Donagi and Witten showed that (for $g \geq 5$) the moduli space of super Riemann surfaces is non-projected i.e. there is no projection $\mathfrak{M}_g \to \mathcal{M}_g$. The proof constructs a non-projected smooth compact curve embedded in $\mathfrak{M}_g$ and further deduces that the some obstruction class $w_k(\mathfrak{M}_g)$ does not vanish and hence $\mathfrak{M}_g$ cannot be projected.
The construction of the compact curve inside $\mathfrak{M}_g$ uses some advance techniques, so we won't discuss it here, we rather look at a closely related example, namely the moduli space of super Riemann surfaces with a marked point $\mathfrak{M}_{g,1}$. There is a canonical projection $\mathfrak{M}_{g,1} \to \mathfrak{M}_g$ which intuitively just forget about the marked point. This moduli space can be understood as the universal curve over $\mathfrak{M}_g$. What does this mean? We would like to think of the moduli space as an actual geometric space rather than the functor of points we used to define it and we would like to think of an element $b \in \mathfrak{M}_g(B)$ and morphism of space $b : B \to \mathfrak{M}_g$ as we know it from algebraic geometry. Given a B-point b, if we want to make sense of b as a morphism $B \to \mathfrak{M}_g$, we should be able to form pullbacks, in particular we can form a Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{M}_{g,1} \arrow[d] \\ B \arrow[r, "b"] & \mathfrak{M}_g \end{tikzcd}$$

Since $\mathfrak{M}_{g,1}$ is a super Riemann surface over $\mathfrak{M}_g$, the pullback $X \to B$ is a super Riemann surface over B. We now remember that an element in $\mathfrak{M}_g(B)$ really corresponds to a super Riemann surface over B and the super Riemann surface corresponding to b is $X \to B$.

We want to construct a non-projected super Riemann surface. We already know, that all super Riemann surface over a bosonic base are split so the simplest case where we can hope to find a non-split example is for $B = \mathbb{C}^{0|1}$. We start with a point $f_\eta \in \mathfrak{M}_g(C^{0|1}$. Remember that for any supermanifold M we have that $Hom_{sMfd}(\mathbb{C}^{0|1}, M) \cong \{(p, v) \,|\, p \in M_{red},\, v \in T_M\{$. In our case a point is given by $(\Sigma, \mathcal{D})$ a super Riemann surface of genus g (or equivalently a spin curve) and a tangent vector $\eta \in H^1(\Sigma_{red}, T_C^{1/2})$. We define $X_\eta := f_\eta^*(\mathfrak{M}_{g,1}) \to \mathbb{C}^{0|1}$ as the pullback of $\mathfrak{M}_{g,1} \to \mathfrak{M}_g$ along f_η^*. X_η has total dimension $(1|2)$, so the obstruction to projectedness is given by $w_2(X_\eta)$.

Proposition 4.1. *X_η is projected if and only if $\eta = 0$, in which case it is actually split.*

Proof. First note that for $\eta = 0$ the map f_η is constant and hence $X_\eta = S \times \mathbb{C}^{0|1}$ so clearly X_η is split.
In the general case, we will use the following result

Lemma 4.2. *A supermanifold S of dimension $(m|2)$ is determined by the triple (M, V, w), where $w = w_2 \in H^1(M, Hom(\wedge^2 T_-, T_+))$, and any such triple arises from some S. A supermanifold of dimension $(m|2)$ is projected if and only if it is split if and only if $w \neq 0$.*

We want to determine

$$w_2(X_\eta) \in H^1((X_\eta)_{red}, Hom(\wedge^2 T_- X_\eta, T_+ X_\eta)) \tag{30}$$

we first need to understand the sheaf $Hom(\wedge^2 T_- X_\eta, T_+ X_\eta)$ better. We first introduce $C := (X_\eta)_{red}$ to keep the notation compact and than identify

$$\begin{gathered} T_+ X_\eta = TC \\ \wedge^2 T_- X_\eta = T_- S \otimes T_- C^{0|1} = T_C^{1/2} \otimes \mathcal{O} = T_C^{1/2} \\ Hom(\wedge^2 T_- X_\eta, T_+ X_\eta) = (T_C^{1/2})^* \otimes TC = T_C^{1/2}. \end{gathered}$$

