Supersymmetric Localizations

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What are localizations?

Let's integrate an object α (a function, differentialform, superfunction, path integrand ..) over some measure space M:

$$\int_M \alpha$$

That's complex; so let us split M into two pieces: $M = A \sqcup B$ where A is "bigger" then B.

$$\int_M \alpha = \int_A \alpha + \int_B \alpha = \int_B \alpha$$

If $\int_A \alpha = 0$ for some miraculous reason. We say that $\int_M \alpha$ is **localized** around B.

What are super symmetric localizations?

The properties of super integration give us a new way of thinking about localitazions!

Let $M = M^{(n|m)}$ be a supermanifold with dimension $(n|m)$. Let f be a function on M and Q a fermionic vectorfield such that:

$$Q(f) = 0$$

Further assume that we can find around a point $p \in M$ local coordinates (x^i, ψ^j) such that: $Q = \frac{\partial}{\partial \psi^1}$

$$\int d\psi^1 ... d\psi^m f = \int d\psi^2 ... d\psi^m \left(\frac{\partial}{\partial \psi^1} f \right) = 0$$

The integral over a neighbourhood of p does not contribute to the integral $\int_M f$!

This leads us to the conclusion:

$$\int_M f = \int_{M-N} f$$

With:

$$N = \left\{ p \in M \middle| Q = \frac{\partial}{\partial \psi^1} \text{ in a neighbourhood of } p \right\}$$

This is a **supersymmetric localization**.

Let M be a compact manifold, G a compact Lie group with algebra $\mathfrak{g}$ and an action $G \times M \mapsto M$ on M.
Let:

$$S(\mathfrak{g}^*) = \left\{ f : \mathfrak{g} \mapsto \mathbb{R} \middle| f \text{ is a polynomial} \right\}$$

With "f is a polynomial" we mean that f is a polynomial over a basis $\{\alpha_1, ..., \alpha_n\}$ of $\mathfrak{g}^*$ for example:

$$f(\cdot) = 5\alpha_1(\cdot)^3 + \alpha_2(\cdot)\alpha_4(\cdot)$$

Let's now consider:

$$f \in S(\mathfrak{g}^*) \otimes \Omega^*(M, \mathbb{R})$$

f takes an element of $\mathfrak{g}$ and returns an ordinary differential form:

$$f : \mathfrak{g} \mapsto \Omega^*(M, \mathbb{R})$$

We want a cohomology theory similar to deRahm's build on the forms $S(\mathfrak{g}^*) \otimes \Omega^*(M, \mathbb{R})$. Unfortunately, that does not work. To get a proper theory we need a subspace of $S(\mathfrak{g}^*) \otimes \Omega^*(M, \mathbb{R})$.

$g \in G$ can act on both sides of f:

$$f(\cdot) \mapsto f(Ad_g(\cdot))$$

$$f(\cdot) \mapsto g^* f(\cdot)$$

Where $Ad_g(h) = ghg^{-1}$ for $g \in G$ and $h \in \mathfrak{g}$. (For matrix groups.)

We need the subspace of *equivariant* f the **equivariant differential forms**:

$$(S(\mathfrak{g}^*)\otimes\Omega^*(M,\mathbb{R}))^G = \left\{ f \in S(\mathfrak{g}^*)\otimes\Omega^*(M,\mathbb{R}) \middle| g^* f(\cdot) = f(Ad_g(\cdot)) \forall g \in G \right\}$$

$$= \Omega^*_G(M)$$

This is the proper space to define the cohomology theory on. We need a nilpotent operator.

A nilpotent operator:

Observation: A $X \in \mathfrak{g}$ generates a vectorfield X_M on M via:

$$X_M(h)(p) = \frac{d}{dt}\bigg|_{t=0} h(exp(-tX)p)$$

For a $p \in M$ and a $h \in C^\infty(M, \mathbb{R})$.

