L_∞ in the algebra and coalgebra picture and the connection to the BV picture

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Graded symmetric tensor algebra

- $X = \bigoplus_{i \in Z} X_i$: integer graded vector space
- graded symmetry: for

$$x_1, x_2, \ldots, x_n \in X x_1 \wedge \ldots \wedge x_n = \epsilon(\sigma, x) x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)}$$

- $S(X) = \bigoplus_{i=1}^{\infty} S^n X$: graded symmetric tensor algebra
- for $\{T_a\}$ a basis of X, S : space of polynomials in T_a

- define the operator: $\Delta : S \to S \otimes S$, by:

$$\Delta(x_1 \wedge \ldots \wedge x_n) = \Sigma_{i=1}^{n-1} \Sigma_{\sigma \in (i,n-i)} \epsilon(\sigma, x)$$
$$(x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \wedge \ldots \wedge x_{\sigma(n)})$$

- examples:
  - $\Delta(x) = 0$
  - $\Delta(x_1 \wedge x_2) = x_1 \otimes x_2 + (-1)^{|x_1||x_2|} x_2 \otimes x_1$
  - $\Delta(x_1 \wedge x_2 \wedge x_3) = x_1 \otimes (x_2 \wedge x_3) + (-1)^{|x_1||x_2|} x_2 \otimes (x_1 \wedge x_3) + (-1)^{|x_3|(|x_1|+|x_2|)} x_3 \otimes (x_1 \wedge x_2) + (x_1 \wedge x_2) \otimes x_3 (-1)^{|x_2||x_3|} (x_1 \wedge x_3) \wedge x_2 + (-1)^{(|x_2|+|x_3|)|x_1|} (x_2 \wedge x_1) \otimes x_1$

- applying the coproduct twice we have coassiativity:

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & S \otimes S \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta \otimes 1} \\ S \otimes S & \xrightarrow{1 \otimes \Delta} & S \otimes S \otimes S \end{array}$$

or equivalently:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

- Remark on Notation:

$$(1 \otimes f)(g \otimes 1) = (-1)^{|f||g|}(g \otimes f)$$

- Define a map $D : S \to S$ of odd degree, by:

  $D(x_1 \wedge \ldots \wedge x_n) = \Sigma_{1 \leq i < J \leq n}^{i+j+1} [\, x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n$

  for

  $$x_1 \wedge \ldots \wedge x_n \in S$$

- examples:

  $$D(x_1 \wedge x_2) = [\, x_1, x_2]$$

  $D(x_1 \wedge x_2 \wedge x_3) = [\, x_1, x_2] \wedge x_3 + [\, x_2, x_3] \wedge x_1 - [\, x_1, x_3] \wedge x_2$

- then the Jacobi identity is included in: $D^2 = 0$

- Combining the coderivation with the coproduct we get the Co-Leibnitz-property:

$$\begin{array}{ccc} S & \xrightarrow{\quad D \quad} & S \\ \Big\downarrow \Delta & & \Big\downarrow \Delta \\ S \otimes S & \xrightarrow{1\otimes\Delta+\Delta\otimes 1} & S \otimes S \end{array}$$

- or equivalently:

$$\Delta D = (1 \otimes D + D \otimes 1)\Delta$$

- A L_∞-Algebra is defined as:
  - A $\mathbb{Z}$ graded vector space
  - equipped with multilinear maps:

$$b_i : X^i \to X,$$

  of intrinsic degree -1 such that $D = \Sigma_{i=1}^{\infty} b_i$ defines a coderivation, with $D^2 = 0$

- in the lowest orders D is then given by $b_1 = \partial$, $b_2 = [.,.]$ and

  $b_i(x_1 \wedge \ldots \wedge x_j) = \Sigma_{\sigma \in (i,j-i)} \epsilon(\sigma, x) b_i(x_{\sigma(1)}, \ldots x_{\sigma(j)}) \wedge (x_{\sigma(i+1)} \wedge \ldots \wedge x_{\sigma(j)})$

- $D^2 = 0$ in the lowest orders is them given by: $b_1^2 = \partial^2 = 0$,
  $b_1 b_2 + b_2 b_1 = 0 \iff \partial[x_1, x_2] = -[\partial x_1, x_2] - (-1)^{|x_1|}[x_1, \partial x_2]$
  and the graded Jacobiator plus its failure :
  $b_1^2 + b_1 b_3 + b_3 b_1 = 0$

Cyclic L_∞ and chain complexes

- equipped with an inner product: $\kappa : X \otimes X \to \mathbb{R}$ the L_∞-Algebra is called cyclic
- the L_∞-Algebra can also be viewed as a homology chain, as the coderivation is of $deg - 1$ and nilpotent

