SuperGeometry Integration

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Tangent Bundles

Definition (Vector Bundle)

A (real) vector bundle of rank n is a triple (E, B, π) of topological spaces E, B and a projection $\pi : E \longrightarrow B$ with:

- Every fiber $\pi^{-1}(p)$, $p \in B$, is a n-dim. real vector space.
- Locally trivial: $\forall U \subset B$ open $\exists \varphi : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ homeo. with
  1. $\pi \circ \varphi = proj_1$,
  2. $\varphi| : \{p\} \times \mathbb{R}^n \longrightarrow \pi^{-1}(p)$ is a vector space iso. $\forall p \in B$.

- Transition function for two local trivializations $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) with $U_\alpha \cap U_\beta \neq \emptyset$:
  $\rho_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$.

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$M^{(n)}$ manifold with tangent spaces T_pM and projection $\pi : T_pM \mapsto p$.

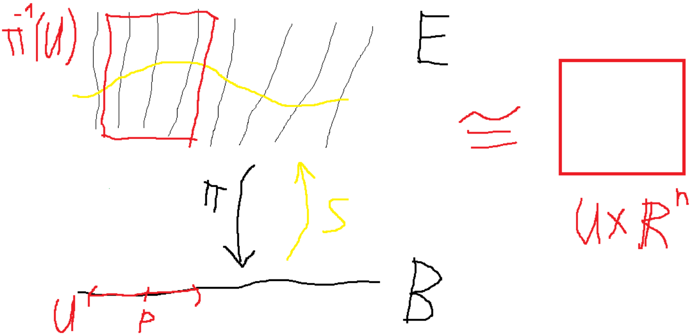
- $TM = \cup_{p \in M} T_pM$, then (TM, M, π) is the tangent bundle.
- $T^*M = \cup_{p \in M} T_p^*M$ is the cotangent bundle.

Tangent Bundles

Definition (Section)

A smooth section s of a vector bundle is a smooth map:
$s : B \longrightarrow E$
with $\pi \circ s = id_B$.

- A vector field X on a manifold M is a section $X : M \longrightarrow TM$.



Exterior Algebra

Definition (Exterior Power)

The k-th exterior power $\bigwedge^k(V)$ of a vector space $V^{(n)}$ is the quotient space:

$\bigwedge^k(V) = \otimes_{j=1}^{k} V \Big/ Lin(v_1 \otimes \ldots \otimes v_k | \exists i \neq j : v_i = v_j)$.

Definition (Exterior Product)

$\wedge : \bigwedge^p(V^*) \otimes \bigwedge^q(V^*) \longrightarrow \bigwedge^{p+q}(V^*)$,
$(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) =$
$\frac{1}{p!q!} \sum_\sigma sgn(\sigma)\omega(v_{\sigma(1)}, .., v_{\sigma(p)})\eta(v_{\sigma(p+1)}, .., v_{\sigma(p+q)})$.

- A basis for $\bigwedge^k(V)$ is given by $\{e_{j_1} \wedge .. \wedge e_{j_k} | 1 \leq j_1 < .. < j_k \leq n\}$.
- $dim \bigwedge^k(V) = \binom{n}{k}$ for $1 \leq k \leq n$, and $\bigwedge^k = \{0\}$ for $k > n$.
- The exterior Algebra of V is $(\oplus_{i \geq 0} \bigwedge^i(V), +, \wedge)$.

Differential Forms

Definition (k-Form)

A smooth differential k-form on a manifold $M^{(n)}$ is a smooth section into the space $\bigwedge^k(T^*M)$.
The space of all k-forms on M is denoted by $\Omega^k(M)$.

- 0-forms are functions f, 1-forms are dual vectors $f_i dx^i$.
- A general k-form is $\omega = \sum_{i_1,..,i_k} f_{i_1,..,i_k} dx^{i_1} \wedge .. \wedge dx^{i_k} = f_I dx^I$.
- Example: $\alpha = x_3 dx^1 \wedge dx^2 - 2x_1 dx^1 \wedge dx^3$ is some 2-form on $\mathbb{R}^3$.