Now define the nilpotent operator

$$D : (S(\mathfrak{g}^*) \otimes \Omega^*(M, \mathbb{R}))^G \mapsto (S(\mathfrak{g}^*) \otimes \Omega^*(M, \mathbb{R}))^G$$

by:

$$D(h \otimes \alpha)(X) = h(X) \otimes d\alpha + h(X) \otimes \iota_{X_M}\alpha$$

or:

$$D = d + \iota_{X_M}$$

This in nilpotent on $\Omega_G^*(M)$ because:

$$D^2 = d^2 + d\iota_{X_M} + \iota_{X_M} d + \iota_{X_M}^2 = 0 + \mathcal{L}_{X_M} + 0 = 0$$

Due to Cartans magic formula.

The Lie derivative is not 0 for general forms but for $\alpha \in \Omega_G^*(M)$:

$$\mathcal{L}_{X_M}\alpha(X) = \lim_{t\to 0} \frac{(1-tX)^*\alpha(X) - \alpha(X)}{t}$$

$$= \lim_{t\to 0} \frac{\alpha((1-tX)X(1+tX)) - \alpha(X)}{t} = \lim_{t\to 0} \frac{\alpha(X + t[X,X]) - \alpha(X)}{t} = 0$$

A form $f = h(\cdot) \otimes \alpha \in \Omega_G^*(M)$ with $h \in S(\mathfrak{g}^*)$ and $\alpha \in \Omega^*(M)$ has the **grading:**

$$deg(f) = deg(\alpha) + 2deg(h)$$

$D = d + \iota_{X_M}$ increases the grading by 1.

Now we can define the **equivariant coboundarys** B_G^* and the **equivariant cocycles** Z_G^*:

$$Z_G^* = kern(D)$$

$$B_G^* = Im(D)$$

And finally the **equivariant cohomology**:

$$H_G^*(M) = \frac{Z_G^*}{B_G^*}$$

Final Remarks:

⊗ There are more models of equivariant cohomology like the **Weil** and **BRST** model.

⊗ The equivariant cohomology is usually defined in a more general way by the means of universal bundles. That is complex because the definition involves infinite dimensional manifolds.

For more detailes see [2] and [1]

Theorem:
Let G be a compact Lie group, $\mathfrak{g}$ its algebra. M is an even dimensional compact manifold with dimension $dim(M) = n = 2l$
The Atiyah Bott Localization formula:

$$\int_M \alpha(X)_{[n]} = (-2\pi)^l \sum_{p \in M_0(X)} \frac{\alpha(X)_{[0]}(p)}{e_F(X)(p)}$$

Where $X \in \mathfrak{g}$, α an equivariantly closed differential form, $M_0(X)$ the set of *isolated* zeros of X_M.

$e_F(X)(p)$ is the so called equivariant Euler class of the normal bundle of the point p.

Lemma: $\alpha(X)_{[n]}$ is d-exact outside $M_0(X)$.

We can find a form θ (with the use of a G invariant metric) such that $\mathcal{L}_{X_M}\theta = 0$ and $D\theta$ is invertibel on $M - M_0(X)$. Then we have on $M - M_0(X)$:

$$\alpha(X)_{[n]} = d\left(\frac{\theta \wedge \alpha(X)}{D\theta}\right)_{[n-1]}$$

because:

$$d\left(\frac{\theta \wedge \alpha(X)}{D\theta}\right)_{[n-1]} = D\left(\frac{\theta \wedge \alpha(X)}{D\theta}\right)_{[n-1]} = \left(\frac{D\theta}{D\theta}\alpha(X)\right)_{[n]} = \alpha(X)_{[n]}$$

(We have no element of order $[n+1]$ so we don't get a contribution by the ι_{X_M})

Lemma:
X_M *rotates* around $p \in M_0(X)$. We can find local coordinates x_i such that:

$$X_M = \lambda_1(x_2\partial_1 - x_1\partial_2) + ...$$

And we define a 1 form θ^p via:

$$\theta^p = \lambda_1^{-1}(x_2 dx_1 - x_1 dx_2) + ...$$

Properties:
$\otimes$ $\pounds_{X_M}\theta^p = 0$
$\otimes$ $\theta^p(X_M) = \Sigma x_i^2 = ||x||^2$ (In the local trivial metric given by the coordinates)
$\otimes$ We can find a θ as above such that near p: $\theta = \theta^p$

