The BV formalism for physical theories

featuring odd symplectic supermanifolds

22.06.21

Tobias Witt

Recall from last time:

$\boxed{\text{Thm. (Schwarz)}}$

- $\mathcal{F}$ an SP-manifold
  i.e. a supermfd. with odd symplectic structure ω and compatible Berezinian μ
- $\mathcal{L}, \mathcal{L}'$ two Lagrangian submanifolds whose bodies define homologous cycles in $|\mathcal{F}|$
- $f \in \mathcal{O}(\mathcal{F})$ a function satisfying $\boxed{\Delta_\mu f = 0}$

Then
$$\int_{\mathcal{L}} \sqrt{\mu}|_{\mathcal{L}}\, f = \int_{\mathcal{L}'} \sqrt{\mu}|_{\mathcal{L}'}\, f$$

Relevant case for physics: $\boxed{f = e^{\frac{1}{g}S} \quad \text{with} \quad g = -i\hbar}$

Questions that will be answered in this talk:

Starting from a physical theory (classical gauge theory) ....

.... what is $\mathcal{F} = \mathcal{F}_{BV}$?

.... what are choices for $\mathcal{L}$ and $\mathcal{L}'$?

.... how do we ensure $\Delta_\mu f = 0$?

Aim: Recover the Fadeev-Popov path integral from this more general BV-formalism.

First spoiler:
$$\boxed{\Delta_\mu e^{\frac{1}{g}S} = 0 \iff \frac{1}{2}\{S,S\} + g\Delta_\mu S = 0 \quad \text{(Quantum Master Equation)}}$$

For $\hbar = 0$ (resp. $g = 0$) this becomes $\boxed{\{S,S\} = 0 \quad \text{(Classical Master Equation)}}$

$$\boxed{\text{Preliminaries about SP-manifolds}}$$

(later we'll have these structures on $\mathcal{F}_{BV}$)

Bookkeeping: $\boxed{\text{Basic structures on P-manifolds}}$

$\mathcal{F}$ (finite-dim) real supermanifold with odd symplectic structure ω (later this will be $\mathcal{F} = \mathcal{F}_{BV}$)

Def For $f \in \mathcal{O}(\mathcal{F})$ we define ....

.... the Hamiltonian vector field $H_f \in \mathrm{Der}(\mathcal{O}_{\mathcal{F}})$ by $\boxed{i_{H_f}\omega = df}$

.... the Poisson bracket with f by $\boxed{\{f, \cdot\} = (-1)^{|f|+1} H_f}$

In the following write $\mathcal{A} := \mathcal{O}(\mathcal{F})$ for the algebra of functions on $\mathcal{F}$
So in our case this is just a $\mathbb{Z}_2$-graded supercommutative algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$

Immediate Observations 0) Since ω is odd (and also d & i_- are parity-reversing operators on $\Omega^\bullet = \mathcal{S}^\bullet(\Pi T^*\mathcal{F})$)
a function f and its Hamiltonian v.f. H_f have opposite parity.

1) $\boxed{\text{The bracket } \{\cdot,\cdot\}: \mathcal{A}_i \otimes \mathcal{A}_j \longrightarrow \mathcal{A}_{i+j+1} \text{ is bilinear, skew-symmetric \& of degree } 1}$

$$\{f,g\} = (-1)^{|f|+1} H_f(g) = (-1)^{|f|+1} \underbrace{\langle H_f, dg\rangle}_{\text{by def. } i_{H_f} dg} = (-1)^{|f|+1} i_{H_f} i_{H_g}\omega = -(-1)^{(|f|+1)(|g|+1)} \{g,f\}$$

$\{\cdot,\cdot\}$	$\lvert g\rvert = 0$	$\lvert g\rvert = 1$
$\lvert f\rvert = 0$	sym.	antisym.
$\lvert f\rvert = 1$	antisym.	antisym.

$$|\{f,g\}| = |i_{H_f}| + |i_{H_g}| + |\omega| = (|H_f|+1) + (|H_g|+1) + 1 = |f| + |g| + 1$$

2) Note that by its definition $\{f,\cdot\} \sim H_f$ is a vector field and thus acts on $(\mathcal{A},\cdot)$ as a derivative of parity $|H_f| = |f|+1$:

$$\boxed{\{f, gh\} = \{f,g\}h + (-1)^{|g|\cdot(|f|+1)} g\{f,h\} \quad (\text{“Leibniz”})}$$

3) Similarly the Jacobi identity tells us that $H_f = (-1)^{|f|+1}\{f,\cdot\}$ is a derivative on $(\mathcal{A},\{\cdot,\cdot\})$ of parity again $|f|+1$:

$$\boxed{H_f\{g,h\} = \{H_f g, h\} + (-1)^{(|g|+1)(|f|+1)}\{g, H_f h\}} \quad (*)$$

#

$\boxed{\text{RK}}$ $\mathcal{A}$ satisfying 1) – 3) is called a <u>Gerstenhaber algebra</u>

$\boxed{\text{Simple auxiliary Lemma}}$ Start from $(*)$ and multiply everything by an overall $(-1)^{|g|+1}$. Taking into account that $|H_f|\cdot|H_g| = (|f|+1)(|g|+1)$ we obtain

$$H_f H_g(h) - (-1)^{|H_f|\cdot|H_g|} H_g H_f(h) = (-1)^{|g|+1}\{H_f g, h\} = (-1)^{|f|+|g|}\{\{f,g\},h\} = -H_{\{f,g\}}(h)$$

Thus,

$\boxed{\text{Jacobi Identity}}$ $\Longleftrightarrow$ $\boxed{f \longmapsto -H_f \text{ preserves the bracket, i.e. } [H_f, H_g] = -H_{\{f,g\}}}$

#

$\boxed{\text{Special Case relevant for us:}}$ $S \in \mathcal{O}(\mathcal{F})_0$ <u>even</u> function on $\mathcal{F}$

Then $Q := \{S,\cdot\} = -H_S$ is an <u>odd</u> vector field.

