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Seminar: p-adic Hodge theory

AG Venjakob

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ORGANISATION: We meet every Thursday at 4 p.m. (s.t.) in SR 8 (Mathematikon).

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Overview

There are at least two ways to motivate p-adic Hodge theory: At first, it provides tools for *classifying p-adic Galois representations* depending on how "nice" they are. A second motivation lies in the *comparison of different cohomology theories*.

Let us start with the following basic example: Let $K/\mathbb{Q}_p$ be a finite extension and $\mathbb{Q}_p(r)$ be the r-th *Tate-twist* of $\mathbb{Q}_p$, i.e. the one-dimensional $\mathbb{Q}_p$-representation defined by the r-th power of the cyclotomic character. These and similar representations appear naturally when studying e.g. *elliptic curves* and contain deep information about their geometry. The philosophy of p-adic Hodge theory is to get rid of the group action on the representations by turning them into objects of linear algebra with some extra structure, e.g. gradings or filtrations. To do this, we use so-called *period rings*, which are large rings that can "absorb" the action of the given representation.

As an example, let's consider the graded ring $\mathbf{B}_{\mathrm{HT}} := \bigoplus_{q\in\mathbb{Z}} \mathbb{C}_p(q)$, where $\mathbb{C}_p$ denotes the completion of $\overline{\mathbb{Q}}_p$. It holds

$$(\mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r))^{G_K} = (\mathbb{C}_p(-r) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r))^{G_K} = K,$$

where the K on the right hand side lives in degree $-r$ of the grading induced by $\mathbf{B}_{\mathrm{HT}}$. Note that this K doesn't carry a G_K-action anymore, but still we can deduce from the *grading*, from which $\mathbb{Q}_p(r)$ it comes.

However, for a larger class of representations this fails: $\mathbf{B}_{\mathrm{HT}}$ cannot "absorb" all G_K-actions, and even if we restrict to the class of the representations V for which $\mathbf{B}_{\mathrm{HT}}$ can, we cannot regain V from the graded vector space $(\mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. A first remedy of this will be to replace *gradings* by the finer structure of *filtrations*. This leads to the construction of the bigger ring $\mathbf{B}_{\mathrm{dR}}$. We can prove that $V \mapsto (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is faithful, but it's not full: The category of filtered K-modules has too many morphisms. So we need to refine our structures once more, which leads to the construction of the rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$. For the class of so-called *crystalline representations*, we then finally obtain the following category equivalence that will be (together with its semi-stable version) the main result of the seminar:

Theorem. *The functors*

$$\begin{aligned} \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) &\longrightarrow \mathbf{MF}^{\varphi,\mathrm{wa}}_K \\ V &\longmapsto (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ \mathrm{Fil}^0(\mathbf{B}_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1} &\longleftarrow D \end{aligned}$$

describe an equivalence between the category $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)$ *of crystalline,* $\mathbb{Q}_p$*-linear representations of* G_K *and the category of weakly admissible, filtered* φ*-modules* $\mathbf{MF}^{\varphi,\mathrm{wa}}_K$ *over the maximal unramified subextension of* $K/\mathbb{Q}_p$*.*

One of many applications of p-adic Hodge theory lies in the comparison of different cohomology theories. Recall from complex geometry that, for a smooth proper variety X over $\mathbb{Q}$, we have the so-called *de Rham isomorphism*

$$H^n_{\mathrm{dR}}(X_{\mathbb{C}}) \xrightarrow{\sim} H^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{C}) = H^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

that is given by sending the class of a differential form ω to the class of the map $\gamma \mapsto \int_\gamma \omega$. Here, $X_{\mathbb{C}} := X \times_{\mathbb{Q}} \mathbb{C}$ denotes the base extension of X to $\mathbb{C}$ and $X(\mathbb{C})$ the set of $\mathbb{C}$-valued points, equipped with its natural topology as a complex manifold.