Hence $w \in H^1(C, T_C^{1/2})$. The claim immediately follows if we can show that $w(X_\eta) = \eta$. The 1-cocycle on $S \times \mathbb{C}^{0|1}$ defining X_η is given by $w_{\alpha\beta}\eta\theta\partial_z$. We can view this as a first-order deformation of $S \times \mathbb{C}^{0|1}$, which deforms $S \times \mathbb{C}^{0|1}$ away from being split.
On the other hand odd deformations are given by vector field ν_f for some odd function f. We want the the deformation to be η dependent, so we choose f and $-u\eta$ for an even function u. We get that

$$\nu_f = -u(z)\eta(\partial_\theta - \theta\partial_z) \tag{31}$$

We forget the the first term since this deformation goes along the S and hence does not affect splitness of $S \times \mathbb{C}^{0|1}$, but the $u(z)\eta\theta\partial_z$ does affect the splitness. If we set $w = u$, this term coincides with the cocycle that characterizes X_η. If the function u is non-zero, we can assume it to be one and we see that the section of $T_C^{1/2}$ is determined by η. □

By construction we get a map $X_\eta \to \mathfrak{M}_{g,1}$ which (for $\eta \neq 0$ is an embedding of supermanifolds since this property is stable under pullbacks. We want to use this fact to proof

Proposition 4.3. *The first obstruction to the splitting of $\mathfrak{M}_{g,1}$:*

$$w := w_2 \in H^1(\mathcal{M}_{spin,g,1}, Hom(\wedge^2 T_- X_\eta, T_+ X_\eta)) \tag{32}$$

does not vanish for $g \geq 2$ (and even spin structure), so the supermanifold $\mathfrak{M}_{g,1}$ is non-projected.

Proof. Fix a spin curve $(C, T_C^{1/2}) \in \mathcal{M}^+_{spin,g}$ and an odd tangent vector $\eta \in H^1(C, T_C^{1/2})$. We have already seen that X_η which is constructed from this data is non-projected. To proof that $\mathfrak{M}$ is non-projected, we use the following result:

Lemma 4.4. *Let M be a supermanifold with submanifold M'. Consider the following diagram of sheaves:*

$$\begin{array}{ccc} & & Hom(\wedge^2 T_- M, T_+ M) \\ & & \downarrow{\scriptstyle \iota} \\ Hom(\wedge^2 T_- M', T_+ M') & \xrightarrow{j} & Hom(\wedge^2 T_- M, T_+ M|_{M'}) \end{array}$$

This induces the following diagram of cohomology groups:

$$\begin{array}{ccc} & & H^1(M_{red}, Hom(\wedge^2 T_- M, T_+ M)) \\ & & \downarrow{\scriptstyle \iota} \\ H^1(M' red, Hom(\wedge^2 T_- M', T_+ M')) & \xrightarrow{j} & H^1(M'_{red}, Hom(\wedge^2 T_- M, T_+ M|_{M'})). \end{array}$$

Then $j(w_2(M')) = \iota(w_2(M))$.

We want to use this lemma with $M = \mathfrak{M}_{g,1}$, $M' = X_\eta$, $M_{red} = \mathcal{M}_{spin,g,1}$, $M'_{red} = C$. We note that $T_- X_\eta$ is a rank 2 vector bundle on C, and we have seen that $\wedge^2 T_- X_\eta \cong T_C^{1/2}$. We know from the proof of 4.1 that $w(X_\eta) = \eta \neq 0$, so in order to show that $w \neq 0$, it suffices to show that

$$j : H^1(C, T_C^{1/2}, T_C)) \to H^1(C, Hom(T_C^{1/2}, T_+\mathfrak{M}_{g,1})) \tag{33}$$

is injective. We start with the sequence of sheaves on C:

$$0 \to T_C \to i^* T_+ \mathfrak{M}_{g,1} \to i^* \pi^* T_+ \mathfrak{M}_g \to 0 \tag{34}$$

