The Berezinian

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27.04.2021

The Berezinian, or superdeterminant, is a landmark of superalgebra and supergeometry. It is named after Felix A. Berezin who was a pioneer in supergeometry and superanalysis. We will see that the Berezinian is the super-analog of the classical determinant.
The construction of the Berezinian via linear superalgebra and the integral of the supertrace is presented. The properties of the Berezinian are discussed and Liouville's theorem is proven. In the end, an outlook to integration on superspaces is given.

1 Introduction

Let us briefly recall some concepts from linear superalgebra and fix the notation for the rest of this report. Let $V = A^{p|q}$ be a super vector space[1] of bosonic and fermionic dimensions $p|q$. A basis of V therefore consists of p even vectors $e_1, ..., e_p$ and q odd vectors $f_1, ..., f_q$, with the whole collection being linearly independent. The basis is abbreviated as $(e_1, ..., e_p|f_1, ..., f_q)$. The super vector space V can be decomposed as $V = V_{\text{even}} \oplus V_{\text{odd}}$, where the summands are of dimension $p|0$ and $0|q$. We can write a linear transformation[2] $T : A^{p|q} \to A^{p|q}$, $T \in \text{End}(A^{p|q})_0$, in matrix form as [10]

$$T = \begin{pmatrix} K & L \\ M & N \end{pmatrix} \in \text{Mat}(A^{p|q})_0 \,, \tag{1.1}$$

where K is a $p \times p$ matrix consisting of even elements of A, L is a $p \times q$ matrix of odd elements, M is a $q \times p$ matrix of odd elements and N is a $q \times q$ matrix of even elements. We say that K and N are even blocks and L and M are odd blocks. Note that the parity of the block matrices are forced by the parity preserving requirements on T. A matrix with such a structure is called graded matrix or (even) supermatrix.

The *supertranspose* of such an even supermatrix T is defined as [7]

$$T^{\text{st}} = \begin{pmatrix} K^{\text{T}} & M^{\text{T}} \\ -L^{\text{T}} & N^{\text{T}} \end{pmatrix} \,, \tag{1.2}$$

where the superscript T denotes the usual transpose of a matrix. Note that $(\text{st})^4 = \text{id}$. Furthermore, for $T, S \in \text{Mat}(A^{p|q})_0$, it satisfies the properties:

$$(T + S)^{\text{st}} = T^{\text{st}} + S^{\text{st}} \,, \tag{1.3}$$

$$(TS)^{\text{st}} = S^{\text{st}} T^{\text{st}} \,. \tag{1.4}$$

[1] $V = A^{p|q}$ is a free A-supermodule, for any superalgebra A of characteristic 0.
[2] We focus on even transformations, which preserve parity, and denote them with a subscript 0.

The *supertrace* of such an even supermatrix T is defined as [7]

$$\mathrm{str}\, T = \mathrm{tr}\, K - \mathrm{tr}\, N\,, \tag{1.5}$$

where tr denotes the usual trace of a matrix. It has the properties:

$$\mathrm{str}(T+S) = \mathrm{str}\, T + \mathrm{str}\, S\,, \tag{1.6}$$
$$\mathrm{str}(TS) = \mathrm{str}(ST)\,, \tag{1.7}$$
$$\mathrm{str}\, T^{\mathrm{st}} = \mathrm{str}\, T\,, \tag{1.8}$$

for even supermatrices, as above. Thus, $\mathrm{str}[T,S] = 0$ and $\mathrm{str}(STS^{-1}) = \mathrm{str}\, T$.

2 Berezinian via Linear Superalgebra

Let us motivate the definition of the Berezinian with the well-known result [8]

$$\det M = e^{\mathrm{tr}\ln M}\,, \tag{2.1}$$

for ordinary square matrices M. This is easily verified by setting $M = 1 + L$:

$$\mathrm{tr}\ln(1+L) = \mathrm{tr}\left(L - \frac{1}{2}L^2 + \frac{1}{3}L^3 - ...\right)\,. \tag{2.2}$$