Let's illustrate this by considering the example of an elliptic curve E over $\mathbb{Q}$: From the theory of elliptic curves it is well-known that there exists a lattice $\Lambda \subseteq \mathbb{C}$ such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. In particular, $E(\mathbb{C})$ is topologically isomorphic to a torus $S^1 \times S^1$ and we have $H^{\mathrm{sing}}_1(E(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. On the other hand, suppose that we have given a Weierstraß equation for E. Then the so-called *invariant differential* is given by $\omega = (X/Z)/d(Y/Z)$. It is a closed but non-exact 1-form. Its class, together with the class of its conjugate $\bar{\omega}$, generate $H^1_{\mathrm{dR}}(E_{\mathbb{C}})$. Integrating ω and $\bar{\omega}$ against α and β gives the so-called *periods*

$$\lambda_1 := \int_\alpha \omega, \ \ \lambda_2 := \int_\beta \omega \in \mathbb{C}$$

and similarly $\bar{\lambda}_1$ and $\bar{\lambda}_2$. If we take the duals α^* and β^* as a $\mathbb{Z}$-basis of $H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z})$ (resp. $\alpha^* \otimes 1$ and $\beta^* \otimes 1$ as a $\mathbb{C}$-basis of $H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$) and ω and $\bar{\omega}$ as a $\mathbb{C}$-basis of $H^1_{\mathrm{dR}}(X_{\mathbb{C}})$, then the de Rham isomorphism from above is described by the matrix

$$\begin{pmatrix} \lambda_1 & \bar{\lambda}_1 \\ \lambda_2 & \bar{\lambda}_2 \end{pmatrix}.$$

If we want to build a bridge from de Rham cohomology to *integral* singular cohomology, one can therefore say: Up to these periods, the image of the $\mathbb{Z}$-span of ω and $\bar{\omega}$ lies in *integral* singular cohomology. Going the other way around from *integral* singular cohomology to de Rham cohomology, one can say: $\mathbb{C}$ is a ring big enough to contain *all* periods to make both cohomologies isomorphic.

Now, let's go over to the p-adic world. Let X be a proper smooth scheme over a finite extension K of $\mathbb{Q}_p$. We can then associate to X its de Rham cohomology groups $H^n_{\mathrm{dR}}(X/K)$ as well as its étale cohomology groups $H^n_{\text{ét}}(X \otimes_K \bar{K})$. In 1988 *Faltings* proved that there also exists a (G_K-equivariant) comparison isomorphism between these cohomologies using the ring $\mathbf{B}_{\mathrm{HT}}$ that we have defined above:

$$\mathbf{B}_{\mathrm{HT}} \otimes_K \mathrm{gr}\, H^n_{\mathrm{dR}}(X/K) \cong \mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} H^n_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p).$$

The ring $\mathbf{B}_{\mathrm{HT}}$ here has a similar purpose as $\mathbb{R}$ in the real case above: It adds enough *periods* to the base ring such that both modules become isomorphic. We can even improve this statement (more precisely: get rid of the $\mathrm{gr}(\cdot)$) by using the other period rings from above!

The seminar

We will start the seminar with some basics about *Galois representations* and give an introduction to *continuous Galois cohomology*, *Witt vectors* and *tilting*. We will then introduce the formalism of *admissible representations*, i.e., the groundwork for our treatment of period rings.

The first of these rings (besides $\mathbf{B}_{\mathrm{HT}}$) is $\mathbf{B}_{\mathrm{dR}}$, then $\mathbf{B}_{\mathrm{cris}}$ and finally $\mathbf{B}_{\mathrm{st}}$. Introducing some more notions and techniques, we can then formulate and sketch the proof of the category equivalences mentioned above.

In the end, we want to take a closer look at some examples from geometry. There are also some optional topics, e.g. about the connection to (φ, Γ)-modules or an overview talk about the comparison of different cohomologies.

