Trees of Groups

Seminar: Groups Acting On Trees

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In this talk every group G acts wihtout inversion of edges on a graph Γ.

Definition 1. *We call a pair* $(\mathcal{G}, \Gamma)$ *a* ***graph of groups*** *if:*

- Γ *is a connected graph*
- $\mathcal{G}$ *is a mapping which assigns to every element* $x \in \mathrm{V}(\Gamma) \cup \mathrm{E}(\Gamma)$ *a group* $\mathcal{G}(x)$.
- *for* $P \in \mathrm{V}(\Gamma)$ *the group* $\mathcal{G}(P) := G_P$ *is called the* ***vertex group*** *at* P
- *for* $e \in \mathrm{E}(\Gamma)$ *the group* $\mathcal{G}(e) := G_e$ *is called the* ***edge group*** *at* e *with* $G_e = G_{\overline{e}}$
- *for each edge* $e \in \mathrm{E}(\Gamma)$ *there is a monomorphism* $G_e \longrightarrow G_{t(e)}$ *donated by* $g \longmapsto g^e$.

If $\Gamma = T$ *is a tree we call* $(\mathcal{G}, T)$ *a* ***tree of groups***.

The **direct limit** of a system of groups given by a graph of groups $(\mathcal{G}, \Gamma)$ we denote with

$$G_\Gamma = \varinjlim(\mathcal{G}, \Gamma)$$

Theorem 2. *Let* $(\mathcal{G}, T)$ *be a tree of groups. Then there exists a graph* X *containing* T *and an action* G_T *on* X *characterized by:*

(1) T *is a fundamental domain for* X mod G_T
(2) $\mathrm{Stab}_{G_T}(P) = G_P$ *for all* $P \in \mathrm{V}(T) \subset \mathrm{V}(X)$
(3) $\mathrm{Stab}_{G_T}(e) = G_e$ *for all* $e \in \mathrm{E}(T) \subset \mathrm{E}(X)$

Moreover the graph X *is a tree.*

By considering the following scenario we can show that the converse of **Thm. 2** is true: Let G be group acting on a graph Γ such that the **fundamental domain** is a tree T. Note that this implies

$$\mathrm{V}(\Gamma) = \bigsqcup_{P \in \mathrm{V}(T)} G.P, \quad \mathrm{E}(\Gamma) = \bigsqcup_{e \in \mathrm{E}(T)} G.e\ .$$

Further let $(\mathcal{G}, T)$ be a **tree of groups** determined by the stabilizers of the action of G on T, i.e.

$$\mathrm{Stab}_G(P) = G_P \quad \forall P \in \mathrm{V}(T) \quad \text{and} \quad \mathrm{Stab}_G(e) = G_e \quad \forall e \in \mathrm{E}(T)$$

with monomorphisms $G_e \longrightarrow G_{t(e)}$ being the inclusions. Set $G_T = \lim(\mathcal{G}, T)$. Because of $G_e, G_P \leq G$ we obtain a map

$$\phi :\ G_T \longrightarrow G$$

Now let X be the graph associated to $(\mathcal{G}, T)$ obtained by **Thm. 2**. Since X contains also T, the identity map $\mathrm{id} : T \longrightarrow T$ extends uniquely to a map

$$\psi : X \longrightarrow \Gamma, \quad gP \longmapsto \phi(g)P$$

Lemma 3. *The map* $\phi : G_T \longrightarrow G$ *ist surjektiv if and only if the graph* Γ *is connected.*

With this in mind we can state

Theorem 4. *With the notations as above, the following statements are equivalent:*

(1) Γ *is a tree*

(2) $\psi : X \longrightarrow \Gamma$ *is an isomorphism*

(3) $\phi : \ G_T \longrightarrow G$ *is an isomorphism*

Lemma 5. *Let* X *be a connected graph and* Γ *be a tree. Is* $f : X \longrightarrow \Gamma$ *a locally injective morphism, then* f *is injective.*

