CONNECTED SUMS AND FINITE ENERGY FOLIATIONS I: CONTACT CONNECTED SUMS

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ABSTRACT. We consider a 3-manifold M equipped with a nondegenerate contact form λ and compatible almost complex structure J. We show that if the data (M, λ, J) admits a stable finite energy foliation, then for a generic choice of distinct points p, $q \in M$, the manifold M' formed by taking the contact connected sum at p and q admits a nondegenerate contact form λ' and compatible almost complex structure J' so that the data (M', λ', J') also admits a stable finite energy foliation. Along the way, we develop some general theory for the study of finite energy foliations.

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1. Introduction and the main result

Let $(M, \xi = \ker\lambda)$ be a closed contact 3-manifold equipped with a nondegenerate contact form λ. Recall a complex structure J on ξ is said to be compatible with the data (M, λ) if $d\lambda(\cdot, J\cdot)$ is a bundle metric on ξ. We denote the set of complex structures J on ξ compatible with (M, λ) by $\mathcal{J}(M, \lambda)$. Given a $J \in \mathcal{J}(M, \lambda)$ we can extend it in the usual way to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ by requiring

$$\tilde{J}\partial_a = X_\lambda \qquad \text{and} \qquad \tilde{J}|_{\pi_M^*\xi} = \pi_M^* J$$

where ∂_a is the coordinate field along $\mathbb{R}$ and $\pi_M : \mathbb{R} \times M \to M$ is the canonical projection onto the second factor.

Given data (M, λ, J) where (M, λ) is a 3-manifold M with contact form λ and $J \in \mathcal{J}(M, \lambda)$ is a compatible complex multiplication on $\xi = \ker\lambda$, a finite-energy pseudoholomorphic map in $\mathbb{R} \times M$ is a quadruple $(\Sigma, j, \Gamma, \tilde{u})$ where (Σ, j) is a compact Riemann surface, $\Gamma \subset \Sigma$ is a finite set, and $\tilde{u} = (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times M$ is a map satisfying

$$\tilde{J} \circ d\tilde{u} = d\tilde{u} \circ j$$

and

$$0 < E(\tilde{u}) < \infty$$

where the energy $E(\tilde{u})$ of the map is defined by

$$E(\tilde{u}) = \sup_{\varphi \in \Xi} \int_{\Sigma\setminus\Gamma} \tilde{u}^* d(\varphi\lambda)$$

where Ξ is the collection of functions defined by

$$\Xi = \{\varphi \in C^\infty(\mathbb{R}, [0, 1]) \mid \varphi'(x) \geq 0\}.$$

A finite-energy pseudoholomorphic curve in $\mathbb{R} \times M$ is then an equivalence class $C = [\Sigma, j, \Gamma, \tilde{u}]$ under the equivalence relation of holomorphic reparametrization. Since the energy of a pseudoholomorphic map is invariant under holomorphic reparametrization of the domain, pseudoholomorphic curves have a well-defined energy. A now well-known result of Hofer [24] tells us that near the (nonremovable) punctures, finite-energy pseudoholomorphic curves are asymptotic to periodic orbits of the Reeb vector field.

A *finite energy foliation* $\mathcal{F}$ for the data (M, λ, J) is a collection of connected finite-energy pseudoholomorphic curves with uniformly bounded energies whose images form a smooth foliation of $\mathbb{R} \times M$. We define the energy $E(\mathcal{F})$ of a foliation $\mathcal{F}$ to be the supremum of the energies of the curves in the foliation, that is

$$E(\mathcal{F}) = \sup_{C \in \mathcal{F}} E(C).$$

A finite energy foliation $\mathcal{F}$ for the data (M, λ, J) is said to be *stable* if:

1. (1) For any $C \in \mathcal{F}$, C is either a trivial cylinder over a periodic orbit or the Fredholm index $\operatorname{ind}(C)$ (see Section 3.4 below) is either 1 or 2.
2. (2) For any two curves C_1, $C_2 \in \mathcal{F}$ with $\operatorname{ind}(C_i) \in \{1, 2\}$, the holomorphic intersection number from [50] (see Section 3.3 below) $C_1 * C_2$ vanishes.

The word "stable" here is meant to connote the fact that both the existence of such a finite energy foliation and its basic structure will persist under suitable sufficiently small perturbations of the data (λ, J). As we will discuss in Section 4 below, a stable finite energy foliation necessarily consists of only punctured spheres and is invariant under the $\mathbb{R}$-action on $\mathbb{R} \times M$ given by shifting the $\mathbb{R}$-coordinate. The $\mathbb{R}$-invariance in turn lets us conclude that the projections of the curves in the foliation to the 3-manifold M are embedded, transverse to the flow of the Reeb vector field, and foliate the complement of a finite collection of periodic orbits in M.

The study of finite energy foliations was initiated by Hofer, Wysocki, and Zehnder in [28] in which they use the existence of a finite energy foliation to construct a global surface of section of disk type for a 3-dimensional strictly convex energy surface in $(\mathbb{R}^4, \sum_{i=1}^2 dx_i \wedge dy_i)$. This work was extended in [31] where the same authors show that any nondegenerate star-shaped hypersurface in $\mathbb{R}^4$ admits a stable finite energy foliation. Using this fact, they then show that there exists a Baire set of star-shaped hypersurfaces in $\mathbb{R}^4$ so that any given hypersurface in this set has either precisely two or infinitely many periodic orbits.

Recently, Bramham has introduced the use of finite energy foliations to the study of area-preserving maps of the disk [9, 8]. Using the foliations that he constructs in [8], Bramham proves in [10] that every smooth, irrational pseudorotation of the 2-disk is the uniform limit of a sequence of maps which are each conjugate to a rotation about the origin. In [11], these foliations are again used to prove there is a dense subset $\mathcal{L}_* \subset \mathcal{L}$ of the Liouville numbers so that a pseudorotation of the disk with rotation number in $\mathcal{L}_*$ has a sequence of iterates which converge uniformly to the identity map and thus such a pseudorotation can't exhibit strong mixing. A discussion of further applications of finite energy foliations to the study of disk maps can be found in the survey [7].

The existence of finite energy foliations has also had applications in contact and symplectic topology. Among these are Hind's work on Lagrangian unknottedness in Stein surfaces [23] and Wendl's work on fillabilty of contact 3-manifolds [60]. Further work either addressing existence of finite energy foliations or in which the existence of finite energy foliations plays a role in dynamical or contact/symplectic topological results can be found in [1, 2, 13, 14, 15, 20, 21, 22, 25, 29, 34, 35, 36, 38, 40, 43, 55, 56, 59, 61].

In the present series of papers we develop abstract tools for extending previously known existence results for stable finite energy foliations. One motivation for this work comes from the study of the planar, circular, restricted three-body problem. Albers, Frauenfelder, van Koert and Paternain show in [5] that near the two massive primaries, the regularized energy levels below and slightly above the first Lagrange point are diffeomorphic respectively to two copies of $\mathbb{RP}^3$ and the connected sum of two copies of $\mathbb{RP}^3$. In [4], Albers, Frauenfelder, Hofer, van Koert, and the first author apply techniques from [28] to construct finite energy foliations for many mass ratios and regularized energy levels below the first Lagrange point. Since many classical techniques to study the restricted three-body problem fail above the first Lagrange point, it's of interest to know whether the existence of finite energy foliations for regularized energy surfaces below the first Lagrange point can be used to deduce the existence of finite energy foliations for regularized energy surfaces above the first Lagrange point.

The results from [5, 4] and the associated problem of attempting to construct a finite energy foliation for regularized energy surfaces above the first Lagrange point naturally lead to the general question of whether the existence of finite energy foliations persist under the formation of contact connected sums as in [41, 54]. Our main theorem, which we state now, answers this question by showing that finite energy foliations do indeed persist after forming the contact connected sum of a contact manifold.

Theorem 1.1. *Let (M, ξ) be a contact 3-manifold with contact structure induced by a nondegenerate contact form λ, and let $J \in \mathcal{J}(M, \lambda)$ be a complex multiplication for which the triple (M, λ, J) admits a stable finite energy foliation $\mathcal{F}$ of energy $E(\mathcal{F})$. Then, there exists an open, dense set $\mathcal{U} \subset M \times M \setminus \Delta(M)$ so that for any $(p, q) \in \mathcal{U}$ the contact manifold (M', ξ') obtained by performing a contact connected sum at (p, q) as in [41, 54] admits a nondegenerate contact form λ' with $\xi' = \ker \lambda'$, a compatible $J \in \mathcal{J}(M', \lambda')$ and a stable finite energy foliation $\mathcal{F}'$ for the data (M', λ', J') with energy $E(\mathcal{F}') = E(\mathcal{F})$.*

We briefly discuss some of the key steps of the proof of this theorem. Given a finite energy foliation $\mathcal{F}$ for the data (M, λ, J) we choose any two distinct points p and $q \in \mathcal{M}$ lying on distinct (up to $\mathbb{R}$-translation) index-2 leaves of the foliation. We then form the connected sum M' by S^2-compactifying $M \setminus \{p, q\}$ and gluing along the newly created boundary. We denote the new manifold by M', the induced inclusion $M \setminus \{p, q\} \hookrightarrow M'$ by i, and the embedded sphere $M' \setminus i(M \setminus \{p, q\})$ by S. It is well known from [41, 54] that the gluing can be done in such a way that M' is a smooth manifold and the induced contact structure continues smoothly across S. We show that, in addition, we can find a contact form λ' and compatible J' which agree respectively with λ and J outside of any desired sufficiently-small neighborhood U of S so that there is precisely one simple periodic orbit $\gamma_0 \subset S$ contained in U and so that γ_0 divides S into two disks, each of which is the projection to M' of an index-1 $\tilde{J}'$-holomorphic plane. Using the fact that curves in a stable finite energy foliation must satisfy so-called automatic transversality conditions, we investigate the boundaries of the moduli spaces of curves surrounding the neighborhood U of S. Using intersection theory arguments, and specifically a result concerning the direction of approach of a curve to an orbit with even Conley–Zehnder index, we show that these families of curves converge to height-2 pseudoholomorphic buildings with one of the planes in S as one of the nontrivial components and that the resulting collection of curves forms a finite energy foliation.

Since the finite energy foliation we construct on the connected sum always contains a pair of rigid (i.e. index-1) planes asymptotic to same periodic orbit, it is natural to ask the question of whether the operation can be reversed anytime one has a foliation with a similar configuration of curves in it. We show in [16] that this in fact can be done. Specifically, assuming the data (M, λ, J) admits a finite energy foliation $\mathcal{F}$ containing two distinct (up to $\mathbb{R}$-translation) index-1 planes asymptotic to the same periodic orbit, the manifold M' obtained by doing surgery on the 2-sphere formed by the orbit and the projections of the planes to M admits a contact form λ' and a compatible $J' \in \mathcal{J}(M', \lambda')$ so that the data (M', λ', J') admits a finite energy foliation $\mathcal{F}'$ with $E(\mathcal{F}') = E(\mathcal{F})$. We further show in [17] that a Weinstein cobordism connecting the two contact manifolds admits a finite energy foliation which is asymptotic to the foliations $\mathcal{F}$ and $\mathcal{F}'$ on the boundaries.

Our main result in this paper can be combined with previous existence results for finite energy foliations to produce new finite energy foliations. We recall that for contact manifolds whose contact structures are supported by planar open book decompositions, Abbas [1] and Wendl [59] have constructed finite energy foliations consisting of curves which correspond via the projection $\mathbb{R} \times M \to M$ to pages of the open book decomposition. Moreover, it is known that performing a contact connected sum at two points in a contact manifold supported by a planar open book decomposition produces a contact manifold whose contact structure is also supported by a planar open book decomposition with one additional binding component.[1] Combining these facts with our construction leads to the following theorem.

Theorem 1.2. *Let (M_0, ξ_0) be a contact manifold with contact structure supported by a planar open book decomposition with b binding components, and let (M_k, ξ_k) denote the contact manifold obtained from performing k successive contact connected sums on (M_0, ξ_0). Then for any integer $\ell \in [1, k]$ there exist a contact form λ_ℓ with $\ker \lambda_\ell = \xi_k$, a compatible complex structure J_ℓ on ξ_k, and a stable finite energy foliation $\mathcal{F}_\ell$ for the data $(M, \lambda_\ell, J_\ell)$ consisting of:*

- *$b + k - \ell$ trivial cylinders over elliptic orbits $\{\gamma_1^e, \dots, \gamma_{b+k-\ell}^e\}$,*
- *ℓ trivial cylinders over even orbits $\{\gamma_1^h, \dots, \gamma_\ell^h\}$,*
- *ℓ pairs of index-1 families of planes, with one such pair of families asymptotic to each of the periodic orbits γ_i^h,*
- *ℓ pairs of index-1 families of curves, one pair for each hyperbolic orbit γ_i^h, having $b + k - \ell$ positive punctures with the collection of elliptic orbits $\{\gamma_1^e, \dots, \gamma_{b+k-\ell}^e\}$ as asymptotic limits and one negative puncture with γ_i^h as an asymptotic limit, and*
- *2ℓ families of index-2 curves in which each curve has $b + k - \ell$ positive punctures with the collection of elliptic orbits $\{\gamma_1^e, \dots, \gamma_{b+k-\ell}^e\}$ as asymptotic limits.*

Proof. Given an integer $\ell \in [1, k]$ we consider the contact manifold $(M_{k-\ell}, \xi_{k-\ell})$ obtained by performing $k - \ell$ successive contact connected sums on (M_0, ξ_0). According to the observation above, since (M_0, ξ_0) is supported by a planar open book decomposition with b binding components, $(M_{k-\ell}, \xi_{k-\ell})$ is supported by a planar open book decomposition with $b + k - \ell$ binding components. The constructions of Abbas [1] and Wendl [59] then provide a finite energy foliation whose leaves are all either trivial cylinders over elliptic orbits or index-2 curves, having only positive punctures, whose projections to M coincide with with the pages of the open book decomposition.

Given this, we then carry out our construction ℓ times on the given open book decomposition. Since varying the points at which the connected sum occurs leads to contactomorphic contact manifolds, we are free to choose the points at each step to lie on an index-2 curve in the foliation. As is shown in Section 6, each time we apply our construction we add one even orbit to the foliation with two planes (modulo the $\mathbb{R}$-action) asymptotic to it and two index-1 curves (modulo the

[1] This can be seen from the following argument which was explained to the second author by O. van Koert. Forming a contact connected sum at two points in a given connected manifold (M, ξ) is the same as forming a contact connected sum of (M, ξ) with $(S^1 \times S^2, \xi_0)$ where ξ_0 is the contact structure arising as the kernel of the S^1-invariant contact form $\lambda_0 = \cos\theta\, dt + \sin^2\theta\, d\phi$ with $t \in \mathbb{R}/\mathbb{Z}$ the coordinate along S^1, and with $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ and $\theta \in [0, \pi]$, respectively, the polar and azimuthal coordinates on S^2. Forming the book connected sum (see e.g. Section 4.5 in [44] or Section 5.2.3 in [53]) of an open book decomposition for (M, ξ) with the open book decomposition for $(S^1 \times S^2, \xi_0)$ by cylinders of the form $S^1 \times \{\phi = c\}$ yields an open book decomposition for the contact connected sum $(M, \xi) \# (S^1 \times S^2, \xi_0)$ with pages having the same genus as those of the given open book decomposition for (M, ξ) and one additional boundary component.

$\mathbb{R}$-action) asymptotic to that orbit with a negative puncture. Moreover the four different height-2 holomorphic buildings that can be formed by these curves are each a boundary component of one of the surrounding index-2 curves, all of which have the same number of punctures and asymptotic behavior as the curves in the original foliation. Since these curves form 1-dimensional families modulo the $\mathbb{R}$-action, and each such family has two boundary components, there must be exactly 2ℓ of them. □

This theorem shows that manifolds of the form (M_k, ξ_k) satisfying the hypotheses above admit at least $k+1$ different stable finite energy foliations: the holomorphic open book decomposition $\mathcal{F}_0$ constructed by Abbas [1] and Wendl [59] and the k foliations $\mathcal{F}_\ell$ constructed by the theorem. An interesting question for further investigation is whether or not the various foliations on the manifold (M_k, ξ_k) are, in language introduced in [55], concordant; that is whether or not two given foliations $\mathcal{F}_i$ and $\mathcal{F}_j$ on (M_k, ξ_k) can be related by a non-$\mathbb{R}$-invariant pseudoholomorphic foliation $\mathcal{F}_{ij}$ of $\mathbb{R} \times M_k$ which is asymptotic at the positive and negative ends to $\mathcal{F}_i$ and $\mathcal{F}_j$ respectively.

Our construction can also be used in combination with the results of Hofer, Wysocki, and Zehnder [31] to make local changes to the structure of a given finite energy foliation on a manifold. Indeed, we recall that forming the contact connected sum of a contact manifold (M, ξ) with a copy of tight S^3 produces a contact structure on M contactomorphic to the original. Given then data (M, λ, J) admitting a foliation, we can apply our construction to the given foliation on M and any of the foliations on tight S^3 constructed by Hofer, Wysocki, and Zehnder in [31] to produce a new foliation on M for a contact form inducing a contact structure contactomorphic to the original. For example, by applying our construction to any number of copies of tight S^3 equipped with an open book decomposition consisting of pseudoholomorphic planes and a planar contact manifold equipped with a holomorphic planar open book decomposition constructed by Abbas [1] and Wendl [59], one can construct a stable finite energy foliation for a given planar contact manifold having any desired number of even Conley–Zehnder index orbits appearing as asymptotic limits of leaves of the foliation.

We make two observations about the new foliations produced by forming a connected sum of a foliation $\mathcal{F}$ for data (M, λ, J) with a Hofer–Wysocki–Zehnder foliation $\mathcal{F}_{HWZ}$ on tight S^3. The first is that the new foliation $\mathcal{F}' = \mathcal{F} \# \mathcal{F}_{HWZ}$ can, with some additional work, be shown to be concordant to the original foliation $\mathcal{F}$. A second observation about this construction is that, since Hofer–Wysocki–Zehnder foliations exist for any nondegenerate contact form on S^3, there is a good deal of freedom in the choice of contact form and almost complex structure on the S^3 portion of the connected sum. Indeed, forming the connected sum of a foliation with a Hofer–Wysocki–Zehnder foliation creates a 3-ball in the manifold bounded by a 2-sphere composed of a pair of pseudoholomorphic planes. The foliation created will persist under arbitrary perturbations of the contact form and almost complex structure which are compactly supported in the interior of this three-ball, provided the resulting contact form is nondegenerate and the almost complex structure is regular. Thus, the existence of a foliation for a given set of data (M, λ, J) is a property which is persistent under a large class of localized perturbations to the data (λ, J).

In regards to the original motivation for this work from the three-body problem, we point out that our main theorem here does not immediately imply the existence of a finite energy foliation for level sets of the Hamiltonian having energies just above the first Lagrange point. However, given the existence of finite energy foliations for level sets below the first Lagrange point, our result allows one to construct a Hamiltonian on the same phase space having an interval of regular level sets which admit finite energy foliations and which are homotopic to a level set of the original Hamiltonian for the three-body problem having energy just above the first Lagrange point. This fact along with some deformation results for finite energy foliations currently being developed by the second author would then allow one to construct a finite energy foliation for any small nondegenerate perturbation of a level set of the original Hamiltonian. In some cases these foliations could then be used to construct foliations for level sets of the Hamiltonian via a limiting argument as in [28].

Finally, we remark that for simplicity of presentation, and for the convenience of being able to quote results from other papers, we have chosen to focus on the case of contact manifolds equipped with a nondegenerate contact form and we don't impose any conditions on the rates of convergence of curves in the foliation to their asymptotic limits. However, with the use of the in-progress work [51], which generalizes the intersection theory of [50] to include exponential weights and Morse–Bott nondegenerate orbits, it is straightforward to adapt our arguments to somewhat more general situations. The essential point is that appropriate generalizations of the necessary results from intersection theory [50] and Fredholm theory [30, 57] are true provided the curves in question approach their asymptotic limits exponentially fast. Given this, one can consider $\mathbb{R}$-invariant finite energy foliations in a manifold with a degenerate contact form, provided all curves converge exponentially to their asymptotic limits. The arguments in the proofs of [50, Theorem 2.4] and [50, Theorem 2.6] show that, for appropriate choices of exponential weights, the exponentially weighted version of the $*$-product, developed in [51], must vanish between any two of the nontrivial curves of such a foliation. Such a foliation would then be called a *weighted, stable finite energy foliation* if all weighted Fredholm indices of nontrivial curves are 1 or 2. Given results that will be proved in [51], it is straightforward to adapt our arguments to work for weighted, stable finite energy foliations.

Remark 1.3. In the recent work [14], de Paulo and Salomão study Hamiltonians H on $\mathbb{R}^4$ having a saddle-center equilibrium point lying on a strictly convex singular subset $S_0 \subset H^{-1}(0)$. They show that for all sufficiently small positive energies E, there is a subset $S_E \subset H^{-1}(E)$ diffeomorphic to the closed three ball so that the symplectization $\mathbb{R} \times S_E$ admits a finite energy foliation. The structure of the finite energy foliation that they construct is the same as that which would result from our construction when taking a connected sum with S^3 equipped with one of the pseudoholomorphic open book decompositions constructed by Hofer, Wysocki, and Zehnder in [28].

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1.1. **Outline of the paper.** While most earlier results on finite energy foliations deal with relatively concrete constructions, the results we prove in the present series of papers deal with finite energy foliations abstractly. The proofs of our results thus require us to develop some general theory for finite energy foliations. To assist in this, we review some background about Reeb dynamics and pseudoholomorphic curves in Sections 2 and 3, with a special focus on giving precise statements and references for facts that we will need in this paper and its sequels.

We start by recalling relevant facts about contact geometry and Reeb dynamics in Section 2, primarily focusing on material concerning properties of the Conley–Zehnder index from [26]. Then in Section 3 we review background on finite-energy pseudoholomorphic curves. First, in Section 3.1 we recall the basic asymptotic convergence to a periodic orbit, established by Hofer in [24], as well as the refined relative asymptotic formula of the second author from [49]. Then in Section 3.2, we recall the compactification of the space of finite-energy pseudoholomorphic curves. Of particular relevance here is the work of Wendl [58] which focuses on what sort of limiting objects can arise as sequences of so-called nicely-embedded curves. After that, in Section 3.3, we review results related to the intersection product for finite-energy pseudoholomorphic curves introduced by the second author in [50]. An adaptation of a result from [50] concerning the direction of approach of curves to even orbits will be key for the proof of our main theorem. Finally, in Section 3.4 we recall facts about the Fredholm theory of embedded finite-energy pseudoholomorphic curves from [30] and review so-called automatic transversality conditions [32, 30, 2, 57] which give topological criteria that guarantee the moduli space of curves is a smooth manifold of dimension equal to the Fredholm index.

General discussion of finite energy foliations begins in Section 4. After giving a definition of stable finite energy foliations we establish some basic properties of stable finite energy foliations that follow from this definition. We then discuss some facts about the structure of the moduli spaces of curves which appear in finite energy foliations. In Section 5 we show that contact connected sums can be formed in a way which gives us properties necessary to prove our main theorem. In order to focus on the main ideas, some of the more straightforward but tedious computations needed to support claims in this section are delayed to Appendix A. Finally in Section 6, we give the proof of our main theorem.

2. Background in contact geometry and Reeb dynamics

In this section we review some basic notions from contact geometry and Reeb dynamics that we will need, and fix some notation. Much of the material from this section, particularly that material pertaining to the Conley–Zehnder index of periodic orbits, is adapted from [26].

Let M be a closed, oriented 3-manifold. Recall that a *contact form* on M is a 1-form λ for which

$$\lambda \wedge d\lambda \text{ is a volume form on } M. \tag{2.1}$$

This condition implies that there is a unique vector field X_λ, called the *Reeb vector field* associated to λ, satisfying the conditions

$$i_{X_\lambda}\lambda = 1 \qquad \text{and} \qquad i_{X_\lambda}d\lambda = 0. \tag{2.2}$$

The *contact structure* ξ determined by λ is defined by $\xi = \ker \lambda$. As a result of condition (2.1) the contact structure is necessarily a 2-plane bundle transverse to X_λ, and $d\lambda$ restricts to a nondegenerate form on ξ. The contact form λ thus determines a splitting

$$TM = \mathbb{R}X_\lambda \oplus (\xi, d\lambda) \tag{2.3}$$

of the tangent space TM of M into a framed line bundle and a symplectic 2-plane bundle. Moreover, the defining conditions (2.2) for X_λ used with the formula $L_X = i_X \circ d + d \circ i_X$ imply that

$$L_{X_\lambda}\lambda = 0 \qquad \text{and} \qquad L_{X_\lambda} d\lambda = 0$$

and thus the flow of X_λ preserves the splitting (2.3).

It will be convenient for our purposes here to think of periodic orbits of the Reeb vector field as maps from the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In particular, for $T > 0$ we consider a T-periodic orbit to be a map $\gamma : S^1 \to M$ satisfying

$$\dot{\gamma}(t) = T \cdot X_\lambda(\gamma(t)).$$

An unparametrized periodic orbit is a collection of parametrized orbits that differ by reparametrization via the S^1-action on the domain. We will generally use the same notation for a parametrized orbit and its associated unparametrized orbit, allowing the context or specific language to distinguish between the two.

Let $\psi_\cdot : \mathbb{R} \times M \to M$ be the flow generated by the Reeb vector field X_λ, that is

$$\dot{\psi}_t(x) = X_\lambda \circ \psi_t(x),$$

and let $\gamma : S^1 \to M$ be a parametrized T-periodic orbit. Since the flow of X_λ preserves the splitting (2.3), we obtain for any $t \in S^1$ a symplectic map

$$d\psi_T(\gamma(t)) \in Sp(\xi_{\gamma(t)}, d\lambda),$$

and, since the group property of the flow and its linearization can be used to show that

$$d\psi_T(\gamma(t)) = [d\psi_{-tT}(\gamma(t))]^{-1} d\psi_T(\gamma(0)) d\psi_{-tT}(\gamma(t)), \tag{2.4}$$

the spectrum of $d\psi_T(\gamma(t))$ is independent of $t \in S^1$. We will thus say that an unparametrized T-periodic orbit γ is *nondegenerate* if for a representative parametrization $\gamma : S^1 \to M$, the map $d\psi_T(\gamma(0))$ does not have 1 in the spectrum. A contact form λ on M is said to *nondegenerate* if all periodic orbits are nondegenerate.

A nondegenerate T-periodic orbit γ is said to be:

- elliptic if $d\psi_T(\gamma(t))$ has complex eigenvalues, or
- hyperbolic if $d\psi_T(\gamma(t))$ has real eigenvalues.

Moreover, γ is said to be:

- odd if γ is elliptic, or if γ is hyperbolic and $d\psi_T(\gamma(t))$ has negative eigenvalues, or
- even γ is hyperbolic and $d\psi_T(\gamma(t))$ has positive eigenvalues.

As a result of (2.4) the designation of a nondegenerate orbit as even/odd, positive/negative is a well-defined property associated to the unparametrized orbit. The parity of a periodic orbit as defined here agrees with the parity of the orbit's Conley–Zehnder index, which we will now define.

Given a trivialization of the contact structure along a nondegenerate periodic orbit, one can assign a number, called the Conley–Zehnder index, to the orbit which can be thought of as a measure of the winding with respect to the given trivialization of the linearized flow along the orbit [12, 47, 45, 26]. We review the key information now. As a starting point we recall information about the Maslov index and Conley–Zehnder index for, respectively, loops and paths in $Sp(1) = Sp(\mathbb{R}^2, \omega_0 = dx \wedge dy)$. We first recall that the fundamental group $\pi_1(Sp(1))$ of the symplectic group is isomorphic to $\mathbb{Z}$ (see e.g. [3, Section 1.2.1]). The *Maslov index* of a (homotopy class of) loop(s) of matrices in $Sp(1)$ based at the identity is, by definition, the isomorphism

$$m : \pi_1(Sp(1)) \to \mathbb{Z}$$

determined by assigning a value of 1 to the (homotopy class of the) loop

$$t \in S^1 = \mathbb{R}/\mathbb{Z} \mapsto \begin{bmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{bmatrix}$$

which is a generator of $\pi_1(Sp(1))$. Given this, we can define the Conley–Zehnder index for (homotopy classes of) paths in $Sp(1)$ that start at the identity and end at a matrix without 1 in the spectrum via the following axiomatic characterization from [26, Theorem 3.2].

Theorem 2.1. *Let*

$$\Sigma(1) = \big\{\Psi \in C^0([0,1], Sp(1)) \,|\, \Psi(0) = I \text{ and } \det(\Psi(1) - I) \neq 0\big\} \tag{2.5}$$

denote the space of continuous paths in $Sp(1)$ which start at the identity and end at a matrix without 1 in the spectrum. There exists a unique map

$$\mu_{cz} : \Sigma(1) \to \mathbb{Z},$$

called the Conley–Zehnder index, determined by the following axioms:

(1) *Homotopy invariance: The Conley–Zehnder index of a path in $\Sigma(1)$ is invariant under homotopies of paths in $\Sigma(1)$.*
(2) *Maslov compatibility: If $\Psi \in \Sigma(1)$ and $g : [0,1] \to Sp(1)$ is a loop based at the identity, then*

$$\mu_{cz}(g\Psi) = 2m(g) + \mu_{cz}(\Psi)$$

where $g\Psi \in \Sigma(1)$ is the path defined by $(g\Psi)(t) = g(t)\Psi(t)$.
(3) *Inverse axiom: If $\Psi \in \Sigma(1)$ and $\Psi^{-1} \in \Sigma(1)$ is the inverse path defined by $\Psi^{-1}(t) = [\Psi(t)]^{-1}$, then*

$$\mu_{cz}(\Psi^{-1}) = -\mu_{cz}(\Psi).$$

Now, let $\gamma : S^1 \to M$ by a nondegenerate, T-periodic orbit, and let $\Phi : S^1 \times \mathbb{R}^2 \to \gamma^*\xi$ by a symplectic trivialization, that is, assume that

$$d\lambda_{\gamma(t)}\left(\Phi(t)\cdot, \Phi(t)\cdot\right) = dx \wedge dy$$

for all $t \in S^1$. Again, recalling that $L_{X_\lambda} d\lambda = 0$, the flow ψ_t of X_λ gives for any $t \in \mathbb{R}$ a symplectic map

$$d\psi_{tT}(\gamma(0)) : (\xi_{\gamma(0)}, d\lambda_{\gamma(0)}) \to (\xi_{\psi_{tT}(\gamma(0))}, d\lambda_{\psi_{tT}(\gamma(0))}) = (\xi_{\gamma(t)}, d\lambda_{\gamma(t)})$$

and thus the map

$$t \in [0,1] \to \Phi^{-1}(t) d\psi_{tT}(\gamma(0))\Phi(0) \tag{2.6}$$

gives a path of matrices in $Sp(1)$ starting at the identity and ending at

$$\Phi^{-1}(1) d\psi_T(\gamma(0))\Phi(0) = \Phi^{-1}(0) d\psi_T(\gamma(0))\Phi(0)$$

which doesn't have 1 in the spectrum by the assumption that γ is nondegenerate. We define the *Conley–Zehnder index* $\mu^{\Phi}(\gamma)$ of the orbit γ relative to the trivialization Φ by

$$\mu^{\Phi}(\gamma) := \mu_{cz}(\Psi) \tag{2.7}$$

with $\Psi \in \Sigma(1)$ the path (2.6), and $\mu_{cz}(\Psi)$ the Conley–Zehnder index of the path Ψ as characterized in Theorem 2.1. We note that, as a result of the homotopy invariance axiom from Theorem 2.1, the Conley–Zehnder index of an orbit is invariant under homotopies of the trivialization. Furthermore, the homotopy invariance axiom can be used to show that the Conley–Zehnder index relative to a given trivialization is independent of the choice of parametrization of the orbit. Finally, we note that, as result of the Maslov compatibility axiom, the parity of the Conley–Zehnder index of an orbit does not depend on the choice of trivialization. Further, this parity can be shown to agree with that defined above in terms of the eigenvalues of the linearized flow.

We will need the characterization of the Conley–Zehnder index in terms of the spectrum of a certain self-adjoint operator acting on sections of the contact structure along the orbit from [26]. Let γ be a parametrized T-periodic orbit, and let $h : S^1 \to \xi$ be a smooth section of the contact structure along γ, i.e. $h(t) \in \xi_{\gamma(t)}$ for all $t \in S^1$. We observe that since h is defined along a flow line of X_λ, it has a well-defined Lie derivative $L_{X_\lambda} h$ defined by

$$L_{X_\lambda} h(t) = \left.\frac{d}{ds}\right|_{s=0} d\psi_{-s}(\gamma(t+s/T))h(t+s/T) \tag{2.8}$$

and, since the flow ψ_t of X_λ preserves the splitting (2.3), $L_{X_\lambda} h$ is also a section of the contact structure along γ. Given any symmetric connection ∇ on TM, we use that $\dot\gamma(t) = T \cdot X_\lambda(\gamma(t))$ to write

$$T \cdot L_{X_\lambda} h = \nabla_t h - T\nabla_h X_\lambda.$$

Thus $\nabla_t \cdot -T\nabla . X_\lambda$ gives a first-order differential operator on $C^\infty(\gamma^*\xi)$ which is independent of choice of symmetric connection.

Next, recall that given a symplectic vector bundle (E, ω) a complex structure J on E is said to be compatible with ω if the bilinear form

$$g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$$

is a metric on E. It is a well known fact the space of compatible almost complex structures on a given symplectic vector bundle is nonempty and contractible in the C^∞ topology (see e.g. Proposition 5 and discussion thereafter in Section 1.3 of [33]). Recalling that $(\xi, d\lambda)$ is a symplectic vector bundle, we define the set $J(M,\lambda) \subset \mathrm{End}(\xi)$ to be the set of complex structures on ξ compatible with $d\lambda$. Given a T-periodic orbit γ and a $J \in J(M,\lambda)$, we define the *asymptotic operator* $\mathbf{A}_{\gamma,J}$ associated to γ and J by

$$\mathbf{A}_{\gamma,J} h = -J(\nabla_t h - T\nabla_h X_\lambda), \tag{2.9}$$

and note that, by the discussion of the previous paragraph, $\mathbf{A}_{\gamma,J}$ gives a first-order differential operator on $C^\infty(\gamma^*\xi)$ which is independent of the choice of symmetric connection used to define it.