Requirements

For this seminar you should have done a basic course on algebraic number theory. In particular you should know how to work with p-adic numbers and their Galois theory. Depending on which of the optional talks will be given in the end, there might be one or two talks for which some algebraic geometry and the theory of (φ, Γ)-modules can be helpful. But for the rest of the seminar we won't use techniques from these areas. In particular, neither knowledge about étale cohomology or de Rham cohomology, nor an attendance at last semester's seminar about (φ, Γ)-modules is required.

The seminar will be held in English. The literature for some of the optional (!) talks at the very end is in French.

The Talks

Talk 1: Overview – ??? (17.10.)

This talk serves as a guideline for the seminar. Go through the talks down below and briefly summarize their main results. Focus on answering the *why* and not the *how*.

Talk 2: ℓ-adic Galois representations and Galois cohomology – ??? (24.10.)

The main objects of p-adic Hodge theory are *continuous p-adic Galois representations*. The goal of this talk is to define them and their *cohomology* and show some basic properties. The main difference to what you might already know from your algebraic number theory course is, that we allow a possibly *non-discrete topology* on G-modules. Without further restriction this is technically quite bad, as the category that we receive doesn't have enough injectives. But as we will see in this talk, there is at least some remedy for this issue.

Start the talk by going through [FO, §2.1.1]: Define *continuous (ℓ-adic) G-representations* and *$\mathbb{Z}_\ell$-representations*. State, but not prove Lemma 2.7. Write down the constructions of Definition 2.9 (i.e. how to build direct sums, duals, etc. of representations). Pick one or two examples from §2.1.2 and present them. In any case define *Tate twists* (the very end of example (1)).

Go through [FO, §1.5] and define *continuous group cohomology*. Point out the main differences to the discrete situation. State Proposition 1.112 (a remedy for the long exact sequence) and Theorem 1.114 (Hilbert 90). If time permits, give a proof of the latter statement.

Talk 3: The formalism of admissible representations – ??? (31.10.)

In this talk we want to learn about the formalism of *B-admissible representations*. The basic idea is, that we want to construct large rings B which can "absorb" the G-action on a certain class of representations. More precisely, for a representation V of this class, we want $B \otimes_F V$ to be *trivial*, i.e. isomorphic to the G-module B^n. If this is the case, we can recover $B \otimes_F V$ from the module $\mathbf{D}(V) := (B \otimes_F V)^G$ and sometimes, in more restricted situations, even V.

Go through [FO, §3.1.1]: Define *B-representations* and what it means if such a representation is *trivial*. Explain the *relationship to cohomology* (Proposition 3.7). Then go through §3.1.2 and define *regular (F, G)-rings* and the functor $\mathbf{D}_B$. Sketch the proof of Theorem 3.14.

Talk 4: The ring $\mathbf{B}_{\mathrm{HT}}$ and graded vector spaces – ??? (7.11.)

In this talk, we will define our first period ring $\mathbf{B}_{\mathrm{HT}} := \bigoplus_{q\in\mathbb{Z}} \mathbb{C}_p(q)$. By definition, this ring can trivialize the action of twists by powers of the cyclotomic character (and more). It is equipped with a *grading* and therefore induces a grading on $\mathbf{D}_{\mathrm{HT}}(V) := (\mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \bigoplus_{q\in\mathbb{Z}} (\mathbb{C}_p(q) \otimes_{\mathbb{Q}_p} V)^{G_K}$. The non-zero indices of the non-zero entries as well as their dimension are important invariants of V.

Start the talk by going through [FO, §6.1]: Define $\mathbf{B}_{\mathrm{HT}}$ and proof that it is $(\mathbb{Q}_p, G_K)$-regular. Define the category of *Hodge-Tate-representations* $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ and explain the statement of *Falting's theorem* in this context. Introduce the *category of graded K-vector spaces* Gr_K and explain the *grading* on $\mathbf{D}_{\mathrm{HT}}(V)$. Define *Hodge-Tate numbers* (more commonly called *Hodge-Tate weights*).

