Super-Spacetimes and Supersymmetric field theories

Some history

- **Mass splitting problem** → There are different species of particles which interact identically with other particles, and differ from each others only in the mass of the particles.

  → It is possible to associate the Poincaré group with an isospin group in such a way that the different masses are due to different charges of the isospin group?

- **Static quark model** → It was discovered a model where the internal 3-flavour symmetry group $SU(3)$ and the non-relativistic spin group $SU(2)$ were non-trivially combined into $SU(6) \supsetneq SU(3) \times SU(2)$. (In this way particles differing by spin fit into a 56-plet of $SU(6)$).

$\boxed{?}$ It is possible to combine internal symmetries groups (describing interactions) and the Poincaré group in a non-trivial way?

$$\Downarrow$$

Exists a consistent QFT with symmetry group

$$G \supsetneq \text{Poinc}(1,3) \times B \quad ?$$

- $\mathcal{H} \approx \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{(n)}$, $\mathcal{H}^{(n)} = \left(\mathcal{H}^{(1)}\right)^{\otimes n}$
- $S : \mathcal{H}_{in} \longrightarrow \mathcal{H}_{fin}$, unitary "S-matrix"
- $\mathcal{U} : \mathcal{H}^{(1)} \longrightarrow \mathcal{H}^{(1)}$, unitary, $[S, \mathcal{U}] = 0$

  $$\mathcal{U}(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \ldots \otimes \mathcal{U}(v_i) \otimes \ldots \otimes v_n$$

  "(unitary representation of a) symmetry"
- Internal symmetry ⇒ symmetry commuting with Poincaré group

Coleman-Mandula thm

Thm [Coleman, Mandula; Phys. Rev. 159 (1967)]

Let G be a connected symmetry group of the S-matrix, and let the following condition hold:

(i) G contains a subgroup locally isomorphic to $\mathrm{Poinc}(1,3) \equiv \mathcal{P}$.

(ii) All particles types correspond to positive-energy representations of $\mathcal{P}$. For any finite $M \in \mathbb{R}_>$, there are only finitely many particles of mass less than M.

(iii) Elastic scattering amplitudes are analytic function of the center of mass energy and the scattering angle, in some neighbourhood of the physical region.

(iv) For any two one-particle momentum eigenstates $|p_1\rangle, |p_2\rangle \in \mathcal{H}^{(1)}$, we have that $(S-\mathbb{1})(|p_1\rangle \otimes |p_2\rangle) \neq 0$ except perhaps certain isolated values of the center of mass energy $(p_1+p_2)^2$. (At almost all energy, any two plane waves scatter).

(v) [Technical assumption on the representation of the generators of G as integral operators in the momentum space]

Then G is locally isomorphic to the direct product of the internal symmetry group and $\mathcal{P}$.

proof: Omitted, see paper and Witten's non-rigorous kinematic argument.

⚠ **Loopholes**

- Theories in 1+1 dimensions admit only foreward and backward scattering $\Rightarrow$ violation of (iii)
- Symmetries of the action which are not captured by the S-matrix are not considered by the theorem (e.g. discrete or spontaneously broken ones).
- Symmetries described by Lie superalgebras evade the theorem $\Rightarrow$ Supersymmetry
  - ↳ Haag, Lopuszanski, Sohnius - Nucl. Phys. B, 88 (1975) 257

Deligne's thm on tensor categories

further reading: arxiv:math/0401347 and www.physicsforums.com/insights/supersymmetry-delignes-theorem

☆ → deep motivation for the study of supersymmetric theories.

Wigner classification → Elementary particles species are identified with irrep of the symmetry group of the theory. Many particle system are obtained tensoring representation spaces.

Deligne's question → Which are the possible symmetry groups whose irrep behave like elementary particles?

Tensor cathegory ≈ Abelian category endowed with tensoring functor ⊗ where objects can be exchanged in such a way that exchanging twice is the identity and every object has a dual under tensoring. Moreover homomeomorphisms between objects form a vector space.