We define an inner product $\langle\cdot,\cdot\rangle_J$ on $C^\infty(\gamma^*\xi)$ by

$$\langle h, k\rangle_J = \int_{S^1} d\lambda_{\gamma(t)}\,(h(t), J(\gamma(t))k(t))\;dt.$$

Recalling that $L_{X_\lambda} d\lambda = 0$, we have for any h, $k \in C^\infty(\gamma^*\xi)$ that

$$\frac{d}{dt}\left[d\lambda_{\gamma(t)}\left(h(t),k(t)\right)\right] = d\lambda_{\gamma(t)}\left(T(L_{X_\lambda}h)(t),k(t)\right) + d\lambda_{\gamma(t)}\left(h(t),T(L_{X_\lambda}k)(t)\right).$$

Using that compatibility of J with $(\xi, d\lambda)$ implies that $d\lambda(J\cdot, J\cdot) = d\lambda$ on $\xi \times \xi$, we can integrate the above equation to give

$$\langle h, \mathbf{A}_{\gamma,J} k\rangle_J = \langle \mathbf{A}_{\gamma,J} h, k\rangle_J.$$

Thus $\mathbf{A}_{\gamma,J}$ is formally self-adjoint, and induces a self-adjoint operator

$$\mathbf{A}_{\gamma,J} : D(\mathbf{A}_{\gamma,J}) = H^1(\gamma^*\xi) \subset L^2(\gamma^*\xi, \langle\cdot,\cdot\rangle_J) \to L^2(\gamma^*\xi, \langle\cdot,\cdot\rangle_J).$$

Since for any value in the resolvent set of $\mathbf{A}_{\gamma,J}$, the associated resolvent operator factors through the compact embedding $H^1(\gamma^*\xi) \hookrightarrow L^2(\gamma^*\xi)$, we know from the spectral theorem for compact self-adjoint operators that the spectrum of $\mathbf{A}_{\gamma,J}$ consists of real, isolated eigenvalues of finite multiplicity accumulating only at $\pm\infty$.

We recall the observation from [27] that $\ker \mathbf{A}_{\gamma,J}$ is trivial if and only if γ is a nondegenerate orbit. Indeed, if h is a section of $\gamma^*\xi$ in the kernel of $\mathbf{A}_{\gamma,J}$ then $L_{X_\lambda} h = 0$ and thus $d\psi_{tT} h(t_0) = h(t_0 + t)$ for any $t \in \mathbb{R}$ and $t_0 \in S^1$. In particular $d\psi_T(\gamma(t_0))h(t_0) = h(t_0 + 1) = h(t_0)$ so $d\psi_T(\gamma(t_0))$ has 1 as an eigenvalue and γ must be a degenerate orbit. Conversely, if the orbit is degenerate, then $d\psi_T(\gamma(t_0))$ has 1 as an eigenvalue. Letting $v_0 \in \xi_{\gamma(0)} \setminus \{0\}$ be a vector with $d\psi_T(\gamma(0))v_0 = v_0$, the map $v : \mathbb{R} \to \xi$ defined by $v(t) = d\psi_{tT}(\gamma(0))v_0 \in \xi_{\gamma(t)}$ will be 1-periodic and satisfy $L_{X_\lambda} v = 0$, thus determining a section of $\gamma^*\xi$ in the kernel of $\mathbf{A}_{\gamma,J}$.

In a unitary trivialization of $(\gamma^*\xi, d\lambda, J)$ — that is, a symplectic trivialization $\Phi : S^1 \times \mathbb{R}^2 \to \gamma^*\xi$ of $(\gamma^*\xi, d\lambda)$ satisfying

$$\Phi \circ J_0 = J \circ \Phi$$

with

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

— the operator $\mathbf{A}_{\gamma,J}$ takes the form

$$\Phi^{-1} \circ \mathbf{A}_{\gamma,J} \circ \Phi = -i\tfrac{d}{dt} - S(t)$$

with $S(t)$ a symmetric matrix. An eigenvector of $\mathbf{A}_{\gamma,J}$ satisfies a linear, first-order ordinary differential equation and therefore never vanishes since it doesn't vanish identically. Hence every eigenvector gives a map from $S^1 \to \mathbb{R}^2 \setminus \{0\}$ and thus has a well-defined winding number. Since $-i\frac{d}{dt} - S(t)$ is a compact perturbation of $-i\frac{d}{dt}$, it can be shown using perturbation theory in [39] that that the winding is monotonic in the eigenvalue and that to any $k \in \mathbb{Z}$ the span of the set of eigenvectors having winding k is two dimensional. These facts are proved in Section 3 of [26], and we restate them here as a lemma.

Lemma 2.2. *Let γ be a T-periodic orbit of X_λ, let $\mathbf{A}_{\gamma,J}$ denote the asymptotic operator of γ, and let $\mathfrak{T}(\gamma^*\xi)$ denote the set of homotopy classes of symplectic trivializations of $(\gamma^*\xi, d\lambda)$. There exists a map $w : \sigma(\mathbf{A}_{\gamma,J}) \times \mathfrak{T}(\gamma^*\xi) \to \mathbb{Z}$ which satisfies*

(1) *If $e : S^1 \to \gamma^*\xi$ is an eigenvector of $\mathbf{A}_{\gamma,J}$ with eigenvalue λ, then $w(\lambda, [\Phi]) = \operatorname{wind}(\Phi^{-1}e)$, that is, $w(\lambda, [\Phi])$ measures the winding with respect to Φ of any eigenvector of $\mathbf{A}_{\gamma,J}$ with eigenvalue λ.*

(2) *For any fixed* $[\Phi] \in \mathfrak{T}(\gamma^*\xi)$ *we have that*

$$w(\lambda, [\Phi]) < w(\mu, [\Phi]) \Rightarrow \lambda < \mu,$$

that is, the winding of eigenvectors of $\mathbf{A}_{\gamma,J}$ *is (not necessarily strictly) monotonic in the eigenvalue.*

(3) *If* $m(\lambda) = \dim\ker(\mathbf{A}_{\gamma,J} - \lambda)$ *denotes the multiplicity of* λ *as an eigenvalue we have for every* $k \in \mathbb{Z}$ *and* $[\Phi] \in \mathfrak{T}(\gamma^*\xi)$ *that*

$$\sum_{\{\lambda \mid w(\lambda,[\Phi])=k\}} m(\lambda) = 2,$$

that is, the span of the set of eigenvectors of $\mathbf{A}_{\gamma,J}$ *with any given winding has dimension* 2*.*

We now describe the characterization of the Conley–Zehnder in terms of the asymptotic operator from [26]. Given a T-periodic orbit γ and a $J \in \mathcal{J}(M, \lambda)$ let $\sigma^-_{max}(\gamma) \in \sigma(\mathbf{A}_{\gamma,J})$ denote the largest negative eigenvalue of $\mathbf{A}_{\gamma,J}$. Given a trivialization Φ of $\gamma^*\xi$, we define

$$\alpha^\Phi(\gamma) = w(\sigma^-_{max}(\gamma); [\Phi]) \tag{2.10}$$

so that $\alpha^\Phi(\gamma)$ is the winding relative to Φ of any eigenvector of $\mathbf{A}_{\gamma,J}$ having the largest possible negative eigenvalue. We define the parity of $p(\gamma)$ of γ by

$$p(\gamma) = \begin{cases} 0 & \text{if } \exists \mu \in \sigma(\mathbf{A}_{\gamma,J}) \cap \mathbb{R}^+ \text{ with } w(\mu, [\Phi]) = \alpha^\Phi(\gamma) \\ 1 & \text{otherwise} \end{cases} \tag{2.11}$$

i.e. the parity is 0 if there is a positive eigenvalue with eigenvectors having winding equal to that of those eigenvectors having largest negative eigenvalue, and the parity is 1 otherwise. The following theorem then gives a formula for the Conley–Zehnder index of γ in terms of the quantities α^Φ and p just defined.

Theorem 2.3 (Hofer–Wysocki–Zehnder [26])**.** *Let* γ *be a* T*-periodic orbit of the Reeb vector field* X_λ*, let* $\Phi : S^1 \times \mathbb{R}^2 \to \gamma^*\xi$ *be a symplectic trivialization of* $(\gamma^*\xi, d\lambda)$ *and let* $\alpha^\Phi(\gamma)$ *and* $p(\gamma)$ *be as defined in* (2.10)*–*(2.11) *above. Then the Conley–Zehnder index of* γ *relative to* Φ *is given by the formula*

$$\mu^\Phi(\gamma) = 2\alpha^\Phi(\gamma) + p(\gamma).$$

Finally we close this section by stating a formula for how the Conley–Zehnder index behaves for iterates of an orbit. This lemma follows from facts about $Sp(1)$ which can be found in e.g. [31, Appendix 8.1] or [3, Section 1.2].

Lemma 2.4. *Let* γ *be a periodic orbit of the Reeb vector field* X_λ *and let* $\Phi : S^1 \times \mathbb{R}^2 \to \gamma^*\xi$ *be a symplectic trivialization. Assume that for each positive integer* m*, the periodic orbit* γ^m *defined by* $\gamma^m(t) = \gamma(mt)$ *is nondegenerate. Then:*

- *If* γ *is a hyperbolic orbit*

$$\mu^\Phi(\gamma^m) = m\mu^\Phi(\gamma).$$

- *If* γ *is an elliptic orbit, there exists an irrational number* θ *so that*

$$\mu^\Phi(\gamma^m) = 2\lfloor m\theta \rfloor + 1.$$

3. Background on pseudoholomorphic curves

In this section we review some basic facts about punctured pseudoholomorphic curves. First, in Section 3.1 we review the basic set-up and review some facts about the asymptotic behavior of finite-energy curves from [24, 27, 42, 49]. Next, in Section 3.2, we recall the compactification of the space of finite-energy curves [6], focusing on a result from [58] concerning the extra properties that can be proved about the compactification when restricting attention to sequences of curves which project to embeddings in the 3-manifold. In Section 3.3 we recall facts about the intersection theory of finite-energy curves from [50]. Of particular importance here is a slight generalization of a result from [50] concerning curves which approach an even orbit in the same direction. Finally, in Section 3.4, we recall facts about the Fredholm theory of embedded finite-energy curves from [30].

3.1. **Basic set-up and asymptotic behavior.** Let (M, λ) be 3-manifold equipped with a nondegenerate contact from, and recall from the previous section that we defined $\mathcal{J}(M, \lambda)$ to be the collection of complex structures on the contact structure ξ compatible with $d\lambda$. Given a $J \in \mathcal{J}(M, \lambda)$ we can extend it in the usual manner to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ by requiring

$$\tilde{J}\partial_a = X_\lambda \qquad \text{and} \qquad \tilde{J}|_{\pi_M^*\xi} = \pi_M^* J \tag{3.1}$$

where a is the coordinate in $\mathbb{R}$, and $\pi_M : \mathbb{R} \times M \to M$ is the canonical projection onto the second factor. We consider quintuples $(\Sigma, j, \Gamma, a, u)$ where

- (Σ, j) is a compact Riemann surface,
- $\Gamma \subset \Sigma$ is a finite set called the set of *punctures*, and
- $\tilde{u} := (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times M$ is a smooth map.

We define the energy of such a quintuple by

$$E(\tilde{u}) = \sup_{\varphi \in \Xi} \int_{\Sigma \setminus \Gamma} \tilde{u}^* d(\varphi\lambda) \tag{3.2}$$

where Ξ is defined by

$$\Xi = \{\varphi \in C^\infty(\mathbb{R}, [0, 1]) \,|\, \varphi'(x) \geq 0\}\,.$$

The data $(\Sigma, j, \Gamma, a, u)$ is said to be a *finite-energy pseudoholomorphic map* if the map $\tilde{u}$ has finite energy and is $\tilde{J}$-holomorphic, that is, if

$$\tilde{J} \circ d\tilde{u} = d\tilde{u} \circ j \tag{3.3}$$

and

$$E(\tilde{u}) < \infty.$$

A *finite-energy pseudoholomorphic curve* is an equivalence class $C = [\Sigma, j, \Gamma, a, u]$ of finite-energy pseudoholomorphic maps $(\Sigma, j, \Gamma, a, u)$ under the equivalence relation of holomorphic reparametrization of the domain. For a given 3-manifold M equipped with a nondegenerate contact form λ, and compatible $J \in \mathcal{J}(M, \lambda)$, we will denote the moduli space of finite-energy $\tilde{J}$-holomorphic curves by $\mathcal{M}(\lambda, J)$.

If $(\Sigma, j, \Gamma, \tilde{u} = (a, u))$ is a $\tilde{J}$-holomorphic map, then we can use the $\mathbb{R}$-invariance of $\tilde{J}$ defined by (3.1) to conclude that the map $\tilde{u}_c := (a + c, u)$ obtained by translating the $\mathbb{R}$-coordinate by a constant is also a $\tilde{J}$-holomorphic map, and it is moreover easily shown that $E(\tilde{u}) = E(\tilde{u}_c)$. Thus there is an $\mathbb{R}$-action on the space of finite-energy $\tilde{J}$-holomorphic curves given by translating the $\mathbb{R}$-coordinate by a constant and, in fact, the M-component u of $\tilde{u} = (a, u)$ determines the $\mathbb{R}$-component a up to a constant. To see this, we define $\pi_\xi : TM = \mathbb{R}X_\lambda \oplus \xi \to \xi$ to be the projection of TM onto ξ determined by the splitting (2.3). It then follows from the definition of $\tilde{J}$ that the equation (3.3) is equivalent to the pair of equations

$$u^*\lambda \circ j = da \tag{3.4a}$$

$$J \circ \pi_\xi \circ du = \pi_\xi \circ du \circ j \tag{3.4b}$$

and from the first of these equations it's clear that the map u determines da, and thus a up to a constant. We will define a *projected (finite-energy) pseudoholomorphic map* to be a quintuple $(\Sigma, j, \Gamma, da, u)$ satisfying equations (3.4) for which the associated map $\tilde{u} = (a, u)$ to $\mathbb{R} \times M$ has finite energy. A *projected (finite-energy) pseudoholomorphic curve* is then an equivalence class $C = [\Sigma, j, \Gamma, da, u]$ of projected pseudoholomorphic maps under the equivalence relation of holomorphic reparametrization of the domain. For a given 3-manifold M equipped with a nondegenerate contact form λ, and compatible $J \in \mathcal{J}(M, \lambda)$, we will denote the moduli space of projected, finite-energy $\tilde{J}$-holomorphic curves by $\mathcal{M}(\lambda, J)/\mathbb{R}$.

In his work on the Weinstein conjecture [24], Hofer showed that near the nonremovable punctures of a finite-energy pseudoholomorphic curves, there are sequences of loops whose images under u converge to periodic orbits of the Reeb vector field. In the case that the periodic orbit of the Reeb vector field is nondegenerate, then more can be said about this convergence. Suppose that λ is a nondegenerate contact form and $(\Sigma, j, \Gamma, a, u)$ is a finite-energy pseudoholomorphic map. Then, for each puncture $z_0 \in \Gamma$ there are three possibilities:

(1) Removable punctures: The map $\tilde{u} = (a, u)$ is bounded near z_0, in which case $\tilde{u}$ admits a smooth, $\tilde{J}$-holomorphic extension over the puncture.

(2) Positive punctures: The function a is bounded from below near z_0 but not from above. In this case there exists a nondegenerate periodic orbit γ with period $T \leq E(\tilde{u})$ and a holomorphic coordinate system

$$\phi : [R, \infty) \times S^1 \subset \mathbb{R} \times S^1 \approx \mathbb{C}/i\mathbb{Z} \to \Sigma \setminus \{z_0\}$$

on a punctured neighborhood of z_0 so that the maps $\tilde{v}_c : [R, \infty) \times S^1 \to \mathbb{R} \times M$ defined by

$$\tilde{v}_c(s, t) = (a(s + c/T, t) - c, u(s + c/T, t))$$

converge in $C^\infty([R, \infty) \times S^1, \mathbb{R} \times M)$ as $c \to \infty$ to the map

$$(s, t) \mapsto (Ts, \gamma(t)).$$

(3) Negative punctures: The function a is bounded from above near z_0 but not from below. In this case there exists a nondegenerate periodic orbit γ with period $T \leq E(\tilde{u})$ and a holomorphic coordinate system

$$\phi : (-\infty, -R] \times S^1 \subset \mathbb{R} \times S^1 \approx \mathbb{C}/i\mathbb{Z} \to \Sigma \setminus \{z_0\}$$

on a punctured neighborhood of z_0 so that the maps $\tilde{v}_c : (-\infty, -R] \times S^1 \to \mathbb{R} \times M$ defined by

$$\tilde{v}_c(s, t) = (a(s - c/T, t) + c, u(s - c/T, t))$$

converge in $C^\infty((-\infty, -R] \times S^1, \mathbb{R} \times M)$ as $c \to \infty$ to the map

$$(s, t) \mapsto (Ts, \gamma(t)).$$

We will henceforth assume that all removable punctures have been removed, and thus that all punctures are either positive or negative punctures at which the curves in question are asymptotic to cylinders over periodic orbits.

We will need more precise information about the asymptotic behavior of curves near a puncture, in particular that the convergence is exponential in nature and the finer behavior of the map (and differences between two maps) can be described in terms of eigenvectors of the asymptotic operator associated to the orbit. Before stating the appropriate result, we first establish some language. Let $(\Sigma, j, \Gamma, a, u)$ be a pseudoholomorphic map and assume that $\tilde{u} = (a, u)$ has a positive puncture at $z_0 \in \Gamma$ where $\tilde{u}$ is asymptotic to a cylinder over the T-periodic orbit γ. A map $U : [R, \infty) \times S^1 \to \gamma^*\xi$ with $U(s, t) \in \xi_{\gamma(t)}$ for all $(s, t) \in [R, \infty) \times S^1$ is called an *asymptotic representative of $\tilde{u}$ near z_0* if there exists a map $\phi : [R, \infty) \times S^1 \to \Sigma \setminus \{z_0\}$ with $\lim_{s\to\infty} \phi(s, t) = z_0$ so that

$$\tilde{u} \circ \phi(s, t) = \Big(Ts, \exp_{\gamma(t)} U(s, t)\Big)$$

where exp is the exponential map of some metric on M.[2] Asymptotic representatives at negative punctures are defined similarly but as maps from negative half-cylinders of the form $(-\infty, -R] \times S^1$. The following theorem, concerning the asymptotic behavior of differences of asymptotic representatives, is proved in [49].

Theorem 3.1. *Let U, $V : [R, \infty)\times S^1 \to \gamma^*\xi$ be smooth maps with $U(s, t)$, $V(s, t) \in \xi_{\gamma(t)}$ representing positive pseudoholomorphic half-cylinders (or, respectively, let U, $V : (-\infty, -R] \times S^1 \to \gamma^*\xi$ be smooth maps with $U(s, t)$, $V(s, t) \in \xi_{\gamma(t)}$ representing negative pseudoholomorphic half-cylinders). Then either $U - V$ vanishes identically or*

$$U(s, t) - V(s, t) = e^{\sigma s}[e(t) + r(s, t)]$$

where

- *σ is a negative (resp. positive) eigenvalue of the asymptotic operator $\mathbf{A}_{\gamma,J}$ (defined in (2.9)),*
- *$e \in \ker(\mathbf{A}_{\gamma,J} - \sigma) \setminus \{0\}$ is an eigenvector of $\mathbf{A}_{\gamma,J}$ with eigenvalue σ, and*
- *$\nabla_s^i \nabla_t^j r(s, t) \to 0$ as $s \to \infty$ (resp. $s \to -\infty$) exponentially for all $(i, j) \in \mathbb{N}^2$.*

The special case of this theorem where $V \equiv 0$ recovers the asymptotic results for single half-cylinders from [27, 42]. As is shown in [26], the asymptotic formula for a single half-cylinder allows one to assign a local invariant to each puncture, known as the asymptotic winding. Indeed, as a result of this formula, the M-portion u of a given pseudoholomorphic map $(\Sigma, j, \Gamma, a, u)$ can be written near some given puncture $z_0 \in \Gamma$

$$u \circ \psi_{z_0}(s, t) = \exp_{\gamma(t)} U_{z_0}(s, t)$$

with the asymptotic representative U satisfying a formula of the form

$$U(s, t) = e^{\sigma s}[e(t) + r(s, t)]$$

with σ, e, and r satisfying the conditions listed above. Since eigenvectors of the asymptotic operator $\mathbf{A}_{\gamma,J}$ are nowhere vanishing, the fact that r converges to 0 as $|s| \to \infty$ implies that $U(s, t)$ is nonvanishing for all sufficiently large $|s|$, or equivalently that in some neighborhood of the puncture, u does not intersect its

[2] In [49] a specific metric is used in the definition of asymptotic representative but that specific choice of metric is not essential for Theorem 3.1 to remain true.

asymptotic limit γ. Choosing a trivialization of $\gamma^*\xi$, we define the asymptotic winding of $\tilde{u}$ at z_0 by

$$\mathrm{wind}^{\Phi}_{\infty}(u; z_0) = \mathrm{wind}(\Phi^{-1}U_{z_0}(s, \cdot))$$

with the right-hand side being well defined and independent of all sufficiently large $|s|$. Using the asymptotic results of [27] and the characterization of the Conley–Zehnder index in terms of the spectrum of $\mathbf{A}_{\gamma,J}$ from [26] (reviewed as Theorem 2.3 above), the following inequality for the asymptotic winding is deduced in [26].

Theorem 3.2. *Let $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ and let $z \in \Gamma$. Then*

$$\pm_z \mathrm{wind}^{\Phi}_{\infty}(u; z) \leq \lfloor \pm_z \mu^{\Phi}(\tilde{u}; z)/2 \rfloor \tag{3.5}$$

where $\pm_z$ is the sign of the puncture z.

3.2. **Compactness.** It is shown in [6] that the space of punctured pseudoholomorphic curves with energy below any given value can be compactified by including more general objects, known as pseudoholomorphic buildings. In [58], it's shown that the space of curves which project to embeddings in the 3-manifold M can be compactified by considering only those buildings whose components are either pairwise disjoint or identical when projected to the 3-manifold M and are all either trivial cylinders or project to embeddings in M. We recall the result here.

We start with some definitions. First, for $i \in \{1, \dots, k\}$, consider a collection of (possibly disconnected) pseudoholomorphic curves $C_i = [\Sigma_i, j_i, \Gamma_i, a_i, u_i] \in \mathcal{M}(\lambda, J)$ and write $\Gamma_i = \Gamma_i^+ \cup \Gamma_i^-$ to indicate the decomposition into positive and negative punctures. Assume there are bijections $I_i : \Gamma_i^- \to \Gamma_{i+1}^+$ between the negative punctures of one curve and the positive punctures of the next in the sequence. We say that the data $(C_1, \dots, C_k; I_1, \dots, I_{k-1})$ form a *height-k non-nodal pseudoholomorphic building* when pairs of punctures identified via the bijections I_i have the same asymptotic limit. We will denote such a height-k pseudoholomorphic building by

$$C_1 \odot_{I_1} \cdots \odot_{I_{k-1}} C_k$$

or simply

$$C_1 \odot \cdots \odot C_k$$

when the specific bijections are not important, and we will refer to the curves C_i as the *levels* of the building. Given a height-k pseudoholomorphic building $C_1 \odot_{I_1} \cdots \odot_{I_{k-1}} C_k$ with $C_i = [\Sigma_i, j_i, \Gamma_i, a_i, u_i]$, we can circle-compactify each of the domain surfaces $\Sigma_i \setminus \Gamma_i$ at the punctures and glue these compactified surfaces together along circles corresponding to punctures identified via the bijections I_i to form a a topological surface with boundary. Due to the asymptotic behavior of the curves, this identification can be done in such a way that the maps u_i extend to the circle-compactifications and glue together to give a continuous map from the glued surface into M. In the event that any of the levels are asymptotic to multiply covered periodic orbits, the operation of gluing the circle-compactified surfaces is only uniquely determined when further choices, namely that of so-called asymptotic markers, are made. The specifics won't be important here, so we won't address this issue any further.

The structure of a non-nodal pseudoholomorphic building $C_1 \odot_{I_1} \cdots \odot_{I_{k-1}} C_k$ can be encoded in a graph with a vertex for each smooth connected component of the domains of the levels C_i and an edge for each pair of punctures identified via the bijections I_i. We will say that a non-nodal pseudoholomorphic building is *connected* if the corresponding graph is connected. This is equivalent to requiring the surface obtained from circle-compactifying and gluing the levels, as described in the previous paragraph, to be connected. The *arithmetic genus* of a connected pseudoholomorphic building is the genus of the glued surface. The arithmetic genus can be computed in terms of the graph modeling the building by the formula

$$g = \#E - \#V + \sum_{v_i \in V} g(v_i) + 1$$

where $\#E$ is the number of edges, $\#V$ is the number of vertices, and $g(v_i)$ is the genus of a given smooth connected surface in the building corresponding to the vertex v_i (see [6, Equation (6)]). In particular, a connected pseudoholomorphic building has arithmetic genus zero precisely when each component has genus 0 and $\#E = \#V - 1$ or, equivalently, precisely when each component has genus zero and the modeling graph is a tree.

Following [58], we refer to a connected pseudoholomorphic curve $C = [\Sigma, j, \Gamma, a, u]$ as a *nicely-embedded pseudoholomorphic curve* if the map $u : \Sigma \setminus \Gamma \to M$ is an embedding, that is, if the curve projects to an embedding in the 3-manifold M. We will say that a non-nodal pseudoholomorphic building is *nicely embedded* if:[3]

(1) Each $C \in \mathcal{M}(\lambda, J)$ occurring as a connected component of the building is either nicely embedded or a trivial cylinder (i.e. a curve of the form $\mathbb{R} \times \gamma$ for some periodic orbit γ).
(2) If C and $D \in \mathcal{M}(\lambda, J)$ occur as connected components of the building, the projections of C and D to M are either identical or disjoint.

We will call a nicely-embedded, non-nodal pseudoholomorphic building, *stable* if no level consists entirely of trivial cylinders.[4] The following theorem is proved as the main theorem in [58]

Theorem 3.3. [58, Theorem 1] *Let $C_k \in \mathcal{M}(\lambda, J)$ be a sequence of nicely-embedded pseudoholomorphic curves with uniformly bounded energy. Then there is a subsequence which converges in the sense of* [6] *to a stable, nicely-embedded pseudoholomorphic building.*

For our purposes, the complete definition of SFT-convergence from [6] is not necessary, but we will need the following facts which we state as a proposition.

Proposition 3.4. *Assume a sequence $C_k = [\Sigma_k, j_k, \Gamma_k, a_k, u_k] \in \mathcal{M}(\lambda, J)$ converges in the sense of* [6] *to a non-nodal pseudoholomorphic building $C_{\infty,1} \odot \cdots \odot C_{\infty,\ell}$ with $C_{\infty,i} = [\Sigma_{\infty,i}, j_{\infty,i}, \Gamma_{\infty,i}, a_{\infty,i}, u_{\infty,i}]$. Then there is a $k_0 \in \mathbb{N}$ so that:*

(1) *For $k \geq k_0$ there exist embeddings $\psi_{k,i} : \Sigma_{\infty,i} \setminus \Gamma_{\infty,i} \to \Sigma_k \setminus \Gamma_k$ and constants $c_{k,i}$ so that*

$$a_k \circ \psi_{k,i} + c_{k,i} \to a_{\infty,i} \text{ in } C^\infty_{loc}(\Sigma_{\infty,i} \setminus \Gamma_{\infty,i}, \mathbb{R})$$

and

$$u_k \circ \psi_{k,i} \to u_{\infty,i} \text{ in } C^\infty_{loc}(\Sigma_{\infty,i} \setminus \Gamma_{\infty,i}, M).$$

[3] The definition in [58] also includes a condition on some of the periodic orbits which connect the levels, but this condition (in fact a slightly stronger condition) is a consequence of the above two conditions. See Lemma 3.6 below.

[4] The general definition of stable from [6] allows for levels which contain only trivial cylinders or constant maps provided the domains of these maps are stable curves, i.e. twice the genus plus the number of special points (marked points and nodes) is greater than or equal to 3. Since we only consider buildings with no nodes or marked points here, our simpler definition is equivalent.

(2) *There exists a punctured surface* $\Sigma_\infty \setminus \Gamma_\infty$ *so that for all* $k \geq k_0$, $\Sigma_k \setminus \Gamma_k$ *is diffeomorphic to* $\Sigma_\infty \setminus \Gamma_\infty$*, and there exist diffeomorphisms* $\psi_k : \Sigma_\infty \setminus \Gamma_\infty \to \Sigma_k \setminus \Gamma_k$ *so that the maps* $u_k \circ \psi_k$ *converge in* $C^0(\Sigma_\infty \setminus \Gamma_\infty, M)$*.*

Finally, we will need to know what sorts of periodic orbits can appear in the SFT-limit of sequences of nicely-embedded curves. We start with a definition. In the following definition, we will use the notation γ^m to denote the m-fold cover a periodic orbit γ.

Definition 3.5. Let γ be a simply covered orbit and let m_+ and m_- be positive integers. We say that (γ, m_+, m_-) is a *bidirectional asymptotic limit* of a given non-nodal pseudoholomorphic building, if there are (possibly identical) nontrivial components C_+, C_- in the building so that γ^{m_+} is a positive asymptotic limit of C_+ and γ^{m_-} is a negative asymptotic limit of C_-.

We remark that nontrivial breaking orbits as defined in [58] always give rise to a bidirectional limit, but the converse is not true.

Lemma 3.6. *Let* (γ, m_+, m_-) *be a bidirectional limit of a nicely-embedded pseudoholomorphic building. Then either:*

- γ *is even and* $m_+ = m_- = 1$*, or*
- γ *is odd, hyperbolic and* $m_+ = m_- = 2$*.*

Proof. This is equivalent to Proposition 4.4 in [58] which references [50] for proof. While this result is easily deduced from facts in [50], this fact is not stated explicitly there, so we outline the proof here.

Let $C_+ = [\Sigma_+, j_+, \Gamma_+, a_+, u_+]$ and $C_- = [\Sigma_-, j_-, \Gamma_-, a_-, u_-]$ be nontrivial components of the building so that C_+ has γ^{m_+} as an asymptotic limit at a puncture $z_+ \in \Gamma_+$ and C_- has γ^{m_-} as an asymptotic limit at a puncture $z_- \in \Gamma_-$. The assumption that $C_\pm$ are nicely embedded and have either identical or disjoint projections to M imply via, as appropriate, either condition 2(c) in [50, Theorem 2.4]/Theorem 3.10 or condition 3(b) in [50, Theorem 2.6]/Theorem 3.13 that[5]

$$\frac{\mathrm{wind}_\infty^\Phi(u_-; z_-)}{m_-} = \frac{-\lfloor -\mu^\Phi(\gamma^{m_-})/2 \rfloor}{m_-} = \frac{\lfloor \mu^\Phi(\gamma^{m_+})/2 \rfloor}{m_+} = \frac{\mathrm{wind}_\infty^\Phi(u_+; z_+)}{m_+}, \tag{3.6}$$

while condition 4(c) of [50, Theorem 2.6]/Theorem 3.13 tells us that

$$\gcd(m_+, \mathrm{wind}_\infty^\Phi(u_+, z_+)) = \gcd(m_-, \mathrm{wind}_\infty^\Phi(u_-, z_-)) = 1 \tag{3.7}$$

for any trivialization Φ of $\xi|_\gamma$. However, it's proved in [50, Theorem 2.4] using the iteration formulas for the Conley–Zehnder index (Lemma 2.4) that

$$\frac{-\lfloor -\mu^\Phi(\gamma^{m_-})/2 \rfloor}{m_-} = \frac{\lfloor \mu^\Phi(\gamma^{m_+})/2 \rfloor}{m_+}$$

if and only if γ^{m_+} and γ^{m_-} are both even orbits. This is equivalent to requiring either that γ is even, or γ is odd hyperbolic and m_+ and m_- are both even. In either

[5]The sign difference between the equations given here and those in [50] are due to a convention difference for computing Conley–Zehnder indices and wind_∞. Here we compute both by always traversing an orbit in the direction of the Reeb vector field, while in [50] both are computed by traversing the orbit in a direction determined by the boundary of the S^1-compactified surface, which means negative asymptotic limits are traversed in the direction opposite of the flow of the Reeb vector field.

case, we can use that the Conley–Zehnder index iterates linearly for hyperbolic orbits (Lemma 2.4). In the case that γ is even we then have from (3.6) that

$$\begin{aligned}\gcd(m_\pm, \operatorname{wind}_\infty^\Phi(u_\pm, z_\pm)) &= \gcd(m_\pm, \lfloor \pm\mu^\Phi(\gamma^{m_\pm})/2\rfloor) \\ &= \gcd(m_\pm, \lfloor \pm m_\pm \mu^\Phi(\gamma)/2\rfloor) \\ &= \gcd(m_\pm, m_\pm\mu^\Phi(\gamma)/2) \\ &= m_\pm \gcd(1, \mu^\Phi(\gamma)/2) \\ &= m_\pm\end{aligned}$$

so we must have $m_\pm = 1$ for (3.7) to hold. On the other hand, if γ is odd hyperbolic and $m_\pm = 2n_\pm$ are even, we have from (3.6) that

$$\begin{aligned}\gcd(m_\pm, \operatorname{wind}_\infty^\Phi(u_\pm, z_\pm)) &= \gcd(m_\pm, \lfloor \pm\mu^\Phi(\gamma^{m_\pm})/2\rfloor) \\ &= \gcd(2n_\pm, \lfloor \pm\mu^\Phi(\gamma^{2n_\pm})/2\rfloor) \\ &= \gcd(2n_\pm, \lfloor \pm 2n_\pm\mu^\Phi(\gamma)/2\rfloor) \\ &= \gcd(2n_\pm, \lfloor \pm n_\pm\mu^\Phi(\gamma)\rfloor) \\ &= \gcd(2n_\pm, n_\pm\mu^\Phi(\gamma)) \\ &= n_\pm \gcd(2, \mu^\Phi(\gamma)) \\ &= n_\pm\end{aligned}$$

where we've used the $\mu^\Phi(\gamma)$ is odd in the last line. This lets us conclude that (3.7) holds precisely when $n_\pm = 1$ and hence $m_\pm = 2n_\pm = 2$. This completes the proof. □

3.3. **Intersection theory.** Here we review some facts about the intersection theory of punctured pseudoholomorphic curves from [50].

