HOLOMORPHIC CURVES IN THE PRESENCE OF HOLOMORPHIC HYPERSURFACE FOLIATIONS

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Abstract. We prove a result which establishes restrictions on the pseudoholomorphic curves which can exist in a stable Hamiltonian manifold in the presence of certain $\mathbb{R}$-invariant foliations of the symplectization by holomorphic hypersurfaces. This result has applications in the first author's work [7, 6] on algebraic torsion in higher dimensional contact manifolds.

Contents

1. Background and main result

Our main results here are motivated by the study of algebraic torsion in contact manifolds [7, 6] and concern finding restrictions on the existence of pseudoholomorphic curves in certain manifolds equipped with stable Hamiltonian structures. The manifolds we consider will be smooth fibrations over a closed, oriented surface and we will assume further that the symplectization admits an $\mathbb{R}$-invariant foliation by pseudoholomorphic hypersurfaces which project to gradient flow lines of a Morse function on the surface. We are interested in identifying conditions which will guarantee that a punctured pseudoholomorphic curve is contained in the image of a leaf of a foliation. Before stating the main results we give some definitions.

Let M^{2n+1} be a closed, orientable manifold. A pair $\mathcal{H} = (\lambda, \omega) \in \Omega^1(M) \times \Omega^2(M)$ is said to be a *stable Hamiltonian structure* on M if

- $\lambda \wedge \omega^n$ is a volume form on M,
- $d\omega = 0$, and
- $d\lambda$ vanishes on the kernel of the map $v \mapsto i_v\omega$.

A stable Hamiltonian structure on M determines a splitting

$$TM = \mathbb{R}X_{\mathcal{H}} \oplus (\xi, \omega|_\xi)$$

of the tangent space of M into a symplectic hyperplane distribution $(\xi = \ker \lambda, \omega|_\xi)$ and a line bundle determined by the span of the *Reeb vector field* $X_{\mathcal{H}}$, which is the unique vector field satisfying

$$\lambda(X_{\mathcal{H}}) = 1 \qquad \text{and} \qquad i_{X_{\mathcal{H}}}\omega = 0.$$

We will refer to the triple (M, λ, ω) as a *stable Hamiltonian manifold.* A stable Hamiltonian structure (λ, ω) on M is said to be *nondegenerate* if all periodic orbits of the Reeb vector field are nondegenerate.

A codimension-2 submanifold $V \subset M$ is said to be a *stable Hamiltonian hypersurface* of M if the pair $\mathcal{H}' := (\lambda', \omega')$ defined by

$$\lambda' := i^*\lambda \qquad \omega' := i^*\omega,$$

where $i : V \hookrightarrow M$ is the inclusion map, is a stable Hamiltonian structure on V. In this case, the hyperplane distribution $\xi' := \ker \lambda'$ is naturally identified via i_* with $TV \cap \xi$. We say a stable Hamiltonian hypersurface

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$V \subset M$ is a *strong stable Hamiltonian hypersurface* if, in addition, V is invariant under the flow of $X_{\mathcal{H}}$. This is easily seen to be equivalent to requiring that the push forward by i_* of the Reeb vector field $X_{\mathcal{H}'}$ of the stable Hamiltonian structure $\mathcal{H}'$ is equal to $X_{\mathcal{H}}$ at all points in V. Along a strong stable Hamiltonian hypersurface $V \subset M$, we thus obtain a splitting of the tangent space of M

$$TM|_V = \mathbb{R}X_{\mathcal{H}} \oplus (\xi', \omega') \oplus (\xi_V^{\perp}, \omega|_{\xi_V^{\perp}})$$

into a line bundle spanned by $X_{\mathcal{H}}$ and two symplectic vector bundles, where

$$\xi_V^{\perp} = \{v \in \xi|_V \,|\, \omega(v, i_* w) = 0 \quad \forall w \in \xi'\}\,, \tag{1}$$

is the symplectic complement of $\xi' \approx TV \cap \xi$ in $\xi|_V$. Moreover, since the flow of $X_{\mathcal{H}}$ preserves λ and ω, it also preserves this splitting. Therefore, given a periodic orbit γ of $X_{\mathcal{H}}$ lying in V and a symplectic trivialization Φ of $\gamma^*\xi_V^{\perp}$, we can assign a normal Conley–Zehnder index $\mu_N^{\Phi}(\gamma)$ by considering the restriction of the linearized flow along γ to the symplectic normal bundle $\xi_V^{\perp}$ of ξ' in ξ. We describe this construction in more detail in Section 2 below.

We consider a manifold M^{2n+1} equipped with a nondegenerate stable Hamiltonian structure $\mathcal{H} = (\lambda, \omega)$, and we will denote by the triple (Σ, p, Y^{2n-1}) a smooth fibration $p : M \to \Sigma$ over a closed, oriented surface Σ with fiber diffeomorphic to Y. Given a Morse function f on Σ we say the fibration (Σ, p, Y) is *f-admissible* if for each critical point $w \in \mathrm{crit}(f)$ of the function, the fiber $Y_w = p^{-1}(w)$ over w is a strong stable Hamiltonian hypersurface. We will denote f-admissible fibrations by quadruples (Σ, p, Y, f).

Given an f-admissible fibration (Σ, p, Y, f) for $(M, \mathcal{H})$ and a critical point $w \in \mathrm{crit}(f)$, we note that at any point $y \in Y_w$, the derivative p_* at y determines a linear isomorphism

$$p_*(y) : (\xi_{Y_w}^{\perp})_y \to T\Sigma_w$$

from the symplectic normal bundle of Y_w in M at $y \in Y_w$ to the tangent space of Σ at w. This map will either be orientation preserving at every point in Y_w or orientation reversing at every point in Y_w. This allows us to define a sign function

$$\mathrm{sign} : \mathrm{crit}(f) \to \{-1, 1\}$$

on the set of critical points of f by requiring

$$\mathrm{sign}(w) = \begin{cases} 1 & \text{if } p_*|_{\xi_{Y_w}^{\perp}} \text{ is everywhere orientation preserving} \\ -1 & \text{if } p_*|_{\xi_{Y_w}^{\perp}} \text{ is everywhere orientation reversing.} \end{cases}$$

Choosing at each $w \in \mathrm{crit}(f)$ with $\mathrm{sign}(w) = 1$ an orientation preserving linear map $\Phi_w : T\Sigma_w \to (\mathbb{R}^2, \omega_0)$ and at each $w \in \mathrm{crit}(f)$ with $\mathrm{sign}(w) = -1$ an orientation reversing linear map $\Phi_w : T\Sigma_w \to (\mathbb{R}^2, \omega_0)$ we obtain a global orientation-preserving trivialization

$$\Phi_w \circ p_*|_{\xi_{Y_w}^{\perp}} \to (\mathbb{R}^2, \omega_0)$$

of $\xi_{Y_w}^{\perp}$ which can be homotoped to a symplectic trivialization. Thus in an f-admissible fibration there is a preferred homotopy class of symplectic trivialization of the symplectic normal bundle to $p^{-1}(\mathrm{crit}(f))$.

Assuming still that M is a stable Hamiltonian manifold, let $\tilde{J}$ be an almost complex structure on $\mathbb{R} \times M$ which is compatible with the stable Hamiltonian structure $\mathcal{H} = (\lambda, \omega)$; that is, $\tilde{J}$ is an $\mathbb{R}$-invariant endomorphism of $T(\mathbb{R} \times M)$ which squares to negative the identity, and with respect to the splitting

$$T(\mathbb{R} \times M) \approx \mathbb{R}\partial_a \oplus TM \approx \mathbb{R}\partial_a \oplus \mathbb{R}X_{\mathcal{H}} \oplus \xi$$

the action of $\tilde{J}$ is given by

$$\tilde{J}\partial_a = X_{\mathcal{H}} \qquad \text{and} \qquad \tilde{J}|_{\pi^*\xi} = \pi^* J \tag{2}$$

where $\pi : \mathbb{R} \times M \to M$ is the canonical projection and $J \in \mathrm{End}(\xi)$ is a complex structure on ξ for which the bilinear form $\omega(\cdot, J\cdot)|_{\xi \times \xi}$ is symmetric and positive definite. We say that a codimension-2 foliation $\mathcal{F}$ of $\mathbb{R} \times M$ is an *$\mathbb{R}$-invariant, asymptotically cylindrical, $\tilde{J}$-holomorphic foliation* if:

- $\mathcal{F}$ is invariant under translations in the $\mathbb{R}$-coordinate,

and if there exists a strong stable Hamiltonian hypersurface $V \subset M$ so that:

- $\mathbb{R} \times V$ has $\tilde{J}$-invariant tangent space,
- the union of leaves of $\mathcal{F}$ fixed by $\mathbb{R}$-translation is equal to $\mathbb{R} \times V \subset \mathbb{R} \times M$, and

- all other leaves of the foliation are $\tilde{J}$-holomorphic hypersurfaces which are asymptotically cylindrical[1] over some collection of components of V, and which project by π to embedded submanifolds smoothly foliating $M \setminus V$.