If S satisfies the CME $\{S,S\} = 0$, we get $\boxed{Q^2 = \frac{1}{2}[Q,Q] = -H_{\{S,S\}} = 0}$ ("cohomological v.f.")

Some more bookkeeping: $\boxed{\text{Basic Structures on SP-manifolds}}$

For X a vector field on a supermanifold $\mathcal{F}$ and $\mu \in \Gamma(\mathrm{Ber}\,\mathcal{F})$ we want to define the divergence $\mathrm{div}_\mu X \in \mathcal{O}(\mathcal{F})$ such that for every compactly supported test function $f \in \mathcal{O}(\mathcal{F})_c$

$$\boxed{\int_{\mathcal{F}} \mu\, X[f] = -\int_{\mathcal{F}} \mu\, \mathrm{div}_\mu(X)\, f}$$

This is achieved by the following construction of [Manin]:

Ch.4 §5 Prop. 3 There is a unique right connection Δ_r on $\mathrm{Ber}\,\mathcal{F}$ such that for any local coord system (u^a) on $\mathcal{F}$:

$$\Delta_r\left(D^*(du) \otimes \frac{\partial}{\partial u^a}\right) = 0 \quad \forall a$$

In local coords:

$$\boxed{\Delta_r\left(D^*(du) \otimes X^a \frac{\partial}{\partial u^a}\right) = -D^*(du)\cdot(-1)^{|X^a||u^a|}\frac{\partial X^a}{\partial u^a} = -D^*(du)\cdot\left[\frac{\partial X^i}{\partial y^i} - (-1)^{|X|}\frac{\partial X^\alpha}{\partial \theta^\alpha}\right]}$$

Note that the RHS is a total derivative, so for X compactly supported in a coord neighbourhood one has

$$\int \Delta_r(D^*(du)\otimes X) = -\int_{\mathbb{R}^{n|m}} d^n y\, d^m\theta \left(\frac{\partial X^i}{\partial y^i} - (-1)^{|X|}\frac{\partial X^\alpha}{\partial \theta^\alpha}\right) = 0 .$$

Moreover as part of the axioms of a right connection one has

$$\Delta_r(D^*(du)\otimes X)\cdot f = \Delta_r(D^*(du)\otimes X\circ f) = D^*(du)\, X[f] + (-1)^{|X||f|}\Delta_r(D^*(du)\otimes fX)$$

So for $f\in\mathcal{O}(\mathcal{F})_c$ a compactly supported test fct.:

$$\boxed{\int \Delta_r(\mu\otimes X)\cdot f = \int \mu\, X[f]}$$

If μ is a nowhere vanishing section of the line bundle $\mathrm{Ber}\,\mathcal{F}$

we have $\Delta_r(\mu \otimes X) = -\mu\, \mathrm{div}_\mu(X)$

where $\boxed{\mathrm{div}_\mu X := -\mu^{-1} \Delta_r(\mu \otimes X) \in \mathcal{O}(\mathcal{F})}$

This expression is defined on local patches as $(D^*(du)\lambda)^{-1}(D^*(du)\eta) := \lambda^{-1}\eta$.

The transition fcts. in denominator and numerator cancel so this glues to a globally def. function!

Def BV Laplacian $\Delta_\mu : \mathcal{O}(\mathcal{F}) \longrightarrow \mathcal{O}(\mathcal{F})$ $\quad \Delta_\mu(f) := \mathrm{div}_\mu H_f$

By direct calculation one can show:

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|} x(\Delta y) + (-1)^{|x|}\{x,y\}$$

"the Poisson bracket measures the defect of Δ being a derivative"

For even $x \in \mathcal{O}(\mathcal{F})_0$ one obtains inductively: $\Delta(x^n) = n\, x^{n-1}\Delta x + \frac{n(n-1)}{2} x^{n-2}\{x,x\}$

So for any polynomial or convergent power series: $\Delta f(x) = f'(x)\Delta x + \frac{1}{2} f''(x)\{x,x\}$

Relevant case for us: $f(x) = e^x$

$$g^2 \Delta_\mu e^{\frac{1}{g}S} = e^{\frac{1}{g}S} \cdot \left(\tfrac{1}{2}\{S,S\} + g\Delta_\mu S\right)$$

so

QME

$$\boxed{\Delta_\mu e^{\frac{1}{g}S} = 0 \iff \tfrac{1}{2}\{S,S\} + g\Delta_\mu S = 0}$$

#

Strategy

Classical gauge theory

- bosonic manifold M ("space of fields")
- compact Lie group acting on M

  $G \times M \xrightarrow{\gamma} M$
- G-inv. action functional $S \in C^\infty(M)^G$

Problem with quantization:

the naive P.I. $Z = \int_M e^{\frac{i}{\hbar}S}$ cannot be evaluated by a stationary phase formula

Reason: critical pts of S are degenerate

Solution attempt: Replace $\int_M e^{\frac{i}{\hbar}S}$ by $\mathrm{Vol}(G) \int_{M/G} e^{\frac{i}{\hbar}S}$ (Faddeev-Popov)

however problematic if M/G is not a manifold!