We continue to assume λ is a nondegenerate contact form on M and $J \in \mathcal{J}(M, \lambda)$ is a compatible almost complex structure. Let $C_1 = [\Sigma, j, \Gamma, a, u]$ and $C_2 = [\Sigma', j', \Gamma', b, v]$ be pseudoholomorphic curves. We write $\Gamma = \Gamma_+ \cup \Gamma_-$ and $\Gamma' = \Gamma'_+ \cup \Gamma'_-$ to indicate the signs of the punctures. We assume that at $z \in \Gamma$, $\tilde{u} = (a, u)$ is asymptotic to $\gamma_z^{m_z}$ where γ_z is a simply-covered, unparametrized periodic orbit, m_z is a positive integer, and $\gamma_z^{m_z}$ denotes the m_z-fold cover of γ_z. Similarly we assume that at $w \in \Gamma'$, $\tilde{v} = (b, v)$ is asymptotic to $\gamma_w^{m_w}$ with γ_w simply covered. We let Φ denote a choice of trivialization of the contact structure along all simply covered periodic orbits of X_λ with covers appearing as asymptotic limits of C_1 or C_2. We define a map $\tilde{v}_\Phi$ by perturbing the M-portion v of the map slightly near the ends by the flow of a section of the contact structure defined near the orbits which has zero winding relative to the trivialization Φ. It can be shown that for suitably small such perturbations, the algebraic intersection number $\operatorname{int}(\tilde{u}, \tilde{v}_\Phi)$ of the maps $\tilde{u}$ and $\tilde{v}_\Phi$ is well defined and depends only on the homotopy classes of the maps $\tilde{u}$, $\tilde{v}$ and the trivialization Φ. We thus define the *relative intersection number* $i^\Phi(C_1, C_2)$ of C_1 and C_2 relative to the trivialization Φ by

$$i^\Phi(C_1, C_2) = \operatorname{int}(\tilde{u}, \tilde{v}_\Phi).$$

For more background on the definition and properties of the relative intersection number, see [37, Section 2.4] or [50, Section 4.1.1]

Given the relative intersection number of two curves, we define the *holomorphic intersection number*[6] by[7]

(3.8)

$$C_1 * C_2 = i^{\Phi}(C_1, C_2) + \sum_{\substack{(z,w)\in\Gamma_+\times\Gamma'_+ \\ \gamma_z=\gamma_w}} m_z m_w \max\left\{\frac{\lfloor \mu^{\Phi}(\gamma_z^{m_z})/2\rfloor}{m_z}, \frac{\lfloor \mu^{\Phi}(\gamma_w^{m_w})/2\rfloor}{m_w}\right\} + \sum_{\substack{(z,w)\in\Gamma_-\times\Gamma'_- \\ \gamma_z=\gamma_w}} m_z m_w \max\left\{\frac{\lfloor -\mu^{\Phi}(\gamma_z^{m_z})/2\rfloor}{m_z}, \frac{\lfloor -\mu^{\Phi}(\gamma_w^{m_w})/2\rfloor}{m_w}\right\}.$$

We note that the sums here are taken over all pairs of ends with the same sign which are asymptotic to coverings of the same underlying simply covered orbit; the quantities in these sums correspond to the negation of the minimum number of intersections that must appear between a pair of such ends when one is perturbed in the prescribed direction (see Section 3.2 and specifically Corollary 3.21 in [50]). As our notation indicates, the holomorphic intersection product of two curves is independent of the choice of trivialization used to define the quantities on the right hand side of (3.8). For proof of this fact and the other basic properties of the holomorphic intersection number collected in the following theorem, we refer the reader to [50].

Theorem 3.7 (Properties of the generalized intersection number)**.** *Let (M, λ, J) be a nondegenerate contact manifold with compatible $J \in \mathcal{J}(M, \lambda)$, and let $\mathcal{M}(\lambda, J)$ denote the moduli space of finite-energy pseudoholomorphic curves in M.*

(1) *If $C = [\Sigma, j, \Gamma, \tilde{u}]$, $D = [\Sigma', j', \Gamma', \tilde{v}] \in \mathcal{M}(\lambda, J)$ are pseudoholomorphic curves then the generalized intersection number $C * D$ depends only on the relative homotopy classes of the maps $\tilde{u}$ and $\tilde{v}$.*

(2) *For any C, $D \in \mathcal{M}(\lambda, J)$*
$$C * D = D * C.$$

(3) *If C_1, C_2, $D \in \mathcal{M}(\lambda, J)$ then*
$$(C_1 + C_2) * D = C_1 * D + C_2 * D$$
where "+" on the left hand side denotes the disjoint union of the curves C_1 and C_2.

(4) *If $C_1 \odot C_2$ and $D_1 \odot D_2$ are asymptotically cylindrical buildings then*
$$(C_1 \odot C_2) * (D_1 \odot D_2) \geq C_1 * D_1 + C_2 * D_2.$$
Moreover, strict inequality occurs if and only if there is a periodic orbit γ so that C_1 has a negative puncture asymptotic to γ^m, D_1 has a negative puncture asymptotic to γ^n, and both γ^m and γ^n are odd orbits.

One of the main motivations for the definition of the holomorphic intersection number is that certain well-known theorems concerning the homological intersection number of holomorphic curves generalize nicely to facts about the holomorphic intersection number, albeit with an additional complication. The first such theorem is a generalization of the fact that for a pair of closed pseudoholomorphic curves having no common components (i.e. no components having identical image) the

[6]This is called the *generalized intersection number* in [50].

[7]This definition appears slightly different from that given in [50] since there Conley–Zehnder indices of orbits at negative punctures are computed by traversing the orbit backwards.

homological intersection number is nonnegative, and equal to zero if and only if the two curves are disjoint. For punctured curves a statement almost as strong can be made, but we have to allow for the possibility that intersections disappear at the punctures when the curves have ends approaching the same orbit. In this case, the disappearance of intersections is traded for a higher degree of "tangency at infinity" with this notion being made precise in terms of the asymptotic relative asymptotic formula from [49] reviewed above as Theorem 3.1. The total measure of "tangency at infinity" between two curves C, D without common components is called the *total asymptotic intersection number* and denoted $\delta_\infty(C,D)$. For a precise definition of the total asymptotic intersection number, and for proof and further discussion, we refer the reader to [50, Theorem 4.4/2.2].

Theorem 3.8. *Let C, $D \in \mathcal{M}(\lambda, J)$ be pseudoholomorphic curves. If C and D have no common components then*

$$C * D = \operatorname{int}(C,D) + \delta_\infty(C,D) \tag{3.9}$$

where $\operatorname{int}(C,D)$ is the algebraic intersection number of C and D, and $\delta_\infty(C,D)$ is the asymptotic intersection index of C and D. In particular

$$C * D \geq \operatorname{int}(C,D) \geq 0,$$

and

$$C * D = 0$$

if and only if C and D don't intersect, and the total asymptotic intersection index vanishes, i.e. $\delta_\infty(C,D) = 0$.

It will be of use to us here to be able to identify situations in which the holomorphic intersection number of two curves vanishes. A set of necessary and sufficient conditions is proved is proved in [50, Corollary 5.9]. We quote that result here with appropriate adjustments to the notation and conventions.

Theorem 3.9. *Let $C = [\Sigma, j, \Gamma, \tilde{u} = (a,u)]$ and $D = [\Sigma', j', \Gamma', \tilde{v} = (b,v)] \in \mathcal{M}(\lambda, J)$ be pseudoholomorphic curves, and assume that no component of C or D lies in a trivial cylinder. Then the following are equivalent:*

(1) *The generalized intersection number $C * D = 0$.*
(2) *All of the following hold:*
  (a) *The map u does not intersect any of the positive asymptotic limits of v.*
  (b) *The map v does not intersect any of the negative asymptotic limits of u.*
  (c) *Let γ be a periodic orbit so that at $z \in \Gamma$, $\tilde{u}$ is asymptotic to γ^{m_z} and at $w \in \Gamma'$, $\tilde{v}$ is asymptotic to γ^{m_w}, and let Φ be a trivialization of $\xi|_\gamma$. Then:*
    (i) *If z and w are both positive punctures,*

$$\operatorname{wind}_\infty^\Phi(\tilde{u}; z) = \lfloor \mu^\Phi(\gamma^{m_z})/2 \rfloor$$

and

$$\frac{\lfloor \mu^\Phi(\gamma^{m_z})/2 \rfloor}{m_z} \geq \frac{\lfloor \mu^\Phi(\gamma^{m_w})/2 \rfloor}{m_w}. \tag{3.10}$$

    (ii) *If z and w are both negative are both negative punctures*

$$-\operatorname{wind}_\infty^\Phi(\tilde{v}; w) = \lfloor -\mu^\Phi(\gamma^{m_w})/2 \rfloor$$

and

$$\frac{\lfloor -\mu^\Phi(\gamma^{m_w})/2\rfloor}{m_w} \geq \frac{\lfloor -\mu^\Phi(\gamma^{m_z})/2\rfloor}{m_z}.$$

(iii) *If z is a negative puncture and w is a positive puncture,*

$$-\operatorname{wind}^\Phi_\infty(\tilde u; z) - \lfloor -\mu^\Phi(\gamma^{m_z})/2\rfloor = \operatorname{wind}^\Phi_\infty(\tilde v; w) - \lfloor \mu^\Phi(\gamma^{m_w})/2\rfloor = 0$$

and γ^{m_z} and γ^{m_w} are both even orbits; or equivalently

$$\frac{\operatorname{wind}^\Phi_\infty(\tilde u; z)}{m_z} = \frac{\operatorname{wind}^\Phi_\infty(\tilde v; w)}{m_w}.$$

(3) *All of the following hold:*
   (a) *The map u does not intersect any of the asymptotic limits of v.*
   (b) *The map v does not intersect any of the asymptotic limits of u.*
   (c) *If γ is a periodic orbit so that at $z \in \Gamma$, $\tilde u$ is asymptotic to γ^{m_z} and at $w \in \Gamma'$, $\tilde v$ is asymptotic to γ^{m_w}, then*

$$\pm_z \operatorname{wind}^\Phi_\infty(\tilde u; z) - \lfloor \pm_z \mu^\Phi(\gamma^{m_z})/2\rfloor = \pm_w \operatorname{wind}^\Phi_\infty(\tilde v; w) - \lfloor \pm_w \mu^\Phi(\gamma^{m_w})/2\rfloor = 0.$$

   Further
   (i) *if γ is elliptic, then z and w are either both positive punctures, or both negative punctures, and*

$$\frac{\lfloor \pm_z \mu^\Phi(\gamma^{m_z})/2\rfloor}{m_z} = \frac{\lfloor \pm_w \mu^\Phi(\gamma^{m_w})/2\rfloor}{m_w}.$$

   (ii) *if γ is odd, hyperbolic then either m_z and m_w are both even, or the punctures have the same sign and $m_z = m_w$.*

It is observed in the discussion following Corollary 5.9 in [50] that if holomorphic intersection number of two connected curves vanishes, then the projections of those curves to the 3-manifold are either disjoint or identical. Indeed, assume that the projections of C and D have neither identical nor disjoint image. Then, arguing as in [26], an intersection point between the projections can be seen as an intersection between one curve C and an $\mathbb{R}$-shift of the other $c \cdot D$. Thus $C * (c \cdot D) > 0$ by Theorem 3.8. But homotopy invariance of the $*$-product from Theorem 3.7 then tells us the $C * D = C * (c \cdot D) > 0$, and we can conclude that $C * D = 0$ if and only if the projections of C and D to M have either identical or disjoint image. While the converse of this statement is not true, a fairly complete set of necessary and sufficient conditions for the projections of two curves to the 3-manifold to not intersect is given in [50, Theorem 2.4/5.12]. We recall that theorem here with appropriate adjustments to the notation and conventions.

Theorem 3.10. *Let $[\Sigma, j, \Gamma, \tilde u = (a, u)]$ and $[\Sigma', j', \Gamma', \tilde v = (b, v)] \in \mathcal{M}(\lambda, J)$ be pseudoholomorphic curves, and assume that no component of $\tilde u$ or $\tilde v$ lies in a trivial cylinder, and that the projected curves u and v do not have identical image on any component of their domains. Then the following are equivalent:*

(1) *The projected curves u and v do not intersect.*
(2) *All of the following hold:*
   (a) *The map u does not intersect any of the positive asymptotic limits of v.*
   (b) *The map v does not intersect any of the negative asymptotic limits of u.*
   (c) *If γ is a periodic orbit so that at $z \in \Gamma$, $\tilde u$ is asymptotic to γ^{m_z} and at $w \in \Gamma'$, $\tilde v$ is asymptotic to γ^{m_w}, then:*

(i) *If z and w are either both positive punctures or both negative punctures then*

$$\frac{\text{wind}_\infty(\tilde{u};z)}{m_z} \geq \frac{\text{wind}_\infty(\tilde{v};w)}{m_w}.$$

(ii) *If z is a negative puncture and w is a positive puncture then*

$$\frac{\text{wind}^\Phi_\infty(\tilde{u};z)}{m_z} = \frac{-\lfloor -\mu^\Phi(\gamma^{m_z})/2\rfloor}{m_z} = \frac{\lfloor \mu^\Phi(\gamma^{m_w})/2\rfloor}{m_w} = \frac{\text{wind}^\Phi_\infty(\tilde{v};w)}{m_w}$$

(this is only possible if γ^{m_z} and γ^{m_w} are both even orbits).

(3) *All of the following hold:*
 (a) *The map u does not intersect any of the asymptotic limits of v.*
 (b) *The map v does not intersect any of the asymptotic limits of u.*
 (c) *If γ is a periodic orbit so that at $z \in \Gamma$, $\tilde{u}$ is asymptotic to γ^{m_z} and at $w \in \Gamma'$, $\tilde{v}$ is asymptotic to γ^{m_w}, then*

$$\frac{\text{wind}_\infty(\tilde{u};z)}{m_z} = \frac{\text{wind}_\infty(\tilde{v};w)}{m_w}.$$

The following corollary will be of use in the proof of our main theorem.

Corollary 3.11. *Let $C = [\Sigma, j, \Gamma, \tilde{u} = (a, u)]$ and $D = [\Sigma', j', \Gamma', \tilde{v} = (b, v)] \in \mathcal{M}(\lambda, J)$ be connected pseudoholomorphic curves. Assume that C or D are not trivial cylinders, that C and D have distinct projections to M, and that at every puncture of C and D the winding bound from* (3.5) *is achieved. Then $C * D = 0$ if and only if the projections of C and D to M are disjoint.*

Proof. In the event that the winding bound from (3.5) is achieved at each puncture, i.e. that

$$\pm_z \text{wind}^\Phi_\infty(\tilde{u}; z) = \lfloor \pm_z \mu^\Phi(\gamma^{m_z})/2\rfloor$$

for all $z \in \Gamma$ and

$$\pm_w \text{wind}^\Phi_\infty(\tilde{v}; w) = \lfloor \pm_w \mu^\Phi(\gamma^{m_w})/2\rfloor$$

for all $w \in \Gamma'$, then condition (2) in Theorem 3.10 and condition (2) in Theorem 3.9 reduce to the same thing. Thus the two theorems together imply that $C * D = 0$ if and only if the projection of C and D to M are disjoint. □

In addition to Theorem 3.8, a second fact which motivates the definition of the holomorphic intersection number is that, like the homological self-intersection number of a closed curve, the holomorphic self-intersection number identifies those relative homotopy classes of simple curves (i.e. those that don't factor through a branched cover) which must be embedded [50, Theorem 2.3]. This result can be combined with Theorem 3.10 and some results and techniques from [26] to state a fairly exhaustive set of necessary and sufficient conditions of the vanishing of the holomorphic self-intersection number of a curve. We will summarize the information we need from this result below after recalling the definitions of some relevant invariants associated to a punctured pseudoholomorphic curve.

Let $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ be a pseudoholomorphic curve, and assume that at $z \in \Gamma$ the map u is asymptotic to a cover of the periodic orbit γ_z. Let Φ denote a trivialization of ξ in a neighborhood of each periodic orbit γ appearing as an asymptotic limit of C, and note that Φ induces a trivialization of $u^*\xi$ in a neighborhood of each puncture. We then define the *total Conley–Zehnder index* of the curve C to be

$$\mu(C) = 2c_1^\Phi(u^*\xi) + \sum_{z\in\Gamma^+} \mu^\Phi(u; z) - \sum_{z\in\Gamma^-} \mu^\Phi(u; z) \tag{3.11}$$

where $c_1^\Phi(u^*\xi)$ is the relative first Chern number, defined to be algebraic count of zeroes of a section of $u^*\xi$ which is nonzero and constant in the trivialization Φ in a neighborhood of each puncture (see [37, Section 2.2] or [50, Section 4.2.1] for more details on the properties of the relative first chern number). As a result of the respective change-of-trivialization formulas for the Conley–Zehnder index and $c_1^\Phi(u^*\xi)$, it follows that the total Conley–Zehnder index of a curve is independent of any choice of trivialization. We then the define the *index* $\mathrm{ind}(C)$ of C by

$$\mathrm{ind}(C) = \mu(C) - \chi(\Sigma) + \#\Gamma \tag{3.12}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and $\#\Gamma$ is the number of puncture of the curve. This index represents the Fredholm index of the operator describing the local deformations of the curve C (the relevant facts from [30] are reviewed in Section 3.4 below).

We now have the following theorem, summarizing relevant information from [50, Corollary 5.17] and the discussion thereafter.

Theorem 3.12. *Let $C = [\Sigma, j, \Gamma, \tilde{u} = (a, u)] \in \mathcal{M}(\lambda, J)$ be a simple, connected pseudoholomorphic curve, and assume that $C * C = 0$ and that C does not lie in a trivial cylinder. Then:*

1. *The map $\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times M$ is an embedding.*
2. *The map $u : \Sigma \setminus \Gamma \to M$ is embedding, everywhere transverse to Reeb flow, which is disjoint from all the asymptotic limits of u.*
3. *For each $z \in \Gamma$, the bound from (3.5) is achieved, i.e.*
$$\pm_z \operatorname{wind}_\infty^\Phi(\tilde{u}; z) = \lfloor \pm_z \mu^\Phi(\tilde{u}; z)/2 \rfloor$$
where $\pm_z$ denotes the sign of the puncture z.
4. *The index $\mathrm{ind}(C)$ satisfies*
$$\mathrm{ind}(C) - \chi(\Sigma) + \#\Gamma_{even} = 0$$
where $\chi(\Sigma)$ is the Euler characteristic of the surface Σ and $\#\Gamma_{even}$ is the number of punctures of C asymptotic to even periodic orbits.

We recall from the discussion following Corollary 5.17 in [50] that for a connected curve $C = [\Sigma, j, \Gamma, a, u]$ satisfying the hypotheses of the previous result, if $C * C = 0$ then the projection of the curve to M is an embedding. Indeed, the result shows that u must be an immersion which doesn't intersect any of its asymptotic limits. Moreover, since for the $\mathbb{R}$-translates $c \cdot C = [\Sigma, j, \Gamma, a + c, u]$, we have

$$0 \le \mathrm{int}(C, c \cdot C) \le C * (c \cdot C) = C * C = 0,$$

it follows from positivity of intersections that $\tilde{u}$ doesn't intersect any of its $\mathbb{R}$-translates, and hence that the projection u is injective. As observed in [26], the asymptotic behavior of the curve then allows us to conclude that u is an embedding. As with the discussion of intersections of curves with distinct projections to the three-manifold, the converse is not true: it is possible for curves to project to embeddings in M but have positive self-intersection number. However, various sets of necessary and sufficient conditions for the projection of a curve of M to be embedded are given in [50, Theorem 2.6/5.20]. We quote that result here, making appropriate adjustments to notation and conventions.

Theorem 3.13. *Let $[\Sigma, j, \Gamma, \tilde{u} = (a, u)] \in \mathcal{M}(\lambda, J)$ be a connected, simple pseudo-holomorphic curve, and assume that $\tilde{u}$ does not have image contained in a trivial cylinder. Then the following are equivalent:*

(1) *The projected map $u : \Sigma \setminus \Gamma \to M$ is an embedding.*
(2) *The algebraic intersection number* $\operatorname{int}(\tilde{u}, \tilde{u}_c)$ *between $\tilde{u}$ and an $\mathbb{R}$-translate $\tilde{u}_c = (a + c, u)$ is zero for all $c \in \mathbb{R} \setminus \{0\}$.*
(3) *All of the following hold:*
  (a) *u does not intersect any of its asymptotic limits.*
  (b) *If γ is a periodic orbit so that u is asymptotic at $z \in \Gamma$ to γ^{m_z} and u is asymptotic at $w \in \Gamma$ to γ^{m_w}, then*
$$\frac{\operatorname{wind}_\infty(\tilde{u}; z)}{m_z} = \frac{\operatorname{wind}_\infty(\tilde{u}; w)}{m_w}.$$
(4) *All of the following hold:*
  (a) *The map $\tilde{u}$ is an embedding.*
  (b) *The projected map u is an immersion which is everywhere transverse to X_λ*
  (c) *For each $z \in \Gamma$, we have*
$$\gcd(m_z, \operatorname{wind}_\infty(\tilde{u}; z)) = 1.$$
  (d) *If γ is a simple periodic orbit so that u is asymptotic at z to γ^{m_z}, u is asymptotic at $w \neq z$ to γ^{m_w}, and the punctures have the same signs, then the relative asymptotic intersection number (see Lemma 3.19 and discussion following in* [50] *for definition) of the ends $[\tilde{u}; z]$ and $[\tilde{u}; w]$ satisfies*
$$i^\Phi_\infty([\tilde{u}; z], [\tilde{u}; w]) = -m_z m_w \max\left\{\frac{\pm_z \operatorname{wind}^\Phi_\infty(\tilde{u}; z)}{m_z}, \frac{\pm_w \operatorname{wind}^\Phi_\infty(\tilde{u}; w)}{m_w}\right\}.$$

The following corollary will be of use in the proof of our main theorem.

Corollary 3.14. *Let $C = [\Sigma, j, \Gamma, \tilde{u} = (a, u)] \in \mathcal{M}(\lambda, J)$ be a connected pseudo-holomorphic curve. Assume C is not a trivial cylinder and that at every puncture of C the winding bound from* (3.5) *is achieved. Then $C * C = 0$ if and only if the map $u : \Sigma \setminus \Gamma \to M$ is an embedding.*

Proof. In the event that the winding bound from (3.5) is achieved at each puncture, i.e. that
$$\pm_z \operatorname{wind}^\Phi_\infty(\tilde{u}; z) = \lfloor \pm_z \mu^\Phi(\gamma^{m_z})/2 \rfloor$$
for all $z \in \Gamma$ then the special case of condition (2) in Theorem 3.9 in which $\tilde{u} = \tilde{v}$ and condition (3) in Theorem 3.13 reduce to the same thing. Thus the two theorems together imply that $C * C = 0$ if and only if the map $u : \Sigma \setminus \Gamma \to M$ is an embedding. □

It is shown in [50] that when two curves have ends approaching coverings of the same hyperbolic orbit, there is a "direction-of-approach" condition that will guarantee the two curves have positive holomorphic intersection number. We review the relevant definitions and results here.

We will first define what it means for two pseudoholomorphic half-cylinders to approach an orbit in the same (or opposite) direction. The definition we use here will apply to any nondegenerate periodic orbit and, when the orbit is even, will be stricter than the definition used in [50]. In exchange for using a slightly stricter definition we will be able to make a slightly stronger conclusion via essentially the same argument used in [50].

Let $\tilde{u}$, $\tilde{v} : [R, \infty) \times S^1 \to \mathbb{R} \times M$ be positive pseudoholomorphic half-cylinders asymptotic to the same nondegenerate periodic orbit γ, and assume that the the asymptotic formulas for asymptotic representatives U, V of $\tilde{u}$ and $\tilde{v}$ respectively are given by

$$\begin{aligned} U(s,t) &= e^{\lambda_u s}[e_u(t) + r_u(s,t)] \\ V(s,t) &= e^{\lambda_v s}[e_v(t) + r_v(s,t)]. \end{aligned}$$

We say that *the half cylinders $\tilde{u}$ and $\tilde{v}$ approach γ in the same direction* if the eigenvectors e_u and e_v are positive scalar multiples of each other, and similarly we say that we say that *the half cylinders $\tilde{u}$ and $\tilde{v}$ approach γ in the opposite direction* if the eigenvectors e_u and e_v are negative scalar multiples of each other. We note that in either case we must have $\lambda_u = \lambda_v$. For a pair of negative half-cylinders asymptotic to the same periodic orbit, the definitions are exactly analogous: the cylinders are said to approach in the same direction if the eigenvectors controlling the approach are positive multiples of each other, and are said to approach in the opposite direction if they are negative scalar multiples of each other.

The notion of approaching an orbit in the same (or opposite) direction is of particular use at an even orbit due to the following lemma, which shows that pairs of pseudoholomorphic ends approaching an even orbit with the same sign and extremal winding (i.e. the bound in (3.5) is achieved) always either approach in the same or opposite direction.

Lemma 3.15. *Let γ be an even periodic orbit. Let $\tilde{u}$, $\tilde{v} : [R, \infty) \times S^1 \to \mathbb{R} \times M$ be either both positive or both negative pseudoholomorphic half-cylinders asymptotic to γ, and assume that*

$$\mathrm{wind}_\infty^\Phi(\tilde{u}) = \mathrm{wind}_\infty^\Phi(\tilde{v}) = \mu^\Phi(\gamma)/2.$$

for any symplectic trivialization Φ of $\gamma^*\xi$. Then $\tilde{u}$ and $\tilde{v}$ either approach γ in the same direction, or opposite direction.

Proof. Let $\lambda_- < 0$ denote the largest negative eigenvalue of $\mathbf{A}_{\gamma,J}$. Then, according to Theorem 2.3 and the fact that the parity of an orbit is equal to the parity of its Conley–Zehnder index, in any symplectic trivialization Φ of $\gamma^*\xi$ the winding of an eigenvector of $\mathbf{A}_{\gamma,J}$ with eigenvalue λ_- is $\lfloor \mu^\Phi(\gamma)/2 \rfloor = \mu^\Phi(\gamma)/2$. It further follows from the same theorem and Lemma 2.2 that eigenvectors of $\mathbf{A}_{\gamma,J}$ having smallest possible positive eigenvalue also have winding equal to $\mu^\Phi(\gamma)/2$. Since Lemma 2.2 tells us that the span of the collection of eigenvectors having winding equal to $\mu^\Phi(\gamma)/2$ is two dimensional, we can conclude that

$$\dim \ker(\mathbf{A}_{\gamma,J} - \lambda_\pm) = 1$$

and that all eigenvectors e of $\mathbf{A}_{\gamma,J}$ with negative (resp. positive) eigenvalue satisfying

$$\mathrm{wind}\, \Phi^{-1} e = \mu^\Phi(\gamma)/2$$

have eigenvalue λ_- (resp. λ_+).

Now, if $\tilde{u}$ and $\tilde{v}$ are positive (resp. negative) half-cylinders and have winding

$$\mathrm{wind}_\infty^\Phi(\tilde{u}) = \mathrm{wind}_\infty^\Phi(\tilde{v}) = \mu^\Phi(\gamma)/2.$$

then the eigenvector controlling the approach of each cylinder must have eigenvalue λ_- (resp. λ_+). Since we've argued above that the eigenspaces $\ker(\mathbf{A}_{\gamma,J} - \lambda_\pm)$ are 1-dimensional, we conclude that the eigenvectors associated to each cylinder are scalar multiples of each other which is equivalent to saying that $\tilde{u}$ and $\tilde{v}$ approach γ in either the same or opposite direction. □

We now state the main theorem concerning the intersection properties of pseudoholomorphic half-cylinders that approach an even orbit with extremal winding in the same direction.

Theorem 3.16. *(c.f. [50, Theorem 5.15]) Let $\tilde{u} = (a, u)$, $\tilde{v} = (b, v) : [R, \infty) \times S^1 \to \mathbb{R} \times M$ be either both positive or both negative pseudoholomorphic half-cylinders asymptotic to an even periodic orbit γ. Assume that $\tilde{u}$ and $\tilde{v}$ have extremal winding, i.e.*

$$\text{wind}^\Phi_\infty(\tilde{u}) = \text{wind}^\Phi_\infty(\tilde{v}) = \mu^\Phi(\gamma)/2, \tag{3.13}$$

and that $\tilde{u}$ and $\tilde{v}$ approach γ in the same direction. Then the projections u, v of the maps $\tilde{u}$, $\tilde{v}$ to the 3-manifold M intersect.

Proof 1. The proof is a combination of the proofs of Theorem 5.14, Theorem 5.15, Lemma 5.10 in [50], and a local version of Theorem 2.2 in [50]/Theorem 3.8 above. If the images of $\tilde{u}$ and $\tilde{v}$ differ by the $\mathbb{R}$-action on some neighborhood of infinity, then the projections to M will be identical on the same neighborhood of infinity so there is nothing more to prove. We thus, without loss of generality, assume that the images of $\tilde{u}$ and $\tilde{v}$ do not differ by the $\mathbb{R}$-action on any neighborhood of infinity. Given this assumption Theorems 5.14-5.15 in [50] shows that the there is a constant $c_0 \in \mathbb{R}$ so that the asymptotic intersection number $\delta_\infty(\tilde{u}, \tilde{v}_{c_0})$ is positive, where $\tilde{v}_c = (b+c, v)$ denotes the half-cylinder obtained by shifting the $\mathbb{R}$-coordinate of $\tilde{v}$ by c. On the other hand, the argument in Lemma 5.10 in [50] shows that $\delta_\infty(\tilde{u}, \tilde{v}_c) = 0$ for all $c \neq c_0$ nearby to c_0 (in fact, the proof there reveals, for all $c \neq c_0$). Since intersections are isolated, we can, after perhaps restricting the domains, define an algebraic intersection number $\text{int}(\tilde{u}, \tilde{v}_{c_0})$, relative intersection number $i^\Phi(\tilde{u}, \tilde{v}_{c_0})$, and holomorphic intersection number $[\tilde{u}] * [\tilde{v}_{c_0}] = i^\Phi(\tilde{u}, \tilde{v}_{c_0}) + \mu^\Phi(\gamma)/2$. Moreover, $i^\Phi(\tilde{u}, \tilde{v}_{c_0})$ and $[\tilde{u}] * [\tilde{v}_{c_0}]$ will be invariant under small perturbations of $\tilde{v}_{c_0}$, and the analogy of formula (3.9)

$$[\tilde{u}] * [\tilde{v}_{c_0}] = \text{int}(\tilde{u}, \tilde{v}_{c_0}) + \delta_\infty(\tilde{u}, \tilde{v}_{c_0})$$

holds as well for the localized versions of the intersection product. As c_0 changes to a sufficiently nearby $c \neq c_0$ in the equation, the local holomorphic intersection product $[\tilde{u}] * [\tilde{v}_c]$ remains unchanged, while we've just argued that the asymptotic intersection number $\delta_\infty(\tilde{u}, \tilde{v}_c)$ changes from a positive number to zero. Thus the algebraic intersection number $\text{int}(\tilde{u}, \tilde{v}_c)$ must increase, and since $\text{int}(\tilde{u}, \tilde{v}_c) \geq 0$ for all c, we conclude that $\text{int}(\tilde{u}, \tilde{v}_c) > 0$ for c very near to c_0. Since $\tilde{u} = (a, u)$ and $\tilde{v}_c = (b + c, v)$ intersecting implies u and v intersect, this completes the proof. □

For the convenience of the reader we provide a self-contained presentation of the above argument below.

Proof 2. For simplicity we will carry out the proof assuming that both cylinders are positive. The proof in the case that both are negative is completely analogous. As in the previous proof, we continue to assume that the images of $\tilde{u}$ and $\tilde{v}$ do not differ by the $\mathbb{R}$-action on any neighborhood of infinity.

Proceeding now with the above assumptions, we let U, $V : [R', \infty) \times S^1 \to \gamma^*\xi$ be asymptotic representatives of $\tilde{u}$ and $\tilde{v}$ respectively, that is U and V satisfy

$$\tilde{u} \circ \phi(s,t) = \Big(Ts, \exp_{\gamma(t)} U(s,t)\Big)$$
$$\tilde{v} \circ \psi(s,t) = \Big(Ts, \exp_{\gamma(t)} V(s,t)\Big)$$

for some proper embeddings ϕ, $\psi : [R', \infty) \times S^1 \to [R, \infty) \times S^1$. We further observe that if $\tilde{v}_c$ is the map $\tilde{v}_c(z) = (b(z) + c, v(z))$ obtained by shifting the $\mathbb{R}$-component of $\tilde{v}$ by c, then

$$\tilde{v}_c \circ \psi_c(s,t) = \Big(Ts, \exp_{\gamma(t)} V_c(s,t)\Big),$$

where $\psi_c(s,t) := \psi(s - c/T, t)$ and $V_c(s,t) := V(s - c/T, t)$.

According to our winding assumption (3.13) and the fact that γ is an even orbit, the asymptotic formulas for U and V must be of the form

$$U(s,t) = e^{\lambda s}[e_u(t) + r_1(s,t)] \tag{3.14}$$

$$V(s,t) = e^{\lambda s}[e_v(t) + r_2(s,t)] \tag{3.15}$$

with λ the largest negative eigenvalue of $\mathbf{A}_{\gamma,J}$, and $r_i(s,t) \to 0$ exponentially in s. Hence the asymptotic formula for $V_c = V(\cdot - c/T, \cdot)$ is of the form

$$\begin{aligned} V_c(s,t) &= e^{\lambda s} e^{-\lambda c/T}[e_v(t) + r_2(s - c/T, t)] \\ &= e^{\lambda s}[K_c e_v(t) + r_c(s,t)] \end{aligned} \tag{3.16}$$

where $K_c = e^{-\lambda c/T} > 0$ and $r_c = K_c r_2(\cdot - c/T, \cdot)$ decays exponentially in s.

We seek to understand how the intersection behavior of two cylinders $\tilde{u}$ and $\tilde{v}_c$ changes as c changes. We first observe that by the assumption that $\tilde{u}$ and $\tilde{v}$ approach γ in the same direction, there is a c_0 so that $e_u = K_{c_0} e_v$. If $\tilde{u}$ and $\tilde{v}_{c_0}$ intersect, the projections u and v intersect, so there is nothing more to prove. We assume then that $\tilde{u}$ and $\tilde{v}_{c_0}$ don't intersect and consider the difference $U(s,t) - V_{c_0}(s,t)$ for $(s,t) \in [R'', +\infty)$ for some large R''. According to Theorem 3.1 we have that

$$U(s,t) - V_{c_0}(s,t) = e^{\lambda_1 s}[e_1(t) + r(s,t)] \tag{3.17}$$

with $r(s,t) \to 0$ exponentially. Meanwhile, direct computation using formulas (3.14) and (3.16) shows that

$$U(s,t) - V_{c_0}(s,t) = e^{\lambda s}[r_1(s,t) - r_{c_0}(s,t)].$$

Since $r_1 - r_{c_0}$ converges exponentially to 0, comparing above two equations shows that $\lambda_1 < \lambda$. Since the orbit is even, Lemma 2.2 with Theorem 2.3 tells us that we must have that $\operatorname{wind} \Phi^{-1} e_1 < \operatorname{wind} \Phi^{-1} e_u = \mu^\Phi(\gamma)/2$ in any trivialization Φ of $\gamma^*\xi$. We thus conclude from this observation and (3.17) that

$$\operatorname{wind} \Phi^{-1}[U(s,\cdot) - V_{c_0}(s,\cdot)] = \operatorname{wind} \Phi^{-1} e_1 < \mu^\Phi(\gamma)/2 \tag{3.18}$$

for all sufficiently large s. Moreover, since we assume that $\tilde{u}$ and $\tilde{v}_{c_0}$ don't intersect, we conclude that (3.18) holds for all $s \in [R'', \infty)$.

Meanwhile, we can choose $c \neq c_0$ sufficiently close to c_0 so that $U(s,t) - V_c(s,t)$ is defined for $(s,t) \in [R'' + 1, \infty) \times S^1$ and so that for all $s \in [R'' + 1, R'' + 2]$

$$\operatorname{wind} \Phi^{-1}[U(s,\cdot) - V_c(s,\cdot)] = \operatorname{wind} \Phi^{-1}[U(s,\cdot) - V_{c_0}(s,\cdot)] < \mu^\Phi(\gamma)/2.$$

On the other hand, computation using (3.14), (3.16) $K_c = e^{-\lambda c/T}$, and $e_u = K_{c_0} e_v$ gives us

$$\begin{aligned} e^{-\lambda s}[U(s,t) - V_c(s,t)] &= e_u(t) - K_c e_v(t) + r_4(s,t) \\ &= e_u(t) - e^{-\lambda(c-c_0)/T} K_{c_0} e_v(t) + r_4(s,t) \\ &= e_u(t) - e^{-\lambda(c-c_0)/T} e_u(t) + r_4(s,t) \\ &= \sigma_{c_0} e_u(t) + r_4(s,t) \end{aligned}$$

where $r_4 := r_1 - r_c$ decays exponentially in s and $\sigma_{c_0} := 1 - e^{-\lambda(c-c_0)/T} \neq 0$. We conclude that for all s sufficiently large,

$$\operatorname{wind} \Phi^{-1}[U(s,t) - V_c(s,t)] = \operatorname{wind} \Phi^{-1} e_u = \mu^\Phi(\gamma)/2.$$

Since the winding of $\Phi^{-1}[U(s,t) - V_c(s,t)]$ changes as s changes, $U(s,t) - V_c(s,t)$ must have at least one zero. This implies that $\tilde{u}$ and $\tilde{v}_c$ must intersect at least once, which in turn implies that the projections u and v intersect. This completes the proof. □

Finally, we close this section with a result concerning intersections between components of a holomorphic building resulting as the limit of a sequences of holomorphic curves having intersection number equal to zero.