We will refer to the stable Hamiltonian hypersurface V as the *binding set* of the foliation $\mathcal{F}$. For brevity, we will from now on refer to $\mathbb{R}$-invariant, asymptotically cylindrical, $\tilde{J}$-holomorphic foliations simply as *holomorphic foliations* or *$\tilde{J}$-holomorphic foliations* when we wish to specify the almost complex structure.

Now assuming the manifold M is equipped with both a holomorphic foliation $\mathcal{F}$ with binding V and an f-admissible fibration (Σ, p, Y, f), we will say that $\mathcal{F}$ *is compatible with* (Σ, p, Y, f) if:

- $p^{-1}(\mathrm{crit}(f))$ is equal to the binding V of the foliation,
- all other leaves of the foliation are diffeomorphic to $\mathbb{R} \times Y$ and admit smooth parametrizations of the form
$$(s, y) \in \mathbb{R} \times Y \mapsto (a(s, y), m(s, y)) \in \mathbb{R} \times M$$
where $p(m(s, y)) = \gamma(s)$ for some solution γ to the gradient flow equation
$$\dot{\gamma}(s) = \nabla f(\gamma(s))$$
with respect to an appropriate metric g_Σ on Σ, and where $a(s, y)$ satisfies
$$\lim_{s\to\pm\infty} a(s, y) = \mathrm{sign}(\lim_{s\to\pm\infty} p(m(s, y))) \pm \infty.$$

We are interested in understanding the punctured pseudoholomorphic curves in $\mathbb{R}\times M$ when M is equipped with a holomorphic foliation compatible with an f-admissible fibration and all asymptotic limits of the given curve lie in the binding of the foliation. We recall that a *asymptotically cylindrical, punctured pseudoholomorphic map* is a quadruple $(S, j, \Gamma, \tilde{u} = (a, u))$ where

- (S, j) is a closed Riemann surface
- $\Gamma \subset S$ is a finite set, called the set of *punctures*
- $\tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ is a $\tilde{J}$-holomorphic map, i.e. $\tilde{u}$ satisfies the equation
$$d\tilde{u} \circ j = \tilde{J}(\tilde{u}) \circ d\tilde{u},$$
and
- For each puncture $z \in \Gamma$ there exists a periodic orbit $\gamma_z^{m_z}$ so that the map $\tilde{u}$ is asymptotic near z to a half-cylinder of the form $\mathbb{R}^+ \times \gamma_z^{m_z}$ or $\mathbb{R}^- \times \gamma_z^{m_z}$. Here γ_z is a simple periodic orbit of $X_{\mathcal{H}}$ and $\gamma_z^{m_z}$ for $m_z \in \mathbb{N}$ denotes the m_z-fold cover of the simple orbit γ_z.

We will use the term *pseudoholomorphic curve* to refer to an equivalence class $C = [S, j, \Gamma, \tilde{u} = (a, u)]$ of such maps under the equivalence relation of holomorphic reparametrization of the domain.

We consider now an asymptotically cylindrical, punctured pseudoholomorphic map, $(S, j, \Gamma, \tilde{u} = (a, u))$ in a stable Hamiltonian manifold $(M, \mathcal{H})$ equipped with an f-admissible fibration (Σ, p, Y, f), and we assume that all asymptotic limits $\gamma_z^{m_z}$ of the map $\tilde{u}$ lie in the strong stable Hamiltonian hypersurface $p^{-1}(\mathrm{crit}(f))$. In this case, the projection of the map $p \circ u : S \setminus \Gamma \to \Sigma$ admits a continuous extension $\bar{v} : S \to \Sigma$ over the punctures. Our main theorem puts restrictions on the degree of this map in the presence of a holomorphic foliation and under some assumptions on the normal Conley–Zehnder indices of the asymptotic limits and tells us that, under these assumptions, the image of the pseudoholomorphic map $\tilde{u}$ is contained in a leaf of the foliation precisely when the degree of this map is zero.

Theorem 1.1. *Let (M, λ, ω) be a nondegenerate stable Hamiltonian manifold equipped with a compatible almost complex structure $\tilde{J}$, an f-admissible fibration (Σ, p, Y, f) and a $\tilde{J}$-holomorphic foliation $\mathcal{F}$ compatible with (Σ, p, Y, f). Let $(S, j, \Gamma, \tilde{u} = (a, u))$ be an asymptotically cylindrical $\tilde{J}$-holomorphic map with all asymptotic limits contained in $p^{-1}(\mathrm{crit}(f))$ and let $\bar{v} : S \to \Sigma$ denote the continuous extension of the map $p \circ u : S \setminus \Gamma \to \Sigma$ over the punctures. Assume moreover that*

$$\mu_N^\Phi(\gamma_z^{m_z}) \in \{-1, 0, 1\} \tag{3}$$

for every $z \in \Gamma$, where Φ is the homotopy class of symplectic trivialization of the symplectic normal bundle to $p^{-1}(\mathrm{crit}(f))$ determined by the fibration. Then:

[1] *Asymptotically cylindrical* will be defined precisely in Section 2.

- *If* $\operatorname{sign}(w) = 1$ *for all* $w \in \operatorname{crit}(f)$ *then the map* $\bar{v}$ *has nonnegative degree and has degree zero precisely when the image of the map* $\tilde{u}$ *is contained in a leaf of the foliation* $\mathcal{F}$*.*
- *If* $\operatorname{sign}(w) = -1$ *for all* $w \in \operatorname{crit}(f)$ *then the map* $\bar{v}$ *has nonpositive degree and has degree zero precisely when the image of the map* $\tilde{u}$ *is contained in a leaf of the foliation* $\mathcal{F}$*.*
- *If the map* $\operatorname{sign} : \operatorname{crit}(f) \to \{-1, 1\}$ *is surjective then the map* $\bar{v}$ *has degree zero and the image of the map* $\tilde{u}$ *is contained in a leaf of the foliation* $\mathcal{F}$*.*

The proof of our main result here relies heavily on some results from the in-preparation work [8] which generalizes some of the results from the 4-dimensional intersection theory studied in [10] to higher dimensions. We will summarize the relevant results in the following section and then apply these results in Section 3 below to prove the main the result.

As an immediate corollary of the above result we have the following.

Corollary 1.2. *With* (M, λ, ω)*,* $\tilde{J}$*,* (Σ, p, Y, f)*,* $\mathcal{F}$*, and* $(S, j, \Gamma, \tilde{u} = (a, u))$ *satisfying all the assumptions of Theorem 1.1 above, assume that* $g(S) < g(\Sigma)$*. Then the image of the map* $\tilde{u}$ *is contained in a leaf of the foliation* $\mathcal{F}$*.*

Proof. As observed before the statement of Theorem 1.1, the assumption that all punctures of $u : S \setminus \Gamma \to M$ are contained in the binding set of the foliation $V = p^{-1}(\operatorname{crit}(f))$ implies that the map $p \circ u : S \setminus \Gamma \to \Sigma$ has a continuous extension $\bar{v} : S \to \Sigma$. A well-known argument from algebraic topology then implies that the degree of the map $\bar{v}$ must be zero. Indeed, since for a closed, oriented surface the cup product pairing

$$H^1(\Sigma) \times H^1(\Sigma) \xrightarrow{\cup} H^2(\Sigma)$$

is nondegenerate, a map $\bar{v} : S \to \Sigma$ with nonzero degree induces an injection

$$\bar{v}^* : H^1(\Sigma) \approx \mathbb{Z}^{2g(\Sigma)} \to H^1(S) \approx \mathbb{Z}^{2g(S)}$$

which is impossible unless $g(\Sigma) \leq g(S)$. Since we assume that $g(S) < g(\Sigma)$ we conclude that $\bar{v}$ has degree zero.