Replace M/G by the action Lie groupoid
$(G \times M \rightrightarrows M)$

infinitesimal version

action Lie algebroid

$E = \mathfrak{g} \times M \longrightarrow M$

trivial vector bundle over M

$\downarrow$ I

BRST package

- $\mathcal{F}_{BRST} = \Pi E$

  carries a canonical $Q^2 = 0$
- G-invariance of S translates into $Q(S) = 0$

$\downarrow$ II

BV-package

- $\mathcal{F}_{BV} := \Pi T^* \mathcal{F}_{BRST}$

  carries a canonical odd sympl. form ω
- by a clever choice of S_{BV}

  BRST data is rephrased as $\{S_{BV}, S_{BV}\} = 0$

  (and also $\Delta_\mu S_{BV} = 0$ for μ chosen properly)

BV-quantization

$Z := \int_{\mathcal{L}} \sqrt{\mu}\big|_{\mathcal{L}} \, e^{\frac{i}{\hbar}S_{BV}}$

I From Lie Algebroids to Q-manifolds

Def A *Lie algebroid* over M is a vector bundle $E \to M$ together with:

- a morphism of vector bundles $\begin{tikzcd} E \arrow[r, "\rho"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd}$ ("anchor map")
- a Lie bracket on its sections $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

satisfying the Leibniz formula $\boxed{[u, fw] = \rho(u)[f]w + f\cdot[u,w]}$ $\forall f \in C^\infty(M)$, $u, w \in \Gamma(E)$

(where $\rho(u) := \rho\circ u \in \Gamma(TM)$ is a vector field on M)

Basic examples

(1) $E = TM$, $\rho = \mathrm{id}_{TM}$

$[\cdot,\cdot]$ = Lie bracket of vector fields

(2) $M = \{pt.\}$

$E = \mathfrak{g}$ Lie algebra

$\rho = 0$

Main example

(3) To a Lie group action $\gamma : G \times M \to M$ we can associate:

$\boxed{\text{action Lie algebroid} \quad E = \mathfrak{g} \times M \xrightarrow[\text{trivial bundle}]{} M}$

- given (T_a) a basis of $\mathfrak{g}$ we can identify $\Gamma(E) = C^\infty(M) \otimes_{\mathbb{R}} \mathfrak{g} \cong C^\infty(M)^{\oplus \dim(\mathfrak{g})}$
- anchor $\rho : \Gamma(E) \longrightarrow \Gamma(TM)$, $s \otimes w \longmapsto s \cdot \nu(w)$ where $\nu(w) \in \Gamma(TM)$ is the fundamental vector field of $w \in \mathfrak{g}$
- bracket on $\Gamma(E)$: $[s^a T_a, t^b T_b]_E = s^a t^b [T_a, T_b]_{\mathfrak{g}} + \big(\rho(s)[t^a] - \rho(t)[s^a]\big) T_a$

Side remark (for (3))

Here the fundamental vector field is defined as follows:

$\mathfrak{g}$ = left-inv. vector fields on G

$\Rightarrow$ every $w \in \mathfrak{g}$ extends to a (vertical) vector field w on $G \times M$

$$\nu(w) : f \longmapsto w(f\circ\gamma)\big|_{\{e\}\times M}$$

is a derivation on $C^\infty(M)$

hence a vector field.

Immediate Observations

Prop. For any Lie algebroid E over M the anchor map $\rho : \Gamma(E) \longrightarrow \Gamma(TM) =: \mathfrak{X}(M)$

is a <u>Lie algebra homomorphism</u>

i.e. $$\rho([s,t]_E) = [\rho(s), \rho(t)] \quad \forall s,t \in \Gamma(E)$$

Proof: Direct calculation using the axioms of a Lie algebroid (Leibniz, Jacobi & antisymmetry)

$[[s,t], f\cdot u] = \ldots\ldots$ see [Waldmann] $\square$

We will use the following

Corollary Let $E = \mathfrak{g} \times M \longrightarrow M$ be the action Lie algebroid.

The map
$$\mathfrak{g} \xrightarrow[\text{constant section}]{} \Gamma(E) \xrightarrow{\rho} \mathfrak{X}(M)$$
$$w \longmapsto 1 \otimes w \longmapsto \underbrace{v(w)}_{\text{fundamental v.f.}}$$

is a <u>Lie algebra homomorphism</u>, i.e. $v([w_1, w_2]_{\mathfrak{g}}) = [v(w_1), v(w_2)]$

Every Lie Algebroid $E \to M$ defines a Q-manifold with body M!

Def A Q-manifold is a supermanifold $\mathcal{F}$ with an odd vector field $Q \in \mathrm{Der}_1(\mathcal{O}_{\mathcal{F}})$ satisfying $Q^2 = 0$. ("cohomological vector field" / "BRST generator")

Construction Given a Lie Algebroid $E \to M$ consider the supermanifold $\mathcal{F} = \Pi E$.

A trivialisation of the bundle E over $\mathcal{U} \subset M$ comes with a basis of local sections $T_a \in \Gamma(\mathcal{U}, E)$ (dual basis $T^a \in \Gamma(\mathcal{U}, E)^*$) and induces a trivialisation of the structure sheaf $\mathcal{O}_{\mathcal{F}}(\mathcal{U}) = \Lambda^\bullet_{C^\infty(\mathcal{U})} \Gamma(\mathcal{U}, E)^* \cong C^\infty(\mathcal{U}) \otimes_{\mathbb{R}} \Lambda^\bullet_{\mathbb{R}} \mathrm{span}(T^a)$

Notation: $\Theta^a := 1 \otimes T^a$ (odd) supercoordinate on $\mathcal{F}$

$f_{ab}^{\ \ c}(x) = T^c([T_a, T_b]_E) \in C^\infty(\mathcal{U})$ structure coefficients of E