Lemma 3.17. *Let C_k, $D_k \in \mathcal{M}(\lambda, J)$ be sequences of holomorphic curves satisfying $C_k \neq D_k$ and $C_k * D_k = 0$ for all k, and assume that C_k and D_k converge (in the sense of* [6]*) respectively to a holomorphic buildings C_∞, D_∞. Then for every component C' of C_∞ and D' of D_∞, the projections of C' and D' to M are either disjoint or identical.*

Proof. Assume that the projections of some components C' and D' of the limit buildings to M are neither disjoint nor identical. Then for some value of $d' \in \mathbb{R}$, C' and the $\mathbb{R}$-translate $d' \cdot D'$ have at least one isolated intersection. But according to the definition of SFT-convergence from [6] there exist sequences of constants c_k, $d_k \in \mathbb{R}$ so that $c_k \cdot C_k$ and $d_k \cdot D_k$ converge respectively in C^∞_{loc} to C' and D' (see Proposition 3.4), and thus $(d_k d') \cdot D_k$ converges in C^∞_{loc} to $d' \cdot D$. But, since C' and D' have at least one isolated intersection, we can conclude from the C^∞_{loc} convergence that $c_k \cdot C_k$ and $(d_k d') \cdot D_k$ have at least one isolated intersection for sufficiently large values of k. Theorem 3.8 then allows us to conclude that $(c_k \cdot C_k) * (d_k d' \cdot D_k) > 0$ for sufficiently large k. But by homotopy invariance of the $*$-product, we have that

$$(c_k \cdot C_k) * (d_k d' \cdot D_k) = C_k * D_k = 0.$$

This contradiction completes the proof. □

3.4. **Fredholm theory and transversality.** We will briefly review the Fredholm theory for embedded (or immersed) pseudoholomorphic curves from [30].

First, given (M, λ) and compatible $J \in \mathcal{J}(M, \lambda)$, we define a metric g_J on M by

$$g_J(v, w) = \lambda(v)\lambda(w) + d\lambda\left(\pi_\xi(v), J\pi_\xi(w)\right)$$

where $\pi_\xi : TM \approx \mathbb{R}X_\lambda \oplus \xi \to \xi$ is the projection to ξ along X_λ. That g_J defined this way is a metric on TM follows from the definition of compatibility of J. We extend this to a metric $\tilde{g}_J$ on $\mathbb{R} \times M$ by defining

$$\tilde{g}_J\left((h, v), (k, w)\right) = h \cdot k + g_J(v, w)$$

where (h, v), $(k, w) \in \mathbb{R} \oplus TM \approx T(\mathbb{R} \times M)$. Compatibility of J with $(\xi, d\lambda)$ and the definition of the extension of J to an almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ implies that $\tilde{g}_J$ is a hermitian metric on the $(\mathbb{R} \times M, \tilde{J})$, that is $\tilde{J}$ is a $\tilde{g}_J$-orthogonal endomorphism of $T(\mathbb{R} \times M)$.

Now, let $C = [\Sigma, j, \Gamma, \tilde{u} = (a, u)] \in \mathcal{M}(\lambda, J)$ be an embedded $\tilde{J}$-holomorphic curve, and choose a model parametrization $(\Sigma, j, \Gamma, \tilde{u} = (a, u))$. Then C has a well-defined normal bundle N_C which can be realized as a subbundle of $T(\mathbb{R} \times M)|_C = \tilde{u}^* T(\mathbb{R} \times M)$ by letting

$$(N_C)_{\tilde{u}(z)} = d\tilde{u}(z)(T_z\Sigma)^{\perp}$$

with $\perp$ denoting the $\tilde{g}_J$ orthogonal complement within $T_{\tilde{u}(z)}(\mathbb{R} \times M)$. We consider curves which are parametrized by mapping sections of the normal bundle N_C of the curve C to $\mathbb{R} \times M$ via the exponential map $\widetilde{\exp}$ of the metric $\tilde{g}_J$, that is, those curves $C' = [\Sigma', j', \Gamma', \tilde{v} = (b, v)] \in \mathcal{M}(\lambda, J)$ for which there exists a smooth map $\psi : \Sigma \setminus \Gamma \to \Sigma' \setminus \Gamma'$ and a smooth section V of N_C so that

$$\tilde{v}(\psi(z)) = \widetilde{\exp}_{\tilde{u}(z)} V(z).$$

In order to to do this, we first recall that the asymptotic behavior of the curve C implies that $\widetilde{\exp}$ is an immersion on some ε-neighborhood N_C^{ε} of the zero section of N_C with respect to the metric on N_C induced from $\tilde{g}_J$ (see e.g. Corollary 2.7 in [52]). We can thus define an almost complex structure $\bar{J}$ on this ε-neighborhood of the zero section of N_C by pulling back $\tilde{J}$ via the exponential map. Give a connection ∇ on N_C, we get a splitting

$$T_{(z,V)} N_C \approx T_z(\Sigma \setminus \Gamma) \oplus (N_C)_z$$

of the tangent space of N_C into horizontal and vertical distributions. This splitting is canonical along the zero section. With respect to the splitting induced by a given connection we can write

$$\bar{J}(z, V) = \begin{bmatrix} i(z, V) & \tilde{\Delta}(z, V) \\ \Delta(z, V) & J(z, V) \end{bmatrix} \tag{3.19}$$

with $i \in \mathrm{End}(T(\Sigma \setminus \Gamma))$, $\tilde{\Delta} \in \mathrm{Hom}(N_C, T(\Sigma \setminus \Gamma))$, $\Delta \in \mathrm{Hom}(T(\Sigma \setminus \Gamma), N_C)$, and $J \in \mathrm{End}(N_C)$. Moreover, along the zero section of N_C we have

$$\bar{J}(z, 0) = \begin{bmatrix} i(z, 0) & \tilde{\Delta}(z, 0) \\ \Delta(z, 0) & J(z, 0) \end{bmatrix} = \begin{bmatrix} j(z) & 0 \\ 0 & J_N(z) \end{bmatrix}$$

with j the complex structure on $T\Sigma$ and J_N the complex structure on N_C induced from $\tilde{J}$. Note that squaring (3.19) and using $\bar{J}^2 = -I$ we get that

$$\Delta \circ i = -J \circ \Delta.$$

Letting $\Delta' : N_C^{\varepsilon} \to \mathrm{Hom}(N_C, \mathrm{Hom}(T(\Sigma \setminus \Gamma), N_C))$ denote the map obtained from differentiating Δ in the fiber direction, we can differentiate the above equation in the fiber direction and use that Δ vanishes along the zero section to conclude that

$$[\Delta'(0)V] \circ i(0) = -J(0) \circ [\Delta'(0)V]$$

or equivalently

$$[\Delta'(0)V] \circ j = -J_N \circ [\Delta'(0)V]$$

for any section V of N_C. Thus, for any section V of N_C, $[\Delta'(0)V] \circ j$ is a j-J_N anti-linear map from $T(\Sigma \setminus \Gamma)$ to N_C.

Definition 3.18. The linearized normal $\bar{\partial}$-operator $\bar{\partial}_N^{\nabla}(C)$ at an embedded curve $C \in \mathcal{M}(\lambda, J)$ relative to the connection ∇ on N_C is the operator $\bar{\partial}_N^{\nabla}(C) : C^\infty(N_C) \to C^\infty(\mathrm{Hom}^{0,1}(T(\Sigma \setminus \Gamma), N_C))$ defined by

$$\bar{\partial}_N^{\nabla}(C)V = \nabla V + J_N \nabla_{j \cdot} V + [\Delta'(0)V] \circ j.$$

The following theorem summarizes results about the linearized normal $\bar{\partial}$-operator proved in [30] using results from [48].

Theorem 3.19. *There exists a measure and metric on* $\Sigma \setminus \Gamma$ *and connection on* N_C *so that the extensions of* $\bar{\partial}_N^{\nabla}(C)$ *to maps*

$$\bar{\partial}_N^{\nabla}(C) : W^{k,p}(N_C) \to W^{k-1,p}(\mathrm{Hom}^{0,1}(T(\Sigma \setminus \Gamma), N_C))$$

and

$$\bar{\partial}_N^{\nabla}(C) : C_0^{k,\alpha}(N_C) \to C_0^{k-1,\alpha}(\mathrm{Hom}^{0,1}(T(\Sigma \setminus \Gamma), N_C))$$

are Fredholm. Moreover each of the above operators has the same kernel, and the Fredholm index $\mathrm{ind}(\bar{\partial}_N^{\nabla}(C))$ *of each of the above operators is given by*

$$\mathrm{ind}(\bar{\partial}_N^{\nabla}(C)) = \mathrm{ind}(C)$$

with $\mathrm{ind}(C)$ *as defined in* (3.12).

In [30], it is shown that the moduli space of pseudoholomorphic curves near a given embedded $C \in \mathcal{M}(\lambda, J)$ can be given as the zero set of a smooth, nonlinear section $H : \mathcal{B} \to \mathcal{E}$ of a Banach space bundle $\mathcal{E}$ defined over an open neighborhood $\mathcal{B}$ of 0 in the Banach algebra $C_0^{k,\alpha}(N_C)$ of $C_0^{k,\alpha}$ sections of the normal bundle N_C of C. Moreover, if $\mathbf{0} \in \mathcal{B}$ denotes the zero section of N_C, there is a natural isomorphism

$$\boldsymbol{\alpha} : C_0^{k-1,\alpha}(\mathrm{Hom}^{0,1}(T(\Sigma \setminus \Gamma), N_C)) \to \mathcal{E}_{\mathbf{0}}$$

so that the linearization $H'(\mathbf{0})$ of the section H at the zero section $\mathbf{0} \in \mathcal{B}$ satisfies

$$H'(\mathbf{0})V = \boldsymbol{\alpha}(\bar{\partial}_N^{\nabla}(C)V)$$

for any $V \in C_0^{k,\alpha}(N_C)$. Thus, in the case that $\bar{\partial}_N^{\nabla}(C)$ is surjective, the implicit function theorem can be applied to conclude that set of curves near $C \in \mathcal{M}(\lambda, J)$ is a smooth manifold with dimension equal to the index (3.12). This leads to the following theorem, summarized from facts proved in [30].

Theorem 3.20. *Let* $C \in \mathcal{M}(\lambda, J)$ *be an embedded pseudoholomorphic curve with parametrization* $(\Sigma, j, \Gamma, \tilde{u} = (a, u))$ *and assume that*

$$\bar{\partial}_N^{\nabla}(C) : C_0^{k,\alpha}(N_C) \to C_0^{k-1,\alpha}(\mathrm{Hom}^{0,1}(T(\Sigma \setminus \Gamma), N_C))$$

is surjective. Then there exists an open neighborhood $B \subset \ker \bar{\partial}_N^{\nabla}(C)$ *of the zero section of* N_C *and a smooth embedding* $E : B \to C_0^{k,\alpha}(N_C)$ *mapping* 0 *to the zero section satisfying:*

1. *For every* $\tau \in B$, $E_\tau : \Sigma \setminus \Gamma \to N_C$ *is smooth section of the normal bundle to* C.
2. *The derivative* $DE_0 : \ker \bar{\partial}_N^{\nabla}(C) \to C_0^{k,\alpha}(N_C)$ *of* $E : B \to C_0^{k,\alpha}(N_C)$ *at* $0 \in B$ *is the inclusion* $\ker \bar{\partial}_N^{\nabla}(C) \hookrightarrow C_0^{k,\alpha}(N_C)$*, i.e.* $DE_0(v) = v$ *for any* $v \in \ker \bar{\partial}_N^{\nabla}(C) \subset C_0^{k,\alpha}(N_C)$.

(3) *For each* $\tau \in B$*, there exists a distinct pseudoholomorphic curve* $C_\tau \in \mathcal{M}(\lambda, J)$ *with parametrization* $(\Sigma_\tau, j_\tau, \Gamma_\tau, \tilde{v}_\tau = (b_\tau, v_\tau))$ *and a diffeomorphism*[8] $\psi_\tau : \Sigma \setminus \Gamma \to \Sigma_\tau \setminus \Gamma_\tau$ *so that*[9]

$$\tilde{v}_\tau \circ \psi_\tau = \widetilde{\exp}(E_\tau).$$

(4) *The map* $F : B \times (\Sigma \setminus \Gamma) \to \mathbb{R} \times M$ *defined by*

$$F(\tau, z) = \widetilde{\exp}_{\tilde{u}(z)} E_\tau(z)$$

is smooth.

We remark that the last claim above is only proved in [30] as part of a theorem (Theorem 5.7) where it's assumed that the original curve C is a pseudoholomorphic plane satisfying some additional properties. The proof of that portion of theorem however applies to any immersed curve C. The key idea is that the sections E_τ are smooth by elliptic regularity and that the map $\tau \mapsto E_\tau$ determines a smooth map from $B \to C_0^{k,\alpha}(N_C)$ for every positive integer k.

It's proven in [30] that for a generic choice of $J \in \mathcal{J}(M, \lambda)$ the linearized normal $\bar{\partial}$-operator $\bar{\partial}_N^\nabla(C)$ at any immersed curve $C \in \mathcal{M}(\lambda, J)$ is surjective. We will not state the precise result since it is not needed in our proof. What is of interest here is the fact that under certain circumstances, the surjectivity of the linearized normal Cauchy–Riemann operator $\bar{\partial}_N^\nabla(C)$ can be guaranteed provided that certain conditions on the topological invariants of the curve C are met. Such so-called automatic transversality conditions were first described Gromov in [19], with proofs in [32] for compact curves (either without boundary or with totally real boundary conditions), and very general results proven in [57] from which the following theorem can be deduced.

Theorem 3.21. *Let* $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ *be immersed. Then the linearized normal* $\bar{\partial}$*-operator* $\bar{\partial}_N^\nabla(C)$ *at* C *is surjective if*

$$\text{ind}(C) \geq -\chi(\Sigma) + \#\Gamma_{even} + 2 = 2g(\Sigma) + \#\Gamma_{even} \tag{3.20}$$

where $\chi(\Sigma)$ *is the Euler characteristic of the surface* Σ*, and* $\#\Gamma_{even}$ *is the number of punctures of the curve which limit to periodic orbits with even Conley–Zehnder index.*

In the event that $\text{ind}(C)$ is even and positive there is a short proof of a special case of this result (the essential case for our proof is $\text{ind}(C) = 2$, but we include the above result since it's also of interest to know that index-1 curves in a stable foliation are regular). We will recall the proof of this special case below since the proof is easy and uses a fact about the zeros of elements of the kernel of the linearized normal $\bar{\partial}$-operator that we will need later. We state this fact in the following lemma.

[8] We caution the reader that, in general, the continuous extension of this diffeomorphism over the punctures is not smooth. We refer the reader to [30] for more details.

[9] We note that in [30], rather than using the exponential map of the metric, a map from the normal bundle of C to $\mathbb{R} \times M$ is constructed by using a special trivialization in a special coordinate system. However, the essential point for the results of [30] to hold is that one has a map from a neighborhood of 0 in N_C to $\mathbb{R} \times M$ satisfying certain asymptotic conditions. That the exponential map $\widetilde{\exp}$ of the metric $\tilde{g}_J$ has the right properties is easily seen from the asymptotic analysis in [52].

Lemma 3.22. *Let $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ be an embedded pseudoholomorphic curve and let $V \in \ker \bar{\partial}_N^{\nabla}(C)$ be a nontrivial element of the kernel of the linearized normal Cauchy–Riemann operator at C. Then all zeroes of V are isolated and have positive local index. Moreover, if $i(V)$ denotes the total algebraic count of zeroes of V, then*

$$0 \leq i(V) \leq \frac{1}{2}\left(\operatorname{ind}(C) - \chi(C) + \#\Gamma_{even}\right)$$

with $\chi(C)$ the Euler characteristic of the curve, and $\#\Gamma_{even}$ the number of asymptotic limits of the curve with even Conley–Zehnder index.

Proof. The proof is a straightforward generalization of arguments in [26, Proposition 5.6, Theorem 5.8], [30, Theorem 2.11], and [31, Theorem 2.7]. We will highlight the main points. As observed in [30, Theorem 2.11], the fact that the zeroes of a nontrivial element of $\ker \bar{\partial}_N^{\nabla}(C)$ are isolated and have positive local index follows from the similarity principle (see e.g. Appendix A.6 in [33]). As in [30, Theorem 2.11], it can be argued that a nontrivial element V of $\ker \bar{\partial}_N^{\nabla}(C)$ satisfies an asymptotic formula of the same form as that given in [27, 42] or Theorem 3.1 above. Thus choosing a trivialization Φ of the contact structure along the asymptotic limit γ_z of a given puncture $z \in \Gamma$ and extending to a trivialization of the normal bundle N_C near the puncture, the section V has a well-defined asymptotic winding number and the argument of Theorem 3.2 applies to show that

$$\pm_z \operatorname{wind}^{\Phi}(V; z) \leq \lfloor \pm_z \mu^{\Phi}(\gamma_z)/2 \rfloor = \tfrac{1}{2}\left(\pm_z \mu^{\Phi}(\gamma_z) - p(\gamma_z)\right)$$

with $\pm_z$ the sign of the puncture z and $p(z)$ is the parity of the orbit γ_z. A straightforward zero-counting argument gives that

$$i(V) = c_1^{\Phi}(N_C) + \sum_{z \in \Gamma} \pm_z \operatorname{wind}^{\Phi}(V; z).$$

Meanwhile, properties of the relative first Chern number imply that

$$c_1^{\Phi}(N_C) - c_1^{\Phi}(\xi|_C) = -\chi(\Sigma \setminus \Gamma) = -\chi(\Sigma) + \#\Gamma$$

(see [37, Proposition 3.1]). Combining the above with formulas (3.11) and (3.12) leads to

$$i(V) \leq \frac{1}{2}\left(\operatorname{ind}(C) - \chi(C) + \#\Gamma_{even}\right)$$

as claimed. □

We now recall the proof of the special case of Theorem 3.21. The idea is that if the kernel of the linearized operator is too big, then one can construct a section of the kernel with too many zeroes. This same argument is applied in the proofs of [30, Theorem 2.11] and [2, Theorem 2.7].

Theorem 3.23. *Let $C = [S^2, i, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ be a an embedded, pseudoholomorphic (punctured) sphere and assume that all punctures of are odd and that $\operatorname{ind}(C) = 2$. Then the linearized normal $\bar{\partial}$-operator $\bar{\partial}_N^{\nabla}(C)$ is surjective.*

Proof. To show that $\bar{\partial}_N^{\nabla}(C)$ is surjective, it suffices to show that

$$\dim \ker \bar{\partial}_N^{\nabla}(C) = \operatorname{ind}(C) = 2.$$

Suppose to the contrary that $\dim \ker \bar{\partial}_N^{\nabla}(C) > 2$. Then we can find three linearly independent vectors V_1, V_2, $V_3 \in \ker \bar{\partial}_N^{\nabla}(C) \subset C^{\infty}(N_C)$. Choosing a point $z_0 \in S^2 \setminus \Gamma$ and using that the normal bundle N_C has (real) dimension 2, we can find constants c_1, c_2, and c_3 so that $\sum_{i=1}^{3} c_i V_i(z_0) = 0$. Thus, the section $V_{\mathbf{c}}$ of N_C defined by $V_{\mathbf{c}} = \sum_{i=1}^{3} c_i V_i$ is a nonzero element of $\ker \bar{\partial}_N^{\nabla}(C)$ which vanishes at z_0 and therefore, according to Lemma 3.22 satisfies $i(V_{\mathbf{c}}) \geq 1$ since all zeroes have positive local index. But Lemma 3.22 also tells us that

$$i(V_{\mathbf{c}}) \leq \frac{1}{2}\left(\operatorname{ind}(C) - \chi(S^2) + \#\Gamma_{even}\right) = \frac{1}{2}(2 - 2 + 0) = 0.$$

We thus have the contradiction $1 \leq i(V_{\mathbf{c}}) \leq 0$ which completes the proof. □

4. Stable finite energy foliations and moduli spaces of foliating curves

In this section we will develop some general theory for finite energy foliations and collect facts about the moduli spaces of curves which make up finite energy foliations. We start with a definition.

Definition 4.1. Let (M, λ, J) be a three manifold equipped with a nondegenerate contact form and compatible $J \in \mathcal{J}(M, \lambda)$. A *stable finite energy foliation* $\mathcal{F}$ of total energy E_0 for the data (M, λ, J) is a collection of simple curves $C \in \mathcal{M}(\lambda, J)$ satisfying:

- For every point $p \in \mathbb{R} \times M$ there is a unique curve $C \in \mathcal{F}$ passing through p.
- Every $C \in \mathcal{F}$ is either a trivial cylinder or satisfies $\operatorname{ind}(C) \in \{1, 2\}$
- For any C_1, $C_2 \in \mathcal{F}$ with $\operatorname{ind}(C_i) \in \{1, 2\}$, $C_1 * C_2 = 0$.
- $E_0 = \sup_{C \in \mathcal{F}} E(C)$.

We note that this definition is a slightly weaker one than that given in the introduction in that we don't explicitly require here that the curves of $\mathcal{F}$ form a smooth foliation of $\mathbb{R} \times M$. We will see below however, that this condition follows from the above assumptions and, thus, the two definitions are in fact equivalent. We observe that the penultimate condition in our definition of stable finite energy foliation above applies when $C_1 = C_2$. The following theorem collects some facts about the moduli spaces of curves satisfying $C * C = 0$ and $\operatorname{ind}(C) \in \{1, 2\}$ that follow from results reviewed in the preceding sections.

Theorem 4.2. *Let $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{M}(\lambda, J)$ be a simple pseudoholomorphic curve, and assume that $C * C = 0$ and $\operatorname{ind}(C) \in \{1, 2\}$. Then:*

(1) *C is embedded.*
(2) *C is nicely embedded; that is, the projection of C to M is an embedding transverse to the Reeb flow and doesn't intersect any of its asymptotic limits.*
(3) *For each $z \in \Gamma$, the bound from (3.5) is achieved, i.e.*

$$\pm_z \operatorname{wind}_{\infty}^{\Phi}(\tilde{u}; z) = \lfloor \pm_z \mu^{\Phi}(\tilde{u}; z)/2 \rfloor$$

where $\pm_z$ denotes the sign of the puncture z.
(4) *The genus $g(\Sigma)$ of the domain is zero, i.e. (Σ, j) is biholomorphic to the Riemann sphere (S^2, i).*
(5) *The number Γ_{even} of punctures of C asymptotic to even orbits is given by*

$$\#\Gamma_{even} = 2 - \operatorname{ind}(C).$$

(6) *The linearized normal Cauchy–Riemann operator $\bar{\partial}_N^{\nabla}(C)$ is surjective.*

(7) *With $n = \mathrm{ind}(C)$, there exists an $\varepsilon > 0$ and an injective immersion*

$$\tilde{F}_C : B^n_\varepsilon(0) \times \Sigma \setminus \Gamma \to \mathbb{R} \times M$$

so that the map $z \mapsto \tilde{F}_C(0, z)$ is a parametrization of the curve C, and so that for every $\tau \in B^n_\varepsilon(0)$, there is a pseudoholomorphic curve

$$C_\tau = [\Sigma_\tau, j_\tau, \Gamma_\tau, \tilde{u}_\tau = (a_\tau, u_\tau)] \in \mathcal{M}(\lambda, J)$$

and a diffeomorphism[10]

$$\psi_\tau : \Sigma \setminus \Gamma \to \Sigma_\tau \setminus \Gamma_\tau$$

so that

$$\tilde{F}_C(\tau, \cdot) = \tilde{u}_\tau \circ \psi_\tau.$$

Proof. The first three claims follow immediately from [50, Corollary 5.17] (relevant portions are reviewed above in Theorem 3.12). The fourth and fifth claims also follow from [50, Corollary 5.17]/Theorem 3.12. Indeed, we get from that result that $C * C = 0$ implies that

$$\mathrm{ind}(C) - \chi(\Sigma) + \#\Gamma_{even} = 0,$$

which, if $\mathrm{ind}(C) \geq 1$, implies that

$$\chi(\Sigma) \geq 1 + \#\Gamma_{even}$$

and thus we must have $\chi(\Sigma) = 2 - 2g(\Sigma) = 2$ or, equivalently, $g(\Sigma) = 0$ establishing the third claim. Substituting $\chi(\Sigma) = 2$ in the above then immediately yields the fifth claim.

Next, given $g(\Sigma) = 0$ and $\#\Gamma_{even} = 2 - \mathrm{ind}(C)$, we have that

$$\begin{aligned} \mathrm{ind}(C) - 2g(\Sigma) - \#\Gamma_{even} &= \mathrm{ind}(C) - \#\Gamma_{even} \\ &= 2(\mathrm{ind}(C) - 1) \end{aligned}$$

which is greater than or equal to zero provided $\mathrm{ind}(C) \geq 1$. Thus

$$\mathrm{ind}(C) \geq 2g(\Sigma) + \#\Gamma_{even}$$

provided $\mathrm{ind}(C) \geq 1$ and Theorem 3.21 then allows us to conclude that the linearized normal $\bar{\partial}$-operator is surjective. (We note that in the $\mathrm{ind}(C) = 2$ case we have $\#\Gamma_{even} = 2 - \mathrm{ind}(C) = 0$ so the special case, Theorem 3.23, of the automatic transversality result holds.)

The final claim is a generalization of Theorem 5.7 in [30], and follows from Theorem 3.20, (a generalization of) Lemma 3.22, and the fact that $C * C = 0$. Indeed, since $\bar{\partial}^{\nabla}_N(C)$ is surjective, Theorem 3.20 holds, and we obtain a neighborhood B of $0 \in \ker \bar{\partial}^{\nabla}_N(C)$ and a smooth map $F : B \times \Sigma \setminus \Gamma \to \mathbb{R} \times M$ so that each of the maps $z \mapsto F(\tau, z)$ parametrizes a distinct pseudoholomorphic curve C_τ homotopic to C. The assumption $C * C = 0$ with homotopy invariance of the holomorphic intersection number implies that $C_{\tau_1} * C_{\tau_2} = 0$ for any τ_1, $\tau_2 \in B$. Hence item (1) above along with Theorem 3.8 imply that the C_τ form a family of pairwise disjoint, embedded/nicely-embedded pseudoholomorphic curves. This in turn implies that the map F is injective since double points of F can be seen as either intersections between two distinct C_τ's or a self-intersection of some given C_τ. We next claim that F is an immersion. This argument proceeds essentially the same as in [30,

[10]As with Theorem 3.20, we again caution the reader here that the continuous extension of this diffeomorphism over the punctures is not, in general, smooth.

Theorem 5.7] which proves a similar result in the special case that the curve C is a plane. We explain the main points here. Since F is given by

$$F(\tau, z) = \widetilde{\exp}_{\tilde{u}(z)} E_\tau(z)$$

and we have previously remarked that $\widetilde{\exp}$ is an immersion on some uniform neighborhood of the zero section of N_C, it suffices to show that the map $(\tau, z) \mapsto E_\tau(z)$ is an immersion. Since E_τ is a smooth section of a vector bundle, it suffices in turn to show that the fiber derivative $D_\tau E_\tau(z)$ at any point z has full rank. Letting $\{v_i\}_{i=1}^{\mathrm{ind}(C)}$ be a basis for $\ker \bar{\partial}_N^\nabla(C)$, it suffices to show that the sections $D_\tau E_\tau(z) v_i$ are pointwise linearly independent. If not, then we could construct a nontrivial section v of N_C in the image of $D_\tau E_\tau$ having a zero at some point. However, it can be shown that sections in the image of $D_\tau E_\tau$ are in the kernel of a linear Fredholm operator L_τ of the same type as $\bar{\partial}_N^\nabla(C)$. In particular, the proof of Lemma 3.22 applies to elements of the kernel of L_τ and shows that any nontrivial section v of N_C in the kernel of L_τ is nonvanishing since $\mathrm{ind}(C) - \chi(C) + \#\Gamma_{even} = 0$. This contradiction completes the proof that, for some sufficiently small neighborhood B of $0 \in \ker \bar{\partial}_N^\nabla(C)$, F is an injective immersion on $B \times \Sigma \setminus \Gamma$. With $n = \mathrm{ind}(C)$, we choose a basis $\{v_i\}_{i=1}^n$ for $\ker \bar{\partial}_N^\nabla(C)$ and get a map $\tilde{F} : B_\varepsilon^n(0) \times \Sigma \setminus \Gamma \to \mathbb{R} \times M$ by defining $\tilde{F}(c_i, z) = F(\sum_i c_i v_i, z)$, which will be an injective immersion provided ε is small enough. □

We next prove a general lemma which says that up to $\mathbb{R}$-translation all but finitely many curves in a stable finite energy foliation have index 2.

Lemma 4.3. *Let $\mathcal{F}$ be a stable finite energy foliation for the data (M, λ, J) (according to Definition 4.1). Then:*

- *$\mathcal{F}$ contains a finite number of trivial cylinders.*
- *Up to $\mathbb{R}$-translation, $\mathcal{F}$ contains a finite number of curves C with* $\mathrm{ind}(C) = 1$.

Proof. First, consider a curve $C = [\Sigma, j, \Gamma_+ \cup \Gamma_-, a, u] \in \mathcal{M}(\lambda, J)$ and assume that at $z_i^+ \in \Gamma_+$, u is asymptotic to an orbit with period T_i^+ and, similarly, that at $z_j^- \in \Gamma_-$, u is asymptotic to an orbit with period T_j^-. Then the asymptotic behavior, the compatibility of J with $d\lambda$, and Stokes' Theorem can be used to show that :

- the energy $E(C)$ of the curve (defined by (3.2) above) is given by

$$E(C) = \sum_{z_i^+ \in \Gamma_+} T_i^+$$

  and
- the $d\lambda$-energy $E_{d\lambda}(C)$, defined by

$$E_{d\lambda}(C) := \int_{\Sigma \setminus \Gamma} u^* d\lambda, \tag{4.1}$$

  is nonnegative and

$$E_{d\lambda}(C) = \sum_{z_i^+ \in \Gamma_+} T_i^+ - \sum_{z_j^- \in \Gamma_-} T_j^-.$$

Thus, in a finite energy foliation, the period of any orbit appearing as an asymptotic limit of a curve of the foliation is bounded above by the energy of the foliation. Since we assume that λ is nondegenerate, one can then use Arzelà–Ascoli and the fact that nondegenerate orbits are isolated to argue that there are only a finite number of unparametrized periodic orbits of X_λ having period less than any given positive number. Thus a stable finite energy foliation can contain only a finite number of trivial cylinders.

To prove the second claim we will argue by contradiction. Suppose that there are an infinite number of index-1 curves in $\mathcal{F}$, each distinct up to $\mathbb{R}$-translation. Then we can find a sequence of curves $C_k \in \mathcal{F}$ with $\text{ind}(C_k) = 1$ and so that no two of the C_k differ by the $\mathbb{R}$-action. Then, applying the main theorem of [58] (reviewed as Theorem 3.3 above), we can pass to a subsequence, still denoted C_k, which converges to a connected, nicely-embedded, non-nodal pseudoholomorphic building C_∞ whose components have indices summing to 1. We will argue below that the limit building C_∞ is simply an embedded curve with $\text{ind}(C_\infty) = 1$. Once we know this, the completeness property [30, Theorem 7.1] implies that for sufficiently large k, the C_k belong to the same connected component of the moduli space as C_∞ and thus differ by an $\mathbb{R}$-shift. This contradiction will complete the proof.

To argue that the building C_∞ consists of just a single embedded curve, we first note that Lemma 3.17 allows us to conclude that all components of the building C_∞ have image identical to curves in $\mathcal{F}$ and further, since C_∞ is a nicely-embedded building, that all nontrivial components of C_∞ are curves in $\mathcal{F}$ (as opposed to possibly being multiple covers of such curves). Since C_∞ consists of only trivial cylinders (which have index 0) and nontrivial curve of $\mathcal{F}$ (which have index at least 1), and since the indices of the components of C_∞ must sum to 1, we can conclude there is precisely one nontrivial component. Moreover, since the building C_∞ is connected, stable, and has no nodes, we can conclude that C_∞ contains no trivial cylinders, and thus consists of just a single, embedded curve belonging to $\mathcal{F}$. This completes the proof. □

We now have the following corollary which shows that stable finite energy foliations are indeed smooth foliations of $\mathbb{R} \times M$ which are invariant under the $\mathbb{R}$-action and project to M to give smooth foliations of the complement of a finite collection of periodic orbits in M. Moreover, the projected leaves of the foliation are transverse to the Reeb flow.

Corollary 4.4. *Let $\mathcal{F}$ be a stable finite energy foliation for the data (M, λ, J) (according to Definition 4.1). Then:*

1. *If $C_0 \in \mathcal{F}$ and $C_1 \in \mathcal{M}(\lambda, J)$ is a simple*[11] *curve which is relatively homotopic to C_0, then $C_1 \in \mathcal{F}$.*
2. *The family of curves $\mathcal{F}$ is invariant under translation in the $\mathbb{R}$-coordinate, i.e. if $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{F}$ then $c \cdot C := [\Sigma, j, \Gamma, a + c, u] \in \mathcal{F}$.*
3. *The curves in $\mathcal{F}$ form a smooth foliation of $\mathbb{R} \times M$.*
4. *There exists a finite collection B of periodic orbits, so that the curves in $\mathcal{F}$ not fixed by the $\mathbb{R}$-action project to M to form a smooth foliation of $M \setminus B$ transverse to the flow.*

[11] The assumption that C_1 is also simple can be eliminated. Indeed, it can be shown that if C_0 is a simple curve with $C_0 * C_0 = 0$ and $\text{ind}(C_0) \in \{1, 2\}$ and if C_1 is homotopic to C_0 then C_1 must also be simple, but we will not need this here.

Proof. To prove the first statement, we will argue by contradiction. Assume, to the contrary, that $C_0 \in \mathcal{F}$, and that C_1 is a simple curve relatively homotopic to C_0 with $C_1 \notin \mathcal{F}$. Then for any given point p in the image of C_1 there is a simple curve $C_p \in \mathcal{F}$ passing through p and thus intersecting C_1. Theorem 3.8 then implies that $C_1 * C_p \geq 1$. However, since C_0 and C_p are both curves in the family $\mathcal{F}$, we have that $C_0 * C_p = 0$ by definition of stable finite energy foliation. Thus the homotopy invariance of the intersection product from Theorem 3.7 gives us the contradiction

$$1 \leq C_1 * C_p = C_0 * C_p = 0.$$

This completes the proof that if $C_0 \in \mathcal{F}$, all simple curves relatively homotopic to C_0 are also in $\mathcal{F}$. The second statement is then an immediate corollary of the first since any curve in $\mathcal{M}(\lambda, J)$ is relatively homotopic to its $\mathbb{R}$-translates.