We note that $i^*\pi^* T_+\mathfrak{M}_g \cong W \otimes \mathcal{O}_C$ with $W = T_{+,C}\mathfrak{M}_g$. We apply $Hom(T^{1/2}, \cdot)$ to this sequence; the cohomology sequence of the resulting exact sequence reads in part

$$W \otimes H^0(C, K_C^{1/2}) \quad \longrightarrow \quad H^1(C, Hom(T_C^{1/2}, T_C)) \quad \xrightarrow{j} \quad H^1(C, Hom(T_C^{1/2}, T_+\mathfrak{M}_{g,1}))$$

For a generic choice of even spin structures on a Riemann surface of genus $g \geq 2$ we have $H^0(C, K^{1/2}) = 0$, so j is injective. $\square$

5 Punctures

To calculate the transition amplitudes in superstring we need to insert vertex operators on our world sheet to create the in- and out-states. In bosonic string theory it's quite clear, how to do that. The vertex operator are inserted at a point of the world sheet. A point is just a Weil divisor of the Riemann surface. In superstring theory we also will insert vertex operators along Divisors. A divisor on a super Riemann surface has codimension $1|0$, so D needs to have dimension $0|1$. The analog of a Weil divisor on a Riemann surface for super Riemann surface is called a Neveu-Schwarz (NS) puncture. Fix a point $(z_0|\theta_0) \in Hom_B(B \times C^{0|1}, \Sigma)$, i.e. we choose a section of $\Sigma \to B$ and a tangent vector in each point of the section. The parameters $z_0|\theta_0$ are the moduli of the NS puncture. So adding an NS puncture increases the dimension of supermoduli space by $1|1$, and the moduli space of super Riemann surfaces of genus g with $\mathfrak{n}_{NS}$ NS punctures has dimension $3g-3+\mathfrak{n}_{NS}|\mathfrak{n}_{NS}$.
In a Riemann surface this point determines a divisor through that point. This divisor is the orbit through $(z_0|\theta_0)$ generated by the odd vector field

D_θ. This vector field generates the coordinate transformation $\theta \to \theta + \alpha$, $z \to z + \alpha\theta$, so the orbit is given by

$$
\begin{aligned}
z &= z_0 + \alpha\theta \\
\theta &= \theta_0 + \alpha
\end{aligned}
$$

Note that when we consider a Riemann surface over the point $\mathbb{R}^{0|0}$ the divisor take a very simple form. The point z is a constant point in Σ_{red}, since the odd parameter α must be proportional to θ and the divisor is just the odd line over z. Over a general base, we get the equation

$$z = z_0 - \theta_0\theta \tag{35}$$

So the divisor is cut out by one even equation.
A Ramond puncture is a more subtle concept. The technical definition is the following

Definition 5.1. A family of super Riemann surfaces with $\mathfrak{n}_R$ Ramond puncture is a family of supermanifolds $\Sigma \to B$ of relative dimension $1|1$ with the additional structure of a rank $0|1$ locally free subsheaf $\mathcal{D} \subset \mathcal{T}_{\Sigma/B}$, an irreducible relative effective Cartier divisor $\mathcal{F}$ of degree $n = \mathfrak{n}_R$ on Σ/B, and an isomorphism $\frac{1}{2}[\cdot,\cdot] : \mathcal{D}^2 \to (\mathcal{T}_{\Sigma/B}/\mathcal{D})(-\mathcal{F})$ of locally free sheaves on Σ.