↳ Categories of finite dimensional representations of groups are tensor categories.

↳ Under which conditions a tensor category is the representation category of some group, and if so, of which Kind of group?

Deligne's thm → Every K-linear tensor category is the representation category of an algebraic super-group.

Deligne - Moscow Math. Journal 2 (2002) no. 2

RmK: Coleman-Mandula thm + Haag-Łopuszański-Sohnius only state that supersymmetry is a way to combine Poincaré with internal symmetries in a non-trivial way. Deligne's thm is stronger, and says that up to very deep modifications of quantum mechanics, Lie supergroups describe the most general symmetries of fundamental physics.

Super Lie groups

Further reading
[K] Kac – Advances in Math. 26, 8-96 (1977)
[V] Varadarajan – Supersymmetry for Matematicians
[QFTS] Deligne et al. – QFT and Strings

Real Lie supergroup $\rightarrow$ Real supermanifold G with multiplication and inverse morphisms

$$\mu : G \times G \rightarrow G \qquad i : G \rightarrow G$$

and $1 : \mathbb{R}^{0|0} \rightarrow G$ defining the unit element, satisfying usual group axioms:

giving group morphisms to a supermanifold

(i) $\mu \circ (I \times \mu) = \mu \circ (\mu \times I)$ (associativity) (I identity morphism)

(ii) $\mu \circ (I \times i) = \mu \circ (i \times I) = 1$ (inverse)

(iii) $\mu \circ (I \times 1) = \mu \circ (1 \times I) = I$ (identity)

using functor of points

$\rightarrow$ Real supermanifold G such that for any supermanifold T, $\mathrm{Hom}(T, G)$ is a group, and for any supermanifold S and morphism $T \rightarrow S$, the corresponding map $\mathrm{Hom}(S, G) \rightarrow \mathrm{Hom}(T, G)$ is a group homomorphism

$\hookrightarrow$ functor of points $T \rightarrow \mathrm{Hom}(T, G)$ is a functor into the category of groups

Actions of super Lie groups, subsupergroups and stabilizers are defined as in the classical case

Example: $\mathbb{R}^{p|q}$ $\to$ Let $(x,\theta) = (x^1,\dots,x^p,\theta^1,\dots,\theta^q)$ be global coordinates on G, then for any supermanifold S the set $\mathrm{Hom}(S,G)$ is in $1{:}1$ correspondence with the set of vectors $(f,g) = (f^1,\dots,f^p,g^1,\dots,g^q)$, the f^i (resp g^α) being even (resp. odd) global sections of the structure sheaf $\mathcal{O}_S$.

On $\mathrm{Hom}(S,G)$ we have the additive group structure

$$(f,g) + (f',g') = (f+f', g+g')$$

So $S \to \mathrm{Hom}(S,G)$ is the group valued functor defining $\mathbb{R}^{p|q}$ as an abelian super Lie group.

Example $GL(p|q)$ $\to$ Take the space of dim $p|q$-matrices $M^{p|q} \simeq \mathbb{R}^{p^2+q^2|2pq}$ with coordinates written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{array}{lll} A=(a_{ij}) \quad D=(d_{\alpha\beta}) & \text{even submatrices} & 1\leq i,j\leq p \\ B=(b_{i\beta}) \quad C=(c_{\alpha j}) & \text{odd submatrices} & 1\leq \alpha,\beta\leq q \end{array}$$

Then $GL(p|q)$ is the open submanifold of $M^{p|q}$ whose reduced part is $GL(p)\times GL(q)$. A morphism in $\mathrm{Hom}(S,M^{p|q})$ is given by a set of global even sections $a_{ij}, d_{\alpha\beta} \in \mathcal{O}(S)_{\bar 0}$, $b_{i\beta}, c_{\alpha j} \in \mathcal{O}(S)_{\bar 1}$, and such a morphism defines a morphism into $GL(p|q)$ iff $\det(a)\det(d) \in \mathcal{O}(S)^*$ units. The group structure of $\mathrm{Hom}(S, GL(p|q))$ is then given by the usual matrix multiplication and $S \to \mathrm{Hom}(S, GL(p|q))$ defines $GL(p|q)$ as a super Lie group.