If U is a matrix with

$$\mathrm{tr}\, L = \mathrm{tr}\left(ULU^{-1}\right) = \mathrm{tr}\,\Lambda = \sum_i \lambda_i\,, \tag{2.3}$$

where Λ is diagonal and λ_i are the eigenvalues of L. Then:

$$\begin{aligned}\mathrm{tr}\ln(1+L) &= \sum_i \lambda_i - \frac{1}{2}\sum_i \lambda_i^2 + \frac{1}{3}\sum_i \lambda_i^3 - ...\\ &= \sum_i \ln(1+\lambda_i) = \ln\prod_i (1+\lambda_i)\\ &= \ln\det(1+\Lambda) = \ln\det(1+L)\,,\end{aligned} \tag{2.4}$$

because of the cyclic property of the trace, so that

$$\mathrm{tr}\ln M = \ln\det M\,. \tag{2.5}$$

If we want to maintain this relation between the supertrace and the superdeterminant, or the Berezinian, we need to make the following definition.

Definition 2.1 (Berezinian)**.**
The determinant of a supermatrix T is the Berezinian, $BerT$, defined by

$$BerT = e^{str\ln T}\,. \tag{2.6}$$

We will see that although the supertrace is defined for all supermatrices, the Berezinian is defined only for the invertible ones.

In the simplest case, when $T \in \mathrm{Mat}(A^{p|q})_0$ is even and has diagonal form

$$T = \begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix}, \tag{2.7}$$

one can find a simple formula for the Berezinian in terms of the usual determinant:

$$\mathrm{Ber}\, T = e^{\mathrm{str} \ln T} = e^{\mathrm{tr} \ln K - \mathrm{tr} \ln N} = \det K \cdot \det N^{-1} . \tag{2.8}$$

Thus, already we must have N invertible. Note that K and N are even blocks, so their determinant is actually defined[1]. Unlike the determinant, the Berezinian does not extend onto the whole algebra $\mathrm{Mat}(A^{p|q})_0$. It is only defined for invertible matrices $T \in \mathrm{GL}(A^{p|q})$. $\mathrm{GL}(p|q, A) = \mathrm{GL}(A^{p|q})$ is defined as the group of even automorphisms of $A^{p|q}$ [1].
In the example above, we only needed N to be invertible. However, as we will see, there is a second formula for the Berezinian in which K has to be invertible. Since we want both formulas to be equivalent, we require K and N to be invertible. The following proposition comes in handy [6].

Proposition 2.1.
Let $T : A^{p|q} \to A^{p|q}$ be an even transformation with the usual block form as in Eq. (1.1). *Then T is invertible, if and only if K and N are invertible.*

We want to derive a similar formula for the Berezinian of a general even supermatrix

$$T = \begin{pmatrix} K & L \\ M & N \end{pmatrix} \in \mathrm{GL}(A^{p|q}) . \tag{2.9}$$

Let us first show a very important property of the Berezinian.

Proposition 2.2.
Let $T, S \in GL(A^{p|q})$ be two graded matrices. The Berezinian is multiplicative:

$$Ber(TS) = Ber T \cdot Ber S . \tag{2.10}$$

Proof. We follow the proof in [8]. Using the definition in Eq. (2.6), we have that

$$\mathrm{Ber}(TS) = e^{\mathrm{str} \ln(TS)} . \tag{2.11}$$

Define the two matrices

$$X := \ln T , \quad Y := \ln S , \tag{2.12}$$

and use the Baker-Campbell-Hausdorff formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[Y,X]]+\dots} = TS . \tag{2.13}$$

Taking the logarithm on both sides, so that

$$\ln(TS) = \ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [Y, X]] + \dots . \tag{2.14}$$

[1] In particular, $\mathrm{Ber}\, T$ is an element of $A_0^\times$, the even invertible elements of A.

Next, we take the supertrace of both sides, so that

$$\begin{aligned}\operatorname{str}\ln(TS) &= \operatorname{str} X + \operatorname{str} Y + \frac{1}{2}\operatorname{str}[X,Y] + ... \\ &= \operatorname{str}\ln T + \operatorname{str}\ln S + \frac{1}{2}\operatorname{str}(XY - YX) + ... \\ &= \operatorname{str}\ln T + \operatorname{str}\ln S\,. \end{aligned} \tag{2.15}$$

All commutator terms vanish because the supertrace obeys Eq. (1.7). Hence

$$\operatorname{Ber}(TS) = e^{\operatorname{str}\ln(TS)} = e^{\operatorname{str}\ln T + \operatorname{str}\ln S} = \operatorname{Ber} T \cdot \operatorname{Ber} S\,. \tag{2.16}$$

□

A direct consequence of the multiplicativity are the following propositions.