At the end of the talk, show [BC, Theorem 2.4.11]: Denote by $\mathrm{Gr}_K^{<\infty}$ the category of finite-dimensional graded K-vector spaces and, for $A = \bigoplus_{i\in\mathbb{Z}} A_i$ a graded vector space, $\mathrm{gr}^0(A) := A_0$. Then the functors

$$\begin{aligned} \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{C}_p}(G_K) &\rightleftarrows \mathrm{Gr}_K^{<\infty} \\ V &\mapsto \mathbf{D}_{\mathrm{HT}}(V) := (\mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{C}_p} V)^{G_K} \\ \mathrm{gr}^0(\mathbf{B}_{\mathrm{HT}} \otimes_K W) =: \mathbf{V}_{\mathrm{HT}}(W) &\mapsfrom W \end{aligned}$$

are quasi-inverses to each other. Show, that we cannot have a similar equivalence for $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$.

Talk 5: Witt vectors and tilting – ??? (14.11.)

To build other period rings of Fontaine, we need to understand *Witt vectors* and *tilting*. Roughly speaking, Witt vectors turn a characteristic p ring A into a characteristic 0 ring $W(A)$, that modulo p reduces to A (under some mild conditions on A). On the other hand, if we *tilt* a characteristic 0 ring A, we receive a "characteristic p version" of A.

Start the talk with [Sch, §1.1]: Define *(ramified) Witt vectors* and summarize their basic properties. Especially focus on explaining Proposition 1.1.21. Next, go through §1.4 of loc. cit. and give an overview over *perfectoid fields* and *tilting*. Mention, that $W(\_)$ and $(\_)^\flat$ are adjoint to each other. While the unit of this adjunction is given by the identity map $(W(A)^\flat$ is canonically isomorphic to A), the counit is given by θ (introduced in Lemma 1.4.18). Explain Lemma 1.4.19 resp. Proposition 2.1.19 in the cyclotomic case, namely that the element

$$\omega := \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1 + [\varepsilon^{1/p}] + \ldots + [\varepsilon^{1/p}]^{p-1} \in W(\hat{\mathbb{Q}}_p(\zeta_{p^\infty})^\flat),$$

with $\varepsilon = (\zeta_{p^n}) \in \hat{\mathbb{Q}}_p(\zeta_{p^\infty})^\flat$ (cf. [Poy, p. 2]) generates $\ker\theta$. State but not prove the *tilting equivalence* ([Sch, Theorem 1.4.24]). Introduce the field $\mathbb{C}_p^\flat$ and explain Proposition 1.4.27 which says, that $\mathbb{C}_p^\flat$ is the completion of the separable closure of the Laurent series ring $\mathbb{F}_p(\!(t)\!)$.

Note, that this talk should give the audience a rough feeling for the above functors. Schneider's book however contains way more details than we need. In particular, he does everything in the "Lubin-Tate setting", i.e. he uses a generalized version of the cyclotomic tower $\mathbb{Q}_p \subseteq \mathbb{Q}_p(\zeta_p) \subseteq \mathbb{Q}_p(\zeta_{p^2}) \subseteq \ldots \subseteq \mathbb{Q}_p(\zeta_{p^\infty})$ – the only tower, we will need in this seminar. You should either roughly explain the Lubin-Tate setting or adapt the notations to the cyclotomic case. In any case, feel free to use other references as well. E.g. [BC, §4.2] and [FO, 1.2] also contain a summary on Witt vectors and tilting (while not giving a name to the latter construction).

Talk 6: The ring $\mathbf{B}_{\mathrm{dR}}$ and filtered vector spaces – ??? (21.11.)