Super Lie algebras

Superalgebra $\to$ Super vector space A that is an associative algebra with unit such that multiplication is a morphism of super vector spaces $A \otimes A \to A$

(In particular this implies $|ab| = |a| + |b|$ for $a, b \in A$ and $|\cdot| : A \to \mathbb{Z}_2$ parity)

$\hookrightarrow$ A superalgebra is commutative if $\mu \circ c_{A,A} = \mu$ for $\mu : A \otimes A \to A$ multiplication and

$c_{V,W} : V \otimes W \to W \otimes V$ commutativity isomorphism, $c_{V,W} : v \otimes w \mapsto (-1)^{|v||w|} w v$

Lie superalgebra $\to$ Super vector space $\mathfrak{g}$ with morphism $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ s.t.

(i) $[\cdot,\cdot] \circ (1 + c_{\mathfrak{g},\mathfrak{g}}) = 0$

Jacobi

(ii) $[\cdot,[\cdot,\cdot]] \circ (1 + \sigma + \sigma^2) = 0$ $\quad$ $\sigma : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ cyclic permutation

$\hookrightarrow$ from the definition follows immediately that

(i) $\mathfrak{g}_0$ is an ordinary Lie algebra for $[\cdot,\cdot]$

(ii) $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module for the action $\begin{aligned} \mathfrak{g}_0 &\to \operatorname{End}(\mathfrak{g}_1) \\ a &\mapsto \operatorname{ad}(a) \end{aligned}$, $\operatorname{ad}(a)(b) = [a,b]$

(iii) $\begin{aligned} [\cdot,\cdot] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 &\to \mathfrak{g}_0 \\ a \otimes b &\mapsto [a,b] \end{aligned}$ is a symmetric $\mathfrak{g}_0$-module map

(iv) $\forall a \in \mathfrak{g}_1$ we have $[a,[a,a]] = 0$

- Given a superalgebra A we can promote it to a Lie superalgebra defining
  $[a,b]=\mu\circ(1-c_{A,A})(a,b)$. The resulting Lie superalgebra is denoted A_L
- If $A=\underline{End}(V)$ the corresponding Lie superalgebra is denoted $\underline{End}(V)_L\equiv gl(V)$
- For $X\in g$, $ad: g\to \underline{End}(g)$, $ad(X)(Y)=[X,Y]$, Jacobi identity becomes
  $[ad X, ad Y]=ad[X,Y]$

Lie superalgebra of a super Lie group

Thm [Thm 7.1.1. [V]]

The Lie algebra $g=Lie(G)$ of a super Lie group (the set of left/right invariant vector fields on G) is spanned by the set of left (resp. right) invariant vector fields

$$X_\tau=\sum_j\left(\frac{\partial\mu^j}{\partial g^K}\right)_e\frac{\partial}{\partial x^j}\qquad\left(\text{resp.}\quad {}_\tau X=\sum_j\left(\frac{\partial\mu'^j}{\partial g^K}\right)_e\frac{\partial}{\partial x^j}\right)$$

where $\tau\in T_eG$, $\mu: G\times G\to G$ is the multiplication, x^j are the coordinates on the point on which X_τ (resp ${}_\tau X$) is considered, g^K are the coordinates at the origin $e\in G$, and $\mu'(x,g)=\mu(g,x)$.