Given that $\det(\bar{v}) = 0$, it follows immediately from Theorem 1.1 that the image of $\tilde{u}$ is contained in a leaf of the foliation. □

These results have applications in work of the first author on algebraic torsion in contact manifolds. In [7, 6], the author introduces a higher-dimensional generalization of the notion of a *spinal open book decomposition* (SOBD), as defined in [5] for dimension 3. This geometric structure supports a unique isotopy class of contact structures in the spirit of Giroux [2] and contains a fibration over a contact manifold with Liouville fibers (the "pages"). Given a SOBD supporting a contact structure, it induces holomorphic foliations in the symplectization of the contact manifold lifting the pages of the SOBD, generalizing the construction of a *holomorphic open book* as e.g. in [1, 11]. In the case where the leaves of the foliation are codimension-2 — which is the case where the intersection-theoretic arguments used here are applicable — it is of the type described above.

In order to develop computational techniques for SFT-type invariants of the contact manifold, in which the control over holomorphic curves is crucial, the above results are important. In the situation described above, one can arrange that the relevant holomorphic curves are asymptotic to periodic orbits in fibers lying over critical points of a Morse function. Moreover, the normal linearized flow along these fibers can be expressed in terms of the Hessian of the Morse function f. In particular, one can show that given a number $T > 0$, every periodic orbit $\gamma \in Y_w$ with period less than T has normal Conley–Zehnder index given by the formula

$$\mu_N^{\Phi}(\gamma) = \operatorname{sign}(w)\,(\operatorname{ind}_w(f) - 1) \in \{-1, 0, 1\}$$

provided the function f is sufficiently C^2 close to a constant, so the hypotheses of our main theorem are met. In the case where the degree of the resulting map $\bar{v}$ is zero — as it is, for example, in the case considered in Corollary 1.2 above — then the above theorem reduces the study of certain holomorphic curves in the ambient symplectization to that of the Liouville completion of the pages, thus reducing the problem by two dimensions. This fact, when combined with the symmetries of the Morse function f, can be exploited to obtain information on the contact structure as is done in [7, 6].

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2. Intersection theory of punctured pseudoholomorphic curves and pseudoholomorphic hypersurfaces

In this section we will review some results from the in-preparation work [8] which are needed in the proof of our main result.

Let $(M^{2n+1}, \mathcal{H} = (\lambda, \omega))$ be a closed, orientable manifold equiped with a nondegenerate stable Hamiltonian structure. Recall from that the introduction that a submanifold $i : V^{2n-1} \hookrightarrow M$ is said to be a stong stable Hamiltonian hypersurface if $\mathcal{H}' = (\lambda', \omega') := (i^*\lambda, i^*\omega)$ is a stable Hamiltonian structure on V and the Reeb vector field $X_{\mathcal{H}}$ of $\mathcal{H}$ is everywhere tangent to V. In this case we have a splitting

$$TM|_V = \mathbb{R}X_{\mathcal{H}} \oplus (\xi', \omega') \oplus (\xi_V^{\perp}, \omega) \approx \mathbb{R}X_{\mathcal{H}'} \oplus (\xi \cap TV, \omega) \oplus (\xi_V^{\perp}, \omega) \tag{4}$$

with $\xi_V^{\perp}$ the symplectic complement to ξ' in $\xi|_V$ as defined in (1), and we note that the first two summands give TV. The linearized flow of $X_{\mathcal{H}}$ along V preserves this splitting along with the symplectic structure on the second two summands.

Let $\gamma : S^1 \approx \mathbb{R}/\mathbb{Z} \to M$ be a T-periodic orbit of $X_{\mathcal{H}}$, i.e. γ satisfies the equation

$$\dot{\gamma}(t) = T \cdot X_{\mathcal{H}}(\gamma(t))$$

for all $t \in S^1$. Assuming that $\gamma(S^1) \subset V$, we can choose a symplectic trivialization of the hyperplane distribution $\xi = \xi' \oplus \xi_V^{\perp}|_{\gamma(S^1)}$ along γ which respects the splitting (4), i.e. one of the form

$$\Phi = \Phi_T \oplus \Phi_N : \xi' \oplus \xi_V^{\perp}|_{\gamma(S^1)} \to S^1 \times (\mathbb{R}^{2n-2}, \omega_0) \oplus (\mathbb{R}^2, \omega_0)$$

with Φ_T and Φ_N symplectic trivialization of $\xi'|_{\gamma(S^1)}$ and $\xi_V^{\perp}|_{\gamma(S^1)}$ respectively. Given such a trivialization we can define the Conley–Zehnder index of the orbit γ viewed as an orbit in M as usual by

$$\mu^{\Phi}(\gamma) := \mu_{CZ}(\Phi(\gamma(t)) \circ d\psi_{Tt}(\gamma(0)) \circ \Phi(\gamma(0))^{-1})$$

where $\psi : \mathbb{R} \times M \to M$ is the flow generated by $X_{\mathcal{H}}$ and where μ_{CZ} on a path of symplectic matrices starting at the identity and ending at a matrix without 1 in the spectrum is as defined in [3, Theorem 3.1]. But since, as observed above, $d\psi_t$ preserves the splitting (4), we can also consider the Conley–Zenhder indices that arise from the restrictions of $d\psi_t$ to $\xi'|_{\gamma}$ and $\xi_V^{\perp}|_{\gamma}$. In particular we define

$$\mu_V^{\Phi_T}(\gamma) := \mu_{CZ}(\Phi_T(\gamma(t)) \circ d\psi_{Tt}(\gamma(0))|_{\xi'} \circ \Phi_T(\gamma(0))^{-1})$$

which is the Conley–Zehnder index of γ viewed as a periodic orbit lying in V, and

$$\mu_N^{\Phi_N}(\gamma) := \mu_{CZ}(\Phi_N(\gamma(t)) \circ d\psi_{Tt}(\gamma(0))|_{\xi_V^{\perp}} \circ \Phi_N(\gamma(0))^{-1})$$

which we will call the *normal Conley–Zehnder index of* γ *relative to* Φ_N. We note that basic properties of Conley–Zehnder indices which can be found, e.g. in [3, Theorem 3.1] show that these quantites are related by

$$\mu^{\Phi}(\gamma) = \mu_V^{\Phi_T}(\gamma) + \mu_N^{\Phi_N}(\gamma).$$

We now recall that a complex structure J on ξ is said to be *compatible* with the stable Hamiltonian structure (λ, ω) if the bilinear form on ξ defined by $\omega(\cdot, J\cdot)|_{\xi\times\xi}$ is symmetric and positive definite. We will denote the set of compatible complex structures on ξ by $\mathcal{J}(M, \xi)$. Given a strong stable Hamiltonian hypersurface $V \subset M$ and a choice of $J \in \mathcal{J}(M, \xi)$ we will say that J is V*-compatible* if J fixes the hyperplane distribution $\xi' = TV \cap \xi$ along V. We will denote the set of such complex structures by $\mathcal{J}(M, V, \xi)$. We claim that a $J \in \mathcal{J}(M, V, \xi)$ necessarily also fixes $\xi_V^{\perp}$. Indeed, compatibility of J with ω implies that $\omega(J\cdot, J\cdot)|_{\xi\times\xi} = \omega|_{\xi\times\xi}$ since we can compute

$$\begin{aligned} \omega(Jv, Jw) &= \omega(w, J(Jv)) && \text{symmetry of } \omega(\cdot, J\cdot)|_{\xi\times\xi} \\ &= \omega(w, -v) && J^2 = -I \\ &= \omega(v, w) \end{aligned}$$

for any sections v, w of ξ. Thus if $v \in \xi|_V$ is ω-orthogonal to every vector in $\xi' = \xi \cap TV$, then Jv is also ω-orthogonal to every vector in ξ' provided that J fixes ξ'. Thus J fixes $\xi_V^{\perp}$ as claimed. We note that since both the linearized flow $d\psi_t$ of $X_{\mathcal{H}}$ and a compatible $J \in \mathcal{J}(M, V, \xi)$ preserve the splitting (4), the asymptotic operator

$$\mathbf{A}_\gamma h(t) := -J \left. \frac{d}{ds} \right|_{s=0} d\psi_{-Ts} h(t+s)$$

of a periodic orbit γ lying in V also preserves the splitting. We will write

$$\mathbf{A}_\gamma = \mathbf{A}_\gamma^T \oplus \mathbf{A}_\gamma^N : W^{1,2}(\xi') \oplus W^{1,2}(\xi_V^\perp) \to L^2(\xi') \oplus L^2(\xi_V^\perp)$$

to indicate the resulting splitting of the operator.