With the tools from the last talk, we can define the more sophisticated period ring $\mathbf{B}_{\mathrm{dR}}$: Let $\tilde{\mathbf{B}} := W(\mathbb{C}_p^\flat)[\frac{1}{p}]$ and $\mathbf{B}_{\mathrm{dR}}^+ := \varprojlim_n \tilde{\mathbf{B}}/(\ker\theta)^n$. In this ring, the power series

$$t := \log[\varepsilon] = \sum_{n\geq 1} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n}$$

converges (as in the last talk: $\varepsilon := (\zeta_{p^n}) \in \hat{\mathbb{Q}}_p(\zeta_{p^\infty})^\flat$). This *period* has the property that G_K acts on it via $g \cdot t = \chi_{\mathrm{cyc}}(g)t$ (the logarithm turns powers into factors). By defining $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^+[\frac{1}{t}]$, we therefore obtain a period ring that again is able to absorb the action of powers of the cyclotomic character.

While $\mathbf{B}_{\mathrm{dR}}$ has a smaller class of admissible representations than $\mathbf{B}_{\mathrm{HT}}$, it gives more structure to the modules $\mathbf{D}_{\mathrm{dR}}(V)$. Namely, $\mathrm{Fil}^i\,\mathbf{B}_{\mathrm{dR}} = t^i\mathbf{B}_{\mathrm{dR}}^+$ defines an exhausting, decreasing filtration on $\mathbf{B}_{\mathrm{dR}}$, which induces one on $\mathbf{D}_{\mathrm{dR}}(V)$. A filtration always gives rise to a grading (just sum up the quotients of the filtration steps, e.g. $\bigoplus_{i\in\mathbb{Z}} t^i\mathbf{B}_{\mathrm{dR}}/t^{i+1}\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{HT}}$) but doing so, we loose information.

A good reference for the talk is [FO, §6.2], but feel free to use other references as well. Start with the construction of $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}$ and their basic properties (§6.2.2). Show that $t := \log[\varepsilon]$ converges in $\mathbf{B}_{\mathrm{dR}}^+$, that G_K acts on t via the cyclotomic character and that $\mathbf{B}_{\mathrm{dR}}^+$ is a complete discrete valuation ring with uniformizer t (§6.2.3). Introduce the category of *de Rham representation* $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$, the category of *filtered K-vector spaces* Fil_K and summarize the results of §6.2.4. Explain, how the filtration on $\mathbf{B}_{\mathrm{dR}}$ induces one on $\mathbf{D}_{\mathrm{dR}}(V)$. Define the functor $\mathrm{gr}\colon \mathrm{Fil}_K \to \mathrm{Gr}_K$ and show that $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{HT}}(G_K)$. Give an example for the properness of this inclusion.

By the end of this talk, we want to prove the following statement (Theorem 6.34 in loc. cit.): The functor

$$\begin{aligned}\mathbf{D}_{\mathrm{dR}}\colon\ \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) &\to \mathrm{Fil}_K\\ V &\mapsto (\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K},\end{aligned}$$

is an exact, faithful tensor functor. In other words, restricted to its essential image, this functor is *almost* a category equivalence, but we have too many homomorphisms in the target category (e.g. $\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p(1), \mathbb{Q}_p) = 0$ while $\mathrm{Hom}_{\mathrm{Fil}_K}(\mathbf{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)), \mathbf{D}_{\mathrm{dR}}(\mathbb{Q}_p)) = \mathbb{Q}_p$). We will remedy this in the subsequent talks by restricting to the smaller categories of crystalline and semi-stable representations and by adding more structure to the modules.

Talk 7: The rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{max}}$ and $\mathbf{B}_{\mathrm{st}}$ – ??? (28.11.)

In this talk, we want to construct finer period rings, that will give rise to an equivalence of categories, as we will see in the next talks. A very good reference for this talk is the survey article [Poy], which contains very much all constructions, statements and examples that you need for this talk.