We next address the third claim above. By Theorem 4.2 above, all curves in $\mathcal{F}$ are embeddings. The fact that the curves of $\mathcal{F}$ form a smooth foliation of $\mathbb{R} \times M$ then follows from an argument similar to that in the paragraphs following Lemma 6.10 in section 6.3 of [31]. We first observe that Lemma 4.3 tells us that the set of points of $\mathbb{R} \times M$ with index-2 curves passing through them is open and dense. For a point $p \in \mathbb{R} \times M$ with an index-2 curve $C \in \mathcal{F}$ passing through it, it follows from the last item in Theorem 4.2 that C belongs to a smoothly varying 2-dimensional family of pseudoholomorphic curves C_τ which foliate a neighborhood of p. Moreover, it follows from the preceding paragraph that each of the curves C_τ is in $\mathcal{F}$, and thus that curves of $\mathcal{F}$ foliate some neighborhood of p.

Next, considering a point p lying on an index-1 curve $C \in \mathcal{F}$, we've already observed that all $\mathbb{R}$-translates of C belong to $\mathcal{F}$. If p_k is a sequence of points converging to p and not lying on an $\mathbb{R}$-translate of C, we can conclude from Lemma 4.3 that for sufficiently large k, p_k lies on an index-2 curve $C_k \in \mathcal{F}$. Moreover, by Theorem 3.3, we can find a sequence of local parametrizations of some subsequence C_{k_j} which converge in C^∞_{loc} to a parametrization of a curve C_∞ passing through p. We claim that $C_\infty = C$. Indeed, if C_∞ doesn't have identical image with C, it must have an isolated intersection with C. This would then allow us to conclude that the C_{k_j} intersect C for sufficiently large j and thus, by Theorem 3.8, that $C_{k_j} * C \geq 1$. This contradicts the fact that $C_{k_j} * C = 0$ by the assumption that C and all C_k are in the family $\mathcal{F}$. We conclude that C_∞ has the same image as C and further, since Theorem 3.3 tells us that C_∞ must be either a trivial cylinder or nicely embedded, that $C_\infty = C$. This allows us to conclude that the curves of $\mathcal{F}$ smoothly foliate some neighborhood of p. The argument for points lying on one of the finitely-many (according to Lemma 4.3) trivial cylinders of $\mathcal{F}$ now proceeds along similar lines with the use compactness and positivity of intersections.

We finally address the last claim. We first define B to be the collection of periodic orbits which appear as asymptotic limits of curves in $\mathcal{F}$. Then B must be a finite set by Lemma 4.3 above. By Theorem 4.2 every curve $C \in \mathcal{F}$ that is not a trivial cylinder projects to an embedding transverse to the flow and disjoint from B. Moreover, it follows from the assumption that $C_1 * C_2 = 0$ for any two nontrivial curves C_1, $C_2 \in \mathcal{F}$ that the projections of C_1 and C_2 to M have either disjoint or identical images (see e.g. the discussion following Corollary 5.9 in [50]). Therefore, we have a unique embedded curve through every point of $M \setminus B$. Moreover, since the curves of $\mathcal{F}$ form a smooth foliation of $\mathbb{R} \times M$ invariant under $\mathbb{R}$-shifting, the projections of these curves to $\mathbb{R} \times M$ will form a smooth foliation of $M \setminus B$ provided the pullback of the coordinate field ∂_a on $\mathbb{R}$ to $\mathbb{R} \times M$ is not tangent to any of the curves. Since, as a result of the definition of $\tilde{J}$, such a tangency can be identified with tangency of the projected curve to the Reeb vector field, there can be no such tangencies. This completes the proof. □

We can also prove a converse to last part of the above result; specifically, the next result shows that as an alternate definition of stable finite energy foliation, one can consider the projections of curves to M which foliate the complement of a finite collection of periodic orbits.

Corollary 4.5. *Let $B \subset M$ be a finite collection of simple periodic orbits, and let $\mathcal{F} \subset \mathcal{M}(\lambda, J)/\mathbb{R}$ be a collection of simple curves $C \in \mathcal{M}(\lambda, J)/\mathbb{R}$ satisfying:*

- *Each $C \in \mathcal{F}$ is disjoint from B.*
- *For each $p \in M \setminus B$ there is a (not necessarily unique) curve $C \in \mathcal{F}$ passing through p.*
- $\operatorname{ind}(C) \in \{1, 2\}$ *for all $C \in \mathcal{F}$.*
- *$C_1 * C_2 = 0$ for all C_1, $C_2 \in \mathcal{F}$.*
- *The energies of the curves in $\mathcal{F}$ are uniformly bounded; that is, $E(\mathcal{F}) := \sup_{C\in\mathcal{F}} E(C)$ is finite.*

Then the collection of curves $\tilde{\mathcal{F}}$ in $\mathcal{M}(\lambda, J)$ consisting of all possible lifts of curves $C \in \mathcal{F}$ to curves in $\mathbb{R} \times M$ together with cylinders over the periodic orbits in B form a finite energy foliation.

Proof. Given a point $p \in M \setminus B$ there is, by assumption, a curve $C \in \mathcal{F}$ passing through it. Considering the set of all possible lifts gives a curve through each point of $\mathbb{R} \times (M \setminus B)$. Moreover, by the assumption that the holomorphic intersection numbers between all such curves is zero, we indeed get a unique curve through each point of $\mathbb{R} \times (M \setminus B)$ by Theorem 3.8. Moreover, by the assumption that the curves of $\mathcal{F}$ are disjoint from B, we obtain a unique curve through each point of $\mathbb{R} \times M$ by including the trivial cylinders over the orbits in B in the collection we consider. The remaining properties of a finite energy foliation from Definition 4.1 are then easily verified from our remaining assumptions. □

In the rest of this section we will focus on the structure of moduli spaces of simple curves satisfying $C * C = 0$ and $\operatorname{ind}(C) = 2$. Because of the important role that such curves play in what follows, it will be convenient to have a term for such curves.

Definition 4.6. A curve $C \in \mathcal{M}(\lambda, J)$ is said to be a *foliating curve* if $C * C = 0$ and $\operatorname{ind}(C) = 2$.

Given a curve $C \in \mathcal{M}(\lambda, J)$ we will use the notation $\mathcal{M}(C)$ to indicate the moduli space of simple curves in the same relative homotopy class as C and $\mathcal{M}_1(C)$ to indicate the moduli space of simple curves with one marked point in the same relative homotopy class as C. We note the results of [30], reviewed in Section 3.4 above, give a local manifold structure on these spaces in the event that linearized normal $\bar{\partial}$-operator is surjective. However, the fact these local manifold structures glue together to give a global manifold structure on the moduli space is only addressed in [30] as a special case of the fact that the local models for the universal moduli space glue together to give a global Banach manifold structure on the universal moduli space. In the event that the curves in question are foliating curves, a simpler argument is possible using Theorem 4.2 above. We state this result as a corollary.

Corollary 4.7. *Let $C \in \mathcal{M}(\lambda, J)$ be a foliating curve, that is, assume that C is simple, $C * C = 0$ and $\operatorname{ind}(C) = 2$. Then $\mathcal{M}(C)$ has the structure of a smooth, 2-dimensional manifold, and $\mathcal{M}_1(C)$ has the structure of a smooth 4-dimensional manifold. Moreover, the evaluation map $ev : \mathcal{M}_1(C) \to \mathbb{R} \times M$ is a smooth embedding, the forgetful map $\mathcal{M}_1(C) \to \mathcal{M}(C)$ is a smooth submersion, and the action of $\mathbb{R}$-shifting a curve defines smooth, free, proper $\mathbb{R}$-actions on $\mathcal{M}_1(C)$ and $\mathcal{M}(C)$.*

Proof. Given $C_i = [\Sigma_i, j_i, \Gamma_i, a_i, u_i] \in \mathcal{M}(C)$ for $i \in \{1, 2\}$, Theorem 4.2 gives a local identification of the moduli space of curves with one marked point with $B^2_{\varepsilon_i}(0) \times \Sigma_i \setminus \Gamma_i$ together with an embedding $\tilde{F}_{C_i} : B^2_{\varepsilon_i}(0) \times \Sigma_i \setminus \Gamma_i \to \mathbb{R} \times M$. Because the maps $\tilde{F}_{C_i}$ are local diffeomorphisms, maps of the form $\tilde{F}^{-1}_{C_2} \circ \tilde{F}_{C_1}$ are smooth when defined, and thus the local identifications of $\mathcal{M}_1(C)$ with sets of the form $B^2_{\varepsilon_i}(0) \times \Sigma_i \setminus \Gamma_i$ piece together to give a global manifold structure on $\mathcal{M}_1(C)$ in which the evaluation map, being locally given by the $\tilde{F}_C$-maps, are smooth immersions. Moreover, since double points of ev can be seen as intersections/self-intersections between curves in $\mathcal{M}(C)$, the fact that $C * C = 0$ implies that ev is an injective map.

Next, we observe that a local manifold structure on $\mathcal{M}(C)$ near the curve C_i is given by projecting

$$\pi_i : B^2_{\varepsilon_i}(0) \times \Sigma_i \setminus \Gamma_i \to B^2_{\varepsilon_i}(0).$$

Since the maps $\tilde{F}^{-1}_{C_2} \circ \tilde{F}_{C_1}$, where defined, are smooth local diffeomorphisms which restrict to diffeomorphisms on the fibers of the projections π_i, a smooth local section s_1 for the projection π_1 is mapped via $\tilde{F}^{-1}_{C_2} \circ \tilde{F}_{C_1}$ to a smooth local section for π_2, and the composition $\pi_2 \circ \tilde{F}^{-1}_{C_2} \circ \tilde{F}_{C_1} \circ s_1$ is independent of the choice of smooth section s_1. Such maps can then be used to construct smooth change-of-coordinate maps giving a global manifold structure on $\mathcal{M}(C)$. Moreover, since the forgetful map $\mathcal{M}_1(C) \to \mathcal{M}(C)$ is given locally by one of the projections π_i defined above, the forgetful map is a smooth submersion in the manifold structure we've constructed.

To see that the $\mathbb{R}$-action is a smooth, free, proper action on $\mathcal{M}_1(C)$ we first observe that the evaluation map $ev : \mathcal{M}_1(C) \to \mathbb{R} \times M$ is $\mathbb{R}$-equivariant. Since the $\mathbb{R}$-action on $\mathbb{R} \times M$ is smooth, free, and proper and ev is an embedding, it follows immediately that the $\mathbb{R}$-action on $\mathcal{M}_1(C)$ is smooth, free, and proper. Moreover, since the forgetful map $\mathcal{M}_1(C) \to \mathcal{M}(C)$ is a smooth $\mathbb{R}$-equivariant submersion, we can conclude that the $\mathbb{R}$ acts smoothly on $\mathcal{M}(C)$ as well by considering smooth local sections $\mathcal{M}(C) \to \mathcal{M}_1(C)$. Freeness of the $\mathbb{R}$-action on $\mathcal{M}(C)$ follows from the well-known fact that only trivial cylinders can be fixed points of the $\mathbb{R}$-action (or, in this case, from the fact that $C * C = 0$ implies that C is disjoint from all of its nontrivial $\mathbb{R}$-translates). Finally, properness follows from $\mathbb{R}$-equivariance of the forgetful map and properness of the action on $\mathcal{M}_1(C)$. □

Corollary 4.8. *Let C be a foliating curve. Then the moduli space $\mathcal{M}_1(C)/\mathbb{R}$ is a smooth 3-manifold and the moduli space $\mathcal{M}(C)/\mathbb{R}$ is a smooth 1-manifold. Moreover, $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ is an embedding, and the forgetful map $\mathcal{M}_1(C)/\mathbb{R} \to \mathcal{M}(C)/\mathbb{R}$ is a smooth submersion.*

Proof. The facts that $\mathcal{M}_1(C)/\mathbb{R}$ is a smooth 3-manifold and that $\mathcal{M}(C)/\mathbb{R}$ is a smooth 1-manifold follow directly from Corollary 4.7 since the $\mathbb{R}$-action on $\mathcal{M}_1(C)$ and $\mathcal{M}(C)$ is free and proper, while the fact that the forgetful map $\mathcal{M}_1(C)/\mathbb{R} \to \mathcal{M}(C)/\mathbb{R}$ is a smooth submersion follows from the fact that the forgetful map

$\mathcal{M}_1(C) \to \mathcal{M}(C)$ is an $\mathbb{R}$-equivariant smooth submersion. Finally, to see that $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ is an embedding, we first observe that it follows from the fact that the evaluation map $ev : \mathcal{M}_1(C) \to \mathbb{R} \times M$ is an $\mathbb{R}$-equivariant immersion that $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ is also immersion. Since $\mathcal{M}_1(C)/\mathbb{R}$ and M are the same dimension, it remains to show that $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ is injective. But since $C * C = 0$, it follows that the projections of distinct curves in $\mathcal{M}(C)$ to M are embedded and have disjoint image unless they differ by the $\mathbb{R}$-action. Since double points of $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ can be seen as intersections/self-intersections of curves in $\mathcal{M}(C)/\mathbb{R}$, we conclude that $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ in injective, and thus an embedding. □

For the following let ψ_t denote the flow of the Reeb vector field.

Corollary 4.9. *Let $C = [\Sigma, j, \Gamma, da, u] \in \mathcal{M}(\lambda, J)/\mathbb{R}$ be a foliating curve. Then given any $p \in u(\Sigma \setminus \Gamma)$, there exists an $\varepsilon > 0$ so that:*

1. *For every $t \in (-\varepsilon, \varepsilon)$ there exists a unique point of $\mathcal{M}(C)/\mathbb{R}$ passing through $\psi_t(p)$.*
2. *The map taking a point $t \in (-\varepsilon, \varepsilon)$ to the unique curve in $\mathcal{M}(C)/\mathbb{R}$ passing through $\psi_t(p)$ is a local diffeomorphism.*

Proof. The first claim follows from Corollary 4.8. Indeed, since the evaluation map $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ is an embedding, the image of an open set around (C, z) contains an open neighborhood U of the point $p := u(z)$. Thus, there exists some $\varepsilon > 0$ so that $\psi_t(p) \in U$ for all $t \in (-\varepsilon, \varepsilon)$, which tells there us there is a point of $\mathcal{M}_1(C)/\mathbb{R}$ mapping via ev to p, which is equivalent to there being a curve in $\mathcal{M}(C)/\mathbb{R}$ passing through p. Moreover, the fact that the evaluation map is injective implies that there is at most one curve in $\mathcal{M}(C)$ passing through any given point in M.

Next we show that the map taking $t \in (-\varepsilon, \varepsilon)$ to the unique curve in $\mathcal{M}(C)/\mathbb{R}$ passing through p is a local diffeomorphism. By construction, the map taking an interval $(-\varepsilon, \varepsilon)$ to $\mathcal{M}(C)/\mathbb{R}$ is given by the composition

$$(-\varepsilon, \varepsilon) \xrightarrow{\psi_\cdot(p)} M \xrightarrow{ev^{-1}} \mathcal{M}_1(C)/\mathbb{R} \longrightarrow \mathcal{M}(C)/\mathbb{R}$$

with the last map the forgetful map. Since the composition of the first two maps gives an embedding of $(-\varepsilon, \varepsilon)$ in $\mathcal{M}_1(C)/\mathbb{R}$ it suffices to show that this embedding is transverse to the fibers of the forgetful map. However, since the embedding $ev : \mathcal{M}_1(C)/\mathbb{R} \to M$ maps the fibers of the forgetful map to nicely-embedded pseudoholomorphic curves, a tangency of the map $t \mapsto ev^{-1}(\psi_t(p))$ to a fiber of the forgetful map corresponds via the embedding ev with a tangency of the map $t \mapsto \psi_t(p)$ to a curve $C' \in \mathcal{M}(C)/\mathbb{R}$, that is, a tangency of the Reeb vector field to a curve $C' \in \mathcal{M}(C)/\mathbb{R}$. Since we know from Theorem 4.2 that the Reeb vector field is everywhere transverse to every curve in $\mathcal{M}(C)/\mathbb{R}$, no such tangency can exist. We've thus shown the map taking a point $t \in (-\varepsilon, \varepsilon)$ to the unique curve passing through p is a local diffeomorphism. □

5. The connected sum construction

This section is devoted to the proof of Theorem 5.1 below, which shows that we can perform a connected sum on a manifold M with contact form λ and obtain a contact form on the surgered manifold which has certain additional properties which will allow us to prove Theorem 1.1. Previous descriptions/constructions of connected sums in contact manifolds can be found in [41, 54]. For our main theorem, we will need the Reeb vector field of the new contact form to have some specific properties not addressed in these previous constructions.

For the statement of the theorem, we will need the following definition. We will say that an open set U in a contact manifold (M, λ) is a *flow-tube neighborhood* of a point $p \in M$ if the closure $\bar{U}$ of U is contained in a coordinate neighborhood in which $\bar{U}$ takes the form

$$\bar{U} = \overline{B_\varepsilon(p)} \times [-\varepsilon, \varepsilon] \subset \mathbb{R}^2 \times \mathbb{R} = \{(x, y)\} \times \{z\}$$

for some $\varepsilon > 0$ and the Reeb vector field takes the form $X_\lambda = \pm\partial_z$.

Theorem 5.1. *Let M a 3-manifold equipped with a nondegenerate contact form λ and let p and q be distinct points in M, and let $\mathcal{O}$ be an open neighborhood of $\{p, q\}$. Then there exist disjoint flow-tube neighborhoods $U \subset \mathcal{O}$ and $V \subset \mathcal{O}$ of p and q respectively, a manifold M' equipped with a contact form λ', and an embedding $i : M \setminus \{p, q\} \to M'$ so that:*

(1) *The contact form λ' on M' is nondegenerate.*
(2) *The pullback $i^*\lambda'$ agrees with λ on $M \setminus \{U \cup V\}$, that is, if ι denotes the composition*
$$M \setminus \{U \cup V\} \hookrightarrow M \setminus \{p, q\} \xrightarrow{i} M'$$
with $M \setminus \{U \cup V\} \hookrightarrow M \setminus \{p, q\}$ the obvious inclusion, then
$$\iota^*\lambda' = \lambda.$$
(3) *The set*
$$M' \setminus i(M \setminus \{p, q\})$$
is diffeomorphic to an embedded 2-sphere in M', and the set
$$N := M' \setminus \overline{i(M \setminus \{U \cup V\})},$$
called the neck, is diffeomorphic to $\mathbb{R} \times S^2$.
(4) *Letting $X_{\lambda'}$ denote the Reeb vector field of the contact form λ', there exists a simple, even periodic orbit $\gamma_0 \subset N$ of $X_{\lambda'}$ contained entirely within N. All other simple periodic orbits of $X_{\lambda'}$ pass through points of $M' \setminus N$.*
(5) *Given any compatible $J \in \mathcal{J}(M', \lambda')$ we can find a $J' \in \mathcal{J}(M', \lambda')$ agreeing with J outside of the neck N for which there exists a pair of (nicely) embedded, disjoint pseudoholomorphic planes $P^\pm = [S^2, i, \{\infty\}, da^\pm, u^\pm] \in \mathcal{M}(\lambda', J')/\mathbb{R}$ asymptotic to γ_0 in opposite directions with extremal winding. Moreover, $P^\pm * P^\pm = 0 = P^+ * P^-$ and the union $P^+ \cup \gamma_0 \cup P^-$ of the planes and the periodic orbit form a (C^1-)smooth[12] sphere in $N \approx \mathbb{R} \times S^2$ which generates $\pi_2(N)$.*
(6) *Let ψ_t denote the flow of X_λ and $\tilde{\psi}_t$ denote the flow of $X_{\lambda'}$. Then:*
   (a) *If p_+ and p_- are points in ∂U and $\gamma_p : [a, b] \subset \mathbb{R} \to \bar{U} \subset M$ is a smooth integral curve-segment of X_λ connecting p_- to p to p_+ within $\bar{U}$, then there exist smooth integral curve-segments $\tilde{\gamma}_{p,\pm}$ of $X_{\lambda'}$ lying in $\bar{N}$ so that*

[12]Our proof will actually provide a C^∞-smooth sphere, but for our main result we need only assume that the two planes approach γ_0 in opposite directions, in which case Theorem 3.1 can be used to show that the resulting sphere is C^1. This is addressed in [16].

- *$\tilde{\gamma}_{p,-}$ connects $i(p_-)$ to the plane P^- and the interior of $\tilde{\gamma}_{p,-}$ lies in $N \setminus \{P^+ \cup \gamma_0 \cup P^-\}$.*
- *$\tilde{\gamma}_{p,+}$ connects the plane P^+ to $i(p_+)$ and the interior of $\tilde{\gamma}_{p,+}$ lies in $N \setminus \{P^+ \cup \gamma_0 \cup P^-\}$.*

(b) *Similarly, if $q_\pm$ are points in ∂V and $\gamma_q : [a', b'] \subset \mathbb{R} \to \bar{V} \subset M$ is a smooth integral curve-segment of X_λ connecting q_- to q to q_+ within $\bar{V}$, then there exist smooth integral curve-segments $\tilde{\gamma}_{q,\pm}$ of $X_{\lambda'}$ lying in $\bar{N}$ so that*

- *$\tilde{\gamma}_{q,-}$ connects $i(q_-)$ to the plane P^+ and the interior of $\tilde{\gamma}_{q,-}$ lies in $N \setminus \{P^+ \cup \gamma_0 \cup P^-\}$.*
- *$\tilde{\gamma}_{p,+}$ connects the plane P^- to $i(q_+)$ and the interior of $\tilde{\gamma}_{q,+}$ lies in $N \setminus \{P^+ \cup \gamma_0 \cup P^-\}$.*

Before proving this theorem, we will describe a contact connected sum on two copies of $\mathbb{R}^3$ equipped with specific contact forms. Since the connected sum operation we describe can be localized into arbitrarily small regions, Darboux's theorem for contact manifolds will then allow us to transfer the construction to any contact 3-manifold. We describe this construction in a series of lemmas. In order to focus on the main points of the construction we delay some details involving longer but more straightforward computations to Appendix A.

We consider $\mathbb{R}^3 = \{(x, y, z)\}$ equipped with the contact forms λ_+ and λ_- defined by

$$\lambda_\pm = \pm dz + \frac{1}{2}(x\, dy - y\, dx)$$

We equip S^2 with polar coordinate $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ and azimuthal coordinate $\theta \in [0, \pi]$ and consider the 1-form λ_1 on $\mathbb{R} \times S^2$ defined by

$$\lambda_1 = 3\cos\theta\, d\rho - \rho \sin\theta\, d\theta + \frac{1}{2}\sin^2\theta\, d\phi$$

where ρ is the $\mathbb{R}$-coordinate. It follows from Lemma A.1 that λ_1 does in fact extend over the $\theta \in \{0, \pi\}$ locus to gives a smooth 1-form on $\mathbb{R} \times S^2$, and further that λ_1 is a contact form on $\mathbb{R} \times S^2$.

Lemma 5.2. *Consider the maps $\Phi_\pm : \mathbb{R}^\pm \times S^2 \to \mathbb{R}^3 \setminus \{0\}$ defined by*

$$\Phi_\pm(\rho, \phi, \theta) = \pm(\rho \sin\theta \cos\phi, \rho \sin\theta \sin\phi, \rho^3 \cos\theta). \tag{5.1}$$

Then Φ_+ and Φ_- are smooth diffeomorphisms satisfying

$$\Phi_\pm^* \lambda_\pm = \rho^2 \lambda_1 \tag{5.2}$$

with λ_+, λ_-, and λ_1 as defined above.

The proof of this lemma involves straightforward computation and we give the details in Lemma A.2 in Appendix A. This lemma shows that we can take a connected sum between these two copies of $\mathbb{R}^3$ in a way which preserves the Reeb flow outside of an arbitrarily small neighborhood of the surgered region. Indeed, according to this lemma, any smooth positive function $f : \mathbb{R} \times S^2 \to \mathbb{R}^+$ gives us a contact form $f\lambda_1$ on $\mathbb{R} \times S^2$ which is contactomorphic on $\mathbb{R}^\pm \times S^2$ to $(\mathbb{R}^3 \setminus \{0\}, \lambda_\pm)$ via the maps $\Phi_\pm$. Furthermore, the Reeb flow of $f\lambda_1$ is conjugate via $\Phi_\pm$ to that of the Reeb vector field(s) for $(\mathbb{R}^3, \lambda_\pm)$ on any region where $f(\rho, p) = \rho^2$. Since we can easily construct smooth positive functions $f : \mathbb{R} \times S^2 \to \mathbb{R}^+$ satisfying $f(\rho, p) = \rho^2$ on an arbitrarily small neighborhood of $\rho = 0$, this shows the Reeb vector fields of

$f\lambda_1$ and those of $\lambda_\pm$ are identified via $\Phi_\pm$ outside of an arbitrarily neighborhood of the surgered region.

To establish that the connected sum operation can be carried out in such a way as to ensure the other properties we will need, further properties on the function f will be required. Before discussing these properties we first establish some properties of the contact form

$$\lambda_f := f\lambda_1$$

and its associated contact structure

$$\xi_1 := \ker \lambda_f = \ker \lambda_1.$$

It will be convenient to define the function

$$g(\theta) := 2\cos^2\theta + 1 = 3\cos^2\theta + \sin^2\theta \tag{5.3}$$

and we note that g defines a smooth function on S^2 as a result of Lemma A.1.

Lemma 5.3. *For $\theta \notin \{0, \pi\}$:*

- *The set*

$$\begin{aligned} \mathcal{B}_{(\rho,\theta,\phi)} &= \left\{ (fg)^{-1}(-3\cot\theta\,\partial_\phi + \frac{1}{2}\sin\theta\,\partial_\rho), 2\rho\csc\theta\,\partial_\phi + \partial_\theta \right\} \\ &=: \{v_1(\rho,\theta,\phi), v_2(\rho,\theta,\phi)\} \end{aligned} \tag{5.4}$$

  is a symplectic basis for $(\xi_1, d\lambda_f)$.
- *The Reeb vector field X_f of the contact form λ_f is given by*

$$\begin{aligned} X_f = [gf^2]^{-1}\Big[&(-\rho f_\rho - 3f_\theta\cot\theta + 2f)\,\partial_\phi \\ &+ (3\cot\theta f_\phi - \frac{1}{2}\sin\theta f_\rho)\,\partial_\theta \\ &+ (\rho f_\phi + \frac{1}{2}\sin\theta f_\theta + f\cos\theta)\,\partial_\rho\Big]. \end{aligned} \tag{5.5}$$

The proof is straightforward computation. Further details are given in Lemmas A.3-A.4 in the Appendix A.

We now have the following lemma which identifies a condition which guarantees a periodic orbit of X_f on the sphere $\rho = 0$.

Lemma 5.4. *The Reeb vector field X_f of λ_f is a constant multiple of ∂_ϕ along the equator $\theta = \pi/2$ of the sphere $\rho = 0$ precisely when $df = 0$ there.*

Proof. From (5.5), we have for $(\rho,\theta,\phi) = (0,\pi/2,\phi)$ that

$$X_f = f^{-2}\left[2f\,\partial_\phi - \frac{1}{2}f_\rho\,\partial_\theta + \frac{1}{2}f_\theta\,\partial_\rho\right].$$

Thus, $X_f(0,\pi/2,\phi)$ is a positive multiple of ∂_ϕ precisely when $f_\rho(0,\pi/2,\phi) = f_\theta(0,\pi/2,\phi) = 0$, in which case the formula for X_f along $(\rho,\theta,\phi) = (0,\pi/2,\phi)$ reduces to $X_f = (2/f)\,\partial_\phi$. Thus $X_f(0,\pi/2,\phi)$ is a constant multiple of ∂_ϕ precisely when $f(0,\pi/2,\phi)$ is constant, which is equivalent to requiring $f_\phi(0,\pi/2,\phi) = 0$. □

By further restricting the function f we can say that the periodic orbit identified in the above lemma is the only (simple) periodic orbit of X_f, and we can arrange that the flow of X_f is tangent to $\mathbb{R} \times \{\theta = 0, \pi\}$.

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a smooth, positive function satisfying*

$$\rho f'(\rho) > 0$$

for all $\rho \neq 0$. Then

- *X_f has a unique (simple) periodic orbit occurring at $\rho = 0$, $\theta = \pi/2$.*
- *Along $\theta = 0$ (resp. $\theta = \pi$) locus, X_f is a positive (resp. negative) multiple of ∂_ρ.*

Proof. If f depends only on the $\mathbb{R}$-coordinate ρ, then the formula (5.5) of the Reeb vector field of λ_f reduces to

$$X_f = [g(\theta)f(\rho)^2]^{-1}\left[(-\rho f' + 2f)\,\partial_\phi - \frac{1}{2}\sin\theta f'\,\partial_\theta + f\cos\theta\,\partial_\rho\right] \tag{5.6}$$

Define the function $Z : \mathbb{R} \times S^2 \to \mathbb{R}$ by

$$Z(\rho, \theta, \phi) = \rho\cos\theta.$$

It follows from Lemma A.1 that Z defined as such extends to a smooth function on all of $\mathbb{R} \times S^2$. Then

$$dZ = \cos\theta\, d\rho - \rho\sin\theta\, d\theta$$

and so

$$dZ(X_f) = (g(\theta)f(\rho)^2)^{-1}\left(f\cos^2\theta + \frac{1}{2}f'\rho\sin^2\theta\right)$$

which is nonnegative everywhere. Therefore Z is monotonic along any flow line of X_f, and any periodic orbit of X_f must be contained in the zero locus of $dZ(X_f)$. But $dZ(X_f) = 0$ precisely when both $f\cos^2\theta$ and $f'\rho\sin^2\theta$ vanish, which, in turn, happens precisely when $\rho = 0$ and $\theta = \pi/2$.

To see the second claim is true, we observe from Lemma A.1 that ∂_ϕ and $\sin\theta\,\partial_\theta$ define smooth vector fields on S^2 which vanish at the north and south poles $\theta \in \{0, \pi\}$. Thus, the formula above for the Reeb vector field tells us that

$$X_f(\rho, \phi, 0) = [g(0)f(\rho)]^{-1}\cos(0)\,\partial_\rho = \frac{1}{3f(\rho)}\partial_\rho$$

and

$$X_f(\rho, \phi, \pi) = [g(\pi)f(\rho)]^{-1}\cos(\pi)\,\partial_\rho = \frac{-1}{3f(\rho)}\partial_\rho$$

which establishes the second claim of the lemma. □

We next compute the Conley–Zehnder index of the periodic orbit guaranteed by the above lemma provided an additional assumption on the function f.

Lemma 5.6. *Assume that $f : \mathbb{R} \to \mathbb{R}^+$ is a smooth positive function satisfying*

$$\rho f'(\rho) > 0$$

for $\rho \neq 0$ and

$$f''(0) > 0.$$

Then relative to the symplectic trivialization

$$\mathcal{B}_{(0,\pi/2,\phi)} = \{v_1(0, \pi/2, \phi), v_2(0, \pi/2, \phi)\} = \left\{\tfrac{1}{2f(0)}\partial_\rho, \partial_\theta\right\}$$

of $(\xi_1, d\lambda_f)$ from (5.4), the Conley–Zehnder index of the unique simple periodic orbit γ_0 of X_f is 0.

Proof. We first observe that the proof of Lemma 5.5 above shows that along the equator

$$X_f(0, \pi/2, \phi) = (2/f(0))\,\partial_\phi$$

and thus the map $\gamma_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times S^2$ given by $\gamma_0(t) = (0, \pi/2, 2\pi t)$ satisfies

$$\dot{\gamma}_0(t) = 2\pi\,\partial_\phi = (f(0)\pi)\,X_f(\gamma_0(t))$$

so γ_0 is a periodic orbit of period $\tau_f := f(0)\pi$.

Let ψ_t denote the flow of X_f, that is ψ_t satisfies

$$\dot{\psi}_t(x) = X_f(\psi_t(x))$$

To compute the Conley–Zehnder $\mu^\Phi(\gamma_0)$ index of γ_0 in the trivialization Φ arising from $\mathcal{B}_{(0,\pi/2,\phi)}$ we need to analyze the behavior of the linearized flow $d\psi_t$ on ξ_1 in the trivialization Φ. Letting

$$\begin{aligned}\Psi(t) &= \Phi(\psi_{t\tau_f}(0,\pi/2,0))^{-1} d\psi_{t\tau_f}(0,\pi/2,0)\Phi(0,\pi/2,0)\\ &= \Phi(0,\pi/2,2\pi t)^{-1} d\psi_{t\tau_f}(0,\pi/2,0)\Phi(0,\pi/2,0)\end{aligned}$$

we can write

$$\Psi(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{bmatrix}$$

where the c_{ij} are defined by

$$d\psi_{t\tau_f}(0,\pi/2,0)v_j(0,\pi/2,0) = \sum_i c_{ij}(t)v_i(0,\pi/2,2\pi t)$$

and satisfy $c_{ij}(0) = \delta_{ij}$. Since the $d\psi_{t\tau_f}(0,\pi/2,0)v_j(0,\pi/2,0)$ defines a section of ξ_1 along γ_0 defined by pushing forward by the linearized flow of $\tau_f X_f$, the Lie derivative $L_{\tau_f X_f} = \tau_f L_{X_f}$ (in the sense of (2.8)) is well defined and vanishes. Taking the Lie derivative $L_{\tau_f X_f}$ then of the above equation gives us

$$\sum_i c'_{ij}(t)v_i(0,\pi/2,2\pi t) + c_{ij}(t)(L_{\tau_f X_f} v_i)(0,\pi/2,2\pi t) = 0.$$

Letting $M(t) = [m_{ij}(t)]$ be the matrix defined by

$$(L_{\tau_f X_f} v_j)(0,\pi/2,2\pi t) = -\sum_i m_{ij}(t)v_i(0,\pi/2,2\pi t),$$

we substitute in the above equation and use that the v_i are a linearly independent to conclude that

$$c'_{ij} - \sum_k m_{ik}c_{kj} = 0$$

or, equivalently, that Ψ satisfies the linear ODE

$$\begin{aligned}\Psi'(t) &= M(t)\Psi(t)\\ \Psi(0) &= I.\end{aligned} \tag{5.7}$$

To find $M(t)$ we extend $v_1(0,\pi/2,2\pi t)$ and $v_2(0,\pi/2,2\pi t)$ to vector fields

$$\begin{aligned}\tilde{v}_1(\rho,\theta,\phi) &= \frac{1}{2f(0)}\,\partial_\rho\\ \tilde{v}_2(\rho,\theta,\phi) &= \partial_\theta\end{aligned}$$

which are locally constant in (ρ, θ, ϕ) coordinates, and use (5.6) to compute

$$\begin{aligned}
-(L_{\tau_f X_f} v_1)(0, \pi/2, 2\pi t) &= \tau_f(v_1 X_f - X_f \tilde{v}_1)(0, \pi/2, 2\pi t) \\
&= \tau_f(v_1 X_f)(0, \pi/2, 2\pi t) \\
&= (f(0)\pi) \frac{1}{2f(0)} \partial_\rho X_f(0, \pi/2, 2\pi t) \\
&= -\frac{\pi f''(0)}{4f(0)^2} \partial_\theta \\
&= -\frac{\pi f''(0)}{4f(0)^2} v_2(0, \pi/2, 2\pi t)
\end{aligned}$$

and

$$\begin{aligned}
-(L_{\tau_f X_f} v_2)(0, \pi/2, 2\pi t) &= \tau_f(v_2 X_f - X_f \tilde{v}_2)(0, \pi/2, 2\pi t) \\
&= \tau_f(v_2 X_f)(0, \pi/2, 2\pi t) \\
&= (f(0)\pi) \partial_\theta X_f(0, \pi/2, 2\pi t) \\
&= -\pi\, \partial_\rho \\
&= -2\pi f(0) v_1(0, \pi/2, 2\pi t).
\end{aligned}$$

We conclude

$$M(t) = \begin{bmatrix} 0 & -2\pi f(0) \\ -\frac{\pi f''(0)}{4f(0)^2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A^2 \\ -B^2 & 0 \end{bmatrix}.$$

with $A = \sqrt{2\pi f(0)}$ and $B = \frac{\sqrt{\pi f''(0)}}{2f(0)}$. Direct computation then shows that the solution to (5.7) is given by

$$\Psi(t) = \begin{bmatrix} \cosh(ABt) & -(A/B)\sinh(ABt) \\ -(B/A)\sinh(ABt) & \cosh(ABt) \end{bmatrix} = C \begin{bmatrix} e^{ABt} & 0 \\ 0 & e^{-ABt} \end{bmatrix} C^{-1}$$

where C is the symplectic matrix

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} A/B & 1 \\ -1 & B/A \end{bmatrix}.$$

A path of symplectic matrices of this form is well-known to have Conley–Zehnder index equal to 0 (see Lemma A.5 below) and thus

$$\mu^\Phi(\gamma_0) = \mu_{cz}(\Psi) = 0$$

as claimed. □

We next show that we can choose a compatible J on a neighborhood of $\rho = 0$ so that the northern/southern hemispheres of the the sphere $\rho = 0$ are projections of pseudoholomorphic planes to $\mathbb{R} \times S^2$ asymptotic to the periodic orbit at the equator.