Continuing to assume that $V \subset M$ is a strong stable Hamiltonian hypersurface and $J \in \mathcal{J}(M, V, \xi)$ is a V-compatible complex structure, we extend J to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ in the usual way, i.e. so that $\tilde{J}$ satisfies (2). We note that for such an almost complex structure, the submanifold $\mathbb{R} \times V$ of $\mathbb{R} \times M$ is $\tilde{J}$-holomorphic since we assume that $X_{\mathcal{H}}$ is tangent to V and that J fixes $\xi' = \xi \cap TV$. Just as one can consider holomorphic curves which are asymptotic to cylinders of the form $\mathbb{R} \times \{\text{periodic orbit}\}$ one can $\tilde{J}$-holomorphic hypersurfaces which are asymptotic to cylindrindical $\tilde{J}$-holomorphic hypersurfaces of the form $\mathbb{R} \times V$ with V a strong stable Hamiltonian hypersurface. Before giving a more precise definition, we introduce some more geometric data on our manifold.

Given a $J \in \mathcal{J}(M, \xi)$ we can define a Riemannian metric

$$g_J(v, w) = \lambda(v)\lambda(w) + \omega(\pi_\xi v, J\pi_\xi w) \tag{5}$$

where $\pi_\xi : TM \approx \mathbb{R}X_{\mathcal{H}} \oplus \xi \to \xi$ is the projection onto ξ along $X_{\mathcal{H}}$. We can extend g_J to a metric $\tilde{g}_J$ on $\mathbb{R} \times M$ by forming the product metric with the standard metric on $\mathbb{R}$, i.e. by defining

$$\tilde{g}_J := da \otimes da + \pi^* g_J. \tag{6}$$

We will denote the exponential maps of g_J and $\tilde{g}_J$ by exp and $\widetilde{\exp}$ respectively and note that these are related by

$$\widetilde{\exp}_{(a,p)}(b, v) = (a + b, \exp_p v).$$

We note that if J is V-compatible for some strong stable Hamiltonian hypersurface $V \subset M$, then the symplectic normal bundle $\xi_V^\perp$ is the g_J-orthogonal complement of TV in $TM|_V$, and that $\pi^*\xi_V^\perp$ is the $\tilde{g}_J$-orthogonal complement of $T(\mathbb{R} \times V)$ in $T(\mathbb{R} \times M)|_{\mathbb{R}\times V}$. We further note that, since V is assumed to be compact, the restrictions of exp and $\widetilde{\exp}$ to $\xi_V^\perp$ and $\pi^*\xi_V^\perp$ respectively are embeddings on some neighborhood of the zero sections.

Now consider a pair V_+, V_- of strong stable Hamiltonian hypersurfaces and assume that $V := V_+ \cup V_-$ is also a strong stable Hamiltonian hypersurface, i.e. that all components of V_+ and V_- are either disjoint or identical. We let $J \in \mathcal{J}(M, V, \xi)$ be a V-compatible J with associated $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$. We are interested in $\tilde{J}$-holomorphic submanifolds which outside of a compact set can be described by exponentially decaying sections of the normal bundles to V_+ and V_-, More precisely, we say that a $\tilde{J}$-holomorphic submanifold $\tilde{V} \subset \mathbb{R} \times M$ is *positively asymptotically cylindrical over* V_+ and *negatively asymptotically cylindrical over* V_- if there exists an $R > 0$ and sections

$$\eta_+ : [R, +\infty) \to C^\infty(\xi_{V_+}^\perp)$$
$$\eta_- : (-\infty, -R] \to C^\infty(\xi_{V_-}^\perp)$$

so that

$$\tilde{V} \cap ([R, +\infty) \times M) = \bigcup_{(a,p)\in[R,+\infty)\times V_+} \widetilde{\exp}_{(a,p)} \eta_+(a, p)$$
$$\tilde{V} \cap ((-\infty, -R] \times M) = \bigcup_{(a,p)\in(-\infty,-R]\times V_-} \widetilde{\exp}_{(a,p)} \eta_-(a, p)$$

and so that there exist constants $M_i > 0$, $d > 0$ satisfying

$$\left| \widetilde{\nabla}^i \eta_\pm(a, p) \right| \le M_i e^{-d|a|}$$

for all $i \in \mathbb{N}$ and $\pm a \in [R, +\infty)$, where $\tilde{\nabla}$ is the extension of a connection ∇ on $\xi_V^\perp$ to a connection $\tilde{\nabla}$ on $\pi^*\xi_V^\perp$ defined by requiring $\tilde{\nabla}_{\partial_a}\eta(a,p) = \partial_a\eta(a,p)$. We will refer to the sections η_+ and η_- respectively as *positive* and *negative asymptotic representatives of* $\tilde{V}$.

Our main goal here is to understand the intersection properties of punctured pseudoholomorphic curves with asymptotically cylindrical pseudoholomorphic hypersurfaces. The main difficulty arises from the non-compactness of the manifolds in question. Indeed a punctured pseudoholomorphic curve whose image is not contained in the $\tilde{J}$-holomorphic hypersurface $\tilde{V}$ may have punctures limiting to a periodic orbits lying in V_+ or V_-. In this case, it's not a priori clear that the intersection number between the curve and the hypersurface is finite. Even assuming this intersection number is finite, it is not homotopy invariant as intersections can be lost or created at infinity. We will see below that these difficulties can be dealt with via higher-dimensional analogs of techniques developed in [10].

Before presenting the relevant results it will be convenient to establish some standard assumptions and notations for the next several definitions and results.

Assumptions 2.1. We assume that:

(a) (M, λ, ω) is a closed, manifold with nondegenerate stable Hamiltonian structure (λ, ω) with $\xi = \ker\lambda$ and $X_{\mathcal{H}}$ the associated Reeb vector field,
(b) $V_+ \subset M$, $V_- \subset M$, and $V = V_+ \cup V_-$ are strong stable Hamiltonian hypersurfaces of M,
(c) $J \in \mathcal{J}(M, V, \xi)$ is a V-compatible complex structure on ξ and $\tilde{J}$ is the $\mathbb{R}$-invariant almost complex structure on $\mathbb{R} \times M$ associated to J (defined by (2),
(d) $\tilde{g}_J$ is the Riemannian metric on $\mathbb{R} \times M$ defined by (6) and $\widetilde{\exp}$ is the associated exponential map,
(e) $\tilde{V} \subset \mathbb{R} \times M$ is a $\tilde{J}$-holomorphic hypersurface which is positively asymptotically cylindrical over V_+ and negatively asymptotically cylindrical over V_-,
(f) $\eta_+ : [R, +\infty) \to C^\infty(\xi_{V_+}^\perp)$ and $\eta_- : (-\infty, -R] \to C^\infty(\xi_{V_-}^\perp)$ are, respectively, positive and negative asymptotic representatives of $\tilde{V}$,
(g) $C = [S, j, \Gamma = \Gamma^+ \cup \Gamma^-, \widetilde{u} = (a, u)]$ is a finite-energy $\tilde{J}$-holomorphic curve, and at $z \in \Gamma$, C is asymptotic to $\gamma_z^{m_z}$ (with $\gamma_z^{m_z}$ indicating the m_z-fold covering of a simple periodic orbit γ_z),
(h) Φ is a trivialization of $\xi_V^\perp$ along every periodic orbit lying in V which occurs as an asymptotic limit of C.

The following theorem can be seen as a generalization of Theorem 2.2 in [9]. Proof will be given in [8].