☒ Formulas above are very handy for calculations of left/right invariant vector fields. For instance in the case $G = \mathbb{R}^{1|1}$ we have

$(x,\theta)(x',\theta') = (x + x' + \theta\theta', \theta + \theta')$ and then

Lie algebra structure

Left inv. : $D_x = \partial_x$ $\quad D_\theta = -\theta\,\partial_x + \partial_\theta$ $\quad\Longrightarrow\quad [D_x, D_\theta] = 2D_x$

Right inv : $D_x = \partial_x$ $\quad D_\theta = \theta\,\partial_x + \partial_\theta$

Brief recup about Spin representations and Clifford modules

<u>Thm</u> The group $SO(n,\mathbb{C})$ is connected. The group $SO(p,q,\mathbb{R})$ is connected iff $p=0$ or $q=0$. Otherwise it has two connected components.

<u>Def</u> Let V be a finite dim vector space over a field K of characteristics 0. A quadratic form is a function $Q: V \to K$ s.t. $Q(x) = \Phi(x,x)$ where Φ is a symmetric bilinear form. If Φ is non-degenerate we say that Q is non-degenerated. A quadratic vector space is a pair (V,Q) where V is a finite dimensional vector space and Q is a non-degenerate quadratic form.

<u>Def</u> The Clifford algebra $C(V)$ of a quadratic vector space (V,Q) is the associative algebra generated by vectors in V with the relations $v^2 = Q(v)\cdot 1 \quad \forall v \in V$.

<u>Rmk</u> Relations for the Clifford algebra are equivalent to $xy + yx = 2\Phi(x,y)\cdot 1$. A pysicist can safely think of the Clifford algebra as the algebra generated by Dirac γ-matrices for a vector space V with metric Φ.

<u>Rmk</u> The Clifford algebra $C(V)$ is a superalgebra, where an element of $C(V)$ has grading $\bar{0}$ (resp $\bar{1}$) if it is a product of an even (resp. odd) number of vectors of V. Its dimension is $2^{\dim(V)}$.

Thm [[V] thm 5.3.3, 5.3.8]

Let K be algebraically closed.

(i) If $\dim(V) = 2m$ is even, $C(V) \cong \underline{\mathrm{End}}(S)$ for S super vector space s.t. $\dim(S) = 2^{m-1} | 2^{m-1}$.

(ii) If $\dim(V) = 2m+1$ is odd, then $C(V) \cong C(V)^+ \otimes \mathrm{center}(C(V))$, $C(V)^+ \cong \mathrm{End}(S_0)$, $\mathrm{center}(C(V)) \cong D$, where $C(V)^+$ is the even part of $C(V)$, S_0 is a 2^m-dim vector space and $D = \mathbb{C}[\varepsilon]$ with $|\varepsilon| = \bar{1}$ and $\varepsilon^2 = 1$.

Rmk For $\dim(V) = 2m$, let S^+ and S^- be the even and the odd part of S respectively. Then $C^+(V) \cong \mathrm{End}(S^+) \oplus \mathrm{End}(S^-)$.

Constr Let $\Gamma^+ = \{ u \in C^{+*} \mid u V u^{-1} \subset V \}$ closed Lie subgroup of C^{+*}.

Take $\alpha : \Gamma^+ \to \mathrm{End}(V)$, $\alpha(u)(v) = u v u^{-1}$ action of Γ^+ on V.

Since $Q(u v u^{-1}) \cdot 1 = (u v u^{-1})^2 \cdot 1 = u v^2 u^{-1} = Q(v) \cdot 1$ we have

$$\alpha : \Gamma^+ \to O(V) \subset \mathrm{End}(V), \quad \mathrm{Ker}\, \alpha = K^*.$$

It can be proved that $\mathrm{Lie}(C^{+*}) = C^+_L$ (superalgebra C^+ with the usual braket) and $\mathrm{Lie}(\Gamma^+) = \{ u \in C^+ \mid uv - vu \in V \ \ \forall v \in V \}$. Then

$$d\alpha : \mathrm{Lie}(\Gamma^+) \to \mathfrak{so}(V), \quad d\alpha(u)(v) = uv - vu, \quad \mathrm{Ker}\, d\alpha = K.$$

It can be proved that we have the following sequence of Lie algebras is exact

$$0 \longrightarrow K \longrightarrow \mathrm{Lie}(\Gamma^+) \xrightarrow{d\alpha} \mathfrak{so}(V) \longrightarrow 0$$

Moreover $d\alpha(xy)(v) = 2M_{xy}v$ for $v \in V$ and $M_{xy} \in \mathfrak{so}(V)$ generator of rotations in the plane spanned by x, y in V.