$\rho(T_a) = \rho_a^i(x) \frac{\partial}{\partial x^i}$ anchor map

CLAIM

$$Q := \frac{1}{2} f_{ab}^{\ \ c}(x)\, \Theta^a \Theta^b \frac{\partial}{\partial \Theta^c} + \Theta^a \rho_a^i(x) \frac{\partial}{\partial x^i}$$

is a cohomological vector field

Examples

① $\boxed{E = TM\,,\ \ \delta = \mathrm{id}\,,\ \ [\cdot,\cdot] = \text{bracket on vector fields}}$

$$\mathcal{O}_{\Pi E} = \Gamma(\Lambda^\bullet T^*M) = \text{differential forms on } M$$

$$Q = \theta^a \frac{\partial}{\partial x^a} = \text{de Rham differential} \quad \left(\text{using } [\partial_a, \partial_b] = 0 \Rightarrow f_{ab}^{\ c}(x) = 0\right)$$

② $\boxed{E = \mathfrak{g} \longrightarrow \text{pt.}\ \ \text{Lie algebra}\ ,\ \ \delta = 0}$

$$\mathcal{O}_{\Pi E}(\text{pt.}) = \Lambda^\bullet \mathfrak{g}^*$$

$$Q = \frac{1}{2} f_{ab}^{\ c}\, \theta^a \theta^b \frac{\partial}{\partial \theta^c} = \text{Chevalley-Eilenberg differential}$$

③ $\boxed{E = \mathfrak{g} \times M \longrightarrow M \quad \text{action Lie algebroid}}$

$\nu : \mathfrak{g} \xrightarrow[\text{linear}]{} \mathfrak{X}(M)$ fundamental vector field (coming from the group action)

$$Q = \frac{1}{2} f_{ab}^{\ c}\, \theta^a \theta^b \frac{\partial}{\partial \theta^c} + \theta^a \nu^i(T_a) \frac{\partial}{\partial x^i} \quad (\text{"BRST differential"})$$

now constant!

For simplicity take $E = \mathfrak{g} \times M \longrightarrow M$ action Lie algebroid (Example ③)

Lemma

$$\boxed{Q^2 = 0} \iff$$

- Jacobi-identity on $\mathfrak{g}$
- $\nu : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism

  i.e. $\nu([w_1, w_2]) = [\nu(w_1), \nu(w_2)]$

Proof: $Q \in \mathrm{Der}_1(\mathcal{O}_{\mathcal{F}})$ is an <u>odd</u> vector field

$\Longrightarrow$ $Q^2 = \frac{1}{2}[Q, Q]$ is a vector field as well

hence can be written as $Q^2 = (Q^2)^i \dfrac{\partial}{\partial x^i} + (Q^2)^a \dfrac{\partial}{\partial \theta^a}$

We calculate

$$(Q^2)^i = Q^2 x^i = \frac{1}{2}\Big(\nu([T_a, T_b]) - [\nu(T_a), \nu(T_b)]\Big)^i$$

$$(Q^2)^a = Q^2 \theta^a = \frac{1}{2}\, \theta^b \theta^c \theta^d \big[[T_b, T_c], T_d\big]^a = \frac{1}{6}\, \theta^b \theta^c \theta^d \Big(\big[[T_b, T_c], T_d\big] + \text{cycl.}\Big)^a$$

$\square$

<u>Hint:</u> The calculation uses $f_{ab}^c\, \nu(T_c)^i \underset{\nu \text{ linear}}{=} \nu(f_{ab}^c T_c)^i = \nu([T_a, T_b])^i$

and $f_{bc}^e f_{ed}^a = \big[[T_b, T_c], T_d\big]^a$

$\boxed{\text{Observation}}$ From $Q = \frac{1}{2} f_{ab}^{c}\, \theta^a \theta^b \frac{\partial}{\partial \theta^c} + \theta^a v_a$ we have $Q(S) = \theta^a v_a(S)$

so $Q(S) = 0 \iff v_a(S) = 0 \;\; \forall a \iff S \in C^\infty(M)^G$

(all fundamental vector fields of the group action annihilate S)

S is G-invariant

Thus, in summary

Classical gauge theory
$G \times M \xrightarrow{\gamma} M$
$S \in C^\infty(M)^G$

can be rephrased as

(Classical) BRST data
$\mathcal{F}_{BRST} = \Pi(\mathfrak{g} \times M \longrightarrow M)$
$Q \in \mathrm{Der}_1(\mathcal{O}_{\mathcal{F}_{BRST}})$ with $Q^2 = 0$
$Q(S) = 0$

II From BRST to BV

Construction of the BV-Space $\mathcal{F}_{BV} = \Pi T^* \mathcal{F}_{BRST}$

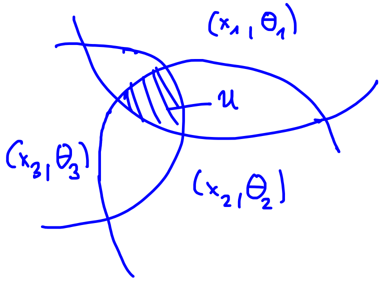
Let $\mathcal{F}$ be a supermfd. with coords $(\underbrace{\overset{0}{x^i}}_{\text{"field"}}, \underbrace{\overset{1}{\theta^j}}_{\text{"ghost"}})$, $\dim \mathcal{F} = k|\ell$

We construct $\Pi T^* \mathcal{F}$ with coords $(\underbrace{\overset{0}{x^i}}_{\text{"field"}}, \underbrace{\overset{1}{\theta^j}}_{\text{"ghost"}}, \underbrace{\overset{1}{p_i}}_{\text{"anti-field"}}, \underbrace{\overset{0}{q_j}}_{\text{"anti-ghost"}})$ $\dim(\Pi T^* \mathcal{F}) = k+\ell \,|\, k+\ell$

$\boxed{\text{Step 1}}$ $\boxed{\text{The body of } \Pi T^* \mathcal{F} \text{ is a (bosonic) vector bundle over the body of } \mathcal{F}}$

- Our Wish: q_j shall transform like $\frac{\partial}{\partial \theta^j}$ (modulo bosonic reduction)