Proposition 2.3.
Let $T \in GL(A^{p|q})$ be a graded matrix. The Berezinian satisfies:

$$Ber T^{-1} = (Ber T)^{-1}\,. \tag{2.17}$$

Proof. This follows by a simple computation:

$$\operatorname{Ber} T^{-1} \cdot \operatorname{Ber} T = \operatorname{Ber}(T^{-1}T) = \operatorname{Ber} 1 = 1 \in A_0\,, \tag{2.18}$$
$$\operatorname{Ber} T \cdot \operatorname{Ber} T^{-1} = \operatorname{Ber}(TT^{-1}) = \operatorname{Ber} 1 = 1 \in A_0\,. \tag{2.19}$$

□

Proposition 2.4.
For block triangular supermatrices of even automorphisms, it holds that

$$Ber\begin{pmatrix} K & L \\ 0 & N \end{pmatrix} = Ber\begin{pmatrix} K & 0 \\ M & N \end{pmatrix} = \det K \cdot \det N^{-1}\,. \tag{2.20}$$

Proof. We give the proof for the block upper-triangular case. The other case is obtained by transposition. Notice that we can write the matrix as a product [6]

$$\begin{pmatrix} K & L \\ 0 & N \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & -N \end{pmatrix}\begin{pmatrix} 1 & K^{-1}L \\ 0 & -1 \end{pmatrix}\,. \tag{2.21}$$

Further, we have

$$\begin{pmatrix} 1 & K^{-1}L \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2}K^{-1}L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2}K^{-1}L \\ 0 & 1 \end{pmatrix}^{-1} \tag{2.22}$$

By multiplicativity, the product of the Berezinian of the conjugating matrices is $1 \in A_0$, so that

$$\operatorname{Ber}\begin{pmatrix} K & L \\ 0 & N \end{pmatrix} = \operatorname{Ber}\begin{pmatrix} K & 0 \\ 0 & -N \end{pmatrix} \cdot \operatorname{Ber}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \operatorname{Ber}\begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix}\,. \tag{2.23}$$

□

Now, let us derive a formula for the Berezinian of a general invertible supermatrix, as in Eq. (2.9). For any $T \in \mathrm{GL}(A^{p|q})$ of this form, we have the easily verified decomposition

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} K - LN^{-1}M & L \\ 0 & N \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N^{-1}M & 1 \end{pmatrix} . \tag{2.24}$$

Using the last proposition and the multiplicativity of the Berezinian, it follows that

$$\mathrm{Ber} \begin{pmatrix} K & L \\ M & N \end{pmatrix} = \det(K - LN^{-1}M) \cdot \det N^{-1} . \tag{2.25}$$

Note that some elements of the supermatrix are odd and therefore anticommuting. Consequently, it is impossible to define its determinant. At the same time, the matrices in the argument of the determinants in Eq. (2.25) consist only of even, hence commuting, elements. The determinants of these matrices are therefore defined [1].

Similarly, the decomposition

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} K & 0 \\ M & N - MK^{-1}L \end{pmatrix} \begin{pmatrix} 1 & K^{-1}L \\ 0 & 1 \end{pmatrix} , \tag{2.26}$$

implies another formula

$$\mathrm{Ber} \begin{pmatrix} K & L \\ M & N \end{pmatrix} = \det K \cdot \det(N - MK^{-1}L)^{-1} . \tag{2.27}$$

Again, the right-hand side of the equation is well-defined. In this case, K has to be invertible. Since we defined the Berezinian for invertible transformations $T \in \mathrm{GL}(A^{p|q})$, both K and N are invertible by Prop. 2.1, and the two formulas are equivalent.
Note that many textbooks take these formulas as the definition of the Berezinian. It is possible to prove multiplicativity from here but it is more technical.

Having established the formulas in Eq. (2.25) and Eq. (2.27), we can show some more properties of the Berezinian.

Proposition 2.5.
The map Ber : $GL(p|q, A) \to GL(1|0, A_0) = A_0^\times$ *is a group homomorphism, which agrees with the determinant when* $q = 0$.