Theorem 2.2. *Assume 2.1 and assume that at $z \in \Gamma^+$, C is asymptotic to a periodic orbit $\gamma_z^{m_z} \subset V_+$. Then there exists an $R' \in \mathbb{R}$, a smooth map*

$$u_T : [R', \infty) \times S^1 \to [R, \infty) \times V_+$$

and a smooth section

$$u_N : [R', \infty) \times S^1 \to u_T^*\pi^*\xi_{V_+}^\perp$$

so that the map

$$(s,t) \mapsto \widetilde{\exp}_{u_T(s,t)} u_N(s,t) \tag{7}$$

parametrizes C near z. Moreover, if we assume that the image of C is not a subset of the asymptotically cylindrical hypersurface $\tilde{V}$, then

$$u_N(s,t) - \eta_+(u_T(s,t)) = e^{\mu s}[e(t) + r(s,t)] \tag{8}$$

for all $(s,t) \in [R', \infty) \times S^1$ where:

- *$\mu < 0$ is a negative eigenvalue of the normal asymptotic operator $\mathbf{A}^N_{\gamma_z^{m_z}}$,*
- *$e \in \ker(\mathbf{A}^N_{\gamma_z^{m_z}} - \mu) \setminus \{0\}$ is an eigenvector with eigenvalue μ, and*
- *$r : [R', \infty) \times S^1 \to u_T^*\pi^*\xi_V^\perp$ is a smooth section satisfying exponential decay estimates of the form*

$$\left|\tilde{\nabla}_s^i \tilde{\nabla}_t^j r(s,t)\right| \le M_{ij} e^{-d|s|} \tag{9}$$

for some positive contants M_{ij}, d and all $(i,j) \in \mathbb{N}^2$

Similarly, if we assume that at $z \in \Gamma^-$, C is asymptotic to a periodic orbit $\gamma_z^{m_z} \subset V_-$., then there exists an $R' \in \mathbb{R}$, a smooth map

$$u_T : (-\infty, R'] \times S^1 \to (-\infty, -R] \times V_-$$

and a smooth section

$$u_N : (-\infty, R'] \times S^1 \to u_T^* \pi^* \xi_{V_-}^{\perp}$$

so that the map

$$(s,t) \mapsto \widetilde{\exp}_{u_T(s,t)} u_N(s,t)$$

parametrizes C near z. Moreover, if the image of C is not contained in $\tilde{V}$, then $u_N(s,t) - \eta_-(u_T(s,t))$ satisfies a formula of the form (8) for all $(s,t) \in (-\infty, R'] \times S^1$, where now:

- *$\mu > 0$ is a positive eigenvalue of the normal asymptotic operator $\mathbf{A}^N_{\gamma_z^{m_z}}$,*
- *$e \in \ker(\mathbf{A}^N_{\gamma_z^{m_z}} - \mu) \setminus \{0\}$, as before, is an eigenvector with eigenvalue μ, and*
- *$r : (-\infty, R'] \to u_T^* \pi^* \xi_V^{\perp}$ is a smooth section satisfying exponential decay estimates of the form (9) for some positive contants M_{ij}, d.*

We note that the bundles of the form $u_T^* \pi^* \xi_V^{\perp}$ ocurring in the statement of this theorem are trivializable since they are complex line bundles over a space which retracts onto S^1. In any trivialization the eigenvector e from formula (8) satisfies a linear, nonsingular ODE, and thus is nowhere vanishing since we assume it is not identically zero. Since the "remainder term" r in the formula (8) converges to zero, we conclude that the functions $u_N(s,t) - \eta_\pm(u_T(s,t))$ are nonvanishing for sufficiently large $|s|$. However, since zeroes of this function can be seen to correspond to intersections between the curve C and the hypersurface $\tilde{V}$ occuring sufficiently close to the punctures of C, we conclude that all intersections between C and $\tilde{V}$ are contained in a compact set. Moreover, since intersections between C and $\tilde{V}$ can be shown to be isolated and of positive local order (see e.g. [4, Lemma 3.4]), we conclude that the algebraic intersection number between C and $\tilde{V}$ is finite:

Corollary 2.3. *Assume 2.1 and assume that no component of the curve C has image contained in the $\tilde{J}$-holomorphic hypersurface $\tilde{V}$. Then the algebraic intersection number $C \cdot \tilde{V}$, defined by summing local intersection indices, is finite and nonnegative, and $C \cdot \tilde{V} = 0$ precisely when C and $\tilde{V}$ do not intersect.*

This corollary deals with the first difficulty in understanding intersections between punctured curves and asymptotically cylindrical hypersurfaces described above, namely the finiteness of the intersection number. A second consequence of the asymptotic formula from Theorem 2.2, again stemming from the fact that the quantities $u_N(s,t) - \eta_\pm(u_T(s,t))$ are nonzero for sufficiently large $|s|$, is that the normal approach of the curve C has a well-defined winding number relative to a trivialization Φ of $\xi_V^{\perp}|_{\gamma_z}$. This winding will be given by the winding of the eigenvector from formula (8) relative to Φ, and for a given puncture z of C, we will denote this quantity by

$$\mathrm{wind}^{\Phi}_{rel}((C; z), \tilde{V}) = \mathrm{wind}(e).$$

Combining this observation with the characterization of the Conley–Zehnder index in terms of the asymptotic operator from [3, Definition 3.9/Theorem 3.10] leads to the following corollary.

Corollary 2.4. *Assume 2.1, and assume that no component of the curve $C = [S, j, \Gamma_+ \cup \Gamma_-, \tilde{u} = (a,u)]$ has image contained in the holomorphic hypersurface $\tilde{V}$. Then:*

- *If $z \in \Gamma$ is a positive puncture at which $\tilde{u}$ limits to $\gamma_z^{m_z} \subset V_+$ then*

$$\mathrm{wind}^{\Phi}_{rel}((C; z), \tilde{V}) \leq \lfloor \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rfloor =: \alpha_N^{\Phi;-}(\gamma_z^{m_z}). \tag{10}$$

- *If $z \in \Gamma$ is a negative puncture at which $\tilde{u}$ limits to $\gamma_z^{m_z} \subset V_-$ then*

$$\mathrm{wind}^{\Phi}_{rel}((C; z), \tilde{V}) \geq \lceil \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rceil =: \alpha_N^{\Phi;+}(\gamma_z^{m_z}). \tag{11}$$

The numbers $\alpha_N^{\Phi;-}(\gamma)$ and $\alpha_N^{\Phi;+}(\gamma)$ are, respectively, the biggest/smallest winding number achieved by an eigenfunction of the normal asymptotic operator of any orbit γ corollary responding to a negative/positive eigenvalue. Observe that we have the formulas

$$\mu^{\Phi}_N(\gamma) = 2\alpha_N^{\Phi;-}(\gamma) + p_N(\gamma) = 2\alpha_N^{\Phi;+}(\gamma) - p_N(\gamma), \tag{12}$$

where $p_N(\gamma) \in \{0,1\}$ is the *normal parity* of the orbit γ (which is independent of the trivialization Φ).

We will see in a moment that this corollary can be used to deal with the second difficulty in understanding intersections between punctured curves and asymptotically cylindrical hypersurfaces described above, namely, the fact that the algebraic intersection number may not be invariant under homotopies. We first introduce some terminology. Assuming again 2.1 and that no component of C is a subset of $\tilde{V}$, we define the asymptotic intersection number at the punctures of C in the following way:

- If for the positive puncture $z \in \Gamma_+$, $\gamma_z^{m_z} \subset V_+$, we define the *asymptotic intersection number* $\delta_\infty((C;z);\tilde{V})$ *of* C *at* z *with* $\tilde{V}$ by

$$\delta_\infty((C;z),\tilde{V}) = \lfloor \mu_N^\Phi(\gamma_z^{m_z})/2 \rfloor - \mathrm{wind}_{rel}^\Phi((C;z),\tilde{V}). \tag{13}$$

- If for the negative puncture $z \in \Gamma_-$, $\gamma_z^{m_z} \subset V_-$, we define the *asymptotic intersection number* $\delta_\infty((C;z);\tilde{V})$ *of* C *at* z *with* $\tilde{V}$ by

$$\delta_\infty((C;z),\tilde{V}) = \mathrm{wind}_{rel}^\Phi((C;z),\tilde{V}) - \lceil \mu_N^\Phi(\gamma_z^{m_z})/2 \rceil. \tag{14}$$

- For all other punctures $z \in \Gamma_\pm$ (i.e. those for which γ_z is not contained in $V_\pm$), we define

$$\delta_\infty((C;z),\tilde{V}) = 0. \tag{15}$$

We then define the *total asymptotic intersection number of* C *with* $\tilde{V}$ by

$$\delta_\infty(C,\tilde{V}) = \sum_{z\in\Gamma} \delta_\infty((C;z),\tilde{V}). \tag{16}$$

We observe that as a result of Corollary 2.4 the local and total asymptotic intersection numbers are always nonnegative.