Lemma 5.7. *Let $f : \mathbb{R} \to \mathbb{R}^+$ satisfy the hypotheses of Lemma 5.6 and let $J \in \mathcal{J}(\mathbb{R} \times S^2, \lambda_f)$ be a compatible almost complex structure. Then, for any open neighborhood U of $\{0\} \times S^2$ there exists a compatible $J' \in \mathcal{J}(\mathbb{R} \times S^2, \lambda_f)$ agreeing* *with J outside of U so that the planes*

$$P^+ = \{\rho = 0, \theta \in [0, \pi/2)\}$$
$$P^- = \{\rho = 0, \theta \in (\pi/2, \pi]\}$$

given from the upper and lower hemispheres of the $\rho = 0$ sphere are projected pseudoholomorphic curves, i.e. elements of $\mathcal{M}(\lambda_f, J')/\mathbb{R}$, approaching their mutual asymptotic limit γ_0 in opposite directions with extremal winding. Moreover,

$$P^+ * P^+ = P^- * P^- = P^+ * P^- = 0.$$

Proof. Considering S^2 as the unit sphere in $\mathbb{R}^3$ we can define diffeomorphisms between $\mathbb{C} = \{x + iy\}$ and $P^\pm$ via radial projection from the origin to the planes $(x, y, \pm 1)$. In standard polar coordinates $x = R\cos\Theta$, $y = R\sin\Theta$ on $\mathbb{C}$, this radial projection map from $P^\pm \to \mathbb{C}$ is given by

$$\begin{aligned} R(\theta, \phi) &= \tan\theta \\ \Theta(\theta, \phi) &= \phi. \end{aligned} \tag{5.8}$$

Since i in these coordinates is given by

$$i(R\,\partial_R) = \partial_\Theta \qquad i(\partial_\Theta) = -R\,\partial_R$$

and a straightforward computation shows that

$$R\,\partial_R = \sin\theta\cos\theta\,\partial_\theta \qquad \partial_\Theta = \partial_\phi$$

under the coordinate change (5.8), we find that the radial projection map induces a smooth complex structure on $T(S^2 \setminus \{\theta = \pi/2\})$ given by

$$\begin{aligned} j\partial_\theta &= \sec\theta\csc\theta\,\partial_\phi \\ j\partial_\phi &= -\cos\theta\sin\theta\,\partial_\theta \end{aligned} \tag{5.9}$$

in which the upper and lower hemispheres $P^\pm$ of S^2 are conformally equivalent to $\mathbb{C}$.

We next claim that if $i_0 : S^2 \to \mathbb{R} \times S^2$ is the inclusion

$$p \in S^2 \mapsto (0, p) \in \mathbb{R} \times S^2$$

then $i_0^*\lambda_f \circ j$ is exact. We have that λ_f along the sphere $\rho = 0$ is given by

$$\lambda_f = f(0)\left[3\cos\theta\,d\rho + \frac{1}{2}\sin^2\theta\,d\phi\right]$$

so the pullback of λ_f to the sphere is given by

$$i_0^*\lambda_f = \frac{1}{2}f(0)\sin^2\theta\,d\phi.$$

From (5.9) we have

$$d\phi \circ j = \sec\theta\csc\theta\,d\theta$$

and hence

$$\begin{aligned} i_0^*\lambda_f \circ j &= \frac{1}{2}f(0)\sin^2\theta\,d\phi \circ j \\ &= \frac{1}{2}f(0)\tan\theta\,d\theta \\ &= d\left(-\frac{1}{2}f(0)\log|\cos\theta|\right). \end{aligned}$$

We note that the function

$$(\theta,\phi) \mapsto -\frac{1}{2} f(0) \log|\cos\theta|$$

is smooth on the upper and lower hemispheres as a result of Lemma A.1.

Next, let

$$\pi_\xi : T(\mathbb{R}\times S^2) = \mathbb{R}X_f \oplus \xi_1 \to \xi_1$$

be the projection onto ξ_1 along X_f, given by the formula

$$\pi_\xi(v) = v - \lambda_f(v)X_f.$$

We claim that along $\rho = 0$, $\pi_\xi|_{TS^2} : TS^2 \to \xi_1$ is an isomorphism away from $\theta = \pi/2$. Along $\rho = 0$, the Reeb vector field is given by

$$\begin{aligned} X_f &= [g(\theta)f(0)^2]^{-1}\,[2f(0)\,\partial_\phi + f(0)\cos\theta\,\partial_\rho] \\ &= [(2\cos^2\theta + 1)f(0)]^{-1}\,[2\,\partial_\phi + \cos\theta\,\partial_\rho] \end{aligned}$$

and λ_f is given by

$$\lambda_f = f(0)\left[3\cos\theta\,d\rho + \frac{1}{2}\sin^2\theta\,d\phi\right].$$

Thus, with v_1 and v_2 as defined by (5.4), straightforward computation shows that

$$\begin{aligned} \pi_\xi(\partial_\theta) &= \partial_\theta - \lambda_f(\partial_\theta)X_f \\ &= \partial_\theta \\ &= v_2(0,\theta,\phi) \end{aligned}$$

and

$$\begin{aligned} \pi_\xi(\partial_\phi) &= \partial_\phi - \lambda_f(\partial_\phi)X_f \\ &= -g(\theta)^{-1}\cos\theta\sin\theta\left[-3\cot\theta\,\partial_\phi + \frac{1}{2}\sin\theta\,\partial_\rho\right] \\ &= -f(0)\cos\theta\sin\theta\,v_1(0,\theta,\phi). \end{aligned}$$

This computation shows that $\pi_\xi|_{TS^2}$ is an isomorphism when $\theta \notin \{0, \pi/2, \pi\}$, and since $TS^2 = \xi_1$ at the north and south poles $\theta \in \{0,\pi\}$, it follows that $\pi_\xi|_{TS^2}$ is an isomorphism away from $\theta = \pi/2$.

Given the results of the previous two paragraphs, we define a compatible almost complex structure J' on $\xi_1|_{\rho=0,\theta\neq\pi/2}$ by

$$J' = \pi_\xi|_{TS^2} \circ j \circ (\pi_\xi|_{TS^2})^{-1}, \tag{5.10}$$

and, as long as J' extends smoothly over the equator $\theta = \pi/2$, we have found a compatible J' along $\rho = 0$ for which $P^\pm$ are projected $\tilde{J}'$-holomorphic curves. Since the space of compatible complex multiplications on a given symplectic vector space is nonempty and contractible, there are no obstructions to extending a compatible J' defined on $\rho = 0$ smoothly to a $J' \in \mathcal{J}(\mathbb{R}\times S^2, \lambda_f)$ which agrees outside of any given open neighborhood of $\{0\}\times S^2$ with any previously chosen $J \in \mathcal{J}(\mathbb{R}\times S^2, \lambda_f)$. The computation of the previous paragraph together with the definition (5.9) of j shows, however, that a J' defined by (5.10) will satisfy

$$\begin{aligned} J'(0,\theta,\phi)v_1(0,\theta,\phi) &= \tfrac{1}{f(0)}\,v_2(0,\theta,\phi) \\ J'(0,\theta,\phi)v_2(0,\theta,\phi) &= -f(0)\,v_1(0,\theta,\phi) \end{aligned}$$

away from the north and south poles $\theta \in \{0,\pi\}$. Since v_1 and v_2 are a smooth basis for ξ_1 on $\theta \notin \{0,\pi\}$, this implies that the J' defined by (5.10) extends smoothly over the equator $\theta = \pi/2$.

It remains to check that $P^{\pm}$ approach γ_0 in opposite directions with extremal winding and that

$$P^{\pm} * P^{\pm} = P^{+} * P^{-} = 0. \tag{5.11}$$

We first claim that the planes approach their asymptotic limit with extremal winding, i.e. that

$$\mathrm{wind}_{\infty}^{\Phi}(P^{\pm}) = \lfloor \mu^{\Phi}(\gamma_0)/2 \rfloor. \tag{5.12}$$

To see this, we observe that large $R =$ constant loops in $\mathbb{C}$ get mapped via the identification (5.8) to $\theta = c$ loops with c some constant close to but not equal to $\pi/2$. It's straightforward to see that such loops lift via the exponential map to sections of $\xi_1|_{\gamma_0}$ which have zero winding relative to the trivialization Φ arising from the framing $\mathcal{B}_{(0,\pi/2,\phi)} = \left\{\frac{1}{2f(0)}\partial_\rho, \partial_\theta\right\}$ from (5.4). Thus $\mathrm{wind}_{\infty}^{\Phi}(P^{\pm}) = 0$. Since we have already computed in Lemma 5.6 that $\mu^{\Phi}(\gamma_0) = 0$, we have confirmed (5.12). Since P^{+} and P^{-} are disjoint and γ_0 is even, it then follows immediately from Lemma 3.15 and Theorem 3.16 that P^{+} and P^{-} approach γ_0 in opposite directions. Finally we prove that all intersection numbers are zero, i.e. that (5.11) holds. We first observe that P^{+} and P^{-} are, by construction, disjoint embeddings. Since we've already confirmed that P^{+} and P^{-} both converge to their unique asymptotic limit with extremal winding, (5.11) is an immediately consequence of Corollaries 3.11 and 3.14. □

We are now prepared to prove Theorem 5.1.

Proof of Theorem 5.1. Recall $(M, \xi = \ker\lambda)$ denotes a contact 3-manifold equipped with a nondegenerate contact form. Given $p \neq q \in M$ and any open neighborhood $\mathcal{O}$ of $\{p,q\}$, we can apply Darboux's theorem for contact manifolds (see e.g. [18, Theorem 2.24]) to find disjoint open neighborhoods O_p, $O_q \subset \mathcal{O}$ of p and q respectively, and embeddings $\phi_p : O_p \to \mathbb{R}^3$, and $\phi_q : O_q \to \mathbb{R}^3$ with $\phi_p(p) = 0$, $\phi_q(q) = 0$ and

$$\begin{aligned} \phi_p^*\lambda_+ &= \lambda \\ \phi_q^*\lambda_- &= \lambda. \end{aligned} \tag{5.13}$$

Choosing an $\varepsilon > 0$ so that $\overline{B_\varepsilon(0)} \times [-\varepsilon,\varepsilon] \subset \phi_p(O_p) \cap \phi_q(O_q)$ gives flow tube neighborhoods

$$U = \phi_p^{-1}\left(B_\varepsilon(0) \times (-\varepsilon,\varepsilon)\right)$$
$$V = \phi_q^{-1}\left(B_\varepsilon(0) \times (-\varepsilon,\varepsilon)\right)$$

of p and q, respectively, identified via ϕ_p and ϕ_q with neighborhoods of 0 in $(\mathbb{R}^3, \lambda_+)$ and $(\mathbb{R}^3, \lambda_-)$, respectively.

Next, with the maps $\Phi_\pm : \mathbb{R}^\pm \times S^2 \to \mathbb{R}^3 \setminus \{0\}$ as defined above in (5.1), choose an $\varepsilon' > 0$ so that

$$\Phi_+\left((0,\varepsilon') \times S^2\right) \subset B_\varepsilon(0) \times (-\varepsilon,\varepsilon)$$
$$\Phi_-\left((-\varepsilon',0) \times S^2\right) \subset B_\varepsilon(0) \times (-\varepsilon,\varepsilon).$$

Given such an $\varepsilon' > 0$ we can find a smooth positive function $f : \mathbb{R} \to \mathbb{R}^+$ satisfying[13]

- $f(\rho) = \rho^2$ for $|\rho| \geq \varepsilon'/2$,
- $\rho f'(\rho)$ for $\rho \neq 0$, and
- $f''(0) > 0$.

As explained following Lemma 5.2 above, this f gives us a contact form $\lambda_f = f\lambda_1$ on $(-\varepsilon', \varepsilon') \times S^2$ so that the maps

$$\Phi_+ : \big((0, \varepsilon') \times S^2, \lambda_f\big) \to (B_\varepsilon(0) \times (-\varepsilon, \varepsilon) \setminus \{0\}, \lambda_+)$$
$$\Phi_- : \big((-\varepsilon', 0) \times S^2, \lambda_f\big) \to (B_\varepsilon(0) \times (-\varepsilon, \varepsilon) \setminus \{0\}, \lambda_-)$$

are contact diffeomorphisms onto their images which, on $(\varepsilon'/2, \varepsilon') \times S^2$ and $(-\varepsilon', -\varepsilon'/2) \times S^2$, satisfy

$$\Phi_\pm^* \lambda_\pm = \rho^2 \lambda_1 = f\lambda_1$$

and hence by (5.13)

$$(\phi_p^{-1} \circ \Phi_+)^* \lambda = f\lambda_1 = \lambda_f$$
$$(\phi_q^{-1} \circ \Phi_-)^* \lambda = f\lambda_1 = \lambda_f. \tag{5.14}$$

We then define

$$M' = \big(M \setminus \{p, q\} \amalg (-\varepsilon', \varepsilon') \times S^2\big) / \sim$$

where $\sim$ is the equivalence relation identifying points in $(-\varepsilon', 0) \times S^2$ and $(0, \varepsilon') \times S^2$ with their respective images in $M \setminus \{p, q\}$ under $\phi_q^{-1} \circ \Phi_-$ and $\phi_p^{-1} \circ \Phi_+$, and we define

$$i : M \setminus \{p, q\} \to M'$$

to be the naturally induced inclusion. With

$$B_p := (\phi_p^{-1} \circ \Phi_+) \big((0, \varepsilon'/2) \times S^2\big)$$
$$B_q := (\phi_q^{-1} \circ \Phi_-) \big((-\varepsilon'/2, 0) \times S^2\big),$$

we define λ' on M' by

$$\lambda' = \begin{cases} \lambda & \text{on } M \setminus \overline{B_p \cup B_q} \\ \lambda_f & \text{on } (-\varepsilon', \varepsilon) \times S^2. \end{cases}$$

It follows from (5.14) and the definition of M' that λ' defines a smooth contact form on M'. Moreover, it is easily verified that items (2) and (3) in theorem are satisfied.

Next, since the neck

$$N := M' \setminus \overline{i(M \setminus \{U \cup V\})}$$

equipped with the contact form λ' can be identified via the maps $\Phi_+^{-1} \circ \phi_p$ and $\Phi_-^{-1} \circ \phi_q$ with a subset of $\mathbb{R} \times S^2$ equipped with the contact form λ_f, it follows immediately from Lemma 5.5 that there is precisely one (simple) periodic orbit γ_0 of the Reeb vector field $X_{\lambda'}$ of λ' contained in the neck. Thus any other (simple) periodic orbit $X_{\lambda'}$ must pass through points of $M' \setminus N$. Moreover, it follows from Lemma 5.6 that γ_0 is an even orbit. Thus item (4) of the theorem is verified. Item (5) meanwhile follows immediately from Lemma 5.7.

To see that Condition (6) holds, we note that since the Reeb vector field of λ_+ is ∂_z, the points $p_\pm := \phi_p^{-1}(0, 0, \pm\varepsilon)$ are points in ∂U which are connected by a flow

[13]Such functions are easy to construct. See Lemma A.6 for an example.

line $\gamma_p(t) = \phi_p^{-1}(0,0,t)$ which is contained in $\bar{U}$ and passes through $p = \phi_p(0,0,0)$. Moreover, since we can easily verify from (5.1) that

$$\Phi_+^{-1}\left(\{(0,0,z)\,|\,z>0\}\right) = \mathbb{R}^+ \times \{\theta = 0\}$$
$$\Phi_+^{-1}\left(\{(0,0,z)\,|\,z<0\}\right) = \mathbb{R}^+ \times \{\theta = \pi\}$$

it follows that

$$\Phi_+^{-1} \circ \phi_p \circ \gamma_p\left((0,\varepsilon)\right) \subset \mathbb{R}^+ \times \{\theta = \pi\}$$
$$\Phi_+^{-1} \circ \phi_p \circ \gamma_p\left((-\varepsilon,0)\right) \subset \mathbb{R}^+ \times \{\theta = 0\}\,.$$

We can then apply the second claim of Lemma 5.5 to conclude that item (6a) of the theorem holds. Item (6b) follows similarly.

At this point, all claims of Theorem 5.1 hold with the possible exception of item (1): nondegeneracy of the contact form λ'. By construction we have that $i^*\lambda' = \lambda$ outside of the region identified with $(-\varepsilon'/2, \varepsilon'/2) \times S^2$ in the construction. Thus any periodic orbit created by this construction (i.e. not identified via i with a periodic orbit of X_λ) must pass through the region $(-\varepsilon'/2, \varepsilon'/2) \times S^2$. Moreover, since γ_0 is the only periodic orbit contained within this region, any other new (and thus potentially nondegenerate) orbit created in the connected sum operation must pass through the boundary $\{\pm\varepsilon'/2\} \times S^2$ of the region. Applying results from [46], we can find a C^∞-small function $h : M' \to \mathbb{R}$ supported in an arbitrarily small neighborhood of these spheres, so that the contact form $e^h\lambda'$ has only nondegenerate periodic orbits. Moreover, since the support and (C^∞-) size of the function can both be chosen arbitrarily small, and since all claims of the theorem remain true under sufficiently C^∞-small perturbations with sufficiently small support in a neighborhood of $\{\pm\varepsilon'/2\} \times S^2$, we can carry out this perturbation of the contact form while maintaining all the claims of the theorem. □

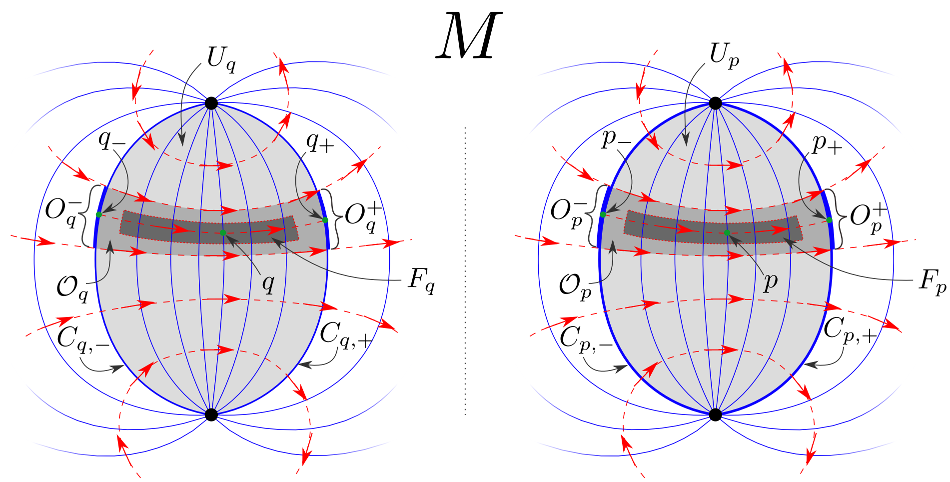
6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Here we will use the alternate definition of finite energy foliation furnished by Corollary 4.5, and will thus work almost exclusively with projections of pseudoholomorphic curves to the 3-manifold. All curves should thus be thought of as equivalence classes of maps to the 3-manifold unless otherwise stated. Since we deal nearly exclusively with simple curves (i.e. those which do not factor though a branched cover of degree 2 or greater) such an equivalence class of maps is entirely determined by the image in M of a representative map from the class. We will thus generally make no distinction between a curve and its image in M.

Our standing assumptions throughout the section will be:

- (M, λ) is a 3-manifold with a nondegenerate contact form λ,
- $J \in \mathcal{J}(M, \lambda)$ is a compatible complex structure on $\xi = \ker\lambda$, and
- $\mathcal{F}$ is a stable finite energy foliation for the data (M, λ, J) with energy $E(\mathcal{F}) = E_0$.

Given the foliation $\mathcal{F}$, we consider a subset $\mathcal{U}$ of $M \times M \setminus \Delta(M)$ defined to be the set of pairs of distinct points $(p,q) \in M \times M \setminus \Delta(M)$ in M with p and q lying on distinct index-2 leaves of the foliation. It is straightforward to use Lemma 4.3 and Corollary 4.8 to argue that $\mathcal{U}$ is an open, dense subset of $M \times M \setminus \Delta(M)$. We will show that the manifold M' formed by taking the connected sum at any given pair



of points $(p, q) \in \mathcal{U}$ admits a contact form λ' and compatible $J' \in \mathcal{J}(M', \lambda')$ so that the data (M', λ', J') admits a stable finite energy foliation $\mathcal{F}'$ with $E(\mathcal{F}') = E(\mathcal{F})$. Moreover, our construction will show that the change in the contact form and almost complex structure can be localized to an arbitrarily small neighborhood of the points p and q; that is, if $i : M \setminus \{p, q\} \to M'$ is the natural inclusion, we can arrange that $i^*\lambda' = \lambda$ and $i^*J' = J$ on the complement of any given neighborhood of $\{p, q\}$.

To start the construction, we choose a pair of points $(p, q) \in \mathcal{U}$; that is, we choose distinct points p and q in M so that

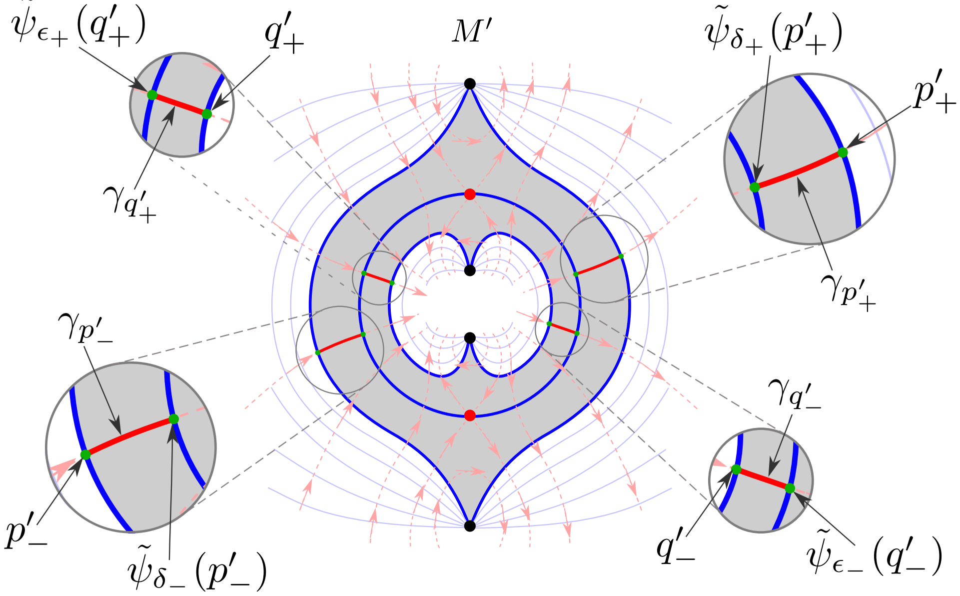
- p lies on a curve $C_p \in \mathcal{F}$ with $\mathrm{ind}(C_p) = 2$,
- q lies on a curve $C_q \in \mathcal{F}$ with $\mathrm{ind}(C_q) = 2$, and
- $C_q \neq C_p$ (in $\mathcal{M}(\lambda, J)/\mathbb{R}$).

We recall from Corollary 4.4 that all curves in the moduli spaces $\mathcal{M}(C_p)/\mathbb{R}$ and $\mathcal{M}(C_q)/\mathbb{R}$ are also in the foliation $\mathcal{F}$ (where $\mathcal{M}(C)$ is the notation introduced in Section 4 to indicate all simple curves in $\mathcal{M}(\lambda, J)$ which are relatively homotopic to C). Letting ψ_t denote the flow generated by the Reeb vector field X_λ associated to λ, we can apply Corollary 4.9 to find an $\varepsilon_0 > 0$ so that

- for each $t \in [-\varepsilon_0, \varepsilon_0]$ there is a unique curve of $\mathcal{M}(C_p)/\mathbb{R}$ passing through the point $\psi_t(p)$, and so that the map taking $t \in [-\varepsilon_0, \varepsilon_0]$ to the unique curve in $\mathcal{M}(C_p)/\mathbb{R}$ passing through $\psi_t(p)$ is an embedding;
- for each $t \in [-\varepsilon_0, \varepsilon_0]$ there is a unique curve of $\mathcal{M}(C_q)/\mathbb{R}$ passing through the point $\psi_t(q)$, and so that the map taking $t \in [-\varepsilon_0, \varepsilon_0]$ to the unique curve in $\mathcal{M}(C_q)/\mathbb{R}$ passing through $\psi_t(q)$ is an embedding;
- the collection of curves passing through the points $\psi_t(p)$ for $t \in [-\varepsilon_0, \varepsilon_0]$ and the collection of curves passing through the points $\psi_t(q)$ for $t \in [-\varepsilon_0, \varepsilon_0]$ are disjoint.

We define points $p_\pm = \psi_{\pm\varepsilon_0}(p)$ and similarly $q_\pm = \psi_{\pm\varepsilon_0}(q)$, and let $C_{p,\pm}$ denote the unique curve in $\mathcal{M}(C_p)/\mathbb{R}$ through $p_\pm$ and similarly let $C_{q,\pm}$ denote the unique curve in $\mathcal{M}(C_q)/\mathbb{R}$ through $q_\pm$.

Given the above, we define an open set U_p to be the union of the images of the curves in $\mathcal{M}(C_p)/\mathbb{R}$ passing through the points $\psi_t(p)$ for $t \in (-\varepsilon_0, \varepsilon_0)$, and



similarly U_q is the image of the curves in $\mathcal{M}(C_q)/\mathbb{R}$ passing through the points $\psi_t(q)$ for $t \in (-\varepsilon_0, \varepsilon_0)$. There exists an open neighborhood O_p^- of p_- in $C_{p,-}$ and a positive function $f_p : O_p^- \to \mathbb{R}^+$ so that

$$\left\{\psi_{f(z)}(z) \,|\, z \in O_p^-\right\}$$

is an open neighborhood of p_+ in $C_{p,+}$ and $\psi_t(z) \in U_p$ for all $t \in (0, f(z))$. Define an open set $\mathcal{O}_p$ by

$$\mathcal{O}_p = \bigcup_{z \in O_p^-} \bigcup_{t \in (0, f(z))} \psi_t(z)$$

and define an open set $\mathcal{O}_q \subset U_q$ analogously. Choosing an open flow tube neighborhood F_p of p with $\bar{F}_p \subset \mathcal{O}_p$ and an open flow tube neighborhood F_q of q with $\bar{F}_q \subset \mathcal{O}_q$, we can apply Theorem 5.1 to find a nondegenerate contact manifold $(M', \xi' = \ker \lambda')$ with compatible $J' \in \mathcal{J}(M', \xi')$, and an embedding $i : M \backslash \{p, q\} \to M'$ so that:

(1) M' is diffeomorphic to the connected sum of M taken at p and q, and the set $M' \setminus i(M \setminus (F_p \cup F_q))$ (called the neck) is diffeomorphic to $\mathbb{R} \times S^2$.
(2) On $M \setminus (F_p \cup F_q)$, $i^*\lambda' = \lambda$ and $i^*J' = J$.
(3) There is precisely one simple (unparametrized) periodic orbit γ_0 contained in the neck. Moreover γ_0 has even Conley–Zehnder index.
(4) There exist two distinct, nonintersecting, nicely-embedded planes $P^\pm \in \mathcal{M}(\lambda', J')/\mathbb{R}$ contained in the neck which are asymptotic to γ_0 in opposite directions (see discussion preceding Lemma 3.15) with extremal winding (see Theorem 3.2). Moreover,

$$P^+ * P^+ = P^- * P^- = P^+ * P^- = 0,$$

and the union of the images P^+, P^- and γ_0 form a C^1-smooth sphere which divides the neck into two pieces, each homeomorphic to $\mathbb{R} \times S^2$.

(5) If $\tilde{\psi}_t$ is the flow of the Reeb vector field $X_{\lambda'}$ then there exist real numbers $\delta_+ < 0 < \delta_-$ and $\varepsilon_+ < 0 < \varepsilon_-$ so that,
   - $\tilde{\psi}_{\delta_+}(i(p_+)) \in P^+$ and $\tilde{\psi}_t(i(p_+)) \notin P^\pm$ for all $t \in (\delta_+, 0]$.
   - $\tilde{\psi}_{\delta_-}(i(p_-)) \in P^-$ and $\tilde{\psi}_t(i(p_-)) \notin P^\pm$ for all $t \in [0, \delta_-)$.
   - $\tilde{\psi}_{\varepsilon_+}(i(q_+)) \in P^-$ and $\tilde{\psi}_t(i(q_+)) \notin P^\pm$ for all $t \in (\varepsilon_+, 0]$.
   - $\tilde{\psi}_{\varepsilon_-}(i(q_-)) \in P^+$ and $\tilde{\psi}_t(i(q_-)) \notin P^\pm$ for all $t \in [0, \varepsilon_-)$.

   Letting $p'_\pm := i(p_\pm)$ and $q'_\pm := i(q_\pm)$, we define embedded flow-line segments $\gamma_{p'_\pm}, \gamma_{q'_\pm} \subset M'$ by:
   - $\gamma_{p'_+} = \left\{\tilde{\psi}_t(p'_+) \,|\, t \in [\delta_+, 0]\right\}$
   - $\gamma_{p'_-} = \left\{\tilde{\psi}_t(p'_-) \,|\, t \in [0, \delta_-]\right\}$
   - $\gamma_{q'_+} = \left\{\tilde{\psi}_t(q'_+) \,|\, t \in [\varepsilon_+, 0]\right\}$
   - $\gamma_{q'_-} = \left\{\tilde{\psi}_t(q'_-) \,|\, t \in [0, \varepsilon_-]\right\}$

Since $i^*\lambda' = \lambda$ and $i^*J' = J$ outside of $F_p \cup F_q$, and since the curves $C_{p,\pm}$ do not meet $F_p \cup F_q$, it follows that $C'_{p,\pm} := i(C_{p,\pm})$ are nicely-embedded finite-energy curves in $\mathcal{M}(\lambda', J')/\mathbb{R}$. We note that since $p \in F_p \subset U_p$ and $q \in F_q \subset U_q$ the connected sum operation yields an open set $U \subset M'$ arising as the connected sum of the sets U_p and U_q taken at p and q. Moreover, the set U is divided into two open subsets by sphere formed by $P^+ \cup \gamma_0 \cup P^-$ in the neck. We call the subset coming from the p-side of the connected sum U'_p and the subset coming from the q-side of the connected sum U'_q. We note that, by construction, the boundary of U'_p consists of the curves $C'_{p,\pm}$ and their asymptotic limits, along with $P^\pm$ and γ_0.

Lemma 6.1. *There are no periodic orbits of $X_{\lambda'}$ contained within U'_p.*

Proof. We first argue that there are no periodic orbits γ of the Reeb vector field X_λ on the unsurgered manifold contained in the set U_p. Indeed, since U_p is foliated by curves in $\mathcal{M}(C_p)/\mathbb{R}$, then γ would intersect some curve $C' \in \mathcal{M}(C_p)/\mathbb{R}$, which implies that $(\mathbb{R}\times\gamma)*C' > 0$. By homotopy invariance of the $*$-product, we conclude $(\mathbb{R}\times\gamma)*C > 0$ for any $C \in \mathcal{M}(C_p)/\mathbb{R}$ and, in particular, $(\mathbb{R}\times\gamma)*C_{p,\pm} > 0$. However, since γ is contained in U_p and thus not an asymptotic limit of $C_{p,\pm}$, this implies that γ intersects $C_{p,\pm}$. This, however, contradicts the assumption that $\gamma \subset U_p$, and we conclude that there are no periodic orbits γ of X_λ contained in U_p.

Now, assume γ is a simple periodic orbit of $X_{\lambda'}$ with $\gamma \subset U'_p$. Then, we claim the previous paragraph shows that γ must enter the neck. Indeed if not, then γ is identified via the map $i : M \setminus \{p, q\} \to M'$ with a periodic orbit of X_λ contained in U_p, of which, we have just argued, there are none. Moreover, since γ_0 is the only periodic orbit contained entirely within the neck, γ must pass through points of U'_p both inside and outside the neck. But by construction — specifically that the connected sum is carried out in flow tubes neighborhoods contained in open sets consisting of flow lines connecting the curves $C_{p,+}$ and $C_{p,-}$ (or $C_{q,+}$ and $C_{q,-}$) — any flow line entering the neck in U'_p must hit $C'_{p,-}$ in backward time, while any flow line exiting the neck in U'_p must hit $C'_{p,+}$ in forward time. Thus γ intersects either $C'_{p,+}$ or $C'_{p,-}$ which contradicts the assumption that $\gamma \subset U'_p$. □

Lemma 6.2. *Let $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$. Then either $C \subset U'_p$ or $C \cap U'_p = \emptyset$. Moreover, the set of curves $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ with $C \subset U'_p$ is nonempty and open.*

Proof. We first observe that, by construction, the boundary of U'_p consists of the pseudoholomorphic curves $C'_{p,\pm}$, $P^\pm$ and their asymptotic limits. Moreover, we know that $C'_{p,\pm} * C'_{p,\pm} = C'_{p,+} * C'_{p,-} = 0$ and, since the curves $P^\pm$ are disjoint from and share no common orbits with the curves $C'_{p,\pm}$, it follows immediately from the definition of the $*$-product that $C'_{p,+} * P^\pm = C'_{p,-} * P^\pm = 0$. Given a curve $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$, homotopy invariance of the intersection number then allows us to conclude that the intersection number of C with each of $C'_{p,\pm}$ and $P^\pm$ is zero. Theorem 3.9/3.10 then lets us further conclude that C doesn't intersect any of the asymptotic limits of the curves in the boundary of U'_p. Thus the curve C can't intersect any of the curves in the boundary of U'_p unless it coincides with that curve. We conclude that either $C \subset U'_p$ or C is disjoint from U'_p.