- $\boxed{\text{Consider change of coords on } \mathcal{F}}$



1) Chain rule: $\frac{\partial \theta_1^i}{\partial \theta_3^j} = \underbrace{\frac{\partial \theta_2^k}{\partial \theta_3^j}}_{\text{even}} \underbrace{\frac{\partial \theta_1^i}{\partial \theta_2^k}}_{\text{even}} + \underbrace{\frac{\partial x_2^k}{\partial \theta_3^j}}_{\text{odd}} \underbrace{\frac{\partial \theta_1^i}{\partial x_2^k}}_{\text{odd}}$ holds in $\mathcal{O}_{\mathcal{F}}(\mathcal{U})$

2) Apply the reduction map $\mathcal{O}_{\mathcal{F}}(\mathcal{U}) \longrightarrow C^\infty(\mathcal{U})$ coming from $\mathcal{F} \longleftarrow \mathcal{F}_{red}$:

$$\operatorname{red}\left(\frac{\partial \theta_1^i}{\partial \theta_3^j}\right) = \operatorname{red}\left(\frac{\partial \theta_2^k}{\partial \theta_3^j}\right) \operatorname{red}\left(\frac{\partial \theta_1^i}{\partial \theta_2^k}\right)$$

$\rightsquigarrow$ has the form of a cocycle condition $T_{31} = T_{32} \circ T_{21}$

3) By glueing these patches we obtain a v.b. with transition functions

$$\psi^*(q_j^2) = \operatorname{red}\left(\frac{\partial \theta_1^i}{\partial \theta_2^j}\right) q_i^1$$

Step 1

Step 2 | The structure sheaf of $\Pi T^* \mathcal{F}$ from glueing locally defined sheaves on $|\Pi T^* \mathcal{F}|$

Technical Rk: The "Chart Thm." (see [CCF]) states that a morphism of superdomains (as locally ringed spaces) is uniquely specified by the pullback of coordinate functions from the target. Accordingly, if two patches overlap on $\mathcal{U} \subset |\Pi T^* \mathcal{F}|$ we can define the transition map $C^\infty|_{\mathcal{U}} \otimes \Lambda^\bullet \mathbb{R}^n \xrightarrow[\cong]{\psi^*} C^\infty|_{\mathcal{U}} \otimes \Lambda^\bullet \mathbb{R}^n$ by its image on generators.

Coordinate transitions on $\Pi T^* \mathcal{F}$

$$(\underbrace{\overset{\text{even}}{x^i}, \overset{\text{odd}}{\xi^j}}_{v^a}, \underbrace{\overset{\text{odd}}{p_i}, \overset{\text{even}}{q_j}}_{\beta_a}) \xrightarrow{\psi} (\underbrace{\overset{\text{even}}{y^i}, \overset{\text{odd}}{\theta^j}}_{u^b}, \underbrace{\overset{\text{odd}}{\tilde{p}_i}, \overset{\text{even}}{\tilde{q}_j}}_{\alpha_b})$$

$$\psi^* u^b := u^b(v) \quad \text{as on } \mathcal{F}$$

$$\psi^* \alpha_b := \frac{\partial v^a}{\partial u^b}(v)\, \beta_a$$

As before, the cocycle condition follows from the (super) Chain rule

$$\frac{\partial u_1^a}{\partial u_3^b} = \frac{\partial u_2^c}{\partial u_3^b} \frac{\partial u_1^a}{\partial u_2^c}$$

cf. [CCF] Prop 4.4.7

\# Step 2

Step 3 | There is a (Darboux-shaped) odd symplectic form ω on $\Pi T^* \mathcal{F}$

Define ω on local patches and check compatibility on overlaps:

$$\omega := (-1)^{|u^b|} du^b\, d\alpha_b = (-1)^{|u^b|} du^b\, d\left(\frac{\partial v^a}{\partial u^b} \beta_a\right) = (-1)^{|u^b|} du^b \left[d\left(\frac{\partial v^a}{\partial u^b}\right) \beta_a + (-1)^{|u^b|+|v^a|} \frac{\partial v^a}{\partial u^b} d\beta_a \right] = (-1)^{|v^a|} \underbrace{du^b \frac{\partial v^a}{\partial u^b}}_{dv^a} \beta_a = (-1)^{|v^a|} dv^a\, d\beta_a$$

Note: The sign is required to make $(-1)^{|u^b|} du^b\, d\left(\frac{\partial v^a}{\partial u^b}\right) = (-1)^{|u^b|} du^b du^c \frac{\partial}{\partial u^c} \frac{\partial v^a}{\partial u^b}$ vanish. The difficulty arises from $\underbrace{dx\, d\xi}_{\text{commute}} \underbrace{\frac{\partial}{\partial x} \frac{\partial}{\partial \xi}}_{\text{commute}} v$

\# Step 3

$\boxed{\text{Remark}}$ ω can be obtained from a globally defined Liouville 1-form:

$$-\lambda := du^a \cdot \alpha_a = du^a \frac{\partial v^b}{\partial u^a} \beta_b = dv^b \beta_b \text{ doesn't depend on the choice of coordinates}$$

and we have $d\lambda = -d(du^a \alpha_a) = -(-1)^{|u^a|+1} du^a \, d\alpha_a = (-1)^{|u^a|} du^a \, d\alpha_a = \omega$

$\boxed{\text{Construction}}$ A similar calculation shows that for $Q = Q^a \frac{\partial}{\partial u^a}$ a vector field on the base $\mathcal{F}$

$$\tilde{Q} := \langle Q, du^a \rangle \alpha_a = Q^a \alpha_a \text{ is a globally defined function on } \Pi T^* \mathcal{F}$$

$\tilde{Q}$ & Q have opposite parity because u^a & α_a have opposite parity.