A first observation is, that we can't extend the *Frobenius operator* of $W(\mathbb{C}_p^\flat)$ to $\mathbf{B}_{\mathrm{dR}}$ as it is not continuous for the $(\ker\theta)$-adic topology (explain why!). We therefore want to complete $W(\mathbb{C}_p^\flat)$ in a more subtle way, such that $t = \log[\varepsilon]$ still converges, but such that we can extend φ. This gives rise to the ring $\mathbf{B}_{\mathrm{cris}}$. However, the topology on $\mathbf{B}_{\mathrm{cris}}$ is quite bad, so in more modern articles authors often prefer to work with the ring $\mathbf{B}_{\mathrm{max}}$, which is slightly larger, but has the same class of admissible representations. Define both rings, the associated functor and the category of *crystalline representations* $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$. Explain, how $\mathbf{B}_{\mathrm{cris}}$ resp. $\mathbf{B}_{\mathrm{max}}$ induces a Frobenius operator on $\mathbf{D}_{\mathrm{cris}}(V)$ (and how it becomes a filtered K_0-module).

Sketch the example of §1.3 in loc. cit. It shows that the category of crystalline representations is too small to include all the representations that e.g. appear in the theory of elliptic curves. We therefore enlarge $\mathbf{B}_{\mathrm{cris}}$ slightly by adjoining another logarithm, namely $\log[\tilde{p}]$, where $\tilde{p} \in \hat{\mathbb{Q}}_p(\zeta_{p^\infty})^\flat$ is an element with $\theta([\tilde{p}]) = p$. The ring we obtain is denoted by $\mathbf{B}_{\mathrm{st}}$, where "st" means "semi-stable". It is equipped with a so-called *monodromy operator* N. Define the category of semi-stable representations and explain, how $\mathbf{B}_{\mathrm{st}}$ induces a monodromy operator on $\mathbf{D}_{\mathrm{st}}(V)$.

Show the properties of Proposition 1.16 and stress that $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$ are K_0- but not K-algebras, where K_0 denotes the maximal unramified subextension of $K/\mathbb{Q}_p$. End the talk with Remark 1.18, i.e. show that $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$. The example from before showed that the first inclusion is proper. Give an example for the properness of the second inclusion.

Talk 8: Filtered φ- and (φ, N)-modules – ??? (5.12.)

We have seen in the previous talk that, for a given (semi-stable) representation V, the module $\mathbf{D}_{\mathrm{st}}(V)$ is equipped with a filtration, a Frobenius and a monodromy operator. In this talk, we want to learn more about the category of such modules.

Start by defining the category $\mathbf{Mod}_{K_0}^{\varphi}$ of *isocrystals* over K_0 ([BC, Definition 7.3.1]). Roughly speaking, these are the objects we obtain by forgetting about all but the Frobenius on $\mathbf{D}_{\mathrm{st}}(V)$. The main goal of the talk is to explain (not prove!) the classification theorem of Dieudonné-Manin ([BC, Theorem 8.1.4] or [FO, Theorem 8.25]). Feel free to mention *Newton polygons* and their *slopes*, but don't go into detail: We will cover these objects in the talk next week.

Now, add the structure of the filtrations: Define the category of *filtered φ-modules* $\mathbf{MF}_K^{\varphi}$ over K_0 ([BC, Definition 7.3.4]) and the category of *filtered (φ, N)-modules* (Definition 8.2.5 in loc. cit.). Don't be confused by the K in the index: The modules still live over K_0, but the *filtrations* are data on the modules obtained by tensoring with K. Show that the monodromy operator on filtered (φ, N)-modules is nilpotent (Lemma 8.2.8 in loc. cit.).

Talk 9: Hodge and Newton polygons – ??? (12.12.)

The Frobenius-operator and the filtration of a general φ-module don't have to fulfill any relations. For a given semi-stable representation V however, the Frobenius and the filtration of $\mathbf{D}_{\mathrm{st}}(V)$ are related by some technical property, which is called *weak admissibility*. It can be described by the use of *Hodge-polygons* (polygons built from the indices of the filtration steps, i.e. the Hodge-Tate weights of a filtered K-module) and *Newton-polygons* (polygons built from the Dieudonné-Manin decomposition of an isocrystal).