We next show that the set of curves $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ with $C \subset U'_p$ is nonempty. Given that we've shown in the previous paragraph that a curve $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ meeting U'_p must be contained in U'_p, it suffices to show there are curves $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ meeting U'_p. This follows from Corollary 4.8. Indeed, since the evaluation map $ev : \mathcal{M}_1(C'_{p,\pm})/\mathbb{R} \to M$ is an embedding, the image of the evaluation map is open. Therefore, given any point $x \in C'_{p,\pm} \subset \partial U'_p$, there is an open set around x in the image of evaluation map. Since an open set around a boundary point of U'_p must meet U'_p, there are points in U'_p in the image of the evaluation map, which is equivalent to there being points in U'_p with curves in $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ passing through them.

Finally we show that the set of curves $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ with $C \subset U'_p$ is open. This follows from Corollary 4.9. Indeed, given a curve $C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ passing through a point $x \in U'_p$ there is an $\varepsilon > 0$ so that for every $t \in (-\varepsilon, \varepsilon)$ there is a unique curve of $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ passing through $\tilde{\psi}_t(x)$, and so that the map taking a point $t \in (-\varepsilon, \varepsilon)$ to the unique curve in $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ passing through $\tilde{\psi}_t(x)$ is an embedding. Since U'_p is open, we can, by shrinking ε if necessary, assume that $\tilde{\psi}_t(x) \in U'_p$ for all $t \in (-\varepsilon, \varepsilon)$. We have thus found a subset I of $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ diffeomorphic to an open interval and containing C so that each curve $C' \in I$ meets U'_p and thus, according to the results of the first paragraph, is contained in U'_p. □

Since, as we've noted above, the images of $C'_{p,\pm}$ are contained in the boundary of the set U'_p, it follows that the curves $C'_{p,\pm}$ are in the boundary of the set $\{C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R} \,|\, C \subset U'_p\}$. We define a submanifold with boundary $\mathcal{M}_{p,\pm} \subset \mathcal{M}(C'_{p,\pm})/\mathbb{R}$ to be the connected component of

$$\{C'_{p,\pm}\} \cup \{C \in \mathcal{M}(C'_{p,\pm})/\mathbb{R} \,|\, C \subset U'_p\}$$

containing $C'_{p,\pm}$. As a result of the above discussion, it is clear that $\mathcal{M}_{p,\pm}$ is diffeomorphic to a half-open interval, and in the following we seek to characterize $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$, where $\overline{\mathcal{M}_{p,\pm}}$ denotes the compactification of $\mathcal{M}_{p,\pm}$ in the SFT topology (relevant information is reviewed in Section 3.2 above). According to our construction — specifically that $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ is a smooth 1-manifold and $\mathcal{M}_{p,\pm}$ can be identified with an embedded half-open subinterval — the set $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ is either an element of $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ or is contained in the boundary of $\mathcal{M}(C'_{p,\pm})/\mathbb{R}$ and thus, according to the main theorem of [58] (reviewed above as Theorem 3.3), consists of stable, nicely-embedded, non-nodal pseudoholomorphic buildings. The next lemma shows that the latter alternative in fact always holds.

Lemma 6.3. *Every element of $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ is a stable, nicely-embedded, non-nodal pseudoholomorphic building with at least two nontrivial components and at least two levels.*

Proof. We prove this for $\mathcal{M}_{p,+}$. The proof for $\mathcal{M}_{p,-}$ is identical.

We first argue that $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ is either a single element of $\mathcal{M}(C'_{p,+})/\mathbb{R}$ or is a subset of the boundary of $\mathcal{M}(C'_{p,+})/\mathbb{R}$. As we discussed above, $\mathcal{M}(C'_{p,+})/\mathbb{R}$ has the structure of a smooth 1-manifold and, by construction, $\mathcal{M}_{p,+}$ is an embedded submanifold with boundary which is diffeomorphic to a half-open interval. Choosing an identification of $\mathcal{M}_{p,+}$ with an embedding $i : [0,1) \hookrightarrow \mathcal{M}(C'_{p,+})/\mathbb{R}$, we can identify the set $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ with limits of SFT-convergent sequences of the form $i(a_k)$ with $a_k \in [0,1)$ an increasing sequence converging to 1. Assume that there exists some such sequence a_k so that $i(a_k)$ converges to a curve $C_\infty \in \mathcal{M}(C'_{p,+})/\mathbb{R}$. Then every open neighborhood of C_∞ in $\mathcal{M}(C'_{p,+})/\mathbb{R}$ meets a set of the form $i\left((1-\varepsilon,1)\right)$ for some $\varepsilon > 0$. Since i is an embedding of a 1-manifold in a 1-manifold, this allows us to conclude that for every sequence $a_k \in [0,1)$ with $a_k \to 1$, $i(a_k)$ converges to C_∞. We conclude that if $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ contains an interior point C_∞ of $\mathcal{M}(C'_{p,+})$ then $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm} = \{C_\infty\}$. Thus $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ is either a single element of $\mathcal{M}(C'_{p,+})/\mathbb{R}$ or is a subset of the boundary of $\mathcal{M}(C'_{p,+})/\mathbb{R}$ as claimed.

To show that $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ can't be a curve in $\mathcal{M}(C'_{p,+})/\mathbb{R}$ we will argue by contradiction and suppose to the contrary that $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ is an element of $\mathcal{M}(C'_{p,+})/\mathbb{R}$. In this case $\overline{\mathcal{M}_{p,+}}$ is diffeomorphic to a closed interval embedded in $\mathcal{M}(C'_{p,+})/\mathbb{R}$ and thus every sequence in $\mathcal{M}_{p,+}$ has a subsequence converging to an element of $\mathcal{M}(C'_{p,+})/\mathbb{R}$. We then claim that every point of U'_p would have a curve from $\mathcal{M}_{p,+}$ passing through it and, since $\mathcal{M}_{p,+}$ consists of foliating curves, that U'_p is homeomorphic to $\mathbb{R} \times (\Sigma \setminus \Gamma)$. Since, by construction, U'_p is homeomorphic to $(\mathbb{R} \times (\Sigma \setminus \Gamma))$ with a point removed, this contradiction would finish the proof.

Supposing then that every sequence in $\mathcal{M}_{p,+}$ has a subsequence converging to an element of $\mathcal{M}(C'_{p,+})/\mathbb{R}$, we claim that the set of points in U'_p with a curve in $\mathcal{M}_{p,+} \setminus \left\{C'_{p,+}\right\}$ passing through them is nonempty and both open and (relatively) closed in U'_p. Both nonemptiness and openness follow immediately from Corollary 4.8; indeed, since the evaluation map $ev : \mathcal{M}_1(C'_{p,+})/\mathbb{R} \to M$ is an embedding it has open image, and therefore the set of points in U'_p with curves from the open subset $\mathcal{M}_{p,+}$ of $\mathcal{M}(C'_{p,+})/\mathbb{R}$ passing through them is nonempty and open. Closedness, in turn, follows from our assumption that all sequences in $\mathcal{M}_{p,+}$ have subsequences converging to a point in $\mathcal{M}(C'_{p,+})/\mathbb{R}$. Indeed, let p_k be a sequence in U'_p converging to a point $p_\infty \in U'_p$, and assume that for each point p_k there is a curve $C_k \in \mathcal{M}_{p,+}$ passing through p_k. Then by assumption, some subsequence of the curves C_k converges to a curve $C_\infty \in \mathcal{M}(C'_{p,+})/\mathbb{R}$ passing through p_∞ and, by the previous lemma, we have that $C_\infty \subset U'_p$ since it passes through the point $p_\infty \in U'_p$. Since Corollary 4.8 implies there is an open set of points around p_∞ having curves of $\mathcal{M}(C'_{p,+})/\mathbb{R}$ passing through them, it follows that C_∞ is in the same connected component of the subset

$$\left\{C \in \mathcal{M}(C'_{p,+})/\mathbb{R} \,|\, C \subset U'_p\right\}$$

of $\mathcal{M}(C_{p,+})$ as the C_k i.e. that $C_\infty \in \mathcal{M}_{p,+}$. This completes the proof that $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ can't be an interior point of $\mathcal{M}(C'_{p,+})/\mathbb{R}$, and therefore must be contained in the (SFT) boundary of $\mathcal{M}(C'_{p,+})/\mathbb{R}$. As noted above, it follows from the main theorem of [58] that $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ consists of stable, non-nodal, nicely-embedded pseudoholomorphic buildings.

Finally we show that any element of $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ has at least two levels and at least two nontrivial components. We first note that an element of $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ is connected since it is the limit of connected curves. Supposing such an element has only one level, then that level must have at least two components (or else we would just have a curve in $\mathcal{M}(C'_{p,+})/\mathbb{R}$). But, since a height-1 pseudoholomorphic building with at least two components and no nodes must be disconnected, this is a contradiction. Given that an element of $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ has at least two levels, stability implies that each level must have at least one nontrivial component. □

We next seek to show that the sets $\overline{\mathcal{M}}_{p,+} \setminus \mathcal{M}_{p,+}$ and $\overline{\mathcal{M}}_{p,-} \setminus \mathcal{M}_{p,-}$ each contain a single non-nodal, nicely-embedded pseudoholomorphic building, and that these buildings consist of exactly one of the planes $P^{\pm}$ and precisely one other nicely-embedded pseudoholomorphic curve Z_p with γ_0 as a negative asymptotic limit.

Proposition 6.4. *Let Γ_p^- denote the set of negative punctures of $C_{p,\pm}$ and assume that at $z \in \Gamma_p^-$, $C_{p,+}$ is asymptotic to the orbit γ_z. Then, there exists a pseudoholomorphic curve $Z_p \in \mathcal{M}(M', \lambda')/\mathbb{R}$ so that*

$$\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+} = Z_p \odot (P^+ \amalg_{z\in\Gamma_p^-} \mathbb{R} \times \gamma_z)$$

and

$$\overline{\mathcal{M}_{p,-}} \setminus \mathcal{M}_{p,-} = Z_p \odot (P^- \amalg_{z\in\Gamma_p^-} \mathbb{R} \times \gamma_z).$$

We note that since, as previously observed, the main theorem of [58] (see Theorem 3.3 above) implies that any curve in $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ must be a nicely-embedded building, it follows immediately from this proposition that Z_p is embedded in M and disjoint from $P^{\pm}$.

We prove the proposition in a series of lemmas. In the following we let

$$C_{\infty,+} \in \overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$$

denote one of the nicely-embedded, non-nodal pseudoholomorphic buildings given by Lemma 6.3 above.

Lemma 6.5. *Let γ be a simple periodic orbit and assume that the m-fold cover γ^m is a positive (resp. negative) asymptotic limit of $C'_{p,+}$. Then there exists a nontrivial component of $C_{\infty,+}$ with γ^m as a positive (resp. negative) asymptotic limit.*

Proof. For simplicity we assume that γ^m is a positive asymptotic limit of $C'_{p,+}$. The argument in the case that it is a negative asymptotic limit is identical. Since $C_{\infty,+}$ can be written as the SFT-limit of a sequence of curves $C_k \in \mathcal{M}_{p,+}$, each homotopic to $C'_{p,+}$, it follows from the definition of SFT-convergence (see Proposition 3.4) that the positive asymptotic limits of the top-most level of $C_{\infty,+}$ agree with the positive asymptotic limits of $C'_{p,+}$ and, similarly, the negative asymptotic limits of the bottom-most level of $C_{\infty,+}$ agree with the negative asymptotic limits of $C'_{p,+}$. Thus there is some component of the top level of $C_{\infty,+}$ with γ^m as a positive asymptotic limit. If this component is nontrivial there is no more to prove. If not, then the component with γ^m as a positive asymptotic limit must be a trivial cylinder and, thus, there is some component on the next level down with γ^m as a positive asymptotic limit. Repeating this argument we find either at least one component of $C_{\infty,+}$ with γ^m as a positive puncture, or we can conclude that there is a trivial cylinder over γ^m on the lowest level of $C_{\infty,+}$ and thus that γ^m is also negative asymptotic limits of $C'_{p,+}$ so that (γ, m, m) is a bidirectional orbit of $C'_{p,+}$. Recalling that $C'_{p,+}$ satisfies $C'_{p,+} * C'_{p,+} = 0$ and $\mathrm{ind}(C'_{p,+}) = 2$ it follows Theorem 4.2 that all asymptotic limits of $C'_{p,+}$, and specifically γ^m, have odd Conley–Zehnder index. But, since $C'_{p,+}$ is nicely embedded, it follows from Lemma 3.6 that γ^m must have even Conley–Zehnder index if it were a bidirectional orbit of $C'_{p,+}$. This contradiction completes the proof. □

Lemma 6.6. *The periodic orbit* $(\gamma_0, 1, 1)$ *is the only possible bidirectional asymptotic limit (see Definition 3.5 above) of* $C_{\infty,+}$. *Moreover, if* γ_0 *appears as a limit of a component of* $C_{\infty,+}$, γ_0 *is necessarily a bidirectional orbit.*

Proof. Since $C_{\infty,+}$ is the limit of a sequence of curves contained entirely within the open set U'_p, it follows from the definition of SFT-convergence (see Proposition 3.4) that any periodic orbit appearing as an asymptotic limit of a component of $C_{\infty,+}$ must be contained in the closure $\bar{U}'_p$ of U'_p. Moreover, Lemma 6.1 tells us that there are no orbits contained within U'_p so any periodic orbit within the closure $\bar{U}'_p$ must touch the boundary. However, by construction, the boundary of U'_p consists of the pseudoholomorphic curves $C'_{p,\pm}$ and $P^{\pm}$ and the periodic orbits which are asymptotic limits of these curves, and any flow line touching the curves $C'_{p,\pm}$ of $P^{\pm}$ necessarily passes through points outside of U'_p. We thus conclude that any orbit contained in $\bar{U}'_p$ is contained within the boundary. Since the periodic orbits in the boundary of U'_p are the asymptotic limits of $C'_{p,+}$ along with γ_0, and since every asymptotic limit of $C'_{p,+}$ is odd and, according to the previous lemma, occurs as an asymptotic limit of some nontrivial component of $C_{\infty,+}$ with the same covering numbers, it follows from Lemma 3.6 that the only possible bidirectional limit is γ_0, which can only occur simply covered since γ_0 is even.

Next, we argue that if γ_0 (or in fact any orbit other than the limits of $C'_{p,+}$) appears as an asymptotic limit of a component of $C_{\infty,+}$, then it must be a bidirectional limit. Recall that since $C_{\infty,+}$ is a limit of pseudoholomorphic spheres, its structure can be modeled by a tree with one vertex for each component, and an edge for each periodic orbit connecting adjacent levels (or, in general, for each node, but we know there are none here). Moreover, the positive asymptotic limits of the top level of $C_{\infty,+}$ and the negative asymptotic limits of the bottom level of $C_{\infty,+}$ agree with those of $C'_{p,+}$. Assuming then that γ_0 appears as an asymptotic limit of a component of $C_{\infty,+}$ but is not a bidirectional limit, we can conclude that either the only curves in $C_{\infty,+}$ having γ_0 as a positive limit are trivial cylinders, or the only curves in $C_{\infty,+}$ having γ_0 as a negative limit are trivial cylinders. In either case, we could, by following a path of vertices corresponding to trivial cylinders, conclude that γ_0 is either a positive asymptotic limit of the top level or a negative asymptotic limit of the bottom level. This contradicts the fact that the positive limits of the top level and the negative limits of the bottom level agree with those of $C'_{p,+}$. □

Lemma 6.7. *All components of* $C_{\infty,+}$ *with* γ_0 *as a positive asymptotic limit must be equal to either* P^+ *or* P^-.

Proof. This follows from Theorem 3.16, [50, Theorem 2.4]/Theorem 3.10, and Lemma 3.17. Indeed if C_k is a sequence in $\mathcal{M}_{p,+}$ converging to $C_{\infty,+}$ then $P^{\pm} * C_k = P^{\pm} * C'_{p,+} = 0$ by homotopy invariance of the intersection number. Thus Lemma 3.17 allows us to conclude that any component of $C_{\infty,+}$ is either identical with or disjoint from $P^{\pm}$.

Now, if there were a component Z of $C_{\infty,+}$ distinct from $P^{\pm}$ with γ_0 as a positive puncture, then Z must approach a simple cover of γ_0 since, according to the previous lemma, γ_0 would have to be a bidirectional limit of $C_{\infty,+}$. Condition (3c) of Theorem 3.10 then tells us that Z must approach with the same winding as $P^{\pm}$. But since P^+ and P^- approach γ_0 in opposite directions with extremal winding, Z necessarily approaches γ_0 with extremal winding and thus, according to Lemma 3.15, in the same direction as either P^+ or P^-. Theorem 3.16 then lets us conclude that Z intersects either P^+ or P^-, which contradicts the fact from the previous paragraph that such a Z must be disjoint from P^+ and P^-. □

Lemma 6.8. *There is precisely one nontrivial component of $C_{\infty,+}$ not equal to P^+ or P^-.*

Proof. We first show that there is at most one nontrivial component of $C_{\infty,+}$ distinct from P^+ and P^-. Recall that the building $C_{\infty,+}$, being the limit of spheres, can be modeled by a tree with vertices corresponding to components of the building and edges corresponding to periodic orbits connecting adjacent levels (or, in general, nodes, but we know there are none in this case). Moreover, we know that all components of the building are either nicely-embedded curves or trivial cylinders. Assuming there are two or more nontrivial components in $C_{\infty,+}$ which are distinct from P^+ and P^-, we can find a sequence of components $(C_1, \ldots, C_n)$ of $C_{\infty,+}$ corresponding to distinct, adjacent vertices in the modeling tree, with C_1 and C_n nontrivial and not equal to P^+ or P^-. Since we assume all the C_i correspond to distinct vertices, we can conclude none of the C_i with $i \notin \{1, n\}$ are planes since planes correspond to univalent vertices in the tree modeling $C_{\infty,+}$. Thus all elements of the sequence $(C_1, \ldots, C_n)$ are either trivial cylinders, or nontrivial components distinct from P^+ and P^-. Moreover, by truncating the sequence if necessary, we can assume without loss of generality that only C_1 and C_n are nontrivial and that all C_i for $1 < i < n$ are trivial cylinders. However, since a sequence of adjacent trivial cylinders in the holomorphic building must be cylinders over the same periodic orbit γ, this allows us to conclude that either C_1 has a positive puncture limiting to γ and C_n has a negative puncture limiting to γ, or that C_1 has a negative puncture limiting to γ and C_n has a positive puncture limiting to γ. Thus γ is a bidirectional orbit, so Lemma 6.6 tells us that $\gamma = \gamma_0$. However, since Lemma 6.7 tells us that the planes P^+ and P^- are the only possible components of the building having γ_0 as a positive asymptotic limit, this contradicts our assumption that C_1 and C_m are distinct from $P^{\pm}$. This completes the argument that there is at most one component of $C_{\infty,+}$ distinct from the $P^{\pm}$.

We next argue there is at least one nontrivial component of $C_{\infty,+}$ distinct from P^+ and P^-. If there are no nontrivial components other than P^+ and P^-, then every component of the building is either equal to P^+, P^- or a trivial cylinder. Since $C_{\infty,+}$ is, as the limit of connected curves, a connected building, this would then let us conclude that γ_0 is the only asymptotic limit of components of the building $C_{\infty,+}$ However, since $C_{\infty,+}$ is a limit of curves in $\mathcal{M}(C'_{p,+})$, the properties of SFT-convergence (see Proposition 3.4) allow us to conclude that the asymptotic limits of the top and bottom levels of $C_{\infty,+}$ agree with those of $C'_{p,+}$. Since $C'_{p,+}$ has only odd asymptotic limits and γ_0 is even, this is a contradiction. Thus there must be at least one nontrivial component in the building not equal to P^+ or P^-. □

Lemma 6.9. *Let $C \in \mathcal{M}_{p,+} \setminus \{C'_{p,+}\}$. Then the intersection numbers of the curve C with the flow segments $\gamma_{p'_\pm}$ are well defined and given by*

$$\gamma_{p'_+} \cdot C = 1 \qquad \gamma_{p'_-} \cdot C = 0.$$

Proof. To see this, we first recall the flow segments have boundary in the boundary U'_p and interior in the interior of U'_p. Since curves $C \in \mathcal{M}_{p,+} \setminus \{C'_{p,+}\}$ limit at the punctures to periodic orbits disjoint from $\gamma_{p'_\pm}$ and, by definition of $\mathcal{M}_{p,+}$, are contained entirely within U'_p, it follows that the intersection number of such a curve with the flow segments $\gamma_{p'_\pm}$ is well defined. Moreover, since the image of $C'_{p,+}$ compactifies to a map that is disjoint from $\gamma_{p'_-}$, it follows that curves in $\mathcal{M}_{p,+}$ nearby to $C'_{p,+}$ are also disjoint from $\gamma_{p'_-}$ and thus

$$\gamma_{p'_-} \cdot C = 0$$

for $C \in \mathcal{M}_{p,+} \setminus \{C'_{p,+}\}$. On the other hand, it follows from Corollary 4.9 — specifically that a flow line passing through $C'_{p,+}$ gives a local diffeomorphism with a neighborhood of $C'_{p,+}$ in $\mathcal{M}(C'_{p,+})$ — that curves in $\mathcal{M}_{p,+}$ nearby to $C'_{p,+}$ have a single transverse intersection with $\gamma_{p'_+}$. Thus

$$\gamma_{p'_+} \cdot C = 1$$

for $C \in \mathcal{M}_{p,+} \setminus \{C'_{p,+}\}$ as claimed. □

We will denote by Z_p the nontrivial component of $C_{\infty,+}$ guaranteed by Lemma 6.8.

Lemma 6.10. *The pseudoholomorphic building $C_{\infty,+}$ is a height-2 building with Z_p on the top level, and P^+ and trivial cylinders over the negative orbits of $C'_{p,\pm}$ on the bottom level, i.e.*

$$C_{\infty,+} = Z_p \odot (P^+ \amalg_{z\in\Gamma_p^-} \mathbb{R} \times \gamma_z)$$

where Γ_p^- is the set of negative punctures of $C'_{p,\pm}$ and γ_z is the asymptotic limit of $C'_{p,\pm}$ at $z \in \Gamma_p^-$.

Proof. We argue that the number of times that $P^\pm$ appears as a component of the building $C_{\infty,+}$ bounds the intersection number $\gamma_{p'_\pm} \cdot C$ for $C \in \mathcal{M}_{p,+} \setminus \{C'_{p,+}\}$ from below. This with the previous lemma will show that P^+ can appear at most once as a component of $C_{\infty,+}$, while P^- can't appear. Assume there are components D_i of $C_{\infty,+}$ with $D_i = P^+$ (mod the $\mathbb{R}$-action), and choose a parametrization $u : \mathbb{C} \to M'$ of P^+ with u mapping $0 \in \mathbb{C}$ to the intersection of $\gamma_{p'_+}$ with P^+. Then according to Proposition 3.4, if $C_k = [\Sigma_k, j_k, \Gamma_k, a_k, u_k] \in \mathcal{M}_{p,+}$ is a sequence which converges in the sense of [6] to $C_{\infty,+}$, then there is a sequence of holomorphic embeddings

$$\phi_k = \amalg_i \phi_{k,i} : \amalg_i \mathbb{D} \to \Sigma_k,$$

with $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ the unit disk in $\mathbb{C}$, so that $u_k \circ \phi_{k,i}$ converges in C^∞ to $u|_{\mathbb{D}}$. Since $u_k \circ \phi_k$ has image in U'_p and $\gamma_{p'_+}$ meets P^+ transversely, it follows from the C^∞ convergence that the pseudoholomorphic disks $u_k \circ \phi_{k,i}$ intersect $\gamma_{p'_+}$ transversely for sufficiently large k. This shows that for large k, C_k has at least one transverse intersection with $\gamma_{p'_+}$ for each component of $C_{\infty,+}$ equal to P^+. However, since $\mathbb{R} \times \gamma_{p'_+}$ is pseudoholomorphic, every intersection of $\gamma_{p'_+}$ with a curve C_k contributes positively to the intersection number $\gamma_{p'_+} \cdot C_k$, so the number of components of $C_{\infty,+}$ equal to P^+ is bounded above by $\gamma_{p'_+} \cdot C_k = 1$. An analogous argument shows that the number of components of $C_{\infty,+}$ equal to P^- is bounded above by $\gamma_{p'_-} \cdot C_k = 0$. Thus P^+ can appear at most once as a component of $C_{\infty,+}$ and P^- does not appear.

Finally, we observe that we have shown in Lemmas 6.3 and 6.6 that $C_{\infty,+}$ has at least two levels, at least two nontrivial components, and precisely one nontrivial component Z_p distinct from P^+ and P^-. This combined with the results of the previous paragraph then shows that $C_{\infty,+}$ has precisely two nontrivial components: Z_p and P^+. Moreover, by stability, $C_{\infty,+}$ must be a height-2 building with Z_p on the top level and P^+ along with cylinders over the other negative orbits of Z_p on the bottom level. □

Remark 6.11. We remark that Theorem 3.16 can also be used to bound the number of planes appearing in $C_{\infty,+}$, but an additional argument is then needed to show that the unique plane appearing in $C_{\infty,+}$ is P^+ and not P^-. Indeed, if the total number of times P^+ and P^- appeared as components of $C_{\infty,+}$ were greater than one, we could argue that the unique nontrivial component of $C_{\infty,+}$ distinct from $P^{\pm}$ guaranteed from Lemma 6.8 must have multiple negative ends approaching γ_0. Moreover, these ends would have to be disjoint in M since all components of $C'_{p,+}$ are nicely embedded. We could then argue, as in Lemma 6.13 below, that these ends would have to approach γ_0 in the same direction. Theorem 3.16 would then yield a contradiction, so we could conclude that there is at most a single plane in the building $C_{\infty,+}$.

Remark 6.12. The work leading up to the proof of Lemma 6.10 can be simplified somewhat if one applies [30, Theorem 1.10] to perturb J' slightly so that all moduli spaces of embedded curves are smooth manifolds of the appropriate dimension as predicted by the index formula (3.12). Since automatic transversality holds for the planes P^+ and P^- such a perturbation of J' could be carried out while maintaining the existence of planes with the properties we need. Given such a J', it would follow immediately from [58, Theorem 2] that $C_{\infty,+}$ is a height-2 building with precisely one nontrivial component on each level. It would then remain to argue, as we have above, that the nontrivial component of the lower level is P^+. However, we do not need to assume that such a generic J' has been chosen since our argument shows that the theorem holds for any J' for which there exist planes P^+ and P^- with the prescribed properties.

Finally, we conclude that $\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$ consists of a single building, completing the proof of the first statement in Proposition 6.4.

Lemma 6.13. *With $C_{\infty,+}$, Z_p, Γ_p^-, and γ_z as above, we have that*

$$\overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+} = C_{\infty,+} = Z_p \odot (P^+ \amalg_{z \in \Gamma_p^-} \mathbb{R} \times \gamma_z).$$

Proof. Let $C'_{\infty,+} \in \overline{\mathcal{M}_{p,+}} \setminus \mathcal{M}_{p,+}$. We seek to show that $C'_{\infty,+} = C_{\infty,+}$. The argument above applies to show that $C_{\infty,+} = Z'_p \odot (P^+ \amalg_{z \in \Gamma_p^-} \mathbb{R} \times \gamma_z)$ for some nicely-embedded curve Z'_p. It only remains to show that $Z'_p = Z_p$. We first argue that Z_p and Z'_p must approach γ_0 in the same direction. To see this, we first note that since

$C_{\infty,+}$ and $C'_{\infty,+}$ are nicely-embedded buildings and thus Z_p and Z'_p are disjoint from $P^{\pm}$, it follows from condition (3c) of Theorem 3.10 that Z'_p and Z_p approach γ_0 with winding equal to that of $P^{\pm}$, and thus equal to $\lfloor \mu^{\Phi}(\gamma_0)/2 \rfloor = \mu^{\Phi}(\gamma_0)/2$. Thus, according to Lemma 3.15, Z'_p and Z_p approach γ_0 in either the same or the opposite direction, with approach governed by a nonzero multiple of an eigenvector e_+ of $\mathbf{A}_{\gamma_0,J}$ with smallest possible positive eigenvalue. However, the boundary of U'_p at γ_0, being given nearby by the planes P^+ and P^-, is tangent to the largest negative eigenspace span$\{e_-\}$ of $\mathbf{A}_{\gamma_0,J}$, since eigenvectors with largest negative eigenvalue govern the approach of P^+ and P^-. Since, according to [26, Lemma 3.5], eigenvectors with the same winding and different eigenvalue are pointwise linearly independent, e_+ is nowhere tangent to the boundary of U'_p. Thus if e_+ points into U'_p, $-e_+$ points into U'_q and vice versa. Since, Z'_p and Z_p both approach γ_0 from within U'_p we can thus conclude that Z'_p and Z_p approach γ_0 in the same direction.

We next claim the fact that Z'_p and Z_p approach γ_0 in the same direction leads to a contradiction unless $Z'_p = Z_p$. Let C_k and C'_k be sequences in $\mathcal{M}_{p,+}$ converging respectively to $C_{\infty,+}$ and $C'_{\infty,+}$. Then, for any j and k, we have that

$$C_j * C'_k = C'_{p,+} * C'_{p,+} = 0$$

by homotopy invariance of the holomorphic intersection product. It then follows from Lemma 3.17 that Z'_p and Z_p are either identical or disjoint. However, since Z'_p and Z_p approach γ_0 in the same direction, it follows from Theorem 3.16 that Z'_p and Z_p must intersect. We thus arrive at a contradiction unless $Z'_p = Z_p$. □

We now complete the proof of Proposition 6.4.

Lemma 6.14. *The set $\overline{\mathcal{M}_{p,-}} \setminus \mathcal{M}_{p,-}$ consists of a single height-2 building with Z_p on the top level, and P^- and trivial cylinders over the negative orbits of $C'_{p,\pm}$ on the bottom layer, i.e.*

$$\overline{\mathcal{M}_{p,-}} \setminus \mathcal{M}_{p,-} = Z_p \odot (P^- \amalg_{z \in \Gamma_p^-} \mathbb{R} \times \gamma_z)$$

where Γ_p^- is the set of negative punctures of $C'_{p,\pm}$ and γ_z is the asymptotic limit of $C'_{p,\pm}$ at $z \in \Gamma_p^-$.

Proof. Let $C_{\infty,-} \in \overline{\mathcal{M}_{p,-}} \setminus \mathcal{M}_{p,-}$. An analogous argument to that in Lemmas 6.6-6.10 as above gives us that $C_{\infty,-} = Z'_p \odot (P^- \amalg_{z \in \Gamma_p^-} \mathbb{R} \times \gamma_z)$ for some nicely-embedded curve Z'_p. It remain only to show that $Z'_p = Z_p$. This follows from an argument analogous to that in Lemma 6.13 above. Indeed, we can argue exactly as in Lemma 6.13 that Z'_p and Z_p approach γ_0 in the same direction. Then, with C_k and C'_k sequences respectively in $\mathcal{M}_{p,+}$ and $\mathcal{M}_{p,-}$ converging respectively to $C_{\infty,+}$ and $C_{\infty,-}$, we have that

$$C_j * C'_k = C'_{p,+} * C'_{p,-} = 0$$

by homotopy invariance of the holomorphic intersection product. We can thus again apply Lemma 3.17 and Theorem 3.16 to arrive at a contradiction unless $Z'_p = Z_p$. □

Next, we define $\mathcal{M}_{q,\pm}$ analogously to $\mathcal{M}_{p,\pm}$, that is, we define $\mathcal{M}_{q,\pm} \subset \mathcal{M}(C_{q,\pm})/\mathbb{R}$ to be the connected component of

$$\{C_{q,\pm}\} \cup \{C \in \mathcal{M}(C_{q,\pm})/\mathbb{R} \,|\, C \subset U'_q\}$$

containing $C_{q,\pm}$. An analogous argument to above shows the following.

Proposition 6.15. *There exists a pseudoholomorphic curve $Z_q \in \mathcal{M}(\lambda', J')/\mathbb{R}$ which is embedded in M and disjoint from $P^\pm$ so that*

$$\overline{\mathcal{M}_{q,+}} \setminus \mathcal{M}_{q,+} = Z_q \odot (P^- \amalg_{z\in\Gamma_q^-} \mathbb{R}\times\gamma_z)$$

and

$$\overline{\mathcal{M}_{q,-}} \setminus \mathcal{M}_{q,-} = Z_q \odot (P^+ \amalg_{z\in\Gamma_q^-} \mathbb{R}\times\gamma_z)$$

where Γ_q^- is the set of negative punctures of $C'_{q,\pm}$ and γ_z is the asymptotic limit of $C'_{q,\pm}$ at $z \in \Gamma_q^-$.

Finally, to complete the argument, we need to show that the curves from the old foliation together with the curves from the compactified moduli spaces from above give a foliation of the surgered manifold with the same energy as the original foliation. We work with the definition of finite energy foliation given by Corollary 4.5. More precisely we consider the collection of simple periodic orbits $B' \subset M'$ defined by

$$B' = i(B) \cup \{\gamma_0\}$$

where B is the set of periodic orbits with covers appearing as asymptotic limits of the original foliation $\mathcal{F}$, and we define a collection of curves $\mathcal{F}' \subset \mathcal{M}(\lambda', J')/\mathbb{R}$ by including:

- The curves in the moduli spaces $\mathcal{M}_{p,+}$, $\mathcal{M}_{p,-}$, $\mathcal{M}_{q,+}$, and $\mathcal{M}_{q,-}$.
- The curves $\{Z_p, Z_q, P^+, P^-\}$ constructed above.
- The push forward via the inclusion $i : M \setminus \{p, q\} \to M'$ of any curve in the original foliation $\mathcal{F}$ which lies in the closure of the complement of the regions U_p and U_q, i.e. if $C = [\Sigma, j, \Gamma, a, u] \in \mathcal{F}$ and $C \subset M \setminus (U_p \cup U_q)$ then we define $i(C) = [\Sigma, j, \Gamma, a, i \circ u]$. Since $i^*J = J'$ outside of U_p and U_q it follows that $i(C)$ is a pseudoholomorphic curve in M'.

We now argue that $\mathcal{F}'$ so defined is a stable finite energy foliation for M'. We need to show that there is a unique curve from $\mathcal{F}'$ through every point of $M' \setminus B'$, that the index of any nontrivial curve in $\mathcal{F}'$ is 1 or 2, that the intersection numbers between any two nontrivial curves in $\mathcal{F}'$ vanish, and that $E(\mathcal{F}') = E(\mathcal{F})$.

We first address the fact that the energies of the two collections of curves are the same.