Continuing to assume 2.1 (but no longer necessarily that no component of the curve C has image contained in $\tilde{V}$), we can use the trivialization Φ of $\xi_V^\perp$ along the asymptotic periodic orbits of C lying in V to construct a perturbation C_Φ of C in the following way. For each puncture $z \in \Gamma$ for which the asymptotic limit $\gamma_z^{m_z}$ lies in V, we first extend Φ to a trivialization $\Phi : \xi_V^\perp|_{U_z} \to U_z \times \mathbb{R}^2$ on some open neighborhood $U_z \subset V$ of the asymptotic limit γ_z. Then we consider the asymptotic parametrization

$$(s,t) \mapsto \widetilde{\exp}_{u_T(s,t)} u_N(s,t)$$

from Theorem 2.2 above for $(s,t) \in [R,+\infty) \times S^1$ or $(-\infty,-R] \times S^1$ as appropriate, where $R > 0$ is chosen large enough so that u_T has image contained in the neighborhood U_z of γ_z on which the trivialization Φ has been extended. We then perturb the map by replacing the above parametrization of C near z by the map

$$(s,t) \mapsto \widetilde{\exp}_{u_T(s,t)} \left(u_N(s,t) + \beta(|s|)\Phi(u_T(s,t))^{-1}\varepsilon \right)$$

where $\beta : [0,\infty) \to [0,1]$ is a smooth cut-off function equal to 0 for $s < |R|+1$ and equal to 1 for $|s| > |R|+2$, and $\varepsilon \neq 0$ is thought of as a number in $\mathbb{C} \approx \mathbb{R}^2$. Given this, we can then define the *relative intersection number* $i^\Phi(C,\tilde{V})$ *of* C *and* $\tilde{V}$ *relative to the the trivialization* Φ by

$$i^\Phi(C,\tilde{V}) := C_\Phi \cdot \tilde{V}.$$

It can be shown that this number is independent of choices made in the construction of C_Φ provided the perturbations are sufficiently small.

Again continuing to assume 2.1, we now define the *holomorphic intersection product of* C *and* $\tilde{V}$ by

$$C * \tilde{V} := i^\Phi(C,\tilde{V}) + \sum_{\substack{z\in\Gamma_+ \\ \gamma_z \subset V_+}} \lfloor \mu_N^\Phi(\gamma_z^{m_z})/2 \rfloor - \sum_{\substack{z\in\Gamma_- \\ \gamma_z \subset V_-}} \lceil \mu_N^\Phi(\gamma_z^{m_z})/2 \rceil$$

The key facts about the holomorphic intersection product are now given in the following theorem which generalizes [10, Theorem 2.2/4.4].

Theorem 2.5 (Generalized positivity of intersections)**.** *With* $\tilde{V}$ *and* C *as in 2.1, assume that* C *is not contained in* $\tilde{V}$*, the holomorphic intersection product* $C * \tilde{V}$ *depends only on the relative homotopy classes of* C *and* $\tilde{V}$*. Moreover, if the image of* C *is not contained in* $\tilde{V}$*, then*

$$C * \tilde{V} = C \cdot \tilde{V} + \delta_\infty(C,\tilde{V}) \geq 0$$

where $C \cdot \tilde{V}$ is the algebraic intersection number, defined by summing local intersection indices, and $\delta_\infty(C, \tilde{V})$ is the total asymptotic intersection number, defined by (13)–(16)*. In particular, $C * \tilde{V} \geq 0$ and equals zero if and only if C and $\tilde{V}$ don't intersect and all asymptotic intersection numbers are zero.*

The proof of this theorem follows along very similar lines to Theorem 2.2/4.4 in [10]. The essential point is that the relative intersection number can be shown to be given by the formula

$$i^\Phi(C, \tilde{V}) = C \cdot \tilde{V} - \sum_{\substack{z \in \Gamma_+ \\ \gamma_z \subset V_+}} \mathrm{wind}^\Phi_{rel}((C; z), \tilde{V}) + \sum_{\substack{z \in \Gamma_- \\ \gamma_z \subset V_-}} \mathrm{wind}^\Phi_{rel}((C; z), \tilde{V}).$$

The result will then follow from Corollary 2.4 above. Detailed proof will be given in [8].

Analogous to the case in four dimensions studied in [10], the $\mathbb{R}$-invariance of the almost complex structure in the set-up here allows one to compute the holomorphic intersection number and (in some cases) the algebraic intersection number of a holomorphic curve and a holomorphic hypersurface with respect to asymptotic winding numbers and intersections of each object with the asymptotic limits of the other.

Before stating the relevant results we will first make some additional assumptions. We will henceforth assume that:

Assumptions 2.6.

(a) V_+ and V_- are disjoint,
(b) $\xi_V^\perp$ of $V = V_+ \cup V_-$ is trivializable,
(c) $\Phi : \xi_V^\perp \to V \times \mathbb{R}^2$ is a global trivialization,
(d) $\tilde{V}$ is connected, and
(e) the projection $\pi(\tilde{V})$ of $\tilde{V}$ to M is an embedded codimension-1 submanifold of $M \setminus V$.

Under these assumptions, $\tilde{V}$ has a well-defined winding $\mathrm{wind}^\Phi_\infty(\tilde{V}, \gamma)$ relative to Φ around any orbit $\gamma \subset V = V_+ \cup V_-$ which can be defined by considering the asymptotic representatives η_+ or η_- as appropriate and computing

$$\mathrm{wind}^\Phi_\infty(\tilde{V}, \gamma) = \lim_{|s| \to \infty} \mathrm{wind}\, \Phi^{-1} \eta_\pm(s, \gamma(\cdot)), \tag{17}$$

or, equivalently by

$$\mathrm{wind}^\Phi_\infty(\tilde{V}, \gamma) = \mathrm{wind}^\Phi_{rel}((\mathbb{R} \times \gamma; \pm\infty), \tilde{V}). \tag{18}$$

As in Corollary 2.4 above, it follows from the asymptotic formula from Theorem 2.2 above and the characterization of the Conley–Zehnder index from [3] that

$$\mathrm{wind}^\Phi_\infty(\tilde{V}, \gamma) \leq \lfloor \mu^\Phi_N(\gamma)/2 \rfloor = \alpha_N^{\Phi;-}(\gamma)$$

if $\gamma \subset V_+$ and

$$\mathrm{wind}^\Phi_\infty(\tilde{V}, \gamma) \geq \lceil \mu^\Phi_N(\gamma)/2 \rceil = \alpha_N^{\Phi;+}(\gamma)$$

if $\gamma \subset V_-$.

The following theorem, which can be seen as the higher dimensional version of [10, Corollary 5.11], gives a computation of the algebraic intersection number of $\tilde{V}$ and $\mathbb{R}$-shifts of the curve C in terms of the asymptotic data, and the intersections of each object with the asymptotic limits of the other. Proof will be given in [8]

Theorem 2.7. *Assume 2.1 and 2.6 and that the curve C is connected and not equal to an orbit cylinder and not contained in $\mathbb{R} \times V$. For $c \in \mathbb{R}$, denote by C_c the curve obtained from translating C in the $\mathbb{R}$-coordinate by c. Then for all but a finite number of value of $c \in \mathbb{R}$, the algebraic intersection number $C_c \cdot \tilde{V}$ is given by the formulas:*

$$
\begin{aligned}
C_c \cdot \tilde{V} = C \cdot (\mathbb{R} \times V_+) \\
&+ \sum_{\substack{z \in \Gamma_+ \\ \gamma_z \in V_+}} \left(\max\left\{ m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z), \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_+) \right\} - \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_+) \right) \\
&+ \sum_{z \in \Gamma_-} m_z (\mathbb{R} \times \gamma_z) \cdot \tilde{V} \\
&+ \sum_{\substack{z \in \Gamma_- \\ \gamma_z \in V_-}} \left(m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z) - \min\left\{ m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z), \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_-) \right\} \right) \\
&+ \sum_{\substack{z \in \Gamma_- \\ \gamma_z \in V_+}} \left(\operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_+) - m_z \operatorname{wind}^\Phi_{rel}(\tilde{V}, \gamma_z) \right) \\
= \sum_{z \in \Gamma_+} m_z (\mathbb{R} \times \gamma_z) \cdot \tilde{V} \\
&+ \sum_{\substack{z \in \Gamma_+ \\ \gamma_z \in V_+}} \left(\max\left\{ m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z), \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_+) \right\} - m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z) \right) \\
&+ C \cdot (\mathbb{R} \times V_-) \\
&+ \sum_{\substack{z \in \Gamma_- \\ \gamma_z \in V_-}} \left(\operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_-) - \min\left\{ m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z), \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_-) \right\} \right) \\
&+ \sum_{\substack{z \in \Gamma_+ \\ \gamma_z \in V_-}} \left(m_z \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma_z) - \operatorname{wind}^\Phi_{rel}((C; z), \mathbb{R} \times V_-) \right)
\end{aligned}
$$

with each of the grouped terms always nonnegative.