Proof. This follows from the multiplicativity of the Berezinian. Note that for $q = 0$:

$$\mathrm{Ber}(K) = \det K . \tag{2.28}$$

□

Moreover, the Berezinian of an automorphism $T \in \mathrm{GL}(A^{p|q})$ does not depend on the choice of the basis. If we choose another basis, the matrix of T changes to $T' = STS^{-1}$, where S is some invertible even matrix. Because of the multiplicativity, we have

$$\mathrm{Ber}\, T' = \mathrm{Ber}(STS^{-1}) = \mathrm{Ber}\, T . \tag{2.29}$$

Proposition 2.6.
Given $T \in GL(A^{p'|q'})$ and $S \in GL(A^{p''|q''})$, we have

$$Ber(T \oplus S) = Ber T \cdot Ber S\,. \tag{2.30}$$

Proof. We follow the proof in [6]. Consider $T \in \mathrm{GL}(A^{p'|q'})$ and $S \in \mathrm{GL}(A^{p''|q''})$ given by

$$T = \begin{pmatrix} K' & L' \\ M' & N' \end{pmatrix}\,, \quad S = \begin{pmatrix} K'' & L'' \\ M'' & N'' \end{pmatrix}\,. \tag{2.31}$$

We can arange that the matrix of $T \oplus S$ is of the block form

$$T \oplus S = \left(\begin{array}{cc|cc} K' & 0 & L' & 0 \\ 0 & K'' & 0 & L'' \\ \hline M' & 0 & N' & 0 \\ 0 & M'' & 0 & N'' \end{array}\right)\,. \tag{2.32}$$

Then, a direct calculation shows that

$$\begin{aligned} \mathrm{Ber}(T \oplus S) &= \det\left(\begin{pmatrix} K' & 0 \\ 0 & K'' \end{pmatrix} - \begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix}\begin{pmatrix} N' & 0 \\ 0 & N'' \end{pmatrix}^{-1}\begin{pmatrix} L' & 0 \\ 0 & L'' \end{pmatrix}\right) \cdot \det\begin{pmatrix} N' & 0 \\ 0 & N'' \end{pmatrix}^{-1} \\ &= \det(K' - M'N'^{-1}L') \cdot \det(K'' - M''N''^{-1}L'') \cdot \det N'^{-1} \cdot \det N''^{-1} \\ &= \mathrm{Ber}\, T \cdot \mathrm{Ber}\, S\,. \end{aligned} \tag{2.33}$$

□

Proposition 2.7.
Let $T \in GL(A^{p|q})$ be a graded matrix. The Berezinian satisfies:

$$Ber T^{st} = Ber T\,. \tag{2.34}$$

Proof. This follows by a simple computation using Eq. (2.25):

$$\begin{aligned} \mathrm{Ber}\, T^{\mathrm{st}} &= \det\left(K^{\mathrm{T}} + M^{\mathrm{T}}(N^{\mathrm{T}})^{-1}L^{\mathrm{T}}\right) \cdot \det\left(N^{\mathrm{T}}\right)^{-1} \\ &= \det(K - LN^{-1}M) \cdot \det N^{-1} \\ &= \mathrm{Ber}\, T\,. \end{aligned} \tag{2.35}$$

□

Consider one last example: The Π-*transpose* of a supermatrix:

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix}^{\Pi} = \begin{pmatrix} N & M \\ L & K \end{pmatrix} \tag{2.36}$$

It satisfies the following properties, for $T, S \in \mathrm{Mat}(A^{p|q})_0$ [7]:

$$(T + S)^{\Pi} = T^{\Pi} + S^{\Pi}\,, \tag{2.37}$$
$$(TS)^{\Pi} = T^{\Pi} S^{\Pi}\,, \tag{2.38}$$
$$\Pi \circ \mathrm{st} \circ \Pi = (\mathrm{st})^4\,,\ \ \Pi^2 = \mathrm{id}\,. \tag{2.39}$$

Proposition 2.8.
Let $T \in GL(A^{p|q})$ be a graded matrix. The Berezinian satisfies:

$$Ber T^{\Pi} = Ber T^{-1} \,. \tag{2.40}$$

Proof. This follows by a simple computation using Eq. (2.25) and Eq. (2.27):

$$\operatorname{Ber} T^{\Pi} = \det(N - MK^{-1}L) \cdot \det K^{-1} = \operatorname{Ber} T^{-1} \,. \tag{2.41}$$

□

3 Liouville's Theorem

There is a connection between the Berezinian and the integral of the supertrace which can be useful in physics.

