Non-Projected and Non-Split Supermanifolds

Jonas Cassel

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1 Supermanifolds, fermionic sheaf and tangent sheaf

We want to recall the definitions for supermanifolds and introduce some new notation.

Definition 1.1. A superspace $M = (|M|, \mathcal{O}_M)$ is a locally ringed space with a sheaf of supercommutative rings $\mathcal{O}_M$. We write $\mathcal{O}_M = \mathcal{O}_0 \oplus \mathcal{O}_1$ for the even and the odd parts of the sheaf. To a superspace, we associate a sheaf of ideals J that is generated by the nilpotent sections of $\mathcal{O}_M$.

From a superspace we can recover a locally ringed space by setting

$$M_{red} := (|M|, \mathcal{O}_M/J) = (|M|, \mathcal{O}_{red}).$$

This space is called the reduced space and its structure sheaf is a sheaf of *commutative* rings. Caution has to be exercised in this construction as a sheaf modulo an ideal is defined as

$$\mathcal{O}_M/J := (U \mapsto \mathcal{O}_M(U)/J(U))^{sh}$$

but we have isomorphisms on the level of stalks

$$(\mathcal{O}_M/J)_x \cong \mathcal{O}_{M,x}/J_x \qquad \forall x \in M.$$

Consequently, we can use theorems from standard ring theory (such as the isomorphism theorems) with impunity.

Definition 1.2. A supermanifold of dimension $n|m$ is a locally decomposable superspace with underlying manifold. This means

- M_{red} is a n-manifold (real smooth or complex holomorphic)
- There is a covering $\{U_i\}$ of $|M|$ and a locally free sheaf $\mathcal{E}$ of rank m on M_{red} such that

$$\mathcal{O}_{U_i} \cong \bigwedge^{\bullet} \mathcal{E}^{\vee}|_{U_i}$$

  as sheaves of graded rings (grading in the exterior algebra as the $\mathbb{Z}$-grading modulo 2).

We can think of the basis vectors of $\mathcal{E}^{\vee}$ as the fermionic coordinates which are additional degrees of freedom on the supermanifold. We take the exterior power of $\mathcal{E}^{\vee}$ since we want these coordinates to be anticommuting. We can recover the local model $\mathcal{E}$ from a given superspace M using the next construction. Defining a supermanifold this way, leaves room for generalizations. For instance, we can define a $\mathbb{C}$-supermanifold by replacing the word manifold by $\mathbb{C}$-variety in the above definition.

Example.
1. The supermanifold $\mathbb{R}^{n|m}$ is the supermanifold with underlying reduced manifold $\mathbb{R}^n$ and locally free sheaf $\mathcal{O}_{\mathbb{R}^n}^{\oplus m}$ which is the trivial rank m sheaf on $\mathbb{R}^n$.
2. The $\mathbb{C}$-supermanifold $\mathbb{P}_{\mathbb{C}}^{n|m}$ is the complex supermanifold with reduced manifold $\mathbb{P}_{\mathbb{C}}^{n}$ modeled by the locally free sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus m}$.

Definition 1.3. Let M be a supermanifold of dimension $n|m$. We define then its fermionic sheaf as

$$\mathcal{F}_M := J/J^2.$$

The fermionic sheaf is a locally free sheaf of $\mathcal{O}_{red}$-modules of rank $0|m$. It is the parity reversed version of the sheaf $\mathcal{E}^{\vee}$

$$\Pi\mathcal{F}_M \cong \mathcal{E}^{\vee}.$$

Remark 1.4. In the following, we encounter symmetric powers of $\mathcal{F}_M$. We use here the conventions of [Man13]. We thus have that the product on $S^{\bullet}\mathcal{F}_M$ is not commutative but it remains supercommutative (i.e. graded commutative). As we have a supercommutative product of $\mathcal{F}_M$ with itself defined, we can identify

$$S^i\mathcal{F}_M \cong J^i/J^{i+1}$$

as $\mathcal{O}_{red}$-modules. By this construction, we can identify the symmetric algebra of $\mathcal{F}_M$ with the exterior algebra of $\mathcal{E}^{\vee}$ as $\mathcal{O}_{red}$-superalgebras

$$S^{\bullet}\mathcal{F}_M \cong \bigwedge^{\bullet} \mathcal{E}^{\vee}.$$

When we label the fermionic coordinates of M by some θ_i, we can think of J as all products and sums of these coordinates and $\mathcal{F}_M$ as the linear span of the θ_i. Therefore, the $\mathcal{F}_M$ and $\mathcal{E}^{\vee}$ can be identified, as $\mathcal{E}^{\vee}$ provides precisely these coordinates as its basis vectors.

Similar as for regular manifolds, we can also associate a tangent sheaf to a supermanifold. This sheaf is defined in terms of algebra derivations.