Let B_ϵ^p be the ϵ ball around the $p \in M_0(X)$

$$\int_M \alpha(X) = \lim_{\epsilon \to 0} \int_{M - \cup B_\epsilon^p} \alpha(X) = \lim_{\epsilon \to 0} \int_{M - \cup B_\epsilon^p} d\left(\frac{\theta \wedge \alpha(X)}{D\theta} \right)$$

$$= -\sum_p \lim_{\epsilon \to 0} \int_{S_\epsilon^p} \frac{\theta \wedge \alpha(X)}{D\theta}$$

Rescaling: $x \mapsto \epsilon x$

- ⊗ ϵ-sphere S_ϵ becomes the unit sphere S_1
- ⊗ $\frac{\theta}{D\theta}$ remains unchanged
- ⊗ $\lim_{\epsilon \to 0} \alpha_\epsilon(X) = \alpha_{[0]}(X)(p)$

$$\int_{S_\epsilon^p} \frac{\theta \wedge \alpha(X)}{D\theta} = \int_{S_1^p} \frac{\theta \wedge \alpha_\epsilon(X)}{D\theta} = \alpha_{[0]}(X)(p) \int_{S_1} \frac{\theta}{D\theta}$$

$$-\int_{S_1} \frac{\theta}{D\theta} = \int_{S_1} \frac{\theta}{1-d\theta} = \int_{S_1} \theta d\theta^{l-1} = \int_{B_1} d\theta^l = RHS$$

And:

$$(d\theta)^l = (-2)^l l!(\lambda_1 ... \lambda_l)^{-1} dx_1 \wedge ... dx_n$$

$2l$ dimensional unit ball has volume $\frac{\pi^l}{l!}$
Define the **equivariant Euler class** (for points!):

$$e_F(X)(p) = \lambda_1 \dots \lambda_l$$

$\Rightarrow$

$$RHS = \frac{(-2\pi)^l}{e_F(X)(p)}$$

$$QED$$

Remark:

$\otimes$ There is also a more general formula for the case in which $M_0(X)$ is a submanifold N and not just a set of isolated points

$\otimes$ We obtain it in in a similar process by integration over $\{M -$ tubular neigbourhood of $N\}$ and manipulationg the integral.

$$\int_M \alpha(X) = \int_N \frac{\alpha(X)}{e_F(X)}$$

Where $e_F(X)$ is the **equivariant Euler form** of the **normal bundle** of N in M. (In our case it was the $e_F(X)(p = \lambda_1 \dots \lambda_l)$)

Duistermaat Heckman Localization

Let (M, ω) be a symplectic manifold of dimension $2l$.

$(\omega \in \Omega^2(M); \quad d\omega = 0$ and ω is invertible)

Let G be a symplectomorphic action $(g^*\omega = \omega)$ on M.
A **moment map** is a function $\mu : \mathfrak{g} \times M \mapsto \mathbb{R}$ such that:

- ⊗ $\mu(X)$ is linear in $X \in \mathfrak{g}$
- ⊗ X_M is the **Hamiltonian** vector field generated by $\mu(X)$

$$d\mu(X)(\cdot) = \omega(X_M, \cdot)$$

- ⊗ $\mu(X)$ is equivariant: $g^*\mu(X) = \mu(Ad_g(X)) \quad \forall g \in G$

Let's add μ and ω:

$$X \mapsto \Omega(X) = \mu(X) - \omega$$

is an equivariant closed differential form.

$$D\Omega(X_M) = d\mu(X) - \omega(X_M, \cdot) = 0$$

We had a nice formula for integrals over equivariantly closed differential forms!
$\Omega(X)$ has maximally degree 2 so we integrate over:

$$e^{i\Omega}{}_{[n]} = e^{i\mu(X)+i\omega}{}_{[n]} = e^{i\mu(X)} e^{i\omega}{}_{[n]} = e^{i\mu(X)} \frac{\omega^l i^l}{l!}$$

Is also equivariantly closed.

Atiyah Bott $\Rightarrow$ **The Duistermaat Heckmann localization**

$$\int_M e^{i\mu(X)} \frac{\omega^l}{l!} = (2\pi i)^l \sum_{p \in M_0(X)} \frac{e^{i\mu(X)}(p)}{e_F(X)(p)}$$

Example: S^2 with S^1 rotation around z localizes to a sum over the poles.

This is the so called **exact stationary phase approximation**.