Theorem 3.1 (Liouville)**.**
Let $M(t) \in Mat(A^{p|q})_0$ and t a real parameter. Let $X(t) \in Mat(A^{p|q})_0$ satisfy the initial value differential equation

$$\frac{d}{dt} X(t) = M(t) X(t) \,, \quad X(0) = 1 \,. \tag{3.1}$$

Then $X(t) \in GL(A^{p|q})$, for all t, and

$$Ber\, X(t) = \exp\left(\int_0^t ds\, str\, M(s)\right) \,. \tag{3.2}$$

Proof. We follow the proof in [1]. Let $\tilde{X}$ be a solution of the differential equation

$$\frac{d}{dt} \tilde{X}(t) = -\tilde{X}(t) M(t) \,, \quad \tilde{X}(0) = 1 \,. \tag{3.3}$$

Then

$$\frac{d}{dt}\left(\tilde{X} X\right) = \frac{d\tilde{X}}{dt} X + \tilde{X} \frac{dX}{dt} = -\tilde{X} M X + \tilde{X} M X = 0 \,. \tag{3.4}$$

The initial conditions imply $\tilde{X} X = 1$, hence $X(t) \in \mathrm{GL}(A^{p|q})$, for all t. Let

$$M(t) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \,, \quad X(t) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \,. \tag{3.5}$$

Set $Y = X_{11} - X_{12} X_{22}^{-1} X_{21}$ and $Z = X_{22}^{-1}$. It follows from Eq. (3.1) that

$$\frac{d}{dt} Y = \left(M_{11} - X_{12} X_{22}^{-1} M_{21}\right) Y \,, \tag{3.6}$$

$$\frac{d}{dt} Z = -Z \left(M_{21} X_{12} X_{22}^{-1} + M_{22}\right) \,. \tag{3.7}$$

From here, by using the classical Liouville theorem, we obtain

$$\frac{d}{dt}\det Y = \mathrm{tr}\left(M_{11} - X_{12}X_{22}^{-1}M_{21}\right)\cdot\det Y\,, \tag{3.8}$$

$$\frac{d}{dt}\det Z = -\mathrm{tr}\left(M_{21}X_{12}X_{22}^{-1} + M_{22}\right)\cdot\det Z\,. \tag{3.9}$$

Further, by the property of the trace and the interchange of two odd matrices:

$$\mathrm{tr}\left(M_{11} - X_{12}X_{22}^{-1}M_{21}\right) = \mathrm{tr}\left(M_{11} + M_{21}X_{12}X_{22}^{-1}\right)\,. \tag{3.10}$$

Hence, by the definition of the Berezinian:

$$\begin{aligned}\frac{d}{dt}\mathrm{Ber}\,X &= \left(\frac{d}{dt}\det Y\right)\cdot\det Z + \det Y\cdot\left(\frac{d}{dt}\det Z\right)\\ &= \mathrm{tr}(M_{11} - M_{22})\cdot\det Y\cdot\det Z\\ &= \mathrm{str}\,M\cdot\mathrm{Ber}\,X\,.\end{aligned} \tag{3.11}$$

Taking $X(0) = 1$ into account, we get Eq. (3.2) after integration with respect to t. $\square$

From this theorem, the multiplicativity and the identity in Eq. (2.6) can be shown [1].

Let $X, Y \in \mathrm{GL}(A^{p|q})$. Connect them with the unit matrix by smooth curves $X(t), Y(t)$: $X(0) = Y(0) = 1$ and $X(1) = X$, $Y(1) = Y$. Set

$$A(t) = \frac{dX(t)}{dt}X^{-1}(t)\,,\quad B(t) = \frac{dY(t)}{dt}Y^{-1}(t)\,. \tag{3.12}$$

Then

$$\frac{d}{dt}(XY) = (A + B_1)\,XY\,,\quad B_1 = XBX^{-1}\,. \tag{3.13}$$

From here

$$\begin{aligned}\mathrm{Ber}(XY) &= \exp\left(\int_0^1 dt\ \mathrm{str}(A + B_1)\right)\\ &= \exp\left(\int_0^1 dt\ \mathrm{str}A + \int_0^1 dt\ \mathrm{str}B\right)\\ &= \mathrm{Ber}\,X\cdot\mathrm{Ber}\,Y\,.\end{aligned} \tag{3.14}$$