The nonnegativity of the terms in the above formulas allows us to establish a convenient set of conditions which will guarantee that the projections $\pi(C)$ and $\pi(\tilde{V})$ of the curve and hypersurface to M do not intersect. Indeed, if $\pi(C)$ and $\pi(\tilde{V})$ are disjoint then C_c and $\tilde{V}$ are disjoint for all values of $c \in \mathbb{R}$ and hence the algebraic intersection number of C_c and $\tilde{V}$ is zero for all values of $c \in \mathbb{R}$. Since the formulas from Theorem 2.7 compute this number (for all but a finite number of values of $c \in \mathbb{R}$) in terms of nonnegative quantities, we can conclude that all terms in the above formulas vanish. We thus obtain the following corollary, which generalizes [10, Theorem 2.4/5.12].

Corollary 2.8. *Assume that all of the hypotheses of Theorem 2.7 hold and that $\pi(C)$ is not contained in $\pi(\tilde{V})$. Then the following are equivalent:*

1. *$\pi(C)$ and $\pi(\tilde{V})$ are disjoint.*
2. *All of the following hold:*
   (a) *None of the asymptotic limits of C intersect $\pi(\tilde{V})$.*
   (b) *$\pi(C)$ does not intersect $V = V_+ \cup V_-$.*
   (c) *For any puncture z at which C has asymptotic limit γ^m lying in $V = V_+ \cup V_-$, $\operatorname{wind}^\Phi_{rel}((C; z), V) = m \operatorname{wind}^\Phi_\infty(\tilde{V}, \gamma)$.*

3. Proof of the main result

We now proceed with the proof of our main result, Theorem 1.1. We consider a manifold M^{2n+1} equipped with a nondegenerate stable Hamiltonian structure (λ, ω) and an f-admissible fibration (Σ, p, Y, f). Recall that this means that $p : M \to \Sigma$ is a smooth fibration over a closed surface with fiber diffeomorphic to Y^{2n-1}, and $V := p^{-1}(\operatorname{crit}(f))$ is a strong stable Hamiltonian hypersurface. We further recall that each critical point w of f is assigned a sign $\operatorname{sign}(w) \in \{-1, +1\}$ according to whether $p_* : \xi^\perp_{Y_w} \to T_w\Sigma$ is everywhere orientation preserving or reversing.

We will let $\tilde{J}$ denote a compatible almost complex structure on $\mathbb{R} \times M$ and will assume that $\mathbb{R} \times V$ has $\tilde{J}$-invariant tangent space. We consider an asymptotically cylindrical, $\tilde{J}$-holomorphic map $\tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ so that all asymptotic limits $\gamma_z^{m_z}$ of $\tilde{u}$ are contained in the binding set $V = p^{-1}(\mathrm{crit}(f))$ of the foliation. As observed in the introduction, the projected map $v := p \circ u : S \setminus \Gamma \to \Sigma$ admits a continuous extension $\bar{v} : S \to \Sigma$ over the punctures. The following lemma computes the degree of this map in terms of the intersection number and relative normal windings of $\tilde{u}$ and any given component of the binding set of the foliation.

Lemma 3.1. *Assume the map $u : S \setminus \Gamma \to M$ does not have image contained in $V = p^{-1}(\mathrm{crit}(f))$. Then, given a point $w \in \mathrm{crit}(f)$, the degree of the map $\bar{v}$ defined above is given by the formula*

$$\begin{aligned}\deg(\bar{v}) &= \mathrm{sign}(w)\left(\tilde{u} \cdot (\mathbb{R} \times Y_w) - \sum_{\substack{z\in\Gamma^+\\ \gamma_z \subset Y_w}} \mathrm{wind}^{\Phi}_{rel}((\tilde{u}; z), Y_w) + \sum_{\substack{z\in\Gamma^-\\ \gamma_z \subset Y_w}} \mathrm{wind}^{\Phi}_{rel}((\tilde{u}; z), Y_w)\right)\\ &= \mathrm{sign}(w)\left(\tilde{u} * (\mathbb{R} \times Y_w) - \sum_{\substack{z\in\Gamma^+\\ \gamma_z \subset Y_w}} \lfloor \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rfloor + \sum_{\substack{z\in\Gamma^-\\ \gamma_z \subset Y_w}} \lceil \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rceil\right)\end{aligned}$$

and thus satisfies

$$\begin{aligned}\mathrm{sign}(w)\deg(\bar{v}) &\geq - \sum_{\substack{z\in\Gamma^+\\ \gamma_z \subset Y_w}} \lfloor \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rfloor + \sum_{\substack{z\in\Gamma^-\\ \gamma_z \subset Y_w}} \lceil \mu^{\Phi}_N(\gamma_z^{m_z})/2 \rceil\\ &= \sum_{\substack{z\in\Gamma^\pm\\ \gamma_z \subset Y_w}} \mp \alpha_N^{\Phi;\mp}(\gamma_z^{m_z})\end{aligned} \tag{19}$$

with equality occurring if and only if u does not intersect Y_w and $\mathrm{wind}^{\Phi}_{rel}((\tilde{u}; z), Y_w) = \alpha_N^{\Phi;\mp}(\gamma_z^{m_z})$ *for every $z \in \Gamma^{\pm}$ with $\gamma_z \subset Y_w$.*

Proof. Given a point $w \in \mathrm{crit}(f)$ it's clear from the definition of the map $\bar{v}$ that points $z \in S$ with $\bar{v}(z) = w$ coincide with intersection points of the map $\tilde{u}$ with $\mathbb{R} \times Y_w$ (or equivalently, intersection points of u with Y_w) and with punctures $z \in \Gamma$ for which the periodic orbit $\gamma_z^{m_z}$ is contained in Y_w. Since we've observed in Corollary 2.3 this number is finite, we can compute the degree of the map $\bar{v} : S \to \Sigma$ by summing the local degree of the map at each point in $\bar{v}^{-1}(w)$.

Assume for the moment that $\mathrm{sign}(w) = 1$. Since in this case the identification of $\xi^{\perp}_{Y_w}$ with $T\Sigma_w$ via the map p_* is orientation preserving, it's clear that the local index of the map $v = p \circ u$ agrees with the intersection number of $\tilde{u}$ with $\mathbb{R} \times Y_w$. Summing over all such intersection points leads to the first term in the given formula for the degree. To compute the local degree at a positive puncture we first choose positive holomorphic cylindrical coordinates $(s, t) \in [0, \infty) \times S^1$ on a deleted neighborhood of the puncture $z \in \Gamma^+$. Because in such a coordinate system the loop $t \mapsto (s, t)$ for fixed s encircles the puncture in the clockwise direction, the local degree of the map $\bar{v}$ at z can be computed by identifying a neighborhood of w with $T_w\Sigma$ and computing

$$-\,\mathrm{wind}\,\bar{v}(s, t) = -\,\mathrm{wind}(p \circ u)(s, t).$$

But considering the asymptotic representation of the normal component of the map $\tilde{u}$ from Theorem 2.2 along with the definition of the normal relative winding, we have that

$$\mathrm{wind}(p \circ u)(s, t) = \mathrm{wind}^{\Phi}_{rel}((\tilde{u}; z), \mathbb{R} \times Y_w)$$

which shows the local degree of the $\bar{v}$ at z is

$$-\,\mathrm{wind}^{\Phi}_{rel}((\tilde{u}; z), \mathbb{R} \times Y_w).$$

At negative punctures, we argue similarly but instead choose negative cylindrical coordinates $(s, t) \in (-\infty, 0] \times S^1$ on a deleted neighborhood of the puncture. Since the loop $t \mapsto (s, t)$ encircles the puncture in the counterclockwise direction, an argument analogous to that given above tells us that the local degree is now given by

$$\operatorname{wind}\bar{v}(s,t) = \operatorname{wind}(p \circ u)(s,t) = \operatorname{wind}_{rel}^{\Phi}((\tilde{u}; z), \mathbb{R} \times Y_w).$$

The first line of the claimed formula for the degree of $\bar{v}$ now follows in the case that $\operatorname{sign}(w) = 1$. The case when $\operatorname{sign}(w) = -1$ is identical with the exception of the fact that the map $p_* : \xi_{Y_w}^{\perp} \to T_w\Sigma$ is now orientation reversing which introduces a factor of -1 into each of the computations.