Thus defining $C^2 = \mathrm{span}\{xy - yx \in \mathrm{Lie}(\Gamma^+) \mid x, y \in V\}$ we get that

$d\alpha : C^2 \xrightarrow{\sim} \mathfrak{so}(V)$ is an isomorphism with inverse

$$\gamma : M_{xy} \longmapsto \frac{1}{4}(xy - yx)$$

Rmk: Interpreting $C(V)$ as the algebra generated by Dirac γ-matrices, the rank 1 elements in $C(V)$ are exactly the matrices $\gamma_0, \gamma_1, \dots, \gamma_{d-1}$ and γ above is the map associating to each element of the Lorentz algebra the corresponding generator in the space of Dirac spinors

$$M_{\mu\nu} \longmapsto \frac{1}{2}\gamma_{\mu\nu} \equiv \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

Note that the splitting $S \cong S^+ \oplus S^-$, $C^+ \cong \mathrm{End}(S^+) \oplus \mathrm{End}(S^-)$ corresponds to the splitting of the $\mathfrak{so}(1,3)$ action on a Dirac spinor into two conjugated actions of $\mathfrak{su}(2)$ on 2 Weyl spinors.

Thm [[V] thm 5.4.2]

If $\dim(V) \geq 3$, then C^2 is the unique subalgebra of $\mathrm{Lie}(\Gamma^+)$ isomorphic to $\mathfrak{so}(V)$, and γ is the only Lie algebra map splitting the exact sequence $0 \to K \to \mathrm{Lie}(\Gamma^+) \to \mathfrak{so}(V) \to 0$

Moreover C^2 generates $C^+(V)$ as an associative algebra.

If $K = \mathbb{C}$ and $G = \exp(C^2) \subset \Gamma^+$ then (G, α) is a double cover of $SO(V)$ and hence $G \cong \mathrm{Spin}(V)$. In this case G is the unique connected subgroup of Γ^+ covering $SO(V)$ and is the universal cover of $SO(V)$.

characterization of G as a subgroup of C^+

Thm [[V] thm 5.4.4]

Let β be the unique antiautomorphism of the ungraded Clifford algebra $C(V)$ which is the identity on V, $\beta : x_1 \cdot \ldots \cdot x_r \longmapsto x_r \cdot \ldots \cdot x_1$, $x_i \in V$.

The map $\varphi : \Gamma \to \mathbb{C}^* \cdot 1$, $x \mapsto x\beta(x)$, is a homomorphism. Let

$$G = \varphi^{-1}(1) \cap \Gamma^+ = \{x \in C^{+*} \mid xVx^{-1} \subset V,\ x\beta(x) = 1\}.$$

If $\dim(V) \geq 2$, then G is an analytic subgroup of C^{+*} of Lie algebra C^2 and (G, α) is a double cover of $SO(V)$. In particular $\mathrm{Spin}(V) \cong G$.

From the complex to the real case

Prop [[V] prop 5.4.5]

Let V arbitrary vector space over $\mathbb{C}$.