Dual of the graded symmetric algebra

- consider an L_∞-Algebra and basis T_a with each element of degree $deg(T_a) = |a|$
- then b_n can be written in terms of some structure constants $C^a_{c_1 \ldots c_n}$ by:

$$b_n(T_{c_1}, \ldots, T_{c_n}) = C^a_{c_1 \ldots c_n} T_a$$

- define a dual basis z^a, of the space $X*$ of linear functions on X, with $z^a(T_b) = \delta^a_b$
- basis of $S(X)$ given by graded symmetric monomials $T_{b_1} \ldots T_{b_m}$
- the dual space $S(X)^*$ is then space of power series in z^a

Dual of the coproduct

- define the product of two functions $f_1(z^a)$ and $f_2(z^a)$:

$$m : S^* \otimes S^* \to S^*$$

$$m(f_1 \otimes f_2) = f_1 f_2$$

- the duality to the coproduct is given by:

$$< m(f_1 \otimes f_2), x >=< f_1 \otimes f_2, \Delta(x) >$$

- associativity:

$$(f_1 f_2) f_3 = f_1 (f_2 f_3) \iff m(m \otimes 1) = m(1 \otimes m)$$

Dual of the coderivation

- Define a derivation on S^* by:

$$Q = \Sigma_{n=1}^{\infty} \frac{1}{n!} C^a_{b_1,\ldots,b_n} z^{b_1} \ldots z^{b_n} \frac{\partial}{\partial z^a}$$

- duality to D, easily seen by using $< z^{b_1} \ldots z^{b_n}, T_{c_1} \ldots T_{c_m} >= n! \delta^{(a_1}_{b_1} \ldots \delta^{a_n)}_{b_n}$ to show $< Qz^a, T_{c_1} \ldots T_{c_m} >=< z^a, \Sigma_{n=1}^{\infty} b_n(T_{c_1} \ldots T_{c_m}) >$
- the nilpotency of D thus also implies the nilpotency of Q

- Q is of $deg 1$
- for $p_1, p_2 \in S^*$, Q as a derivation satisfies the Leibniz rule

$$\begin{aligned} Q(p_1 p_2) &= Q(p_1)p_2 + (-1)^{deg(p_1)} p_1 Q(p_2) \\ \iff Qm &= m(Q \otimes 1 + 1 \otimes Q) \end{aligned}$$

Cylic L_∞, antibrackets and classical master equations

- an L_∞-Algebra is called cyclic when equipped with an inner product:

$$\kappa(x_1, b_n(x_2, x_3, ..., x_{n+1})) = (-1)^{|x_1||x_2|}\kappa(x_2, b_n(x_1, x_3, ..., x_{n+1}))$$

- only non-zero component between spaces X_n and X_{-n+1} and $\kappa_{ab} = \kappa(T_a, T_b) = (-1)^{(a+1)(b+1)}\kappa_{ba}$, we have $\kappa_{ab} = \kappa_{ba}$
- assuming non-degeneracy and with the inverse κ^{ab}, the antibracket can be defined:

$$(f, g) = (-1)^{(degf)(z^a)}\frac{\partial f}{\partial z^a}\kappa^{ab}\frac{\partial g}{\partial z^b}$$

- thus the BV master action can be defined:

$$\Theta = \Sigma_{n=1}^{\infty}\frac{1}{(n+1)!}C_{ab_1...b_n}z^a z^{b_1}...z^{b_n}$$

- then Q can be written as $Q = (\Theta, .)$ and the nilpotency, containing all L_∞ relations, as the classical master equation:

$$(\Theta, \Theta) = 0$$

- assume BV Supermanifold M with fields and anti-fields providing local (Darboux) coordinates: $\Phi^a = (\phi^i, \phi^*_i)$
- an odd symplectic form is then given by:

$$\omega = \frac{1}{2} d\Phi^a \wedge \omega_{ab} d\Phi^b = (-1)^i d\phi^i \wedge d\phi^*_i$$

- thus the antibracket is (with $\frac{\partial_r}{\partial \Phi^a} = (-1)^{a(f+1)} \frac{\partial f}{\partial \Phi^a}$):

$$(f, g) = \frac{\partial_r f}{\partial \phi^i} \frac{\partial g}{\partial \phi^*_i} - \frac{\partial_r f}{\partial \phi^*_i} \frac{\partial g}{\partial \phi^i}$$

- Q is a hamiltonian vector field with hamiltonian function Θ
- the components are given by: $Q_a = \omega_{ab} Q^b = \partial_a \Theta$
- given a solution (of the classical master equation) around $\Phi = 0$ the vector field Q can be expanded around it:

$$Q(\Phi) = \Sigma_{n=1}^{\infty} \frac{1}{n!} C^a_{b_1 \ldots b_n} \Phi^{b_1} \ldots \Phi^{b_n} \frac{\partial}{\partial^a}$$