Denote by $\Pi T^* \mathcal{F} \xrightarrow{p} \mathcal{F}$, $p^* u^a = u^a$ the bundle projection

Then $Q \longmapsto \tilde{Q}$ is an $\mathcal{O}_{\mathcal{F}}$-linear map $T\mathcal{F} \longrightarrow p_* \mathcal{O}_{\Pi T^* \mathcal{F}}$

$\boxed{\text{Theorem}}$ Given

Classical BRST data
- $\mathcal{F}$ supermanifold
- $Q \in \mathrm{Der}_1(\mathcal{O}_{\mathcal{F}})$, $Q^2 = 0$ ("cohomological v.f.")
- even, Q-inv. function $S \in \mathcal{O}(\mathcal{F})_0$, $Q(S) = 0$ ("action")

the BV-action $\boxed{S_{BV} := p^* S + \tilde{Q}} \in \mathcal{O}(\Pi T^* \mathcal{F})_0$

satisfies the Classical Master Equation

$$\{ S_{BV}, S_{BV} \} = 0$$

Proof: One can check that given a derivative D, $D_r f := (-1)^{(|f|+1)|D|} Df$ satisfies the Leibniz rule of a right derivative.

in principle optional

Recall the definitions $i_{H_f} \omega = df$ (Hamiltonian v.f.) and $\{ f, \cdot \} = (-1)^{|f|+1} H_f$ (Poisson bracket).

With these conventions the Poisson bracket of $\omega = (-1)^{|u^a|} du^a \, d\alpha_a$ can be written as

$$\boxed{\{ f, \cdot \} = \frac{\partial_r f}{\partial \alpha_a} \frac{\partial}{\partial u^a} - \frac{\partial_r f}{\partial u^a} \frac{\partial}{\partial \alpha_a}}.$$

Let's see what happens to S_{BV} →

Observations 1) $(p^*S)(u,\alpha) = S(u)$ doesn't depend on α

$\Rightarrow \{p^*S, \cdot\} = -\frac{\partial_r S}{\partial u^a}\frac{\partial}{\partial \alpha_a}$ annihilates everything of the form $p^*(\dots)$

In particular $\{p^*S, p^*S\} = 0$.

2) $-H_{\tilde{Q}} = \{\tilde{Q}, \cdot\} = Q^a\frac{\partial}{\partial u^a} - (-1)^{|u^a|}\frac{\partial Q^b}{\partial u^a}\alpha_b\frac{\partial}{\partial \alpha_a}$

The first summand shows that:

$H_{-\tilde{Q}}$ is a "Lift of Q" in the sense that $\boxed{H_{-\tilde{Q}}\circ p^* = p^*\circ Q}$

Thus, $\{\tilde{Q}, p^*S\} = p^*(Q(S))$.

3) For $A, B \in \mathrm{Der}(\mathcal{O}_{\mathcal{F}})$ one has

$$\boxed{\{\tilde{A}, \tilde{B}\} = A[B^b]\alpha_b - (-1)^{|A||B|}B[A^b]\alpha_b = \widetilde{[A,B]}}$$

Putting these pieces together we get

$$\{S_{BV}, S_{BV}\} = \underbrace{\{p^*S, p^*S\}}_{=0} + 2\{\tilde{Q}, p^*S\} + \{\tilde{Q},\tilde{Q}\} = 2\,p^*(Q(S)) + \widetilde{[Q,Q]} \overset{\swarrow}{=} 0$$

$\boxed{\begin{array}{c}\text{BRST}\\ \text{data}\\ Q(S)=0\\ Q^2=0\end{array}}$

$\square$ Thm.

Remark 1) & 2) show that $H_{-S_{BV}}\circ p^* = (H_{-p^*S} + H_{-\tilde{Q}})\circ p^* = p^*\circ Q$

so $\Pi T^*\mathcal{F} \xrightarrow{p} \mathcal{F}$ becomes a morphism of Q-manifolds

if we define $Q_{BV} := \{S_{BV}, \cdot\} = -H_{S_{BV}}$ (cohomological v.f. on $\Pi T^*\mathcal{F}$) #RK

Where we are now: ROADMAP

$\boxed{\text{classical gauge system on an ordinary mfd. } M}$ $\rightsquigarrow$ $\boxed{\text{consider the action Lie algebroid } E = \mathfrak{g} \times M \longrightarrow M \text{ associated with } \gamma}$

- $G \times M \xrightarrow{\gamma} M$ group action
- $S \in C^\infty(M)^G$ G-inv. action

$\Big\downarrow$ induces

$\boxed{\text{BRST data}}$ $\rightsquigarrow$ $\boxed{\text{BV-data}}$

BRST data:

- $\mathcal{F}_{BRST} = \Pi E$
- $Q \in \mathrm{Der}_1(\mathcal{O}_{\mathcal{F}_{BRST}})$, $Q^2 = 0$
- $Q(S) = 0$

BV-data:

- $\mathcal{F}_{BV} = \Pi T^* \mathcal{F}_{BRST}$
- ω odd symplectic form
- $S_{BV} = p^* S + \tilde{Q}$ satisfies $\{S_{BV}, S_{BV}\} = 0$

$\boxed{\text{RK / BLACK BOX}}$ see [Mnev]

One can show that the two summands in the Quantum Master Equation $\{S_{BV}, S_{BV}\} + g\, \Delta_{\mu_{BV}} S_{BV} = 0$ vanish individually if μ_{BV} is chosen properly.

In fact, $\Delta_{\mu_{BV}} S_{BV} = 0$ comes from $\mathrm{div}_{\mu_{BRST}} Q = 0$.

When starting from Fadeev-Popov-data the latter equation is in part due to the symmetry group G being compact.

<u>By the above RK</u> we have now found S_{BV} = BV-action such that $\Delta_\mu \left(e^{\frac{1}{g} S_{BV}} \right) = 0$.