We can use Duistermaat Heckman to proof that the stationary phase approximation of **certain** path integrals is exact!

Atiyah Bott localizations in susy language:

We saw in the last talk that integration of a differential form over a (bosonic) manifold is a special case of supergeometric integration:

$$(x^i, dx^i) \to (x^i, \psi^i)$$

$$\int_M dx^{i_1} \wedge \dots \wedge dx^{i_n} \alpha_{i_1 \dots i_n} = \int_{\Pi TM} dx^1 \dots dx^n d\psi^1 \dots d\psi^n \alpha(x, \psi)$$

$$= (-2\pi)^l \sum_{p \in M_0(X)} \frac{\alpha(X)_{[0]}(p)}{e_F(X)(p)}$$

$$\int_M dx^{i_1} \wedge ... \wedge dx^{i_n} \alpha_{i_1 ... i_n} = \int_{\Pi TM} dx^1 ... dx^n d\psi^1 ... d\psi^n \alpha(x, \psi)$$

$$D = d + \iota_{X_M} = \psi^i \frac{\partial}{\partial x^i} + X^i_M(p) \frac{\partial}{\partial \psi^i}$$

$$D^2 = \pounds_{X_M}$$

- $\alpha(x, \psi)$ is a superfunction
- D is a fermionic vectorfield
- $D^2 = \pounds_{X_M}$ is a bosonic vectorfield

$D\alpha = 0$ is one of the conditions of the supersymmetric localization from the beginning of the talk.

Is the Atiyah Bott localization formula a special case of the supersymmetric localization formula?

We have to check whether we can write:

$$D = d + \iota_{X_M} = \psi^i \frac{\partial}{\partial x^i} + X_M^i(p) \frac{\partial}{\partial \psi^i}$$

as:

$$D = \frac{\partial}{\partial {\psi'}^1}$$

Is this possible?

Problem: That's not possible!

Theorem:
Let Q be a fermionic vectorfield on a sumpermanifold M that does not vanish at x_0. Then we can write Q in local coordinates (x^i, ψ^j) around x_0 as:
$\otimes \quad Q = \frac{\partial}{\partial \psi^1} \quad \Leftrightarrow \quad Q^2 = 0$
The proof can be found in [1], chapter 4 §4 section 2.

But: $D^2 = \pounds_{X_M} \neq 0$

$\Rightarrow$ Atiyah Bott is **not** a special case of the susy localization from the beginning of the talk!

We now have susy localizations with fermionic vectorfields Q of the form ∂_{ψ^1} and $D = \psi^i \partial_{x^i} + X^i_M \partial_{\psi^i}$.

$\Rightarrow$ There could be a more general localization formula for fermionic vectorfields!

To formulate the localization theorem for supermanifolds we have to define more structures on them.

Reminder: Facts about supermanifolds:

$\otimes$ A *real* supermanifold $M = M^{(n_+|n_-)}$ can be seen as a outer product bundle $\Pi\alpha(N)$ over some bosonic manifold N.

$\otimes$ $N = m(M)$ is called the **body** of M

$\otimes$ We can find local coordinates $(x_1, \ldots, x_{n_+}, \psi_1, \ldots \psi_{n_-})$

$\otimes$ A **bosonic** vector field A on M has a **number part** m(A)

$$A = A^i(x,\psi)\partial_{x^i} + A^\iota(x,\psi)\partial_{\psi^\iota} \Rightarrow m(A) = A^i(x,0)\partial_{x^i}$$

$\otimes$ The **number part of an fermionic vector field** Q is a section in the bundle determined by $(q^1(x,0),\ldots,q^{n_-}(x,0))$.

$$Q = \kappa^i(x,\psi)\partial_{x^i} + q^\iota(x,\psi)\partial_{\psi^\iota}$$

$\otimes$ Diffeomorphisms on M are automorphisms on $\alpha(N)$

$\otimes$ Bosonic vectorfiels A on M are infinitesimal diffeomorphisms on $M \quad \Leftrightarrow \quad$ infinitesimal automorphisms $\bar{A}$ on αN

$\otimes$ A bosonic vectorfield is **compact**, if it is generated by the action of a 1 parameter subgroup of a **compact** Lie group.