Definition 1.5. Let A be a superalgebra over a field k. A superderivation on A is then a homogeneous, k-linear map $D : A \to A$ of parity $|D|$, that satisfies

$$D(ab) = D(a)b + (-1)^{|D||a|}aD(b).$$

for any homogeneous element $a \in A$ and $b \in A$. We can also formulate a sheaf version of derivations. Let M be a supermanifold. A $\mathcal{O}_M$-superderivation is a sheaf morphism $\mathcal{D} : \mathcal{O}_M \to \mathcal{O}_M$ such that $\mathcal{D}(U)$ is a superderivation on $\mathcal{O}_M(U)$ for $U \subseteq |M|$ open.

Definition 1.6. The tangent sheaf $\mathcal{T}_M$ of a supermanifold M is defined as

$$\mathcal{T}_M(U) := \{\text{Superderivations on } \mathcal{O}_U\}.$$

$\mathcal{T}_M$ is a locally free sheaf of rank $n|m$ where $n|m$ is the dimension of M.

All these definitions remain valid for a regular manifold, by deleting the prefix "super" every time and setting $|D| = 0$. We will use the tangent sheaf $\mathcal{T}_{red}$ later on which is the tangent sheaf of the reduced manifold M_{red}. This sheaf represents the derivations of $\mathcal{O}_{red}$.

We can also define superderivations with different target space than $\mathcal{O}_M$ itself. For any $\mathcal{O}_M$-module B we can define (super)derivations $\mathcal{O}_M \to B$. These B-valued derivations are represented by the sheaf $\mathcal{T}_M \otimes_{\mathcal{O}_M} B$.

2 Splitting and Projecting Supermanifolds

We want to analyze the structure of supermanifolds. We define what it means to be split and projected for supermanifold and we define obstructions to splitting.

2.1 Projectedness

We have a natural closed embedding $i : M_{\mathrm{red}} \to M$ which is the identity on the $|M|$ with the sheaf morphism

$$i^\sharp : \mathcal{O}_M \to i_*\mathcal{O}_{red} = \mathcal{O}_M/J_M$$

induced by the quotient projection. The kernel of this projection is given by J_M. By surjectivity of $i^\sharp$ we have an exact sequence

$$0 \to J_M \to \mathcal{O}_M \xrightarrow{i^\sharp} \mathcal{O}_M/J_M \to 0$$

of $\mathcal{O}_M$-modules. We call this exact sequence the structural sequence of the supermanifold.

A short exact sequence like the one above is said to be split, if there exists a morphism $\pi^\sharp : \mathcal{O}_{red} \to \mathcal{O}_M$ such that

$$i^\sharp \circ \pi^\sharp = \mathrm{id}_{\mathcal{O}_{red}}.$$

The morphism of sheaves $\pi^\sharp$ induces a morphism $\pi = (\mathrm{id}, \pi^\sharp) : M \to M_{red}$ of manifolds that also fulfills $i \circ \pi = \mathrm{id}$. If such a morphism exists, we get an isomorphism

$$\mathcal{O}_M \cong \mathcal{O}_M/J_M \oplus J_M$$

which is an isomorphism of $\mathcal{O}_M$-modules (and not as sheaves of superalgebras). The existence of such an isomorphism is equivalent to the splitting of the structural sequence. This idea leads to the following definition

Definition 2.1. The supermanifold M is called *projected*, if its structural sequence splits.

We want to establish criteria for a supermanifold to not be projected. Before we do this, we look at split supermanifolds.

2.2 Split supermanifolds

For a given supermanifold M of odd dimension m, there is a J_M-adic filtration of $\mathcal{O}_M$ by

$$\mathcal{O}_M =: J_M^0 \supseteq J_M \supseteq J_M^2 \supseteq ... \supseteq J_M^m \supseteq J_M^{m+1} = 0$$

Given this sequence, we can define a sheaf of $\mathbb{Z}_2$ graded rings by

$$\operatorname{Gr} \mathcal{O}_M := \mathcal{O}_{red} \oplus J_M/J_M^2 \oplus ... \oplus J_M^{m-1}/J_M^m \oplus J_M^m$$

The $\mathbb{Z}_2$ grading results from reducing the $\mathbb{Z}$ grading modulo 2. This sheaf allows us to define a new supermanifold $\operatorname{Gr} M := (|M|, \operatorname{Gr} \mathcal{O}_M)$. We call this supermanifold, the split supermanifold associated to M. This leads to the definition

Definition 2.2. A supermanifold M is said to be split, if there is an isomorphism $\mathcal{O}_M \cong \operatorname{Gr} \mathcal{O}_M$.