For now we are interested in a single divisor D. Locally D should be generated by a non-zero sections of TM so locally

$$D^*_\theta = a(z)\partial_\theta + b(z)\theta\partial_z \tag{36}$$

Since v has to be non-zero everywhere, we can again assume that $a(z) = 1$. The simplest divisors of this form are given by $b(z) = z^k$ for $k \geq 0$, i.e.

$$D^*_\theta = \partial_\theta + z^k\theta\partial_z \tag{37}$$

We see that $v_k^2 = z^k\partial_z$ which vanishes at the divisor $z = 0$ to order k. The basic case $k = 1$ is called a Ramond puncture. The divisor $\mathcal{F}$ is the divisor in which the superconformal structure degenerates. In the example above this would be the divisor $z = 0$.
We can generalize this by setting $b(z) = \Pi(z - z_i)$. So $D^*_\theta = b(z)\partial_z$, and the super conformal structure degenerates precisely at the divisor $\mathcal{F}_i$ given by $z = z_i$. So $\mathcal{D}^2 \cong T\Sigma/\mathcal{D} \otimes \mathcal{O}(-\mathcal{F})$.

5.1 Relation between punctures and string states

Let consider the propagation of a closed string. That gives us a cylinder. This cylinder can be mapped to the plain by the conformal transformation $z = e^\rho$. In the first description the string propagates around a puncture, in the second description we think of $z = 0$ as a marked point at which an operator is inserted.
On a super Riemann surface we map the supertube to $\mathbb{C}^{1|1}$ by $z = e^\rho$ and $\theta = e^{\rho/2}\zeta$. We have that $D_\theta = e^{-\rho/2}(\partial_\eta + \zeta\partial_\rho)$. They are subject to the equivalence relation

$$\rho \cong \rho + 2\pi\sqrt{-1},\ \zeta \to -\zeta \tag{38}$$

This is precisely the behavior we'd expect from fermions in the NS sector. On the other hand consider the divisor $D_\theta^* = \partial_\theta + z\theta\partial_z$ at $z = 0$ in this case we have $z = e^\rho$ and $\theta = \zeta$. We have that $D_\theta^* = \zeta + \theta\partial_\rho$ and hence $\rho|\zeta$ are superconformal coordinates. This time we have

$$\rho \cong \rho + 2\pi\sqrt{-1},\ \zeta \to \zeta \tag{39}$$

and hence the strings propagation in the Re ρ direction will be in the Ramond sector. What we have encountered here are the two possible spin structures on the purely bosonic cylinder.

6 Perturbative Superstring Theory

So far, we have considered a super Riemann surface purely as a complex supermanifold of dimension $1|1$ with some additional structure. In string theory we usually consider both holomorphic and antiholomorphic degrees of freedom. The solution of this problem is to define a string worldsheet Σ to be a smooth supermanifold that is embedded in a product $\Sigma_L \times \Sigma_R$ of holomorphic Riemann surfaces or super Riemann surfaces.
For heterotic string theory Σ_R is a super Riemann surface and Σ_L is an ordinary Riemann surface. For Type II superstrings, both Σ_R and Σ_L are super Riemann surfaces.
The odd dimension of Σ is the same as that of $\Sigma_L \times \Sigma_R$, and its even dimension is 2. The basic example is that the reduced spaces Σ_L and Σ_R are complex conjugates and Σ_{red} is the diagonal in $\Sigma_{L,red} \times \Sigma_{R,red}$. Then Σ is obtained by a slight thickening in the fermionic direction. Since the $\Sigma_L \times \Sigma_R$ is a product, its Berezinian is $Ber(\Sigma_L \times \Sigma_R) \otimes Ber(\Sigma_L) \otimes Ber(\Sigma_R)$. And $Ber(\Sigma_L \times \Sigma_R)|_\Sigma \cong$