Example 6. *'*
Let K_4 *be the* ***Klein four group*** *with representation* $< a, b | a^2 = b^2 = (ab)^2 = 1 >$.
Let T *be a tree which is a* Path_2 *with vertices* $\{P_1, P_2, P_3\}$ *and edges* $\{e_1, e_2\}$. *Construct* $(\mathcal{G}, T)$ *by assigning*

$$P_1 \longmapsto K_4 =< a, b| \dots >, \quad P_2 \longmapsto K_4 =< c, d| \dots >, \quad P_3 \longmapsto K_4 =< x, y| \dots >,$$

$$e_1 \longmapsto \mathbb{Z}_2 =< w|w^2 >, \quad e_2 \longmapsto \mathbb{Z}_2 =< z|z^2 >$$

with monomorphisms $G_{e_1} \longrightarrow G_{P_1}$, $w \longmapsto b$, $G_{e_1} \longrightarrow G_{P_2}$, $w \longmapsto c$ *and* $G_{e_2} \longrightarrow G_{P_2}$, $z \longmapsto d$, $G_{e_2} \longrightarrow G_{P_3}$, $z \longmapsto x$. *The theorems give us a group action of* $G_T = (K_4 *_{\mathbb{Z}_2} K_4) *_{\mathbb{Z}_2} K_4$ *on a tree* X *with fundamental domain* T. *Even though the group* G_T *depends highly on the inclusions of the edge groups into the corresponding vertex groups, the tree* X *is by* Thm. 4 *unique up to isomorphism.*

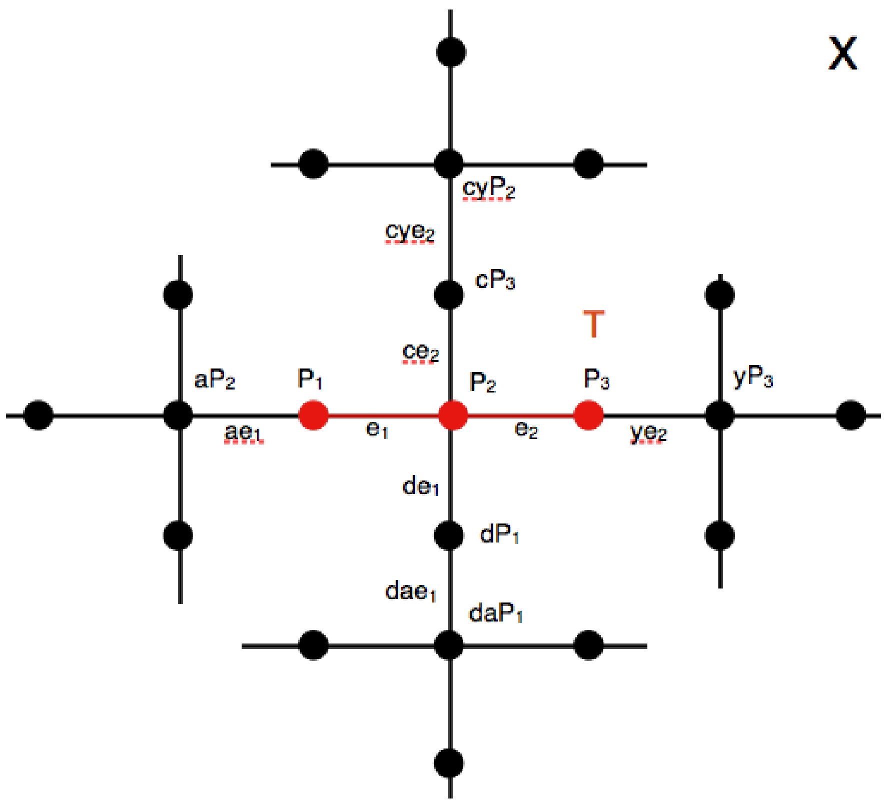

FIGURE 1. The tree X with fundamental domain T