The second summand $\mathcal{F}_M = J_M/J_M^2$ is the fermionic sheaf of M. This corresponds thus to $\mathcal{E}$ from the local model of the supermanifold M (from Def.1.1). The higher summands are the symmetric powers of $\mathcal{F}_M$

$$S^i\mathcal{F} \cong J^i/J^{i+1}$$

This shows that a split supermanifold is completely defined by its reduced manifold M_{red} and the fermionic sheaf $\mathcal{F}_M$. The symmetric powers of $\mathcal{F}_M$ can be identified with the exterior powers of $\mathcal{E}^\vee$. Thus, for a split supermanifold, we have an isomorphism

$$\mathcal{O}_M \cong \bigwedge^\bullet \mathcal{E}^\vee$$

In the definition of a supermanifold, we only demanded the existence of such an isomorphism locally.

We note that we have an isomorphism of sheaves of abelian groupsin the case that the supermanifold is split

$$J_M \cong \bigoplus_{i=1}^m J_M^i/J_M^{i+1}.$$

Thus, split implies projected. The converse is not true; the isomorphism

$$\mathcal{O}_M \cong \mathcal{O}_{red} \oplus J_M$$

need not respect the grading for a projected supermanifold. Being split is stronger, as we have an isomorphism of sheaves of graded rings in this case.

2.3 Obstruction theory

We want to discuss how it can be detected whether a supermanifold is projected or not. In the most general case, this question is very much open, but if the odd dimension is low enough, we can get a good grip on the subject. The theory presented here is taken from [Man13]. For the rest of this subsection, let M be a supermanifold of dimension $n|m$.

In order to project M we need a projection $\pi : M \to M_{red}$ that fulfills $\pi \circ i = \mathrm{id}$. We construct such a map recursively starting from $\mathrm{id} : M_{red} \to M_{red}$ via extension of the domain. In the i-th recursion step, we construct a projection map $\mathcal{O}_M/J \to \mathcal{O}_M/J_M^i$ and we give a criterion whether we can perform the next recursion step.

The recursion is initiated at $i = 1$ by the identity map $\mathrm{id} : O_M/J_M \to O_M/J_M$. We give now an algorithm for the recursion step.

- Assume we have a projection $\pi^i : \mathcal{O}_M/J_M \to \mathcal{O}_M/J^i_M$. With the embedding $i^\sharp_i : \mathcal{O}_M/J^i_M \to \mathcal{O}_M/J_M$, π^i satisfies
$$i^\sharp_i \circ \pi^i = \mathrm{id}.$$
In the next recursion step we want to construct a projection $\pi^{i+1} : \mathcal{O}_M/J_M \to \mathcal{O}_M/J^{i+1}_M$.

- We know that M is locally split by definition. Choose a covering $\mathcal{U}$ of $|M|$ such that $M|_U$ is split for $U \in \mathcal{U}$
$$\mathcal{O}_U \cong \mathcal{O}_U/J_U \oplus ... \oplus J^{m-1}_U/J^m_U \oplus J^m_U.$$

- Define then the maps
$$\pi^{i+1}_U := \mathcal{O}_U/J_U \xrightarrow{\pi^i|_U} \mathcal{O}_U/J^i_U \xrightarrow{\iota_1} \mathcal{O}_U/J^i_U \oplus J^i_U/J^{i+1}_U \cong \mathcal{O}_U/J^{i+1}_U$$
where $\iota_1 : \mathcal{O}_U/J^i_U \to \mathcal{O}_U/J^i_U \oplus J^i_U/J^{i+1}_U$ is the direct product inclusion of the first factor. These morphisms are extensions of $\pi^i|_U$ as we have
$$p_1 \circ \pi^{i+1}_U = \pi^i|_U$$
for $p_1 : \mathcal{O}_U/J^i_U \oplus J^i_U/J^{i+1}_U \to \mathcal{O}_U/J^i_U$ the first factor projection.

- We get
$$i^\sharp_{i+1}|_U \circ \pi^{i+1}_U = i^\sharp_i|_U \circ \pi^i|_U = \mathrm{id}$$
because the image of π^{i+1}_U is contained in $\mathcal{O}_U/J^i_U$ and there the action is given by $\pi^i|_U$. We conclude that π^{i+1}_U is a projection.

- The π^{i+1}_U are only defined on the subsheaves $\mathcal{O}_U/J_U$. We can glue the maps to a global morphism if they agree on intersections. Take then $U, V \in \mathcal{U}$ and set $W := U \cap V$ and define
$$\omega^{i+1}_{UV} := \pi^{i+1}_U|_W - \pi^{i+1}_V|_W : \mathcal{O}_W/J_W \to \mathcal{O}_W/J^{i+1}_W.$$
We can refine the target space of ω^{i+1}_{UV} to $\ker p_1|_W$ which means
$$\omega^{i+1}_{UV} : \mathcal{O}_W/J_W \to J^i_W/J^{i+1}_W = S^i\mathcal{F}_W.$$

- The set $\omega^{i+1}_M := \{\omega^i_{UV}\}$ is a well-defined cohomology class
$$\omega^{i+1}_M \in H^1(M_{red}, (\mathcal{T}_{M_{red}} \otimes S^i\mathcal{F}_M)_0).$$
We show this in Appendix A.