$\otimes$ The set of compact vector fields on M is denoted by $\mathcal{K}(M)$

Superdivergence:

⊗ The *bosonic* divergence $div(X)$ of a *bosonic* vectorfield X on a *bosonic* manifold M tells us how the volume form dV on the manifold changes with the flow of X.

⊗ That's hard to define for supermanifolds!

⊗ Lets just define the **super divergence** div_{dV} by its action unter superintegration:

$$\int_M dV Q(f) = -\int_M dV div_{dV}(Q) f \quad \forall f$$

Where M is a supermanifold, dV is a supervolume form on M, Q is a fermionic vectorfield and f a test function.

Supersymmetric localization Theorem:
Let M be a *compact supermanifold* with volumeform dV. Let Q be a fermionic vectorfield on M such that:

$$div_{dV} Q = 0$$

$$Q^2 \in \mathcal{K}(M)$$

For any neighbourhood U of $M_0(Q)$ exists an bosonic, Q invariant function g_0, that is equal to 1 in a neighbourhood $O \subset U$ of $M_0(Q)$ and vanishes outside. For every function h with $Q(h) = 0$ on M and every g_0 that satisfies this condition we have:

$$\int_M dV h = \int_M dV g_0 \cdot h$$

Special cases

∂_{ψ^1} **Localization:**

⊗ $\partial^2_{\psi^1} \in \mathcal{K}(M)$

$$\left(\frac{\partial}{\partial\psi^1}\right)^2 = 0 \in \mathcal{K}(M)$$

⊗ $div_{dV}(\partial_{\psi^1}) = 0$

$$\int d\psi^1 \dots d\psi^{n-} div_{dV}\left(\frac{\partial}{\partial\psi^1}\right)\phi =$$

$$-\int d\psi^1 \dots d\psi^{n-} \frac{\partial}{\partial\psi^1}\phi = -\int d\psi^2 \dots d\psi^{n-} \left(\frac{\partial}{\partial\psi^1}\right)^2 \phi = 0$$

For every function ϕ.

Atiyah Bott Localization:

⊗ $div_{dV}(D) = 0$

$$\int_M dV\, div_{dV}(D)\phi = -\int_M dV\, D(\phi)$$

$$= -\int_{m(M)} d\phi + \iota_X(\phi) = 0$$

due to stokes and the nonexistence of a $n+1$ form on $m(M)$

⊗ $D^2 = \pounds_X \in \mathcal{K}(\Pi TM)$

Because X is generated by the action of a 1 parameter subgroup of a compact Lie group and so is $\pounds_X$.

Lemma 1:
$\exists$ a fermionic, Q^2 invariant function σ on M such that:

$$m(Q\sigma)(x) \neq 0 \quad \forall x \notin M_0(Q)$$

Proof:
$\otimes$ Local coordinates: $z = (x^i, \psi^\alpha)$

$$Q = \sum_{i=1}^{n_+} a^i_\alpha(z)\psi^\alpha \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n_-} b^\alpha(z)\frac{\partial}{\partial \psi^\alpha}$$

Where: $a^i_\alpha(z) = a^i_\alpha(x) + \dots$ and $b^\alpha(z) = b^\alpha(x) + \dots$
Where the dots denote higher orders in ψ^α

$\otimes$ We can write in a similar manner:

$$Q^2 = k^i(z)\frac{\partial}{\partial x^i} + l^\alpha_\beta \psi^\beta \frac{\partial}{\partial \psi^\alpha}$$

With some coefficients $k^i(z) = k^i(x) + \dots$ and $l^\alpha_\beta(z) = l^\alpha_\beta(x) + \dots$ depending on a^i_α and b^α

$\otimes \quad [Q, Q^2] = 0 \quad \Rightarrow$

$$k^i(x)\frac{\partial b^\alpha(x)}{\partial x^i} - l^\alpha_\beta(x) b^\beta(x) = 0$$

The section in α generated by the number part $b^\alpha(x)$ of Q is therefore invariant under the infinitesimal automorphism $\overline{Q^2}$

⊗ We assumed that $\overline{Q^2}$ is generated by a 1 parameter subgroup of a compact group G.

⊗ We can find a G invariant metric $g_{\alpha\beta}$ on the fibers of the bundle α because G is **compact**.