Definition (Exterior Derivative)

We define the exterior derivative (de Rham differential) d by its action on a general k-form $\omega = f_I dx_I$:
$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$,
$d\omega = \sum_j \frac{\partial f_I}{\partial x_j} dx^j \wedge dx^I$.

de Rham Complex

The de Rham differential satisfies:

1. $d^2 = 0$, or more precisely $d(d\omega) = 0$ for any form ω.
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for ω k-form.

Example

$\alpha = x_3 dx^1 \wedge dx^2 - 2x_1 dx^1 \wedge dx^3$.
$d\alpha = dx^3 \wedge dx^1 \wedge dx^2 - 2dx^1 \wedge dx^1 \wedge dx^3 = dx^1 \wedge dx^2 \wedge dx^3$.

Some terminology:

- $\omega \in \Omega^k(M)$ is called closed if $d\omega = 0$.
- $\omega \in \Omega^k(M)$ is called exact if $\exists \eta \in \Omega^{k-1}(M)$ with $\omega = d\eta$.

de Rham Complex

The de Rham complex is following the sequence:

$$0 \longrightarrow \Omega^0(M) \longrightarrow \Omega^1(M) \longrightarrow ... \longrightarrow \Omega^n(M) \longrightarrow 0,$$

where $\Omega^k(M) = 0 \forall k > n$ because of the antisymmetry of $\wedge$.
We take a closer look at $d^2 = 0$. This implies:

$$\Omega^{k-1}(M) \longrightarrow \Omega^k(M) \longrightarrow \Omega^{k+1}(M) : \eta \mapsto d\eta \mapsto 0.$$

Therefore the de Rham complex satisfies $Im(d_{k-1}) \subset Ker(d_k)$ for every k and we define the k-th (de Rham) Cohomology Group as:
$H^k(M) = {}^{Ker(d_k)} \Big/ {}_{Im(d_{k-1})}$.

Definition (Pullback)

Let $\varphi : M \longrightarrow N$ between two manifolds. This induces a map:
$\varphi^* : \Omega^k(N) \longrightarrow \Omega^k(M)$ given by:
$\varphi^*\omega_p(X_1, .., X_k) = \omega_{\varphi(p)}(d_p\varphi(X_1), .., d_p\varphi(X_k))$.

Pullbacks are "nice":

- For $\psi : N \longrightarrow \mathbb{R}$ is $\varphi^*\psi = \psi \circ \varphi$.
- $\varphi^* \circ d = d \circ \varphi^*$.
- $\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta$.

Example

$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^3$.
$\alpha = x_3 dx^1 \wedge dx^2 - 2x_1 dx^1 \wedge dx^3 \in \Omega^2(\mathbb{R}^3)$.
$\varphi^*\alpha = \varphi_3 d\varphi^1 \wedge d\varphi^2 - 2\varphi_1 d\varphi^1 \wedge d\varphi^3 \in \Omega^2(\mathbb{R}^n)$ with $\varphi_i = x_i(\varphi)$.

Integration of Bosonic Forms

Let $U \subset \mathbb{R}^n$ be open and oriented. Let $\omega = f(x_1, .., x_n) dx^1 \wedge .. \wedge dx^n$ be an n-form with $supp(\omega) \subset U$ compact.

Definition

The integral of ω over U is defined as the Lebesgue integral:
$\int_U \omega = \int_{\mathbb{R}^n} f(x_1, .., x_n) dx_1 .. dx_n$.

Choose a partition of unity (h_i) where each $supp(h_i) \subset U_i$ for some chart (U_i, φ_i).

Definition

$\int_M \omega = \sum_i \int_{U_i} h_i \cdot (\varphi_i^{-1})^* \omega$.

Supermanifolds

Let $M^{p|q}$ be a supermanifold with p even coordinates $x = (x_1, .., x_p)$ and q odd coordinates $\theta = (\theta_1, .., \theta_q)$.