- **If the class ω^{i+1}_M vanishes**, the local projections π^{i+1}_U glue to a global one
$$\pi^{i+1} : \mathcal{O}_M/J_M \to \mathcal{O}_M/J^{i+1}_M \quad \text{with} \quad i^\sharp_{i+1} \circ \pi^{i+1} = \mathrm{id}.$$

After $m+1$ successful steps (assuming that the classes ω^i_M always vanish) we get a projection
$$\pi := \pi^{m+1} : \mathcal{O}_M/J_M \to \mathcal{O}_M/J^{m+1}_M = \mathcal{O}_M \quad \text{with} \quad i^\sharp \circ \pi = \mathrm{id}.$$

This gives us thus the following criterion on projectedness (see [DW15], p.14).

Lemma 2.3. *A supermanifold M of odd dimension m is projected if and only if all obstruction classes ω^i_M vanish in the cohomology groups $H^1(M_{red}, (\mathcal{T}_{M_{red}} \otimes S^i\mathcal{F}_M)_0)$ for all $i = 1, ..., m$.*

Remark 2.4.

- The classes ω_M^i for i odd vanish for $H^1((\mathcal{T}_{red} \otimes S^i\mathcal{F}_M)_0) = 0$ in this case. This is because for odd i the target space $S^i\mathcal{F}_M$ is a purely odd sheaf. Since $\mathcal{O}_{red}$ is purely even, there are no even derivations from $\mathcal{O}_{red}$ to $S^i\mathcal{F}_M$.
- The classes constructed here can also be interpreted in terms of non-abelian cohomology. In Sec. 4 we describe how this is done.

We can now apply this criterion in specific situations

2.4 Applications

Theory for smooth supermanifolds For smooth differentiable supermanifolds, we have the following result [Man13].

Lemma 2.5. *Any locally free sheaf on a differentiable supermanifold M is acyclic.*

This derives from the existence of a smooth partition of unity on smooth supermanifolds. This property is lost in the case of complex analytic supermanifolds, as the partition of unity cannot be chosen to be holomorphic. Being an acyclic sheaf means that the cohomology in degrees higher than 0 vanishes. The sheaf $\mathcal{T}_{M_{red}} \otimes \mathcal{S}^i\mathcal{F}_M$ is a locally free sheaf on M_{red} which means that we can conclude

$$H^1(\mathcal{T}_{M_{red}} \otimes S^i\mathcal{F}_M) = 0 \qquad \forall i.$$

This implies that all obstruction classes ω_M^i vanish. Therefore, we get that every smooth differentiable supermanifold is projected.

Remark 2.6. One has now projectedness of any smooth differentiable supermanifold. With a little extra work, one can now extend this result to also show that every such supermanifold is also split. This is Batchelor's theorem.

Theory in odd dimension 1 In the smooth case we could answer the question about being split for all odd dimensions. In the complex analytic case we need a more careful treatment. In odd dimension 1 nothing can go wrong, though.

Theorem 2.7. *Let M be a complex supermanifold of odd dimension 1. Then M is defined up to isomorphism by the pair $(M_{red}, \mathcal{F}_M)$.*

Proof. We prove that any such supermanifold is split. We have $\mathcal{O}_M = \mathcal{O}_0 \oplus \mathcal{O}_1$. Because the odd dimension is 1, we have $J_M^2 = 0$ and $\mathcal{F}_M = J_M/J_M^2 = J_M$ and $\mathcal{O}_1 \cong J_M$. This implies

$$\mathcal{O}_M/J_M = \frac{\mathcal{O}_0 \oplus \mathcal{O}_1}{\mathcal{O}_1} \cong \mathcal{O}_0$$

Thus, we get

$$\mathcal{O}_M = \mathcal{O}_0 \oplus \mathcal{O}_1 \cong \mathcal{O}_M/J_M \oplus J_M. \tag{2.1}$$

□

Theory in odd dimension 2 In odd dimension 2, there is also the possibility of not projecting, i.e. the case that ω_M^2 is non-zero. In this case, the supermanifold is *not* determined by the reduced manifold and fermionic sheaf. But we can ask the question how much more information we need to reconstruct the supermanifold. The answer is given by the following theorem.