The second line in the claimed formula for the degree of $\bar{v}$ now follows from the definition of the holomorphic intersection product, the definition of the asymptotic intersection numbers and Theorem 2.5

Finally, the inequality (19) and the claim about when equality is achieved is an immediate consequence of local positivity of intersections and the bounds (10) and (11) or, equivalently, as a consequence of Theorem 2.5. $\square$

Now, in addition to the assumptions preceding Lemma 3.1 we assume that M is equipped with a holomorphic foliation $\mathcal{F}$ compatible with the f-admissible fibration (Σ, p, Y, f). Recall this means that $\mathcal{F}$ is an $\mathbb{R}$-invariant foliation of $\mathbb{R} \times M$ all of whose leaves have $\tilde{J}$-invariant tangent spaces and are diffeomorphic to $\mathbb{R} \times Y$. Moreover the leaves of the foliation fixed by the $\mathbb{R}$-action are precisely those contained in $\mathbb{R} \times V = \mathbb{R} \times p^{-1}(\operatorname{crit}(f))$ and all other leaves project via $\mathbb{R} \times M \xrightarrow{\pi} M$ to embeddings smoothly foliating $M \setminus V$, and the leaves of this foliation project under $M \xrightarrow{p} \Sigma$ to flow lines of the gradient of f with respect to some metric on Σ. We observe that the assumption that leaves of the foliation project to gradient flow lines implies that with respect to a trivialization Φ of $\xi_V^{\perp}$ in the preferred homotopy class determined by the fibration, the windings $\operatorname{wind}_{\infty}^{\Phi}(\tilde{V}, \gamma)$, defined by (17) or (18), vanish for any leaf $\tilde{V}$ of the foliation not fixed by the $\mathbb{R}$-action and any periodic orbit γ lying in the one of the asymptotic limits of $\tilde{V}$.

We now proceed with the proof of our main theorem, Theorem 1.1. We will continue to let $(S, j, \Gamma, \tilde{u} = (a, u))$ denote a punctured pseudoholomoprhic curve with all punctures limiting to periodic orbits in $V = p^{-1}(\operatorname{crit}(f))$, and we let $\bar{v} : S \to \Sigma$ denote the continuous extension of the map $p \circ u : S \setminus \Gamma \to \Sigma$. We further assume that at each puncture $z \in \Gamma$ the normal Conley–Zehnder index of the asymptotic limits $\gamma_z^{m_z}$ satisfies

$$\mu_N^{\Phi}(\gamma_z^{m_z}) \in \{-1, 0, 1\} \tag{20}$$

with Φ still denoting a symplectic trivialization of $\xi_V^{\perp}$ in the preferred homotopy class determined by the f-admissible fibration (Σ, j, p, f). We then claim that:

- If $\operatorname{sign}(w) = 1$ for all $w \in \operatorname{crit}(f)$ then the map $\bar{v}$ has nonnegative degree and has degree zero precisely when the image of the map $\tilde{u}$ is contained in a leaf of the foliation $\mathcal{F}$.
- If $\operatorname{sign}(w) = -1$ for all $w \in \operatorname{crit}(f)$ then the map $\bar{v}$ has nonpositive degree and has degree zero precisely when the image of the map $\tilde{u}$ is contained in a leaf of the foliation $\mathcal{F}$.
- If the map $\operatorname{sign} : \operatorname{crit}(f) \to \{-1, 1\}$ is surjective then the map $\bar{v}$ has degree zero and the image of the map $\tilde{u}$ is contained in a leaf of the foliation $\mathcal{F}$.

Proof of 1.1. The condition (20) implies that

$$\alpha_N^{\Phi;-}(\gamma_z^{m_z}) = \lfloor \mu_N^{\Phi}(\gamma_z^{m_z})/2 \rfloor \le \lfloor 1/2 \rfloor = 0$$

for all $z \in \Gamma^+$ and

$$\alpha_N^{\Phi;+}(\gamma_z^{m_z}) = \lceil \mu_N^{\Phi}(\gamma_z^{m_z})/2 \rceil \ge \lceil -1/2 \rceil = 0$$

for all $z \in \Gamma^{-1}$, which together are equivalent to

$$\mp \alpha_N^{\Phi;\mp}(\gamma_z^{m_z}) \ge 0 \tag{21}$$

for all $z \in \Gamma^{\pm}$.

Assuming that $\operatorname{sign}(w) = 1$ for all $w \in \operatorname{crit}(f)$, using (19) from Lemma 3.1 with (21) shows that $\deg \bar{v} \ge 0$ and that $\deg \bar{v} = 0$ precisely when

$$\begin{gathered} u(\Sigma \setminus \Gamma) \text{ and } Y_w \text{ are disjoint for all } w \in \operatorname{crit}(f), \text{ and} \\ \operatorname{wind}_{rel}^{\Phi}((C; z), Y_{w_z}) = \mp\alpha^{\Phi;\mp}(\gamma_z^{m_z}) = 0 \text{ for all } z \in \Gamma^{\pm}. \end{gathered} \tag{22}$$

Similarly if $\operatorname{sign}(w) = -1$ for all $w \in \operatorname{crit}(f)$, the same argument shows that $\deg \bar{v} \le 0$ and that $\deg \bar{v} = 0$ precisely when (22) holds.

In the third case that there exist points w_+, $w_- \in \mathrm{crit}(f)$ with $\mathrm{sign}(w_+) = 1$ and $\mathrm{sign}(w_-) = -1$, applying (19) and (21) with $w = w_+$ yields $\deg(\bar{v}) \geq 0$ while applying (19) and (21) with $w = w_-$ yields $\deg(\bar{v}) \leq 0$. We conclude that $\deg(\bar{v}) = 0$ which once again happens precisely when (22) holds.

To complete the proof of all three cases, it remains to show that $\deg(\bar{v}) = 0$ precisely when the image of the map $\tilde{u}$ is contained in a leaf of the foliation $\mathcal{F}$. Assume that $\deg(\bar{v}) = 0$. Since $\mathcal{F}$ is a foliation, and we assume that the image of u is not contained in the binding $V = \mathrm{crit}(f)$, there exists at least one leaf $\tilde{Y}$ not fixed by the $\mathbb{R}$-action for which $\tilde{u}$ intersects $\tilde{Y}$. Assume $\tilde{Y}$ projects via $p \circ \pi$ to a gradient flow line between critical points w_1 and w_2 so that $\tilde{Y}$ is asymptotically cylindrical over $Y_{w_1} \cup Y_{w_2}$, and recall that we've noted above that with respect to the global trivialization Φ for $\xi_V^\perp$ we've chosen, $\mathrm{wind}_\infty^\Phi(\tilde{Y}, \gamma) = 0$ for all periodic orbits $\gamma \in Y_{w_1} \cup Y_{w_2}$. We would now like to apply Corollary 2.8 above to prove the image of $\tilde{u}$ is contained in $\tilde{Y}$. Assume to the contrary that the image of $\tilde{u}$ is not contained in $\tilde{Y}$. By the assumptions that all asymptotic limits of the map $\tilde{u}$ are periodic orbits in the binding $V = p^{-1}(\mathrm{crit}(f))$ we know that none of these limits intersect $\pi(\tilde{Y})$ since $\tilde{Y}$ is assumed to project to an embedding in $M \setminus V$. Moreover, since we have already observed that in each case, $\deg(\bar{v}) = 0$ is true precisely when (22) holds, we can conclude that $u(S \setminus \Gamma)$ does not intersect the asymptotic limit set $Y_{w_1} \cup Y_{w_2}$ and that for each $z \in \Gamma^\pm$ with $\gamma_z^{m_z} \subset Y_{w_1} \cup Y_{w_2}$ we'll have that $\mathrm{wind}_{rel}^\Phi((u; z), Y_{w_1} \cup Y_{w_2}) = 0 = m_z \, \mathrm{wind}_\infty^\Phi(\tilde{Y}, \gamma_z)$. Corollary 2.8 now lets us conclude that the image of $\tilde{u}$ is disjoint from $\tilde{Y}$ in contradiction to the assumption that they intersect. We thus conclude that the image of $\tilde{u}$ is contained in $\tilde{Y}$ as desired. $\square$

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