- $M_{red} = M|_{\theta_1 = .. = \theta_q = 0}$ is the (purely bosonic) reduced manifold.
- Recall that fermionic coordinates are infinitesimal.
- Let $U \subset M$. U is called open iff $U_{red} = U \cap M_{red}$ is open in $\mathbb{R}^p$.

Idea of the Berizinian Integral

Start with a superspace $\mathbb{R}^{p|q}$ with $x = (x_1, .., x_p)$ bosonic and $\theta = (\theta_1, .., \theta_q)$ fermionic coordinates.

- We want to integrate a function $g(x_1, ..., \theta_q)$.
- Write down some measure: $[dx^1, ..|.., d\theta^q]$.
- Expand g in powers of θs:
  $g(x, \theta) = g_0(x) + g_1^i(x)\theta_i + .. + g_q(x)\theta_1..\theta_q$.

We assume that g_q is compactly supported (or vanishes fast enough at infinity).

Definition (Berizinian Integral)

$\int_{\mathbb{R}^{p|q}}[dx^1, ..|.., d\theta^q]g(x, \theta) = \int_{\mathbb{R}^p} dx^1..dx^p g_q(x).$

The Berizinian Bundle

What is the integral measure?

Definition (Berizinian Bundle)

We define a line bundle $Ber(M)$ locally:

- Each coordinate system $(x|\theta) = (x_1, .., x_p|\theta_1, .., \theta_q)$ is a local trivialization which we call $[dx^1..|..d\theta^q]$.
- The transition functions between two trivializations (x, θ) and (x', θ') are given by the Berenzian:
  $[dx^1..|..d\theta^q] = Ber(\frac{\partial(x|\theta)}{\partial(x'|\theta')})[dx'^1..|..d\theta'^q]$.
- The fibres are one-dimensional.

Reminder

For $V = V_{even} \oplus V_{odd}$ a matrix $W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Hom(V, V)$, where A, D even and B, C odd, has the Berizinian:
$Ber(W) = det(A - BD^{-1}C)det^{-1}(D)$.

The Berizinian Bundle

Let σ be a section of $Ber(M)$ that is supported locally in $(U, (x|\theta))$:

- $\sigma = g(x, \theta)[dx^1..|..d\theta^q]$.

Definition

$\int_U \sigma = \int_{\mathbb{R}^{p|q}} [dx^1, ..|.., d\theta^q] g(x, \theta)$.

Let now σ be a general section.

- We define the integral over M piece-wise as above.
- Choose a partition of unity (h_i) where each $supp(h_i) \subset U_i$ for some chart (U_i, φ_i).

Definition

$\int_M \sigma = \sum_i \int_{U_i} h_i \cdot \sigma$.

Remark

Remark:

- You CAN think of sections in $Ber(M)$ as the supersymmetric equivalent of a top form.
- We have defined no such thing as a k-form yet! $[dx^1..|..d\theta^q]$ is irreducible, and not a form.
- Careful about transformation properties.
  - $Ber(\cdot) = \lambda^{-1}$ for the transformation $\theta \mapsto \lambda\theta$.
  - $Ber(\cdot) = (-1)$ for swapping two θs.

ALGEBRAIC CONSTRUCTION OF FORMS.

Construction of Forms via Clifford Algebra

Let V be an odd vector space $\cong \mathbb{R}^{0|p}$.

- $(\zeta^1, .., \zeta^p)$ basis of V.
- $(\eta_1, .., \eta_p)$ basis of V^*.

Consider the space $V \oplus V^*$.
Introduce canonical bilinear form $\langle \cdot, \cdot \rangle$:

- $\langle \zeta^i, \zeta^j \rangle = \langle \eta_i, \eta_j \rangle = 0$.
- $\langle \zeta^i, \eta_j \rangle = \langle \eta_j, \zeta^i \rangle = \delta^i_j$.

Quantisation:

- Vectors η_i, $\zeta^j \longrightarrow$ Operators η_i, ζ^j.
- Bilinear form $\langle \cdot, \cdot \rangle \longrightarrow$ Anticommutator $\{\cdot, \cdot\}$ with: $\{A, B\} = AB + BA$.