Theorem 2.8. *Let M be a complex supermanifold of dimension $n|2$. Then M is defined up to isomorphism by the triple $(M_{red}, \mathcal{F}_M, \omega_M)$ where $\mathcal{F}_M$ is a locally free rank $0|2$ sheaf of $\mathcal{O}_{M_{red}}$-modules and $\omega_M \in H^1(T_{M_{red}} \otimes S^2\mathcal{F}_M)$.*

Proof. M is augmented in two ways compared to M_{red}. Firstly, the even part of the structure sheaf is the reduced structure sheaf $\mathcal{O}_{red}$ extended by even products of fermionic coordinates. The odd part of the $\mathcal{O}_M$ is given by $\mathcal{F}_M$ itself. Thus we set $\mathcal{O}_1 := \mathcal{F}_M$. We realize the bosonization as an extension of $\mathcal{O}_{M_{red}}$ by $S^2\mathcal{F}_M$. This means that we want to define $\mathcal{O}_0$ such that it fits in an exact sequence

$$0 \to S^2\mathcal{F}_M \to \mathcal{O}_0 \xrightarrow{i_0^\sharp} \mathcal{O}_{red} \to 0. \tag{$\star$}$$

Furthermore, we want this sequence to be locally split.

We start by using a cover $\{U_i\}$ such that we have ω_{ij} on the intersection $U_i \cap U_j$ as representatives of ω. Then define the sheaves

$$\mathcal{O}_{U_i,0} := \mathcal{O}_{M_{red}} \oplus S^2\mathcal{F}_M|_{U_i}.$$

This setup guarantees that $\mathcal{O}_0$ sits locally split exact in the above sequence ($\star$). We define transitions between these sheaves by

$$\psi_{ij} : \mathcal{O}_{U_i,0}|_{U_i \cap U_j} \to \mathcal{O}_{U_j,0}|_{U_i \cap U_j}; \quad (a, b) \mapsto (a, b + \omega_{ij}(a))$$

This prescription gives an isomorphism of sheaves. The set of isomorphisms $\{\psi_{ij}\}$ satisfies the cocycle condition because the ω_{ij} do. Thus, the isomorphisms constitute a gluing datum; this means the sheaves can be glued together along these isomorphisms. If we change the ω_{ij}'s by a coboundary, the resulting glued sheaf is isomorphic to the sheaf glued by the ω_{ij}. Therefore, the isomorphism class of the resulting sheaf is well-defined.

The supermanifold M constructed this way is split (and hence projected) if and only if $\omega = 0$. □

This proof tells us how to realize the transition functions for the non-projected manifold explicitly. Namely, if we have coordinates z_l, w_k on U_i, U_j for $\mathcal{O}_{red}$ with transition $z_l(w_k)$ on the intersection $U_i \cap U_j$ and coordinates θ_a, η_b for $S^2\mathcal{F}_M$ we get transitions

$$z_l(w_k|\eta_b) = z_l(w_k) + \omega_{ij}(w_k)$$

where $\omega_{ij}(w_k)$ lands in $S^2\mathcal{F}_M$ and contains only products of the coordinates η_b. This means that we get a fermionic correction to the even transition functions. The odd transition functions remain the same as for $\mathcal{F}_M$.

3 Superextensions of $\mathbb{P}^1$ of odd dimension 2

The goal of this section is to describe all possible supermanifolds of odd dimension 2 with $\mathbb{P}^1$ as underlying space. Before we start with the results, we recall some results on projective space. These results can be found in Hartshorne's Algebraic Geometry in chapters II and III [Har13].

Facts. We recall about $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{C}}$:

1. $\mathbb{P}^1$ is a complex manifold.
2. The group of all line bundles on $\mathbb{P}^1$ is isomorphic to $\mathbb{Z}$ with isomorphism
$$\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^1); n \mapsto \mathcal{O}_{\mathbb{P}^1}(n).$$
3. The tangent sheaf $\mathcal{T}_{\mathbb{P}^1}$ is an invertible sheaf and we have
$$\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2).$$
4. $\mathbb{P}^1$ is covered by two affine charts U and V. When we consider
$$\mathbb{P}^1 = \{[x_0 : x_1] | (x_0, x_1) \neq (0,0)\}$$
as a set, we can write
$$U = \{[x_0 : x_1] | x_0 \neq 0\} \quad \text{and} \quad V = \{[x_0 : x_1] | x_1 \neq 0\}.$$
5. The first cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}^1}(-k), k \in \mathbb{Z}$ are given by
$$H^1(\mathcal{O}_{\mathbb{P}^1}(-k)) \cong \begin{cases} \mathbb{C}^{k-1} & , k \geq 2 \\ 0 & , \text{else.} \end{cases}$$

We examine the non-projected superextensions of $\mathbb{P}^1$ of odd dimension 2. The situation is that we have given a locally free sheaf $\mathcal{F}_M$ of $\mathcal{O}_{\mathbb{P}^1}$ modules of rank $0|2$ and a cohomology class $\omega \in H^1(T_{\mathbb{P}^1} \otimes S^2\mathcal{F}_M)$ and we want to obtain the associated supermanifold.