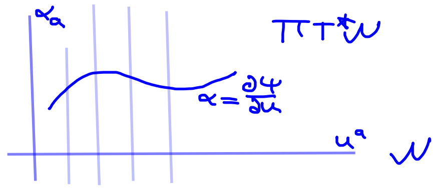
To complete the gauge fixing procedure according to the Schwarz thm. we have to construct Lagrangian submanifolds in $\mathcal{F}_{BV}$.

Useful construction: $\boxed{\text{Graph Lagrangians}}$

$\mathcal{W}$ supermanifold, $\Psi \in \mathcal{O}(\mathcal{W})_1$ <u>odd</u> function

CLAIM:

$$\boxed{\begin{array}{ccc} \underset{\substack{\text{coords}\\ u^a}}{\mathcal{W}} \xrightarrow{\ \iota_\Psi\ } \underset{\substack{\text{coords}\\ (u^a,\alpha_a)}}{\Pi T^*\mathcal{W}} & & \begin{array}{l} \iota_\Psi^* u^a = u^a \\ \iota_\Psi^* \alpha_a = \dfrac{\partial \Psi}{\partial u^a} \end{array} \end{array}}$$



defines a <u>Lagrangian submfd</u> of $(\Pi T^*\mathcal{W}, \omega_{\text{standard}})$, denoted by $\mathcal{L}_\Psi$

$\boxed{\text{RKs}}$

- Since Ψ is odd, we have $\left|\frac{\partial\Psi}{\partial u^a}\right| = |u^a|+1 = |\alpha_a|$ as required.
  So ι_Ψ gives a well-defined (parity-preserving) morphism of supermanifolds
- ι_Ψ is a section of the projection map : $\quad p \circ \iota_\Psi = \mathrm{id}_{\mathcal{W}}$ $(\Rightarrow \iota_\Psi^* \circ p^* = \mathrm{id})$
- choosing $\Psi = 0$ yields the zero-section

<u>Proof of Claim</u> Using the naturality of d we have

$$\iota^*\omega = (-1)^{|u^a|} d(\iota^* u^a)\, d(\iota^*\alpha_a) = (-1)^{|u^a|} du^a\, d\left(\frac{\partial\Psi}{\partial u^a}\right) = (-1)^{|u^a|} du^a du^b \frac{\partial^2\Psi}{\partial u^b \partial u^a} = 0$$

Moreover, $\dim \mathcal{W} = k|n-k$ and $\dim \Pi T^*\mathcal{W} = n|n$ #

$\boxed{\text{The body of } \mathcal{L}_\Psi \text{ can be read off from the function } \Psi:}$

Formal argument:

$$\begin{tikzcd} (x,\xi,p,q) & \Pi T^*\mathcal{W} & \mathcal{W} \arrow[l, "\iota_\Psi"'] \arrow[dl, phantom, "C"] & (x,\xi) \\ (x,q) & {|\Pi T^*\mathcal{W}|} \arrow[u] & {|\mathcal{W}|} \arrow[l, "|\iota_\Psi|"', "\text{section of the vector bundle}"] \arrow[u] & x \end{tikzcd}$$

$\Longrightarrow$

$$\boxed{\begin{array}{l} \iota_{red}^* x^i = x^i \\ \iota_{red}^* q_j = red\left(\dfrac{\partial\Psi}{\partial\xi^j}\right) \end{array}}$$

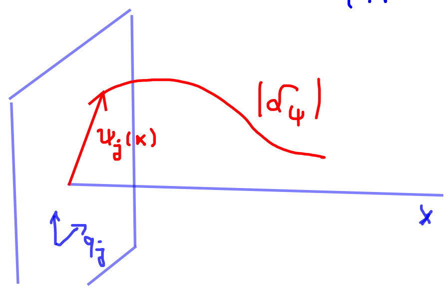
(we used that $red(x^i) = x^i$, $red(q_j) = q_j$ and by commutativity of the diagram $\iota_{red}^* \circ red = red \circ \iota^*$)

Write $\Psi(x,\xi) = \Psi_j(x)\xi^j +$ higher orders in ξ

Then $q_j = red\left(\frac{\partial\Psi}{\partial\xi^j}\right) = \Psi_j(x)$

parametrizes a section of $|\Pi T^*\mathcal{W}|$ as a vector bundle over $|\mathcal{W}|$.

This describes $|\mathcal{L}_\Psi|$.



Gauge fixing for physical theories coming from the BRST construction

$$\underset{(u^a,\ \alpha_a)}{\mathcal{F}_{BV} = \Pi T^* \mathcal{F}_{BRST}} \quad \underset{\substack{\sigma_\psi \\ \text{graph Lagrangian}}}{\overset{\substack{\text{bundle projection} \\ p}}{\rightleftarrows}} \quad \underset{(u^a)}{\mathcal{F}_{BRST}}$$

$$p \circ \sigma_\psi = \mathrm{id}_{\mathcal{F}_{BRST}}$$

$$S_{BV} = p^* S + \tilde{Q}$$

Calculate: $\sigma_\psi^* \, p^* S = S \qquad \sigma_\psi^* \tilde{Q} = Q^a \, \sigma_\psi^* \alpha_a = Q^a \frac{\partial \psi}{\partial u^a} = Q[\psi]$

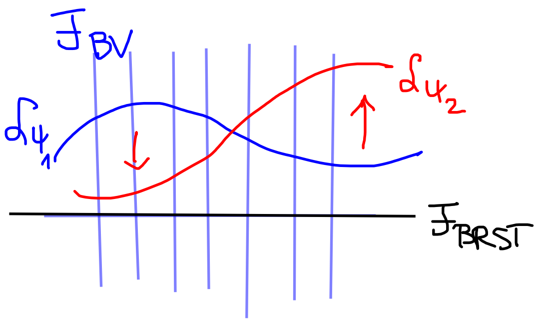
Therefore:

$$\boxed{\sigma_\psi^* S_{BV} = S + \underbrace{Q[\psi]}_{\text{gauge fixing fermion}}}$$