In this talk, you can follow [BC, §8.1]: Start by defining the *Hodge polygon* and *Hodge number*. Show their behavior under determinants, tensor products, duals and short exact sequences. Give at least one example. Then do the same for the *Newton polygon* and *Newton number* associated to an isocrystal over K_0. End the talk with a definition and explanation of *weak admissibility* (Definition 8.2.1).

Talk 10: The category equivalences $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \xrightarrow{\sim} \mathbf{MF}_K^{\varphi,\mathrm{wa}}$ and $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \xrightarrow{\sim} \mathbf{MF}_K^{\varphi,N,\mathrm{wa}}$ – ??? (19.12.)

The goal of this talk is to sketch a proof of the above category equivalences (the functors for the crystalline case were described in the introduction). For simplicity, in this description we only consider the semi-stable case and do not always mention the crystalline case separately. Then there are three statements that we have to show:

(i) The above functors are well-defined. (yet, we havn't seen that, for a semi-stable V, its filtered (φ, N)-module is weakly admissible, i.e. "admissible implies weakly admissible").

(ii) The functor $\mathbf{D}_{\mathrm{st}} \colon \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathbf{MF}_K^{\varphi,N,\mathrm{wa}}$ is fully faithful. Here $\mathbf{MF}_K^{\varphi,N,\mathrm{wa}}$ denotes the subcategory of those filtered (φ, N)-modules that are weakly admissible.

(iii) The functor $\mathbf{D}_{\mathrm{st}}$ is essentially surjective.

Stament (iii) is a very deep result ("Theorem B") that we cannot show in this seminar (but we might talk about some ideas of the proof in an optional talk). Statement (i) is Theorem 9.3.4, statement (ii) is Proposition 9.1.11 and 9.2.14 in [BC]. Gather all the results that we need and then give a detailed proof of both statements. In particular, explain the quasi-inverse functor $\mathbf{V}_{\mathrm{st}}$ and give an example.

Talk 11: (Optional) Relation to (φ, Γ)-modules – ??? (9.1.)

In [Ber] Berger explains how to recover $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{st}}(V)$ from the (φ, Γ)-module (over the Robba ring) associated to V. Define the functors $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{D}_{\mathrm{st}}$ for (φ, Γ)-modules and explain how they can be compared to the functors on Galois representations. Outline that these constructions can be used to show Theorem A and B.

Talk 12: (Optional) Theorem A: $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) = \mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(G_K)$ (p-adic monodromy conjecture) – ??? (16.1.)

Introduce the category of potentially semi-stable p-adic Galois representations, i.e. representations that become semi-stable after restricting the action of G_K to the action of $G_{K'}$ for a finite field extension K'/K. In this talk, we want to learn about the p-adic monodromy conjecture (which is a theorem now). It states that every de Rham representation is potentially semi-stable (the other implication is relatively easy, cf. [FO, Proposition 8.48]). Sketch the main ideas of the proof of [Ber, Corollaire 5.22].

Talk 13: (Optional) Theorem B: Weak admissibility implies admissibility – ??? (23.1.)

As a result of the previous talks, we were able to show that there exists a fully faithful functor $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \hookrightarrow \mathbf{MF}^{\varphi,\mathrm{wa}}_K$. That the essential image of this functor is actually the full category is the statement of *Theorem B*. Sketch the main ideas of [Col] (in particular Remarque 2.43) that lead to a proof of this theorem.

Talk 14: (Optional) Comparison isomorphisms between different cohomology theories and the Fontaine-Mazur conjecture – ??? (30.1.)

So far we have used the comparison isomorphisms for different cohomologies only as a motivation, but we have never seen any details. Use this talk to give a survey about étale cohomology and de Rham cohomology of varieties, and explain the rough ideas behind Falting's theorem. A good survey paper is [Niz], but feel free to use other references as well. Explain the *Fontaine-Mazur conjecture* (see e.g. [Car, §4]) and its relation to the comparison of cohomologies.

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