GAUS AG:
Emerton-Gee-Stacks

AG Venjakob

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ORGANISATION: The dates are preliminary. We meet every Thursday at 11 (c.t.) in SR8.
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Talk 1: Overview (19.10. – Marlon Kocher)

The goal of this talk is to briefly explain the ideas of the seminar. Give an informal introduction to stacks and an informal definition of the Emerton-Gee stack $\mathcal{X}_d$ ([EG22b, Definition 5.1.1]). Give an idea of how the moduli space perspective helps in solving the problems of the seminar (e.g. the existence of crystalline lifts, see talk 9).

Talk 2: Moduli of Galois Representations (26.10. – Alireza Shavali)

The goal of this talk is to give some background on moduli of Galois representations and complications that arise. Following [EG22b, Lecture 1] give an overview of the geometry of the scheme V parametrising Weil-Deligne Representations for $d = 1$. For $d = 2$ explain in what sense the geometry of V is complicated and explain how imposing the Fontaine-Laffaille condition serves as an improvement (cf. p.8 of [EG22b].) Explain [EG22b, Proposition 1.4.1]. We will learn more details on stacks in a subsequent talk. To simplify the presentation define the quotient stacks $[X/G]$ in terms of their T-points i.e. $[X/G](T) = \{(P, f) \mid P \text{ is a } G\text{-torsor over } T \text{ with a } G\text{-map } f : P \to G\}$. Sketch a proof of Proposition 1.4.1.

Talk 3: Introduction to Stacks (2.11. – Gautier Ponsinet)

The goal of this talk is to give an introduction to stacks. A good starting point is [EG22b, Lecture 2]. Ellaborate on the terminology from [Sta]. In particular define *category fibered in groupoids*, explain [Sta, Example 4.37.1] and mention [Sta, Lemma 4.37.3]. Explain the notions of (formal, resp. Ind) algebraic spaces/stacks and put an emphasis on presenting examples ([Eme, Examples 4.18, 5.7-5.9, 7.7, 7.8]). (See also [EG22a, Appendix A]).

Talk 4: Scheme theoretic images (9.11. – Anna Blanco)

Motivate and define formal objects [Sta, Definition 90.7] and versal rings ([Sta, Definition 107.2.2], [EG22a, p. 224 - 229)]). Discuss the notion of a scheme theoretic image of morphisms of stacks (see [EG22a, p. 220f] and [EG20]) and explain [EG22b, Theorem 2.5.4] (without a detailed proof).

Talk 5: φ- and (φ, Γ)-modules with A-coefficients (16.11. – Marvin Schneider)

Cover the content of [EG22b, Lecture 3]. The goal is to explain the equivalences of categories for φ- and (φ, Γ)-modules with A-coefficients. Ideally, sketch how these equivalences are proved.

Talk 6: Moduli stacks of étale φ-modules (23.11.)

In this talk we introduce the moduli stacks of étale φ-modules. [EG22b, Lecture 4] serves as a good outline of the talk. Fill in details from [EG22a] and [EG20] where you deem appropriate.

Talk 7: Moduli stacks of (φ, Γ)-modules I - construction of the stack (30.11. – Max Witzelsperger)

In the first part of this talk, construct the moduli stack $\mathcal{X}_d$ of projective étale (φ, Γ)-modules of rank d and give an overview of basic properties ([EG22b, section 5.1]). Introduce the moduli stack of weak Wach modules ([EG22b, section 5.2]) but don't go into detail in this part. State that $\mathcal{X}_d$ is an Ind-algebraic stack ([EG22b, section 5.3]).

In the second part, go through [EG22a, section 5.5] and focus on Theorem 5.5.12 and Corollary 5.5.18, the latter stating that $\mathcal{X}_d$ is a Noetherian formal algebraic stack.

Talk 8: Moduli stacks of (φ, Γ)-modules II - the geometry (7.12. – Rustam Steingart)

In the first part of this talk we want to study the irreducible components of the reduced Emerton-Gee stack $\mathcal{X}_{d,\mathrm{red}}$. Go through [EG22a, section 6.5] with a focus on Theorem 6.5.1.

In the second part of the talk, go through [EG22a, section 6.6] and state the main theorem of this section, which tells us that finite type points of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ are in bijection with the isomorphism classes of continuous G_K-representations over $\overline{\mathbf{F}}_p$ and that the closed points of the underlying topological space correspond to semi-simple representations.

Talk 9: Moduli stacks of (φ, Γ)-modules III - the rank one case (14.12. – Otmar Venjakob)

In this talk we study the geometry of $\mathcal{X}_1$. The goal is to prove [EG22a, Proposition 7.2.17] which asserts that we have

$$[\mathrm{Spf}(\mathcal{O}[\![I_K^{ab}]\!]) \times \widehat{\mathbb{G}_m}/\widehat{\mathbb{G}_m}] \cong \mathcal{X}_1.$$

Explain which problems arise for $d > 1$ (cf. [EG22a, Remark 7.2.19]).

Talk 10: Moduli stacks of (φ, Γ)-modules IV - trianguline rank two case (21.12.)

To understand $\mathcal{X}_2$ we will first study extensions of rank 1 modules. Recall that extensions are controlled by the Herr complex [EG22a, Lemma 5.1.2]. Following [EG22b, Lecture 7] describe the irreducible components of $\mathcal{X}_{2,red}$ labeled by Serre weights. State [EG22b, Theorem 7.3.3] without proof.

Talk 11: Crystalline Stacks (11.1.)

Go through [EG22b, Lecture 6]. You could start with a short reminder on Breuil-Kisin(-Fargue) modules (Definition 3.3.2 in (loc. cit.)) and then move on with section 6.1. Define the stack $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}$ which is roughly speaken the scheme-theoretic image (see talk 4) in $\mathcal{X}_d$ of a stack of Breuil-Kisin-Fargue modules with a certain descent condition. Present its basic properties and especially state Theorem 6.2.4. Give an overview of section 6.3 in (loc. cit.) but skip the details.

Talk 12: Crystalline Lifts (18.1.)

The goal of this talk is to give an idea of the proof of [EG22b, Theorem 8.1.1]. You could start by explaining an elementary proof for irreducible $\overline{\rho}\colon G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ and what problems occur in extending it to the arbitrary situation. Explain the example on p. 41 in (loc. cit.). Write down Theorem 8.1.4 in (loc. cit.). Explain how to derive the main theorem from it and sketch its proof.

Talk 13: Categorical Langlands (25.1.)

Give an overview of [EG22b, Lecture 10]. The goal of this talk is to explain [EG22b, rough conjecture 10.4.1]. Feel free to also take a look at (the introduction of) [EGH] and present some of the ideas.

Literatur

[EG20] Emerton, M., Gee, T. 2020. *"Scheme-theoretic images" of morphisms of stacks.* http://arxiv.org/abs/1506.06146v7.

[EG22a] Emerton, M., Gee, T. 2022. Moduli stacks of étale (phi,Gamma)-modules and the existence of crystalline lifts. arXiv e-prints. doi:10.48550/arXiv.1908.07185

[EG22b] Emerton, M., Gee, T. 2022. *Moduli stacks of (phi,Gamma)-modules: a survey.* https://arxiv.org/abs/2012.12719.

[EGH] Emerton, M., Gee, T., Hellmann, E. 2023. *An introduction to the categorical p-adic Langlands program*, https://arxiv.org/pdf/2210.01404.pdf

[Eme] Emerton, M. *Formal algebraic stacks*, Available online: https://www.math.uchicago.edu/~emerton/pdffiles/formal-stacks.pdf.

[Sta] Stacks Project