Such a superextension is non-split if $\omega \neq 0$ and we state now when this is possible. We can remark first that $S^2\mathcal{F}_M$ is a locally free sheaf of rank $1|0$. This means that it is an invertible sheaf implying that there is some $k \in \mathbb{Z}$ such that $S^2\mathcal{F}_M \cong \mathcal{O}_{\mathbb{P}^1}(k)$. In total we get
$$\mathcal{T}_{\mathbb{P}^1} \otimes S^2\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(k+2).$$
Therefore, non-vanishing cohomology is only possible for $k \leq -4$. We can restrict ourselves to the case that $S^2\mathcal{F}_M \cong \mathcal{O}_{\mathbb{P}^1}(-l)$ for $l \geq 4$. This allows us to classify the superextensions of $\mathbb{P}^1$.

Theorem 3.1. *All non-projected supermanifolds of odd dimension 2 over $\mathbb{P}^1$ are completely characterized by a locally free sheaf $\mathcal{F}$ of rank $0|2$ such that $S^2\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(-l)$ for $l \geq 4$ and $\omega \in H^1(\mathcal{O}_{\mathbb{P}^1}(2-l))$ such that $\omega \neq 0$. Furthermore, we can characterize the locally free sheaf $\mathcal{F}$ by two integers $a, b \in \mathbb{Z}$*
$$\mathcal{F} \cong \Pi\mathcal{O}_{\mathbb{P}^1}(a) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(b)$$
with $a + b = -l$.

Proof. The only thing that remains to be proven is the classification of the locally free sheaf $\mathcal{F}$. This follows from the Grothendieck splitting theorem [Har13]. Further
$$S^2\mathcal{F} \cong \bigwedge^2 \Pi\mathcal{F} \cong \bigwedge^2 (\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathcal{O}_{\mathbb{P}^1}(a+b).$$
□

We define $\mathbb{P}^1_\omega(a,b)$ as the supermanifold uniquely defined by the triple $(\mathbb{P}^1, \mathcal{F}, \omega)$ with $\mathcal{F} = \Pi\mathcal{O}_{\mathbb{P}^1}(a) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(b)$. This supermanifold is split if and only if $\omega = 0$.

Given these explicit descriptions of $\mathbb{P}^1_\omega(a,b)$, we can describe this supermanifold also more concretely in terms of transition functions. The coordinates of $\mathbb{P}^1_\omega(a,b)$ are given on the charts U and V. On these charts we get the bosonic coordinates

$$\text{On } U : z := \frac{x_1}{x_0} \quad \text{and} \quad \text{On } V : w := \frac{x_0}{x_1}.$$

Thus we have the transition $z = \frac{1}{w}$ on $U \cap V$. The fermionic coordinates can be described by the local bases of $\mathcal{F} = \Pi\mathcal{O}_{\mathbb{P}^1}(a) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(b)$. The local bases are deduced from non-vanishing local sections. These sections are given by $\frac{1}{x_i^{-k}}$ for $\mathcal{O}_{\mathbb{P}^1}(k)$ for $i = 0$ on U and $i = 1$ on V. This entails

$$\text{On } U : \theta_1 := \frac{1}{x_0^{-a}} \quad \text{and} \quad \theta_2 := \frac{1}{x_0^{-b}}.$$

and

$$\text{On } V : \eta_1 := \frac{1}{x_1^{-a}} \quad \text{and} \quad \eta_2 := \frac{1}{x_1^{-b}}.$$

Thus, we get the transition functions on $U \cap V$

$$\theta_1 = \frac{\eta_1}{w^{-a}} \quad \text{and} \quad \theta_2 = \frac{\eta_2}{w^{-b}}.$$

Furthermore, we can give the cohomology class $\omega \in \mathbb{C}^{l-3}$ as a vector $(\lambda_1, ..., \lambda_{l-3})$.

Theorem 3.2. *The transition functions of $\mathbb{P}^1_\omega(a,b)$ are given by*

$$\begin{aligned} z &= \frac{1}{w} + \sum_{j=1}^{l-3} \lambda_j \frac{\eta_1\eta_2}{w^{j+2}}, \\ \theta_1 &= \frac{\eta_1}{w^{-a}}, \\ \theta_2 &= \frac{\eta_2}{w^{-b}}. \end{aligned}$$

The proof of this relies on the explicit structure of $\mathbb{P}^1$. It can be found in [Noj18].

4 Obstruction theory in terms of non-abelian cohomology

We present here the approach to obstructions of splitting via non-abelian cohomology as presented in [DW15]. This is based on the insights from [Gre82]. As a reference for non-abelian cohomology, we give [GW10].

Classification of supermanifolds of fixed odd dimension It is our first goal to classify all supermanifolds of fixed odd dimension.