Lemma 6.16. *With $\mathcal{F}' \subset \mathcal{M}(\lambda', J')/\mathbb{R}$ the collection of curves defined above, we have that*

$$E(\mathcal{F}') := \sup_{C\in\mathcal{F}'} E(C) = \sup_{C\in\mathcal{F}} E(C) =: E(\mathcal{F}).$$

Proof. We recall from the proof of Lemma 4.3 that the energy $E(C)$ of a curve $C = [\Sigma, j, \Gamma, da, u] \in \mathcal{M}(\lambda, J)/\mathbb{R}$, defined by (3.2), is given by the sum of the periods of the orbits that are asymptotic limits of the positive punctures of C. Since all curves $C \in \mathcal{F}$ have asymptotic limits in the region where $i^*\lambda' = \lambda$ and, by construction, either satisfy $i(C) \in \mathcal{F}'$ or are homotopic to a curve satisfying this, every curve in $\mathcal{F}$ has energy equal to that of some curve in $\mathcal{F}'$. We conclude $E(\mathcal{F}) \le E(\mathcal{F}')$.

Conversely, every curve in $C' \in \mathcal{F}' \setminus \{Z_p, Z_q, P^+, P^-\}$ is, by construction, either of the form $C' = i(C)$ for some $C \in \mathcal{F}$ or is homotopic to a curve of this form. Thus every curve $C' \in \mathcal{F}' \setminus \{Z_p, Z_q, P^+, P^-\}$ has energy equal to that of some curve in

$\mathcal{F}$ and we conclude that $E(C') \le E(\mathcal{F})$. Moreover, by Propositions 6.4 and 6.15, Z_p and Z_q have positive punctures identical respectively to $C'_{p,\pm} = i(C_{p,\pm})$ and $C'_{q,\pm} = i(C_{q,\pm})$ and thus $E(Z_p) \le E(\mathcal{F})$ and $E(Z_q) \le E(\mathcal{F})$ as well. Finally, we recall from the proof of Lemma 4.3 that the $d\lambda$-energy of a curve, defined by (4.1), is always nonnegative and is given by the difference between the sums of periods of the positive asymptotic limits and those of the negative asymptotic limits. Thus, if follows immediately from Propositions 6.4 that $E(P^{\pm}) \le E(Z_p) \le E(\mathcal{F})$. We conclude that $E(\mathcal{F}') \le E(\mathcal{F})$ and, with the previous paragraph, this completes the proof. □

We next address the fact that all nontrivial curves in $\mathcal{F}'$ have index 1 or 2.

Lemma 6.17. *Let $C \in \mathcal{F}'$. Then* $\operatorname{ind}(C) \in \{1, 2\}$.

Proof. Except for $C \in \{Z_p, Z_q, P^+, P^-\}$, this is immediate from the fact that all nontrivial curves in $\mathcal{F}$ have index 1 or 2. To see that $\operatorname{ind}(P^{\pm}) = 1$ we use the fact $P^{\pm} * P^{\pm} = 0$. Then according to Theorem 3.12

$$\begin{aligned}\operatorname{ind}(P^{\pm}) &= \chi(S^2) - \#\Gamma_{even}\\ &= 2 - 1 = 1.\end{aligned}$$

The fact that $\operatorname{ind}(Z_p) = \operatorname{ind}(Z_q) = 1$ then follows from Propositions 6.4 and 6.15. Indeed, since the pseudoholomorphic buildings $\overline{\mathcal{M}_{p,\pm}} \setminus \mathcal{M}_{p,\pm}$ and $\overline{\mathcal{M}_{q,\pm}} \setminus \mathcal{M}_{q,\pm}$ have no nodes, the sum of the indices of the nontrivial components must add to the index of a curve in $\mathcal{M}_{p,\pm}$ or $\mathcal{M}_{q,\pm}$. Since such curves have index 2, and we have just shown that $P^{\pm}$ have index 1, it follows that $\operatorname{ind}(Z_p) = \operatorname{ind}(Z_q) = 1$ as claimed. □

We next address the intersection numbers. We start by showing that any two distinct curves in $\mathcal{F}'$ are disjoint.

Lemma 6.18. *Let C_1, $C_2 \in \mathcal{F}'$ be distinct curves. Then C_1 and C_2 are disjoint.*

Proof. Since any two nontrivial curves in the original foliation $\mathcal{F}$ have vanishing intersection number, it follows that any two nontrivial curves in

$$C_1,\, C_2 \in \mathcal{F}' \setminus \left\{Z_p, Z_q, P^+, P^-\right\}$$

have vanishing intersection number. Thus any two such distinct curves are disjoint in M. Moreover, since the curves Z_p, Z_q, P^+, and P^- occur as components of limiting buildings of sequences of curves in $\mathcal{F}' \setminus \{Z_p, Z_q, P^+, P^-\}$ we can immediately conclude from Lemma 3.17 that any two distinct, nontrivial curves in $\mathcal{F}'$ are disjoint. □

Lemma 6.19. *Let C_1, $C_2 \in \mathcal{F}'$. Then $C_1 * C_2 = 0$.*

Proof. As noted in the proof of the above lemma, this is immediate for any two curves

$$C_1,\, C_2 \in \mathcal{F}' \setminus \left\{Z_p, Z_q, P^+, P^-\right\}.$$

It remains to show that $C_1 * C_2 = 0$ when one or both of C_1, C_2 is equal to one of Z_p, Z_q, P^+, or P^-. We first observe that $\operatorname{ind}(Z_p) = \operatorname{ind}(Z_q) = 1$ and Z_p and Z_q each have precisely one puncture asymptotic to an even orbit. We thus have for $C \in \{Z_p, Z_q\}$ that

$$\operatorname{ind}(C) - \chi(S^2) + \#\Gamma_{even} = 1 - 2 + 1 = 0,$$

so it then follows from facts in [26] that the bound in inequality (3.5) is achieved at each puncture of Z_p and Z_q (see also discussion preceding Lemma 2.6 in [2] and equation 5.1 in [50]). Meanwhile, we know that the bound in inequality (3.5) is achieved at each puncture of every other curve in $\mathcal{F}'$ from Theorem 3.12. Thus, by Corollary 3.11 and the previous lemma, we can conclude that $C_1 * C_2 = 0$ for any distinct C_1, $C_2 \in \mathcal{F}'$. Since we already know that $P^{\pm} * P^{\pm} = 0$ by Theorem 5.1, it remains to show $Z_p * Z_p = Z_q * Z_q = 0$. However, since Z_p and Z_q are embedded in M' and, as observed above, have extremal winding at each puncture, Corollary 3.14 implies that $Z_p * Z_p = Z_q * Z_q = 0$. This completes the proof that $C_1 * C_2 = 0$ for any two curves C_1, $C_2 \in \mathcal{F}'$. □

It remains to show that there is a curve of $\mathcal{F}'$ through every point of $M' \setminus B'$.

Lemma 6.20. *For every $x \in M' \setminus B'$ there is a curve $C \in \mathcal{F}'$ passing through x.*

Proof. By construction, since $i^* J' = J$ outside of $U_p \cup U_q$ and since the boundaries of U_p and U_q are made up of curves in the foliation $\mathcal{F}$, it suffices to show there is a curve through every point of $U'_p \setminus Z_p$ and $U'_q \setminus Z_q$. We will prove this for $U'_p \setminus Z_p$. The argument for $U'_q \setminus Z_q$ is identical.

We first define a subset $U'_{p,+}$ (resp. $U'_{p,-}$) of $U'_p \setminus Z_p$ to be the sets of points in $U'_p \setminus Z_p$ having a curve of $\mathcal{M}_{p,+}$ (resp. $\mathcal{M}_{p,-}$) passing through them. Then $U'_{p,+}$ and $U'_{p,-}$ are each nonempty (by construction of $\mathcal{M}_{p,\pm}$), open (by Corollary 4.8), and (relatively) closed (by compactness and Proposition 6.4). Thus, $U'_{p,+}$ and $U'_{p,-}$ each form a connected component of $U'_p \setminus Z_p$. Since U'_p is connected and Z_p is an embedded submanifold, $U'_p \setminus Z_p$ has at most two connected components. Thus, the proof is completed unless $U'_{p,+} = U'_{p,-}$. However, if $U'_{p,+} = U'_{p,-}$ there is a point $x' \in U'_p \setminus Z_p$ with curves $C_+ \in \mathcal{M}_{p,+}$ and $C_- \in \mathcal{M}_{p,-}$ passing through x'. Since $C_+ * C_- = C'_{p,+} * C'_{p,-} = 0$ by homotopy invariance of the intersection number and Theorem 3.8, we must have that $C_+ = C_-$, so we have found a curve belonging to both $\mathcal{M}_{p,+}$ and $\mathcal{M}_{p,-}$. But we've shown in Lemma 6.9 that the intersection numbers $\gamma_{p'_\pm} \cdot C$ of curves $C \in \mathcal{M}_{p,+}$ with the flow segments $\gamma_{p'_\pm}$ are well defined and satisfy

$$\gamma_{p'_+} \cdot C = 1 \qquad \gamma_{p'_-} \cdot C = 0,$$

while an analogous argument shows that for curves $C \in \mathcal{M}_{p,-}$ the intersection numbers $\gamma_{p'_\pm} \cdot C$ are well defined and satisfy

$$\gamma_{p'_+} \cdot C = 0 \qquad \gamma_{p'_-} \cdot C = 1.$$

Thus, a curve belonging to both $\mathcal{M}_{p,+}$ and $\mathcal{M}_{p,-}$ would lead to a contradiction, and this completes the proof that there is at least one curve of $\mathcal{F}'$ through every point of $M' \setminus B'$. □

Appendix A. Additional details for Section 5

In this section we collect some of the more straightforward but tedious computations supporting claims made in Section 5.

Lemma A.1. *Consider S^2 equipped with polar coordinate $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ and azimuthal coordinate $\theta \in [0, \pi]$. The following define smooth tensor fields on S^2:*

(1) $\sin^2 \theta$
(2) $\cos \theta$
(3) $\sin \theta \, d\theta$

(4) ∂_ϕ *(and* ∂_ϕ *vanishes for* $\theta \in \{0, \pi\}$*)*
(5) $\sin^2\theta \, d\phi$
(6) $\sin\theta \, \partial_\theta$ *(and* $\sin\theta\,\partial_\theta$ *vanishes for* $\theta \in \{0, \pi\}$*)*

Proof. Considering S^2 embedded as the unit sphere in $\mathbb{R}^3$, the smooth change of coordinates on the upper and lower hemispheres obtained by projecting onto the xy-plane is given by

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi.$$

In these coordinates we have

$$\sin^2\theta = x^2 + y^2$$

which is clearly smooth. Meanwhile

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - x^2 - y^2}$$

which is also smooth near $(x, y) = 0$, and hence the 1-form

$$-d(\cos\theta) = \sin\theta \, d\theta$$

is also smooth.

Next we have that

$$\begin{aligned} \partial_\phi &= x_\phi \, \partial_x + y_\phi \, \partial_y \\ &= -y \, \partial_x + x \, \partial_y \end{aligned}$$

which is smooth and vanishes when $x = y = 0$ (i.e. when $\theta \in \{0, \pi\}$). Meanwhile, using that the standard round metric g on S^2 with total area 4π is given by

$$g = d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi$$

we have that

$$g(\partial_\phi, \cdot) = \sin^2\theta \, d\phi$$

so $\sin^2\theta \, d\phi$ is smooth since it is dual to a smooth vector field. Similarly

$$g(\sin\theta \, \partial_\theta, \cdot) = \sin\theta \, d\theta$$

so $\sin\theta \, \partial_\theta$ is smooth since it is dual to a smooth 1-form. Moreover,

$$g(\sin\theta \, \partial_\theta, \sin\theta \, \partial_\theta) = \sin^2\theta = x^2 + y^2$$

shows that $\sin\theta \, \partial_\theta$ vanishes when $x = y = 0$ i.e. when $\theta \in \{0, \pi\}$. □

Lemma A.2 (Lemma 5.2)**.** *Consider the maps* $\Phi_\pm : \mathbb{R}^\pm \times S^2 \to \mathbb{R}^3 \setminus \{0\}$ *defined by*

$$\Phi_\pm(\rho, \phi, \theta) = \pm(\rho \sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho^3\cos\theta).$$

Then Φ_+ *and* Φ_- *are smooth diffeomorphisms satisfying*

$$\Phi_\pm^* \lambda_\pm = \rho^2 \lambda_1$$

with λ_+*,* λ_-*, and* λ_1 *as defined in Section 5.*

Proof. We first claim that $\Phi_\pm$ is bijective. Since the two maps differ by negation on $\mathbb{R}^3 \setminus \{0\}$ it suffices to show that Φ_+ is bijective. We first observe that Φ_+ maps the set $\mathbb{R}^+ \times \{\theta = 0, \pi\}$ bijectively to complement of the origin on the z-axis. We thus consider a point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^3 \setminus \{0\}$ not in the z-axis, and seek to find a unique solution to

$$x_0 = \rho \sin\theta \cos\phi \tag{A.1}$$

$$y_0 = \rho \sin\theta \sin\phi \tag{A.2}$$

$$z_0 = \rho^3 \cos\theta \tag{A.3}$$

with $(\rho, \phi, \theta) \in \mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z} \times (0, \pi)$. Squaring and summing (A.1)-(A.2) gives

$$x_0^2 + y_0^2 = \rho^2 \sin^2\theta \tag{A.4}$$

and the assumption that p_0 is not in the z-axis implies that $x_0^2 + y_0^2 > 0$. Combining (A.4) and (A.3) leads to

$$\cot\theta \csc^2\theta = \frac{z_0}{(x_0^2 + y_0^2)^{3/2}}$$

which has a unique solution with $\theta_0 \in (0, \pi)$ since the derivative of $\cot\theta\csc^2\theta$ is everywhere negative and $\lim_{\theta \to k\pi^\pm} = \pm\infty$ for $k \in \mathbb{Z}$. Substituting this θ_0 in (A.3) gives a unique $\rho_0 > 0$. Finally substituting these values of ρ_0 and θ_0 into equations (A.1)-(A.2) gives a unique value of $\phi_0 \in \mathbb{R}/2\pi\mathbb{Z}$ for which (A.1)-(A.3) are satisfied. This completes the proof that Φ_+ and Φ_- are bijective.

To show the $\Phi_\pm$ are diffeomorphisms, it remains to show that $\Phi_\pm$ are immersions. Again it suffices to show this for Φ_+. For $\theta \notin \{0, \pi\}$ we have that

$$D\Phi_+(\rho, \phi, \theta) = \begin{bmatrix} \sin\theta\cos\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\cos\theta\sin\phi \\ 3\rho^2\cos\theta & 0 & -\rho^3\sin\theta \end{bmatrix}$$

from which we can compute that

$$\det D\Phi_+(\rho, \phi, \theta) = -\rho^4 \sin\theta(1 + 2\cos^2\theta)$$

which is nonzero for $\theta \notin \{0, \pi\}$. Meanwhile, in a neighborhood of $\theta \in \{0, \pi\}$ we can make the change of coordinates

$$X = \sin\theta\cos\phi$$
$$Y = \sin\theta\sin\phi$$

to write

$$\Phi_+(\rho, X, Y) = (\rho X, \rho Y, \rho^3\sqrt{1 - X^2 - Y^2}).$$

Thus,

$$D\Phi_+(\rho, X, Y) = \begin{bmatrix} X & \rho & 0 \\ Y & 0 & \rho \\ 3\rho^2\sqrt{1 - X^2 - Y^2} & -\rho^3\frac{X}{\sqrt{1-X^2-Y^2}} & -\rho^3\frac{Y}{\sqrt{1-X^2-Y^2}} \end{bmatrix}$$

and

$$\det D\Phi_+(\rho, X, Y) = \frac{\rho^4(3 - 2X^2 - 2Y^2)}{\sqrt{1 - X^2 - Y^2}}$$

which is nonzero along the set $X = Y = 0$ as required. Thus $\Phi_\pm$ are immersions.

Finally recall that $\lambda_\pm$ were defined by

$$\lambda_\pm = \pm dz + \frac{1}{2}(x\,dy - y\,dx).$$

The maps $\Phi_\pm : (\rho, \phi, \theta) \in \mathbb{R}^\pm \times S^2 \to (x, y, z) \in \mathbb{R}^3$ defined by

$$\begin{aligned}
x &= \pm\rho \sin\theta\cos\phi & dx &= \pm\left(\sin\theta\cos\phi\,d\rho + \rho\cos\theta\cos\phi\,d\theta - \rho\sin\theta\sin\phi\,d\phi\right)\\
y &= \pm\rho \sin\theta\sin\phi & dy &= \pm\left(\sin\theta\sin\phi\,d\rho + \rho\cos\theta\sin\phi\,d\theta + \rho\sin\theta\cos\phi\,d\phi\right)\\
z &= \pm\rho^3\cos\theta & dz &= \pm\left(3\rho^2\cos\theta\,d\rho - \rho^3\sin\theta\,d\theta\right)
\end{aligned}$$

so

$$\begin{aligned}
y\,dx &= \rho\sin^2\theta\sin\phi\cos\phi\,d\rho + \rho^2\sin\theta\cos\theta\sin\phi\cos\phi\,d\theta - \rho^2\sin^2\theta\sin^2\phi\,d\phi\\
x\,dy &= \rho\sin^2\theta\sin\phi\cos\phi\,d\rho + \rho^2\sin\theta\cos\theta\sin\phi\cos\phi\,d\theta + \rho^2\sin^2\theta\cos^2\phi\,d\phi
\end{aligned}$$

and hence

$$x\,dy - y\,dx = \rho^2\sin^2\theta\,d\phi.$$

We then find that

$$\begin{aligned}
\Phi_\pm^*\lambda_\pm &= 3\rho^2\cos\theta\,d\rho - \rho^3\sin\theta\,d\theta + \frac{1}{2}\rho^2\sin^2\theta\,d\phi\\
&= \rho^2\left(3\cos\theta\,d\rho - \rho\sin\theta\,d\theta + \frac{1}{2}\sin^2\theta\,d\phi\right)\\
&= \rho^2\lambda_1
\end{aligned}$$

as required. □

We next compute the Reeb vector field of the contact form $\lambda_f = f\lambda_1$ on $\mathbb{R}\times S^2$. We start with a general lemma.

Lemma A.3. *Let (M, λ) be a contact 3-manifold, let f be a smooth positive function, and let $\tilde{X}_f$ be the unique section of $\xi = \ker\lambda$ satisfying*

$$i_{X_f}d\lambda = df - df(X_\lambda)\lambda \tag{A.5}$$

where X_λ is the Reeb vector field. Then the Reeb vector field of the contact form associated to $f\lambda$ is given by

$$X_{f\lambda} = \tfrac{1}{f}X_\lambda + \tfrac{1}{f^2}\tilde{X}_f.$$

Moreover, if $\{v_1, v_2\}$ is a basis for ξ, then we have that

$$\tilde{X}_f = [d\lambda(v_1, v_2)]^{-1}(df(v_2)v_1 - df(v_1)v_2). \tag{A.6}$$

Proof. Let $\tilde{X}_f$ be the unique section of ξ satisfying $i_{\tilde{X}_f}d\lambda = df - df(X_\lambda)\lambda$. Then

$$df(\tilde{X}_f) = (i_{\tilde{X}_f})^2 d\lambda + f(X_\lambda)i_{\tilde{X}_f}\lambda = 0$$

since $\tilde{X}_f \in \xi = \ker\lambda$. We then find that

$$\begin{aligned}
i_{X_\lambda}d(f\lambda) &= i_{X_\lambda}df\wedge\lambda + i_{X_\lambda}(f\,d\lambda)\\
&= i_{X_\lambda}df\wedge\lambda\\
&= df(X_\lambda)\lambda - df
\end{aligned}$$

while

$$\begin{aligned} i_{\tilde{X}_f} d(f\lambda) &= i_{\tilde{X}_f} df \wedge \lambda + i_{\tilde{X}_f}(f d\lambda) \\ &= i_{\tilde{X}_f}(f d\lambda) \\ &= f\, i_{\tilde{X}_f} d\lambda \end{aligned}$$

so

$$\begin{aligned} i_{\frac{1}{f}X_\lambda + \frac{1}{f^2}\tilde{X}_f} d(f\lambda) &= \tfrac{1}{f} i_{X_\lambda} d(f\lambda) + \tfrac{1}{f^2} i_{\tilde{X}_f} d(f\lambda) \\ &= \tfrac{1}{f}[df(X_\lambda)\lambda - df] + \tfrac{1}{f^2}[f\, i_{\tilde{X}_f} d\lambda] \\ &= \tfrac{1}{f}[df(X_\lambda)\lambda - df + i_{\tilde{X}_f} d\lambda] \\ &= 0 \end{aligned}$$

by definition of $\tilde{X}_f$. Furthermore

$$\begin{aligned} i_{\frac{1}{f}X_\lambda + \frac{1}{f^2}\tilde{X}_f}(f\lambda) &= \tfrac{1}{f} i_{X_\lambda}(f\lambda) + \tfrac{1}{f^2} i_{\tilde{X}_f}(f\lambda) \\ &= i_{X_\lambda}\lambda + \tfrac{1}{f} i_{\tilde{X}_f}\lambda \\ &= 1 + 0 = 1. \end{aligned}$$

Thus $\frac{1}{f}X_\lambda + \frac{1}{f^2}\tilde{X}_f$ is the Reeb vector field of $f\lambda$ as claimed.

Next, to verify the second claim, we define $\tilde{X}_f$ by (A.6) and verify that this $\tilde{X}_f$ satisfies (A.5). Computing, we have that

$$i_{\tilde{X}_f} d\lambda - df - df(X_\lambda)\lambda = [d\lambda(v_1, v_2)]^{-1}[df(v_2)d\lambda(v_1, \cdot) - df(v_1)d\lambda(v_2, \cdot)] - df - df(X_\lambda)\lambda.$$

Using that $i_{v_i}\lambda = 0$, $i_{X_\lambda}\lambda = 1$ and $i_{X_\lambda} d\lambda = 0$ we see that each of the vectors X_λ, v_1, and v_2 yields 0 when evaluated on the right hand side of this equation above. Since these vectors form a basis for the tangent space, it follows that this quantity vanishes on TM and thus (A.5) is satisfied. $\square$

Lemma A.4 (Lemma 5.3)**.** *Recalling the definition*

$$g(\theta) = 2\cos^2\theta + 1 = 3\cos^2\theta + \sin^2\theta$$

from (5.3)*, we have for* $\theta \notin \{0, \pi\}$*:*

- *The set*

$$\begin{aligned} \mathcal{B}_{(\rho,\theta,\phi)} &= \left\{ (fg)^{-1}(-3\cot\theta\,\partial_\phi + \frac{1}{2}\sin\theta\,\partial_\rho), 2\rho\csc\theta\,\partial_\phi + \partial_\theta \right\} \\ &=: \{v_1(\rho,\theta,\phi), v_2(\rho,\theta,\phi)\} \end{aligned}$$

  is a symplectic basis for $(\xi_1, d\lambda_f)$*.*
- *The Reeb vector field* X_f *of the contact form* λ_f *is given by*

$$\begin{aligned} X_f = [gf^2]^{-1}\Big[&(-\rho f_\rho - 3 f_\theta \cot\theta + 2f)\,\partial_\phi \\ &+ (3\cot\theta f_\phi - \frac{1}{2}\sin\theta f_\rho)\,\partial_\theta \\ &+ (\rho f_\phi + \frac{1}{2}\sin\theta f_\theta + f\cos\theta)\,\partial_\rho\Big]. \end{aligned}$$

Proof. The vectors v_1 and v_2 are clearly linearly independent for $\theta \notin \{0, \pi\}$ since v_1 has nonzero ∂_ρ component and vanishing ∂_θ component, while the opposite is true of v_2. Recalling that

$$\begin{aligned} \lambda_1 &= 3\cos\theta\, d\rho - \rho\sin\theta\, d\theta + \frac{1}{2}\sin^2\theta\, d\phi \\ d\lambda_1 &= (2\sin\theta\, d\rho - \cos\theta\sin\theta\, d\phi)\wedge d\theta \end{aligned}$$

we immediately find that

$$\lambda_1(v_1) = (fg)^{-1}\left[(3\cos\theta)(\tfrac{1}{2}\sin\theta) + (\tfrac{1}{2}\sin^2\theta)(-3\cot\theta)\right] = 0$$

and

$$\lambda_1(v_2) = -\rho\sin\theta + (\tfrac{1}{2}\sin^2\theta)(2\rho\csc\theta) = 0$$

so v_1 and $v_2 \in \xi_1$. Finally, we have that

$$\begin{aligned} i_{v_1} d\lambda_1 &= (fg)^{-1}(\sin^2\theta + 3\cot\theta\cos\theta\sin\theta)\, d\theta \\ &= (fg)^{-1}(\sin^2\theta + 3\cos^2\theta)\, d\theta \\ &= 1/f\, d\theta \end{aligned}$$

and thus

$$\begin{aligned} d\lambda_1(v_1, v_2) &= i_{v_2}(i_{v_1}\lambda_1) \\ &= i_{v_2}(1/f\, d\theta) \\ &= 1/f. \end{aligned} \tag{A.7}$$

Since $\lambda_1(v_1) = \lambda_1(v_2) = 0$, it follows that

$$d\lambda_f(v_1, v_2) = f d\lambda_1(v_1, v_2) = 1$$

and thus $\{v_1, v_2\}$ is a symplectic basis for $(\xi_1, d\lambda_f)$ for $\theta \notin \{0, \pi\}$.

Next to compute the Reeb vector field of λ_f we first observe that the vector field X_1 defined by

$$X_1 = g(\theta)^{-1}\left(\cos\theta\,\partial_\rho + 2\,\partial_\phi\right).$$

satisfies

$$\lambda_1(X_1) = g(\theta)^{-1}\left[3\cos^2\theta + \sin^2\theta\right] = 1$$

and

$$i_{X_1} d\lambda_1 = g(\theta)^{-1}\left[2\sin\theta\cos\theta - 2\sin\theta\cos\theta\right] d\theta = 0$$

so X_1 is the Reeb vector field of λ_1. According to Lemma A.3, the Reeb vector field of X_f of λ_f is then given by

$$X_f = \tfrac{1}{f}X_1 + \tfrac{1}{f^2}\tilde{X}_f$$

with

$$\begin{aligned} \tilde{X}_f &= [d\lambda_1(v_1, v_2)]^{-1}(df(v_2)v_1 - df(v_1)v_2) \\ &= f(df(v_2)v_1 - df(v_1)v_2) \end{aligned}$$

where we've applied (A.7) in the second line. Computing, we have that

$$\begin{aligned} f\, df(v_2)v_1 &= g(\theta)^{-1}\left(2\rho\csc\theta f_\phi + f_\theta\right)\left(-3\cot\theta\,\partial_\phi + \frac{1}{2}\sin\theta\,\partial_\rho\right) \\ &= g(\theta)^{-1}\left[\left(-6\rho\csc\theta\cot\theta f_\phi - 3\cot\theta f_\theta\right)\partial_\phi + \left(\rho f_\phi + \frac{1}{2}\sin\theta f_\theta\right)\partial_\rho\right] \end{aligned}$$

and

$$\begin{aligned} f\,df(v_1)v_2 &= g(\theta)^{-1}\left(-3\cot\theta f_\phi + \frac{1}{2}\sin\theta f_\rho\right)(2\rho\csc\theta\,\partial_\phi + \partial_\theta) \\ &= g(\theta)^{-1}\left[(-6\rho\csc\theta\cot\theta f_\phi + \rho f_\rho)\,\partial_\phi + \left(-3\cot\theta f_\phi + \frac{1}{2}\sin\theta f_\rho\right)\partial_\theta\right]. \end{aligned}$$

Combining the above we conclude that

$$\begin{aligned} X_f = [gf^2]^{-1}\Big[&(-\rho f_\rho - 3f_\theta\cot\theta + 2f)\,\partial_\phi \\ &+ (3\cot\theta f_\phi - \frac{1}{2}\sin\theta f_\rho)\,\partial_\theta \\ &+ (\rho f_\phi + \frac{1}{2}\sin\theta f_\theta + f\cos\theta)\,\partial_\rho\Big] \end{aligned}$$

as claimed. □

Lemma A.5. *Let $C \in Sp(1)$ be a symplectic matrix and let $k \neq 0$ be a constant. Then the path $\Psi_{C,k} : [0,1] \to Sp(1)$ defined by*

$$\Psi_{C,k}(t) = C\begin{bmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{bmatrix} C^{-1}$$

has Conley–Zehnder index $\mu_{cz}(\Psi_{C,k}) = 0$.

Proof. We first consider the case $C = I$, i.e. the path given by

$$\Psi_{I,k} = \begin{bmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{bmatrix}.$$

The path $\Psi_{I,k}$ is easily seen to be homotopic within $\Sigma(1)$ to its inverse

$$\Psi_{I,k}^{-1} = \Psi_{I,-k} = \begin{bmatrix} e^{-kt} & 0 \\ 0 & e^{kt} \end{bmatrix}.$$

via the homotopy

$$\Psi_s(t) := R(s\pi/2)\Psi_{I,k}(t)R(s\pi/2)^{-1} = R(s\pi/2)\Psi_{I,k}(t)R(-s\pi/2)$$

where $R(\theta)$ is the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The homotopy axiom from Theorem 2.1 then implies that

$$\mu_{cz}(\Psi_{I,k}) = \mu_{cz}(\Psi_{I,k}^{-1})$$

while the inverse axiom implies that

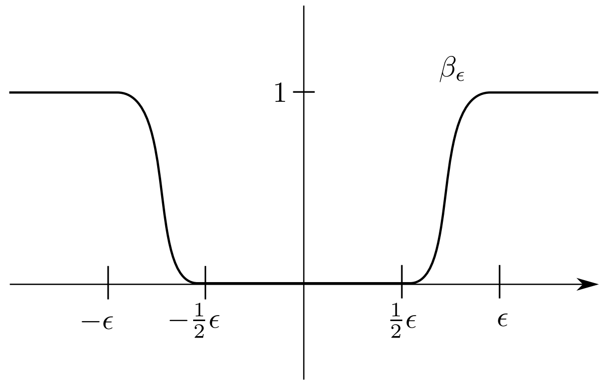
$$\mu_{cz}(\Psi_{I,k}) = -\mu_{cz}(\Psi_{I,k}^{-1}).$$

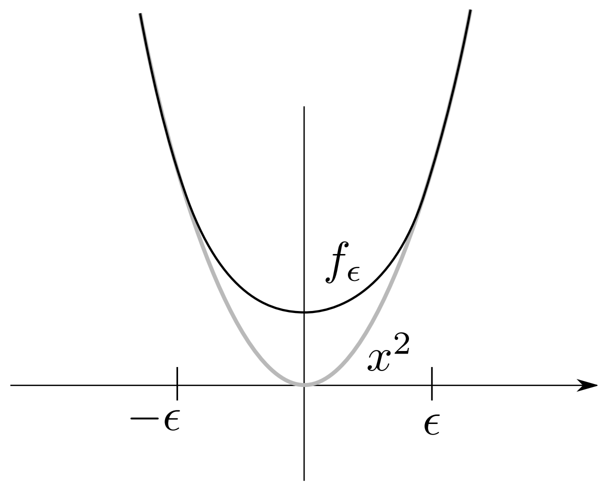
We conclude that

$$\mu_{cz}(\Psi_{I,k}) = 0.$$

For $C \neq I$, we can use the fact that the symplectic group is path connected to find a path $C_s \in Sp(1)$ with $C_0 = C$ and $C_1 = I$. We then construct a homotopy $\Psi_s \in \Sigma(1)$ defined by

$$\Psi_s(t) := \Psi_{C_s,k}(t) = C_s\begin{bmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{bmatrix} C_s^{-1}$$





with $\Psi_0 = \Psi_{C,k}$ and $\Psi_1 = \Psi_{I,k}$. The homotopy invariance axiom of Theorem 2.1 and the result of the previous paragraph then imply

$$\mu_{cz}(\Psi_{C,k}) = \mu_{cz}(\Psi_{I,k}) = 0$$

as claimed. □

Lemma A.6. *For any $\varepsilon > 0$, there exists a smooth, positive function $f_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ satisfying:*

(1) $f_\varepsilon(x) = x^2$ *whenever* $|x| \geq \varepsilon$,
(2) $xf_\varepsilon'(x) > 0$ *for all* $x \neq 0$, *and*
(3) $f_\varepsilon''(0) > 0$.

Proof. Given $\varepsilon > 0$ choose a smooth function $\beta_\varepsilon : \mathbb{R} \to [0, 1]$ satisfying

- $\beta_\varepsilon(x) = 1$ for $|x| \geq \varepsilon$,
- $\beta_\varepsilon(x) = 0$ for $|x| \leq \varepsilon/2$, and
- $x\beta_\varepsilon'(x) \geq 0$ for $|x| \in (\varepsilon/2, \varepsilon)$ (and thus all $x \in \mathbb{R}$).

Since β_ε' is compactly supported and vanishes for $|x| \leq \varepsilon/2$ it follows that the function $x \mapsto x^{-1}\beta'(x)$ is smooth, compactly supported, and thus bounded. We can thus find a $c_\varepsilon > 0$ so that

$$\frac{1}{c_\varepsilon} \geq 2 \max\left(x^{-1}\beta'_\varepsilon(x)\right)$$

and hence

$$\begin{aligned} 1 - x^{-1}\beta'_\varepsilon(x)c_\varepsilon &\geq 1 - \left(\max x^{-1}\beta'_\varepsilon(x)\right) c_\varepsilon \\ &\geq 1 - \frac{1}{2} = \frac{1}{2} \end{aligned} \tag{A.8}$$

for all $x \in \mathbb{R}$. Defining

$$\begin{aligned} f_\varepsilon(x) &= \beta_\varepsilon(x)x^2 + (1-\beta_\varepsilon(x))(\tfrac{1}{2}x^2 + c_\varepsilon) \\ &= \tfrac{1}{2}(\beta_\varepsilon(x)+1)x^2 + (1-\beta_\varepsilon(x))c_\varepsilon \end{aligned}$$

it's immediately clear that f_ε is smooth, positive, and satisfies the first and third conditions listed in the lemma.

To check that f_ε satisfies the second condition we compute using $x\beta'_\varepsilon(x) \geq 0$, $\beta_\varepsilon(x) \geq 0$, and (A.8), and find

$$\begin{aligned} xf'_\varepsilon(x) &= x\left[\tfrac{1}{2}\beta'_\varepsilon(x)x^2 + (\beta_\varepsilon(x)+1)x - \beta'_\varepsilon(x)c_\varepsilon\right] \\ &= \tfrac{1}{2}[x\beta'_\varepsilon(x)]x^2 + (\beta_\varepsilon(x)+1)x^2 - x\beta'_\varepsilon(x)c_\varepsilon \\ &\geq x^2 - x\beta'_\varepsilon(x)c_\varepsilon \\ &= x^2(1 - x^{-1}\beta'_\varepsilon(x)c_\varepsilon) \\ &\geq \tfrac{1}{2}x^2. \end{aligned}$$

Since $\frac{1}{2}x^2 > 0$ for $x \neq 0$, $xf'_\varepsilon(x) > 0$ for all $x \neq 0$, and this completes the proof. □

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