$Ber(\Sigma)$. We recall that for a super Riemann surface X, the Berezinian $Ber(X) \cong \mathcal{D}^{-1}$. So a section is locally given by $[dz||d\theta]D_\theta\phi$. So a section of Σ in heterotic string theory is given by

$$[d\tilde{z}; dz|d\theta]\partial_{\tilde{z}}\phi D_\theta\phi \tag{40}$$

To make the for of the Lagrangian as readable as possible we set

$$\mathcal{D}(\tilde{z}, z|\theta) = -i[d\tilde{z}; dz|d\theta] \tag{41}$$

6.1 Lagrangians

We merely want to give a (very) brief overview of the constructions of Langrangians and action functionals from the supergeometry point of view. A thorough discussion of the constructions of Lagrangians can be found in [7]. For the necessary background in perturbative superstring theory and a discussion of the RNS-formalism we refer to [2]. To formulate the heterotic string on $\mathbb{R}^{10}$, four contributions to the Lagrangian are important.
The first contribution is the well known RNS-Lagrangian:

$$I_X = \frac{1}{2\pi\alpha'}\int \mathcal{D}(\tilde{z}, z|\theta)\sum_{IJ}\eta_{IJ}\partial_{\tilde{z}}X^I D_\theta X^J, \tag{42}$$

In Type II superstring theory the action we start with is given by

$$I_X = \frac{1}{2\pi\alpha'}\int \mathcal{D}(\tilde{z}, z|\theta, \tilde{\theta})\sum_{IJ}\eta_{IJ}D_{\tilde{\theta}}X^I D_\theta X^J. \tag{43}$$

The holomorphic ghosts are a section C of $\Pi S = \Pi\mathcal{D}^2$ where $\mathcal{S}$ is the sheaf of super conformal vector fields on Σ_R, thus $D_{\tilde{\theta}}C$ makes sense as a section of $Ber(\Sigma_L)$. The holomorphic antighosts are a section B of $\Pi\mathcal{D}^{-3}$. Hence $BD_{\tilde{\theta}}C$ is a section of $Ber(\Sigma_L)\otimes\mathcal{D}^{-1} \cong Ber(\Sigma_L \times \Sigma_R)$ so it can be integrated over Σ:

$$I_{B,C} = \frac{1}{2\pi}\int \mathcal{D}(\tilde{z}, z|\theta, \tilde{\theta})B\mathcal{D}_{\tilde{\theta}}C \tag{44}$$

The antiholomorphic ghosts are sections $\tilde{C}$ of $\Pi\tilde{\mathcal{D}}^{-2}$,and the corresponding antighosts are a section $\tilde{B}$ of $\Pi\tilde{\mathcal{D}}^{-3}$. And

$$I_{\tilde{B},\tilde{C}} = \frac{1}{2\pi}\int \mathcal{D}(\tilde{z}, z|\theta, \tilde{\theta})\tilde{B}\mathcal{D}_\theta\tilde{C}. \tag{45}$$

For heterotic superstring theory one replaces $D_{\tilde{\theta}}$ by $\partial_{\tilde{z}}$, and leave the form of the Lagrangian unchanged.
Let's compare I_X to the classical $\mathcal{N}=1$ supergravity action in 2 dimensions

$$S[x,\psi;g,\chi]=\frac{1}{4\pi}\int d\mu_g\big[\frac{1}{2}(\partial_m x^\mu\partial_n x_\mu+\psi^\mu\gamma^a e_a^m\partial_m\psi_\mu \\ -\psi^\mu\gamma^a\gamma^b\chi_a e_b^m\partial_m x^\mu-\frac{1}{4}\psi^\mu\gamma^a\gamma^b\chi_a(\chi_b\psi_\mu)\big]$$

The first line is our familiar RNS-action written in local coordinates. The second line parametrizes deformations of the pair (g_{mn},χ_m^σ) which is a super analog of a Riemannian metric and we denote the space of all such pairs on a curve C of genus g by $s\mathrm{Met}(C)$. The moduli space of super Riemann surfaces then arises as the quotient

$$\mathfrak{M}_g=s\mathrm{Met}(C)/(sDiff(C)\ltimes sWeyl(C)\times\mathrm{Lorentz}(C)). \tag{46}$$

Since this action is not free the quotient won't be a supermanifold but rather an orbifold.

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