Let M be a supermanifold with reduced space M_{red}. Let now $\mathcal{F}_M$ be the fermionic sheaf of M and $\mathcal{E}$ the locally free sheaf on M_{red} that is given by $\Pi\mathcal{F}_M = \mathcal{E}^\vee$. We define then $S(M_{red}, \mathcal{E})$ as the split supermanifold with reduced space M_{red} and local model $\mathcal{E}$. Thus, we can think of M as a gluing of patches from $S(M_{red}, \mathcal{E})$ via isomorphisms. These isomorphisms are captured in the sheaf

$$\mathrm{Isom}(S(M_{red}, \mathcal{E}), M) : U \mapsto \left\{\text{sheaf isomorphisms } \mathcal{O}_U \xrightarrow{\sim} \wedge^\bullet \mathcal{E}^\vee|_U\right\}$$

In the special case of $M = S(M_{red}, \mathcal{E})$ (the split supermanifold associated to M, also denoted by $\mathrm{Gr}M$), we get

$$\mathrm{Isom}(S(M_{red}, \mathcal{E}), S(M_{red}, \mathcal{E})) = \mathrm{Aut}(\wedge^\bullet \mathcal{E}^\vee) = \mathrm{Aut}(\wedge^\bullet \mathcal{E})$$

We can identify automorphisms of $\wedge^\bullet \mathcal{E}$ with the automorphisms $\wedge^\bullet \mathcal{E}^\vee$ by identifying a bundle automorphisms with its inverse transpose. Since M is locally isomorphic to $S(M_{red}, \mathcal{E})$, we have that $\mathrm{Isom}(S(M_{red}, \mathcal{E}), M)$ is locally isomorphic to $\mathrm{Aut}(\wedge^\bullet \mathcal{E})$. We note here that the sheaf of groups $\mathrm{Aut}(\wedge^\bullet \mathcal{E})$ is not abelian. The supermanifold M is defined by giving local isomorphisms $\mathcal{O}_U \to \wedge^\bullet \mathcal{E}^\vee|_U$ which are compatible on intersections. By that, we mean that the isomorphisms satisfy the cocycle condition such that we can glue the manifold M along them. Furthermore, changing these isomorphisms by coboundaries does not change the resulting manifold up to isomorphism. This means that an isomorphism class of supermanifolds corresponds uniquely to an element in $H^1(M_{red}, \mathrm{Aut}(\wedge^\bullet \mathcal{E}))$. The odd dimension of such a supermanifold is $\mathrm{rank}\, \mathcal{E}$. The set $H^1(M_{red}, \mathrm{Aut}(\wedge^\bullet \mathcal{E}))$ suffices to describe all supermanifolds of odd dimension of $\mathrm{rank}\, \mathcal{E}$ as all locally free sheaves of the same rank are locally isomorphic. Since $\mathrm{Aut}(\wedge^\bullet \mathcal{E})$ is a sheaf of non-abelian groups, the cohomology is not a group but only a pointed set. What we have achieved here is the classification of supermanifolds of fixed reduced manifold and fixed odd dimension.

Refinement of this classification In the theory of splitting, we need to refine this classification by classifying the supermanifolds M with given reduced space M_{red} and fixed isomorphism class of the vector bundle $\mathcal{E}$. This setting is different from the former, as we have here a given vector bundle on not just an arbitrary bundle of fixed rank. In order to classify these supermanifolds, we look at the exact sequence of groups

$$1 \to G \to \mathrm{Aut}(\wedge^\bullet \mathcal{E}) \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{E}) \to 1$$

The map α sends a $\wedge^\bullet \mathcal{E}$-automorphism to the induced automorphism of $\mathcal{E} \cong J/J^2$ (when interpreting $\mathcal{E}$ as a local model and J as the induced nilpotent sheaf). The sheaf of groups G on M_{red} can then be identified as those automorphisms of $\wedge^\bullet \mathcal{E}$ that preserve $\mathcal{E}$.

We can then look at the long exact sequence of cohomology

$$1 \to H^0(G) \to H^0(\mathrm{Aut}(\wedge^\bullet \mathcal{E})) \to H^0(\mathrm{Aut}(\mathcal{E})) \to H^1(G) \to ...$$

This exact sequence says that the set $H^0(\mathrm{Aut}(\mathcal{E}))$ acts on $H^1(G)$, where $H^0(\mathrm{Aut}(\mathcal{E}))$ are just the global bundle automorphisms of $\mathcal{E}$. The quotient $H^1(G)/H^0(\mathrm{Aut}(\mathcal{E}))$ (i.e. the orbit space of $H^1(G)$ under this action) can be identified with the set of isomorphism classes of supermanifolds with given reduced space M_{red} and fixed bundle $\mathcal{E}$. The set $H^1(G)$ specifies all supermanifolds coming from a fixed bundle $\mathcal{E}$ and to allow also bundles isomorphic to $\mathcal{E}$ we have to divide out the automorphisms of $\mathcal{E}$. We summarize our findings by the following lemma.

Lemma 4.1. *A given supermanifold M with reduced manifold M_{red} and bundle $\mathcal{E}$ induces a unique class $\omega_M \in H^1(G)$. The supermanifold is split if and only if this class vanishes*

Obstruction classes The task of determining whether a given supermanifold is split or not is thus to compute the class ω_M. Here we give a criterion when the obstruction class vanishes.