- the master action is then:

$$\Theta = \Sigma_{n=2}^{\infty} \frac{1}{n!} C_{b_1 \ldots b_n} \Phi^{b_1} \ldots \Phi^{b_n}$$

- and the coefficients:

$$C_{b_1 \ldots b_n} = \frac{\partial^n \Theta}{\partial^{b_n} \ldots \partial^{b_1}} |_{\Phi=0}$$

  give structure constants that fulfill generalised Jacobi identities

Link to Field Theories

- with a graded vertor space, isomorphic to the tangent space of the BV-manifold at $\Phi = 0$, with a basis T_a, the structure constants and the two form ω we get a cyclicL_∞-Algebra
- the connection of BV and the algebraic formulation is given by the replacement $\Phi^a \to z^a$ and thus $Q(\Phi) \to Q(z^a)$
- while $Q(\Phi)$ is given on the BV manifold M $Q(z)$ is given on S^*
- the brackets of fields can then be defines with $\Phi = \Phi^a T_a$, as:

$$B_n(\Phi_1, ..., \Phi_n) = \Phi_1^{c_1} ... \Phi_n^{c_n} b_n(T_{c_1}, ..., T_{c_n})$$

- the BV manifold is locally isomorphic to a super vectors pace $V = \mathbb{R}^{m|n}$
- for a function on V, $f = \Sigma_n a_{b_1 \dots b_n} \Phi^{b_1} \dots \Phi^{b_n}$, consider the map

$$\Lambda : \Sigma_n a_{b_1 \dots b_n} \Phi^{b_1} \dots \Phi^{b_n} \to \Sigma_n a_{b_1 \dots b_n} z^{b_1} \dots z^{b_n}$$

- then we have $Q(z) = \Lambda Q(\Phi) \Lambda^{-1}$
- Λ is the map taking the BV structure, like the antibracket, master equation and the two form to S^*

- assuming the functional integral

$$\int_{\Sigma} d\Phi^{a} \exp \frac{i}{h}(\Theta)$$

- with the gauge function Ψ the antifields are then given by: $\phi_i^* = \frac{\partial \Psi}{\partial \phi^i}$
- with $\Delta = \frac{\partial_r}{\partial \phi^i} \frac{\partial_l}{\partial \phi_i^*}$ the quantum master equation is given by:

$$(\Theta, \Theta) = 2ih\Delta\Theta$$

- Consider field theory, given by the action

$$S[\phi] = \frac{1}{2}A_{ij}\phi^i\phi^j + \Sigma_{k=2}^{\infty}\frac{1}{k!}A_{i_1\ldots i_k}\phi^{i_1}\ldots\phi^{i_k}$$

- with the fields and antifields $(\phi^i, \phi_i^*) \in \mathbb{R}^{n|n}$ the *BV* structure is then given by the symplectic form $\kappa^i_j = \delta^i_j$ and the antibracket:

$$(f, g) = \frac{\partial_r f}{\partial\phi^i}\frac{\partial g}{\partial\phi_i^*} - \frac{\partial_r f}{\partial\phi_i^*}\frac{\partial g}{\partial\phi^i}$$

- master action and the corresponding homologous vector field are given by:

$$\Theta = \Sigma_{k=2}^{\infty} \frac{1}{k!} A_{j_1 \ldots j_k} \phi^{j_1} \ldots \phi^{j_k}$$

,

$$Q = \Sigma_{k=1}^{\infty} \frac{1}{k!} A_{ij_1 \ldots j_k} \phi^{j_1} \ldots \phi^{j_k} \frac{\partial}{\partial \phi_i^*},$$

- as there are no gauge symmetries the L_∞ the is on $X_0 \oplus X_{-1}$, with Basis elements T_i and T^{*i}
- we can get to the coalgebra picture by considering the equations of motion:

$$\frac{\partial S}{\partial \phi^i} = A_{ij} \phi^j + \Sigma_{k=2}^{\infty} \frac{1}{k!} A_{ij_1 \ldots j_k} \phi^{j_1} \ldots \phi^{j_k} = 0$$

- the nonzero brackets are given by:

$$b_n(T_{i_1}, \ldots, t_{i_n}) = A_{i_1 \ldots i_n i_{n+1}} T^{*i_{n+1}}$$

- we can also get to the algebra picture by $\phi \to \zeta$, with a dual basis $z^a = (\zeta^i, \zeta_i^*)$:

$$\Theta = \Sigma_{k=2}^{\infty} \frac{1}{k!} A_{j_1 \ldots j_k} \zeta^{j_1} \ldots \zeta^{j_k}$$

,

$$Q = \Sigma_{k=1}^{\infty} \frac{1}{k!} A_{ij_1 \ldots j_k} \zeta^{j_1} \ldots \zeta^{j_k} \frac{\partial}{\partial \phi_i^*}$$

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