Then $\mathrm{Spin}(V) = \{ x = v_1 \cdot \ldots \cdot v_{2r} \mid v_i \in V,\ Q(v_i) = 1 \}$

Costr: Consider V vector space over $\mathbb{R}$. Let $V_{\mathbb{C}}$ be its complexification, and let $x \longmapsto x^{conj}$ be the unique conjugation on $C(V_{\mathbb{C}})$ extending the conjugation on $V_{\mathbb{C}}$ whose fixed points are elements of $C(V)$. The conjugation commutes with β and so it leaves $\mathrm{Spin}(V_{\mathbb{C}})$ invariant. We define

$$\mathrm{Spin}(V) = \{ x \in \mathrm{Spin}(V_{\mathbb{C}}) \mid x = x^{conj} \}.$$

Thm [[V] thm 5.4.7]

Let V be a real quadratic vector space and let $\mathrm{Spin}(V)$ as before.

If $\dim(V) \geqslant 3$ $\mathrm{Spin}(V)$ is the double cover of the component $SO(V)^{\circ}$ of $SO(V)$ connected to the identity.

If $V = \mathbb{R}^{p,q}$, then $\mathrm{Spin}(p,q) \equiv \mathrm{Spin}(V)$ is characterized as the unique double cover of $SO(V)^{\circ}$ when one of $p, q \leqslant 1$, and as the unique double cover that is nontrivial over both $SO(p)$ and $SO(q)$ when $p, q \geqslant 2$

Prop [[V] prop 5.4.8]

For $p, q \geqslant 0$ we have

$$\mathrm{Spin}(p,q) = \{ v_1 \cdots v_{2a} w_1 \cdots w_{2b} \mid v_i, w_j \in V,\ Q(v_i) = 1,\ Q(w_j) = -1 \}$$

Summary and further comments

- We obtained an embedding $\mathrm{Spin}(V) \hookrightarrow C^+$. In particular, we have a bijection between simple C^+-modules and certain irreducible $\mathrm{Spin}(V)$-modules. These are the spin and semi-spin representations (for dim V even and odd respectively).
- Spin modules are irreducible C^+-modules (called Clifford modules).
- The algebra C^+ turns out to be semisimple, and so the restriction of any C^+-module to a $\mathrm{Spin}(V)$-module is a direct sum of spin modules. Restrictions of C^+-modules to $\mathrm{Spin}(V)$-modules are called spinorial modules.

Super Poincaré algebra

If we consider the more general setting of Lie superalgebras, which are the restrictions analogous to the ones introduced by Coleman-Mandula thm for ordinary Lie algebras?

$\rightsquigarrow$ Nucl. Phys. B, 88 (1975); see also Sohnius - Phys. Rep. 128 (1985)

Haag-Lopuszanski-Sohnius $\longrightarrow$ The most general Lie superalgebra containing the Poincaré group and an internal symmetries group B is generated by

- P_μ, $M_{\mu\nu}$ generators of Poincaré
- B_ℓ (Poincaré) scalar generators of B
- Z_{ij} central generators

$\Big\}$ $\mathbb{Z}_2$ grade $\bar{0}$

- Q^i_A rank 1 spinors $\}$ $\mathbb{Z}_2$ grade $\bar{1}$

In $d=4$: $\rightsquigarrow$ $S = \underset{SU(2)}{S^+} \oplus \underset{\overline{SU(2)}}{S^-}$

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho})$$

$$[B_r, B_s] = c^t_{rs}B_t$$

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0$$

$$[Z_{ij}, \text{anything}] = 0$$

$$[Q^i_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \qquad [\bar{Q}^i_{\dot\alpha}, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}^i_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$$

$$[Q^i_\alpha, P_\mu] = [\bar{Q}^i_{\dot\alpha}, P_\mu] = 0$$

$$[Q^i_\alpha, B_r] = (b_r)^i{}_j Q^j_\alpha \qquad [\bar{Q}^i_{\dot\alpha}, B_r] = -\bar{Q}^j_{\dot\alpha}(b_r)^i{}_j$$

$$[Q^i_\alpha, \bar{Q}^j_{\dot\beta}] = 2\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu$$

$$[Q^i_\alpha, Q^j_\beta] = 2\epsilon_{\alpha\beta}Z^{ij} \qquad [\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}] = -2\epsilon_{\dot\alpha\dot\beta}Z^{ij}$$

- Neglecting for a moment the generators B_r and Z^{ij} the so called super Poincaré algebra. It is a Lie superalgebra $\mathfrak{g}$ s.t.