⊗ Define:

$$\sigma(z) = g_{\alpha\beta}(x) b^{\alpha}(x) \psi^{\beta}$$

⊗ $b^{\alpha}(x)$ and $g_{\alpha\beta}$ are $\overline{Q^2}$ invariant $\Rightarrow \quad Q^2\sigma = 0$

⊗ $m(Q\sigma)(x) = g_{\alpha\beta}(x) b^{\alpha}(x) b^{\beta}(x) \neq 0 \quad \forall x \notin M_0(Q)$

That completes the proof of the Lemma 1.

⊗ Lets define:

$$\beta(z) = \frac{\sigma(z)}{Q\sigma(z)} \quad \forall z \notin M_0(Q)$$

⊗ $Q\beta = 1$ (Does this look familiar? Compare to $\frac{\theta}{D\theta}$)

Lemma 2:

We can find a partition of unity $\sum g_i = 1$ such that:

$$supp(g_0) \subset U$$

$$g_0|_O = 1$$

$$Qg_n = 0 \text{ and } g_n = Q(\rho_n) \text{ if } n \neq 0$$

Where $O \subset U$ are the neigbourhoods of $M_0(Q)$ from the Theorem.

Proof:

$\otimes$ Choose an open covering (U_n) such that:

$$M_0(Q) \subset U_0 \text{ and } M_0(Q) \cap U_n = \emptyset \quad \forall n > 0$$

$\otimes$ Choose a partition of unity f_n on this set.

$\otimes$ The partition f_n is G-invariant. (We can always choose this because G is **compact**). It is also $\overline{Q^2}$ invariant.

$\otimes$ Define:

$$g_n = Q(\beta f_n) \quad \forall n > 0$$

$$g_0 = 1 - \sum_{n>0} g_n$$

$$g_n = Q(\beta f_n) \quad \forall n > 0$$
$$g_0 = 1 - \sum_{n>0} g_n$$

Does this satisfy all conditions?

$$\sum_{n>0} g_n = \sum_{n>0} Q(\beta f_n) = \sum_{n>0} f_n + \beta Q(\sum_{n>0} f_n)$$

That's 0 in a neighbourhood of $M_0(Q)$ and 1 in $M - U_0$

Further: $Q(g_n) = Q(Q(\beta f_n)) = 0$ since $Q^2(f_n) = 0$

Proof of the Theorem:
Let h be an function that is invariant under Q.

$$\int_M dV h = \sum_n \int_M dV g_n h = \sum_{n>0} \int_M dV Q(\rho_n) h + \int_M dV g_0 h$$

$$= \sum_{n>0} \int_M dV Q(\rho_n h) + \int_M dV g_o h = \int_M dV g_o h$$

Because $div_{dV}(Q) = 0$

Is this independend from the choise of g_0?

$\otimes$ Assume that we have another function $\widetilde{g_0}$ with the same properties as g_0

$$g_0 - \widetilde{g_0} = 0 \text{ in a small nbhd of } M_0(Q)$$

$\Rightarrow$

$$(g_0 - \widetilde{g_0}) = Q(\beta(g_0 - \widetilde{g_0}))$$

$\Rightarrow$

$$\int_M dV g_0 h - \int_M dV \widetilde{g_0} h = \int_M dV Q(\beta(g_0 - \widetilde{g_0}))h = \int_M dV Q(\beta h(g_0 - \widetilde{g_0}))$$

$$= 0$$

That completes the proof of the supersymmetric localization theorem.

A more general result can be found in [5]

Some Applications

⊗ Wittens proof of the Morse inequalities [6]

⊗ Calculation of **some** QFT partition functions and even some correlation functions [4]

Summary:

$\otimes$ The Atiyah Bott theorem can be proven with the means of equivariant differentialforms.

$\otimes$ The Duistermaat Heckman theorem is a corollary of the Atyiah Bott theorem. It provides the exactness of the stationary phase approximations in some cases.

$\otimes$ The Atiyah Bott theorem can be rephrased as a localization theorem on certain supermanifolds.

$\otimes$ The Atiyah Bott is a special case of a localization theorem on general supermanifolds.

$\otimes$ The proof of the general theorem shows astonishing parallels to the proof of Atiyah Bott.

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