Construction of Forms via Clifford Algebra

We want to construct a module $\mathcal{S}$ for the Clifford Algebra.

- Take a vector $|\downarrow\rangle$ that is annihilated by the η_i.
- Basis for $\mathcal{S}$ is then given by acting on $|\downarrow\rangle$ with the ζ^j: $\{\zeta^{i_1}..\zeta^{i_k}|\downarrow\rangle | k \in [|0, p|]\}$.

Remark:

A corresponding state $|\uparrow\rangle$ that is annihilated by the ζ^js is then given by $\zeta^1..\zeta^p|\downarrow\rangle$.
Alternatively one can start the construction with $|\uparrow\rangle$ and deriving $|\downarrow\rangle$ by acting on it with the η_i.

Construction of Forms via Clifford Algebra

Let $M^{(p)}$ be a bosonic manifold.

- We denote ΠTM as the tangent bundle of M with twisted fibers: $(x_1, .., x_p)$ on $M \longrightarrow (x_1, ..|.., dx^p)$ on ΠTM where $(dx^1, .., dx^p)$ are odd.
- Expand a function on ΠTM in powers of dx^i as before: $f(x|dx) = f_0(x) + f_{1\,i}(x)dx^i + .. + f_p(x)dx^1..dx^p$.

Remark:

A k-order term $f_{k\,I}(x)dx^I$, $I = (i_1, .., i_k)$, is a differential k-forms.
The space of functions on ΠTM is the space of differential forms on M.

Construction of Forms via Clifford Algebra

For $a \in M$ we define a Clifford Algebra by specifying the operators η_i and ζ^j.

Definition

Let f be a function on ΠTM.
$\zeta^j : f \mapsto dx^j \wedge f \equiv dx^j f$.
$\eta_i : f \mapsto \frac{\partial}{\partial dx^i}(f)$.

Sanity check:

- $\langle \zeta^i, \zeta^j \rangle = \langle \eta_i, \eta_j \rangle = 0$.
- $\langle \zeta^i, \eta_j \rangle = \langle \eta_j, \zeta^i \rangle = \delta^i_j$.

Construction of Forms via Clifford Algebra

Remark:

The exterior derivative is recovered via the definition:
$d = \zeta^j \partial_j = \sum_j dx^j \frac{\partial}{\partial x^j}$.

Sanity check:

- Degree: $+1$.
- $d^2 = 0$.
- Leibniz rule.

Construction of Forms via Weyl Algebra

Let W be an even vector space $\cong \mathbb{R}^{q|0}$.

- $(\alpha^1, .., \alpha^q)$ basis of W.
- $(\beta_1, .., \beta_q)$ basis of W^*.

Consider the space $W \oplus W^*$.
Introduce canonical bilinear form $\langle \cdot, \cdot \rangle$:

- $\langle \alpha^i, \alpha^j \rangle = \langle \beta_i, \beta_j \rangle = 0$.
- $\langle \alpha^i, \beta_j \rangle = \langle \beta_j, \alpha^i \rangle = \delta^i_j$.

Quantisation:

- Vectors β_i, $\alpha^j \longrightarrow$ Operators β_i, α^j.
- Bilinear form $\langle \cdot, \cdot \rangle \longrightarrow$ Commutator $[\cdot, \cdot]$ with: $[A, B] = AB - BA$.

Construction of Forms via Weyl Algebra

We want to construct a module $\mathcal{V}$ for the Weyl Algebra.

- Take a vector $|\downarrow\rangle$ that is annihilated by the β_i.
- Basis for $\mathcal{V}$ is then given by acting on $|\downarrow\rangle$ with the α^j: $\{\alpha^{i_1}..\alpha^{i_k}|\downarrow\rangle | k \geq 0\}$.

Remark:

The basis is not finite! This is a symptom of the fact that the α^j are commuting and will be important later.
One can again construct a module $\mathcal{V}'$ with $|\uparrow\rangle$ acting on it with the β_i, but the two modules are not equivalent.