(i) $\mathfrak{g}^0$ is the ordinary Poincaré algebra $\overset{V}{\overset{\equiv}{\mathbb{R}^{1,d-1}}} \oplus so(1,d-1)$

(ii) $\mathfrak{g}^1$ is a real spinorial module S for $Spin(1,d-1)$. It can be reduced into N real spin modules. $\longrightarrow$ Extended supersymmetry

In Minkowski signature always exists a symmetric equivariant pairing

$\Gamma: S \times S \longrightarrow V$, equivariant w.r.t. $Spin(V)$.

We regard S as a $\mathfrak{g}^0$-module, where V acts trivially on S. Then defining $[s_1, s_2] = \Gamma(s_1, s_2)$, $s_1, s_2 \in S$, we have a super Lie algebra, becouse of $[s,[s,s]] = [s, v] = 0$.

- If S is a real spin module, Γ is unique up to scalars, and the sign of the scalar can be choosen so that for $s \in S \setminus \{0\}$

$$(v, \Gamma(s,s)) > 0 \qquad \forall v \in V^+$$ timelike

If $S = \bigoplus_{i=1}^{N} S_i$, for S_i real spin modules, $\Gamma_i : S_i \otimes S_i \to \mathbb{R}$ positive as before, then $\Gamma: \Sigma_j s_j, \Sigma_k t_k \longrightarrow \Sigma_i \Gamma_i(s_i, t_i)$, is positive as well. Positivity is crucial to get a positive energy of the physical system.

- Fixed $\Gamma : S \otimes S \to V$, $S = \bigoplus_{i=1}^{N} S_i$ for S_i irreducible spin modules, we can choose a basis (Q_A^i) for S, $1 \leq i \leq N$, so that $[Q_a^i, Q_b^j] = \delta^{ij} \Gamma_{ab}^{\mu} P_\mu$
- The introduction of the generators B_ℓ in the Lie algebra is trivial. Regarding the central charges Z^{ij}, they arises when the symmetric part $S \otimes S$ contains some copies of the trivial representation, i.e there is a symmetric pairing $S \otimes S \to \mathbb{R}^c$ for some c. Then we can form a new Lie superalgebra by adding $\mathbb{R}^{c'}$ to the even part of the super Poincaré algebra, for any $c' < c$: $\mathfrak{g} = V \oplus \mathbb{R}^{c'} \oplus \mathfrak{so}(1, d-1) \oplus S$. This is said a central extension of the Lie superalgebra.

Super Spacetimes

Idea $\Rightarrow$ The simpler way to make a theory invariant under a given symmetry is to define the theory over a space whose group of isometries contains such symmetries (in this way the action, given by the integration of the lagrangian density over the whole spacetime, is invariant).

■ Consider the Lie superalgebra $\tilde{g} = \mathbb{R}^{1,d-1} \oplus S$ obtained from the super Poincaré algebra removing the $so(1,d-1)$ generators. It is a supersymmetric extension of the abelian spacetime translation algebra $\mathbb{R}^{1,d-1}$, but $\tilde{g}$ is not abelian since $\Gamma \neq 0$. However, $[a,[b,c]] = 0 \quad \forall a,b,c \in \tilde{g}$.

The corresponding supes Lie group $L = \exp(\tilde{g})$ is called superspacetime.