Schwarz Thm. $\Longrightarrow$

$$\int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}(S + Q[\psi_1])} = \int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}(S + Q[\psi_2])}$$

freedom of gauge fixing



Finally: Fadeev-Popov from BV

$$\mathcal{F}_{BRST} = \overset{(x^i,\, c^a)}{\mathcal{F}_{BRST,\,min}} \times \overset{(\lambda_a,\, \bar{c}_a)}{\mathcal{F}_{aux}}$$

coming from action Lie algebroid $\Pi(\mathfrak{g}\times M \to M)$

just a supervector space of auxilliary fields $\mathfrak{g}^* \oplus \Pi\mathfrak{g}^*$

$$|\mathcal{F}_{BRST}| = M \times \mathfrak{g}^* = (x^i, \lambda_a)$$

$$\mathcal{F}_{BV} = \Pi T^* \mathcal{F}_{BRST} = (\underbrace{\overset{0}{x^i}}_{\text{fields}}, \underbrace{\overset{1}{c^a}}_{\text{ghosts}}, \underbrace{\overset{0}{\lambda_a}, \overset{1}{\bar{c}_a}}_{\text{auxilliary fields}} \,\vdots\, \underbrace{\overset{1}{x_i^+}}_{\text{anti-fields}}, \underbrace{\overset{0}{c_a^+}}_{\text{anti-ghosts}}, \underbrace{\overset{1}{\lambda^{+a}}, \overset{0}{\bar{c}^{+a}}}_{\text{anti-auxilliary fields}})$$

Cohomological v.f. on $\mathcal{F}_{BRST}$: $Q = Q_{min} + Q_{aux}$

from Lie algebroid $Q = \frac{1}{2} f^c_{ab}\, c^a c^b \frac{\partial}{\partial c^c} + c^a v_a$

$\lambda_a \frac{\partial}{\partial \bar{c}_a}$ some Koszul differential

Assume we are given a "gauge fixing function" $\phi : M \longrightarrow \mathfrak{g}$

Then we take the gauge fixing fermion to be $\Psi = \underbrace{\langle \bar{c}, \phi(x) \rangle}_{\text{pairing } \langle \mathfrak{g}^*, \mathfrak{g}\rangle} = \bar{c}_a \phi^a(x)$

$$\Rightarrow \quad Q_{min}[\Psi] = c^a\, \bar{c}_b\, v_a[\phi^b]$$

$$Q_{aux}[\Psi] = \lambda_a \phi^a(x) = \langle \lambda, \phi(x) \rangle$$

This has a familiar interpretation ↪

$$\boxed{2^*_\Psi S_{BV} = S + Q[\Psi] = S + \langle \lambda, \phi(x) \rangle - \langle \bar{c}, FP(x) c \rangle}$$

with $FP(x) = d\phi_x \circ d\gamma_{(e,x)} \big|_{T_eG \subset T_{(e,x)}(G\times M)} : \mathfrak{g} \longrightarrow \mathfrak{g}$, $FP(x)^a{}_b = v_b[\phi^a]$

being the differential of $G \times M \xrightarrow{\gamma} M \xrightarrow{\phi} \mathfrak{g}$

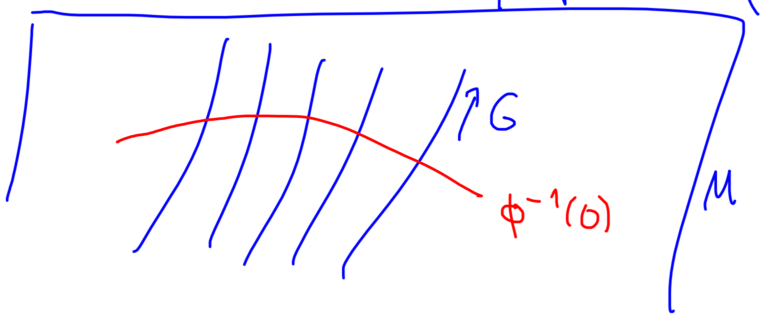
has the effect of constraining the path integral to an integral over the gauge slice $\phi^{-1}(0)$:

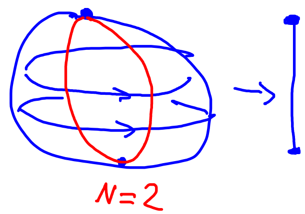
$$\int d^n\lambda \; e^{i\langle \lambda, \phi(x)\rangle} = \delta(\phi(x))$$

$$\int D^n c \, D^n \bar{c} \; e^{i\langle \bar{c}, FP(x) c\rangle} = \det(FP(x))$$

In good cases $\phi^{-1}(0)$ will be an N-fold cover of M/G and one has the formula
$$\int_M \mu_M \, e^{\frac{i}{\hbar}S} = \frac{\mathrm{Vol}(G)}{N} \int_M \mu_M \, \delta(\phi(x)) \det(FP(x)) \, e^{\frac{i}{\hbar}S}.$$

As one chooses $\phi^{-1}(0)$ transverse to the gauge orbits, critical pts. of $S|_{\phi^{-1}(0)}$ will be non-degenerate so one can evaluate the path integral perturbatively by a stationary phase formula. (Details see [Mnev] Lecture 15)





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For generalisations....

....to the $\mathbb{Z}$-graded setting:

[Vys] J. Vysoky: Global Theory of Graded Manifolds

....to infinite-dim. manifolds (relevant for AKSZ sigma models)

[Kob] S. Kobayashi: Manifolds over function algebras and mapping spaces

(discusses the bosonic theory only, but possibly a good inspiration)