Construction of Forms via Weyl Algebra

Again we choose α^j to be multiplications and β_i derivatives:

- $\alpha^j : f \mapsto \alpha^j f$,
- $\beta_i : f \mapsto \frac{\partial}{\partial \alpha^i}(f)$.

Sanity check:

- $\langle \alpha^i, \alpha^j \rangle = \langle \beta_i, \beta_j \rangle = 0$.
- $\langle \alpha^i, \beta_j \rangle = \langle \beta_j, \alpha^i \rangle = \delta^i_j$.

Construction of Forms via Weyl Algebra

The different modules support different functions.

$\mathcal{V}$

$|\downarrow\rangle$ is annihilated by derivatives β_i:
$\implies \psi = 1$ is a ground state.
$\implies$ polynomials in α_i are basis elements.

$\mathcal{V}'$

$|\uparrow\rangle$ is annihilated by multiplication with α^j:
$\implies$ distributions supported at the origin $\alpha^j = 0$ are ground states. $\implies$ basis for $\mathcal{V}'$ is: $\{\frac{\partial}{\partial\alpha^{j_1}}..\frac{\partial}{\partial\alpha^{j_k}}\delta^{(q)}(\alpha^1..\alpha^q)|k \geq 0\}$.

Construction of Forms via Weyl Algebra

Let $M^{(p)}$ be a fermionic manifold, $M \cong \mathbb{R}^{0|q}$.

Definition

- $\alpha^j \equiv d\theta^j$ are called one-forms and considered even.
- The exterior derivative is defined on $\mathcal{V}$ and $\mathcal{V}'$ by: $d = \alpha^j \partial_{\theta^j} = \sum_j d\theta^j \frac{\partial}{\partial \theta^j}$.

Sanity check:

- This trivially fulfills $d^2 = 0$.
- The wedge product is a simple multiplication: $\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, $\mathcal{V} \times \mathcal{V}' \longrightarrow \mathcal{V}'$.
- There is no way to multiply two elements of $\mathcal{V}'$.

Forms on Supermanifolds

Let $M^{(p|q)}$ be a general supermanifold.

Definition

A form ω on M is a function on ΠTM: $\omega(x, d\theta|\theta, dx)$.

1. Differential forms are functions with polynomial dependence on $d\theta^i$. The space of differential forms on M is called $\Omega^*(M)$.
2. Integral forms are functions whose dependence on all $d\theta^i$ is a Dirac-delta distribution supported at $d\theta^i = 0$. The space of integral forms on M is called $\Omega^*_{int}(M)$.

Definition

The exterior derivative is the following vector field on ΠTM:
$d = dx^i \frac{\partial}{\partial x^i} + d\theta^j \frac{\partial}{\partial \theta^j}$.

Remark

Remark:

1. There is no top differential form, we can not integrate $\omega \in \Omega^*(M)$! (For positive fermionic dimensions.)
2. We can integrate $\omega \in \Omega^*_{int}(M)$. A top integral form is: $f(x|\theta)dx^1..dx^p\delta(d\theta^1..d\theta^q)$.
   However there is no bottom form as every $\frac{\partial}{\partial d\theta^j}$ increases the codimension by 1.

$$0 \longrightarrow \Omega^0(M) \longrightarrow \Omega^1(M).. \longrightarrow \Omega^p(M).. \longrightarrow \Omega^N(M) \longrightarrow ..$$

$$.. \longrightarrow \Omega^{-1}_{int}(M) \longrightarrow \Omega^0_{int}(M) \longrightarrow \Omega^1_{int}(M).. \longrightarrow \Omega^p_{int}(M) \longrightarrow 0.$$

Integration

We want to specify the integral over the $d\theta$ variable.

Definition

- Abuse of notation: The measure is denoted by $[d(d\theta)]$.
- Transformation properties of the measure imply:
  - $\delta(\lambda d\theta^i) = \lambda^{-1}\delta(d\theta^i)$.
  - $\delta(d\theta^i)\delta(d\theta^j) = -\delta(d\theta^j)\delta(d\theta^i)$.
- $\int g(d\theta)\frac{\partial}{\partial d\theta^i}[d(d\theta)]$ is defined by "partial integration".