Using the BCH formula we get

$$\exp(A)\exp(B) = \exp\left(A + B + \tfrac{1}{2}[A,B]\right) \quad \forall A,B \in \tilde{g}.$$

Thus we can identify L with $\tilde{g}$ and define the group law

$$A \circ B = A + B + \tfrac{1}{2}[A,B]$$

From the algebraic point of view, L could be characterized as follows. Take (B_μ), (F_a) bases of $\mathbb{R}^{1,d-1}$ and S, respectively, then for any supermanifold T, $\mathrm{Hom}(T, \tilde{\mathfrak{g}})$ can be identified with (β_μ, τ_a) where β_μ and τ_a are elements of $\mathcal{O}(T)$ that are even and odd, respectively. Equivalently

$$\mathrm{Hom}(T, \tilde{\mathfrak{g}}) = (\tilde{\mathfrak{g}} \otimes \mathcal{O}(T))_0 = V \otimes \mathcal{O}(T)_0 \oplus S \otimes \mathcal{O}(T)_1$$

Clearly $\mathrm{Hom}(T, \tilde{\mathfrak{g}})$ is a Lie algebra with the only non-trivial bracket $[s_1 \otimes \tau_1, s_2 \otimes \tau_2] = -\Gamma(s_1, s_2)\tau_1\tau_2$ where $\tau_1, \tau_2 \in \mathcal{O}(T)_1$, $s_1, s_2 \in S$.

We now take $\mathrm{Hom}(T, L) = \mathrm{Hom}(T, \tilde{\mathfrak{g}})$ and we define a binary operation on $\mathrm{Hom}(T, L)$ by $A \circ B = A + B + \frac{1}{2}[A, B]$ for all $A, B \in \mathrm{Hom}(T, \tilde{\mathfrak{g}})$. The Lie algebra structure on $\mathrm{Hom}(T, \tilde{\mathfrak{g}})$ implies that this is a group law.

- From the group multiplication written in coordinates we can apply the previous theorem on left/right invariant vector fields on supergroups. Using $(x,\theta)\cdot(x',\theta') = (x'',\theta'')$ where

$$x''^{\mu} = x^{\mu} + x'^{\mu} - \frac{1}{2}\Gamma^{\mu}_{ab}\,\theta^{a}\theta'^{b} \qquad \theta''^{a} = \theta^{a} + \theta'^{a}$$

we get

left inv. v.f. $D_\mu = \partial_\mu \qquad D_a = \frac{1}{2}\Gamma^{\mu}_{ab}\,\theta^{b}\partial_\mu + \partial_a$

right inv v.f. $D_\mu = \partial_\mu \qquad D_a = -\frac{1}{2}\Gamma^{\mu}_{ab}\,\theta^{b}\partial_\mu + \partial_a$

It follows that (for both left/right inv. v.f.)

$$[D_a, D_b] = \Gamma^{\mu}_{ab}\,\partial_\mu$$

thus making the identification $P_\mu \longleftrightarrow D_\mu$, $Q_a \longleftrightarrow D_a$ we get a representation of the super Lie algebra $\tilde{g}$ in terms of left/right invariant vector fields.

- In the extended case $S = \bigoplus S_i$, the multiplication property becomes

$$x''^{\mu} = x^{\mu} + x'^{\mu} - \frac{1}{2}\sum_i \Gamma^{\mu}_{ab}\,\theta^{ai}\theta^{bi} \qquad \theta''^{ai} = \theta^{ai} + \theta'^{ai}$$

Outro

Given some superspacetime M and a bundle $E \to M$, one could ask for the analogue of Poincaré invariant field equations in the super context. More in general one could be interested in the G-invariant super differential operators D for a given super Lie group G and in the solutions of the equations $D\psi = 0$ where ψ is a global section of the structure sheaf.

This gives an extension of Klein-Gordon and Dirac operators, and ultimately leads to the formulation of the supersymmetric analogue of classical field theory.

Moreover one could extract from a superfield its component fields obtaining several component fields. This leads to the notion of multiplet and to the idea that a superparticle defines a multiplet of ordinary particles.

One could also impose superfield equations to the metric of an unspecified supermanifold obtaining in this way a supergravity theory.

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