And to recover Eq. (2.6), set $M(t) = M$, $X(t) = \exp(tM)$ and $t = 1$ to get

$$\mathrm{Ber}\left(\exp M\right) = \exp\left(\mathrm{str}\,M\right)\,. \tag{3.15}$$

4 Integration on Superspaces

The Berezinian is closely connected to integration on superspaces and takes the role of a Jacobian determinant. In this Chapter, we will only give an outlook for this topic. We will learn more about this in the next Talks. For an introduction, see also Ref. [10].

Recall the classical case first. For a finite-dimensional vector space V, say with dimension $p > 0$, the 1-dimensional top exterior power $\Lambda^p V$ is called the *determinant* of V, denoted by $\det V$. Let $(e_1, ..., e_p)$ be a basis of V, then $\det V$ is spanned by the element $e_1 \wedge ... \wedge e_p$. If we perform a change of coordinates to a second basis $(e'_1, ..., e'_p)$ of V by a linear transformation M, then the corresponding two elements in $\det V$ are related by

$$e'_1 \wedge ... \wedge e'_p = \det M \cdot e_1 \wedge ... \wedge e_p \,. \tag{4.1}$$

For a finite-dimensional super vector space V of dimension $p|q$, the Berezinian of V, denoted by $\operatorname{Ber} V$, is defined analogously. For every basis $(e_1, ..., e_p|f_1, ..., f_q)$ of V, there is a corresponding element in $\operatorname{Ber} V$ that we denote by $[e_1 \; ... \; e_p|f_1 \; ... \; f_q]$. Note that the definition of an exterior algebra on a super vector space is more subtle than on an ordinary vector space, since there are also odd elements. If we perform a change of coordinates to a second basis $(e'_1, ..., e'_p|f'_1, ..., f'_q)$ by a linear transformation T, then the corresponding two elements in $\operatorname{Ber} V$ are related by

$$[e'_1 \; ... \; e'_p|f'_1 \; ... \; f'_q] = \operatorname{Ber} T \cdot [e_1 \; ... \; e_p|f_1 \; ... \; f_q] \,. \tag{4.2}$$

Thus, the Berezinian of V is a 1-dimensional vector space that one can think of as the space of densities on V. The best example are maybe differential forms:

On $V = \mathbb{R}^p$, one can integrate over a p-dimensional subspace $M \subseteq \mathbb{R}^p$, with local coordinates $(x_1, ..., x_p)$, by using the transformation rule

$$\int_{\Phi(M)} dx'_1 \wedge ... \wedge dx'_p = \int_M |\det J| \cdot dx_1 \wedge ... \wedge dx_p \,, \tag{4.3}$$

where $\Phi : M \to \Phi(M) \subseteq \mathbb{R}^p$ is a diffeomorphism and $J = D\Phi(x)$ is the Jacobian of Φ.

On $V = \mathbb{R}^{p|q}$, one can integrate over a $p|q$-dimensional subspace $M \subseteq \mathbb{R}^{p|q}$, with local coordinates $(x_1, ..., x_p|\theta_1, ..., \theta_q)$, by using the transformation rule

$$\int_{\Phi(M)} [dx'_1 \; ... \; dx'_p|d\theta'_1 \; ... \; d\theta'_q] = \int_M |\operatorname{Ber} J| \cdot [dx_1 \; ... \; dx_p|d\theta_1 \; ... \; d\theta_q] \,, \tag{4.4}$$

where $\Phi : M \to \Phi(M) \subseteq \mathbb{R}^{p|q}$ is a diffeomorphism and $J = D\Phi(x)$ is the Jacobian of Φ. We will understand this formula in further detail, when we study the Berezin integral.

Now, the formula in Eq. (2.8) for the Berezinian of a diagonal supermatrix is very intuitive, since the first factor comes from the transformation property of the ordinary differentials and the second factor stems from the transformation property of the Grassmannian-type differentials.

We mention that the explicit form of the generators of the Berezinian module can be constructed via homological algebra and the Koszul complex. For more information, see Ref. [6, 9].

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