Example ($\mathbb{R}^{0|1}$ with one coordinate θ)

- $\int [d(d\theta)]\frac{\partial}{\partial d\theta}\delta(d\theta) \equiv 0$.
- $\int [d(d\theta)] d\theta \frac{\partial}{\partial d\theta}\delta(d\theta)$

$= - \int [d(d\theta)]\frac{\partial}{\partial d\theta}(d\theta)\delta(d\theta) +$ (total derivative) $= -1$.

Integration

With this we can define the integral over the total space:

Definition

Let $\omega \in \Omega^*_{int}(M)$.
$\int_M \omega = \int_{\Pi TM} \omega(x, d\theta | \theta, dx)$.

Remark:

- This is a Berenzian integral over the odd coordinates θ, dx.
- The integration over $d\theta$ is distributional.
- The remaining even coordinates x get integrated ordinarily.

Integration

Example ($M = \mathbb{R}^{3|2}$)

Let $d\alpha = dx^1 dx^2 dx^3$.
Let $f(x_1, x_2, x_3)$ be a function with $\int_{\mathbb{R}^3} f = 1$.
Consider the following function on M:
$\omega = (x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)(1 + \theta_1\theta_2)\delta(d\theta^1)\delta(d\theta^2)$.

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ω is an integral form.
We calculate:

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ω is an integral form.
We calculate:
$\int_M \omega = \int_{\Pi TM} (x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)(1 + \theta_1\theta_2)\delta(d\theta^1)\delta(d\theta^2)$

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We calculate:

$$\int_M \omega = \int_{\Pi TM} (x_1 x_2 dx^3 + f(x_1, x_2, x_3)d\alpha)(1 + \theta_1\theta_2)\delta(d\theta^1)\delta(d\theta^2)$$
$$= \int_{\Pi T\mathbb{R}^{3|0}} \int [d(d\theta^1), d(d\theta^2)](x_1 x_2 dx^3 + f(x_1, x_2, x_3)d\alpha)\delta(d\theta^1)\delta(d\theta^2)$$

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$$= \int_{\Pi T\mathbb{R}^{3|0}} \int [d(d\theta^1), d(d\theta^2)](x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)\delta(d\theta^1)\delta(d\theta^2)$$
$$= \int_{\Pi T\mathbb{R}^{3|0}} (x_1 x_2 dx^3 + f(x_1, x_2, x_3) dx^1 dx^2 dx^3)$$

Integration

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We calculate:

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$$= \int_{\Pi T\mathbb{R}^{3|0}} \int [d(d\theta^1), d(d\theta^2)](x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)\delta(d\theta^1)\delta(d\theta^2)$$
$$= \int_{\Pi T\mathbb{R}^{3|0}} (x_1 x_2 dx^3 + f(x_1, x_2, x_3) dx^1 dx^2 dx^3)$$
$$= \int_{\mathbb{R}^3} f(x_1, x_2, x_3)$$

Integration

Example ($M = \mathbb{R}^{3|2}$)

Let $d\alpha = dx^1 dx^2 dx^3$.
Let $f(x_1, x_2, x_3)$ be a function with $\int_{\mathbb{R}^3} f = 1$.
Consider the following function on M:
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We calculate:

$$\int_M \omega = \int_{\Pi TM} (x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)(1 + \theta_1\theta_2)\delta(d\theta^1)\delta(d\theta^2)$$
$$= \int_{\Pi T\mathbb{R}^{3|0}} \int [d(d\theta^1), d(d\theta^2)](x_1 x_2 dx^3 + f(x_1, x_2, x_3) d\alpha)\delta(d\theta^1)\delta(d\theta^2)$$
$$= \int_{\Pi T\mathbb{R}^{3|0}} (x_1 x_2 dx^3 + f(x_1, x_2, x_3) dx^1 dx^2 dx^3)$$
$$= \int_{\mathbb{R}^3} f(x_1, x_2, x_3)$$
$$= 1.$$

References

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