Although the interpretation of $H^1(G)$ is clear, we would like carry out the computations in the domain of abelian cohomology. In order to do so, we define the subgroups

$$G^i := \Big\{ g \in G | g(x) - x \in J^i \ \forall x \in \wedge^\bullet \mathcal{E} \Big\}.$$

These are normal subgroups and we have the filtration

$$G = G^2 \trianglerighteq ... \trianglerighteq G^{m+1} = 1$$

The quotients G^i/G^{i+1} are abelian and we have the following isomorphisms of sheaves of abelian groups ($k \in \mathbb{N}$)

$$G^{2k}/G^{2k+1} \cong \mathcal{T}_{M_{red}} \otimes \wedge^{2k}\mathcal{E} \quad \text{and} \quad G^{2k+1}/G^{2k+2} \cong \mathfrak{Hom}_{\mathcal{O}_{M_{red}}}(\mathcal{E}, \wedge^{2k+1}\mathcal{E}).$$

The inclusions $G^i \hookrightarrow G$ induce maps $H^1(G^i) \to H^1(G)$. We can formulate a criterion for the vanishing of the class ω_M.

Lemma 4.2. *The class $\omega_M \in H^1(G)$ vanishes if and only if this class is the image of some class $\phi_i \in H^1(G^i)$ for all $i \geq 2$.*

Assume now that ω_M is the image of some class $\phi_i \in H^1(G^i)$. We can then determine whether there is also a class $\phi_{i+1} \in H^1(G^{i+1})$ such that ω_M is also the image of this class. We look at the exact sequence

$$H^1(G^{i+1}) \to H^1(G^i) \xrightarrow{\omega} H^1(G^i/G^{i+1}).$$

The class ϕ_{i+1} exists if the image $\omega(\phi_i)$ vanishes in $H^1(G^i/G^{i+1})$. This defines the i-th obstruction class of splitting

$$\Omega^i_M := \omega(\phi_i) \in H^1(G^i/G^{i+1}).$$

The supermanifold M is split if and only if all obstruction classes Ω^i_M vanish.

Remark 4.3. In a similar way, we can also define obstructions to projecting, but looking at the subgroup of G that preserves projections. This leads to the concept of obstruction classes for projecting, and these classes are then precisely the classes ω^i_M that we defined earlier.

A Appendix

Lemma A.1. *ω^i_M is well-defined in $H^1(M_{red}, (T_{red} \otimes S^i\mathcal{F}_M)_0)$*

Proof.

- ω^i_M defines an even derivation: Let f, g be two sections in $\mathcal{O}_{red}(W)$ with $W \subseteq U \cap V$ open. Then

$$\omega^i_{UV}(fg) = \pi^i_U(fg) - \pi^i_V(fg) = \pi^i_U(f)\pi^i_U(g) - \pi^i_V(f)\pi^i_V(g) \tag{A.1}$$
$$= \pi^i_U(f)(\pi^1_U(g) - \pi^i_V(g)) + (\pi^i_U(f) - \pi^i_V(f))\pi^i_V(g) \tag{A.2}$$
$$= \pi^i_U(f)\omega^i_{UV}(g) + \omega^i_{UV}(f)\pi^i_V(g). \tag{A.3}$$

  But we consider π^i_U and π^i_V define the same $\mathcal{O}_{red}(W)$-module structure on $S^i\mathcal{F}_M(W)$. Thus, we have

$$\omega^i_{UV}(fg) = f\omega_{UV}(g) - \omega_{UV}(f)g$$

  which means that ω_{UV} is an even derivation on $S^i\mathcal{F}_M$. This implies that ω^i_M is a collection of sections in the sheaf $\mathcal{T}_{red} \otimes S^i\mathcal{F}_M$.

- ω^i_M is a cocycle: We have

$$\omega^i_M = \mathrm{d}\Big(\Big\{\pi^i_U\Big\}\Big) \quad \Rightarrow \quad \mathrm{d}\omega^i_M = \mathrm{d}^2(\Big\{\pi^i_U\Big\}) = 0$$

  where d is the Cech-coboundary operator.

- ω^i_M is unchanged by a different choice of π^i_U: If we have different projections $\{\pi'_U\}$ we get

$$\omega'_{UV} = \pi'_U - \pi'_V.$$

Then we have

$$\psi_U := \pi^i_U - \pi'_U : \mathcal{O}_{U,red} \to S^2\mathcal{F}_U$$

is again a derivation on $\mathcal{O}_{U,red}$. Thus, we get

$$\omega'_{UV} = \omega^i_{UV} + \psi_U - \psi_V.$$

The additional term $\psi_U - \psi_V$ is a coboundary, which means that

$$[\omega^i_M] = [\omega'] \in H^1(\mathcal{T}_{red} \otimes S^i\mathcal{F}_M).$$

□

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