Talk 2: Trees

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1 Graphs

Definition 1.1. *i) A* ***graph*** Γ *consists of a set* $V = V(\Gamma)$*, a set* $E = E(\Gamma)$ *and two maps*

$$E \to V \times V, \qquad e \mapsto (o(e), t(e))$$
$$E \to E, \qquad e \mapsto \bar{e}$$

which satisfy the following condition:

$$\forall e \in E : \bar{\bar{e}} = e, \bar{e} \neq e \text{ and } o(e) = t(\bar{e})$$

ii) An ***orientation*** *of a graph* Γ *is a subset* E_+ *of* $E = E(\Gamma)$ *such that* E *is the disjoint union of* E_+ *and* $\overline{E_+}$*. An* ***oriented graph*** *is given by the set* $V = V(\Gamma)$*, an orientation* E_+ *and a map* $E_+ \to V \times V$*.*

iii) A ***morphism*** φ *of graphs (from a graph* Γ_1 *to a graph* Γ_2*) consists of two maps*

$$\varphi_V : V_1 \to V_2 \qquad \varphi_E : E_1 \to E_2$$

where $V_i = V(\Gamma_i)$ *and* $E_i = E(\Gamma_i)$ $(i \in \{1, 2\})$ *and the following conditions hold:*

$$\forall e \in E_1 : \qquad \varphi_V(o(e)) = o(\varphi_E(e))$$
$$\varphi_V(t(e)) = t(\varphi_E(e))$$
$$\varphi_E(\bar{e}) = \overline{\varphi_E(e)}$$

Definition 1.2. *Let* $n \in \mathbb{N}$*.*

i) The oriented graph $\boldsymbol{Path_n}$ *is given by* $V = \{0, ..., n\}$*, the orientation* $E_+ = \{[i, i+1] | 0 \leq i < n\}$ *and* $o([i, i+1]) = i$ *as well as* $t([i, i+1]) = i + 1$*.*

ii) A ***path*** *(of length n) in a graph* Γ *is a morphism* c *of* $Path_n$ *into* Γ*. When* $c([i, +1]) = \overline{c([i+1, i+2])}$ *(for some* $0 \leq i < n - 1$*) we call the pair* $(c([i, i+1]), c([i+1, i+2]))$ *a* ***backtracking****.*

iii) Let Γ *be a graph and* c *a path of length* n *in* Γ*. The vertices* $P_0 := c(0)$ *and* $P_n := c(n)$ *are called the* ***extremities*** *of* c *and* c *is called* ***a path from*** P_0 ***to*** P_n*.*

*iv) A graph Γ is said to be **connected** if for any $v_1, v_2 \in V(\Gamma)$ there exists an $n \in \mathbb{N}$, and a path c of length n in Γ whose extremities are v_1 and v_2.*

Proposition 1.3. *Let Γ be a graph and $v_1, v_2 \in V(\Gamma)$. If there exists a path c with extremities v_1, v_2 then there exists a path c' with the same extremities and without backtracking.*

Definition 1.4. *Let $n \in \mathbb{N}^+$.*

*i) The oriented graph **Circ**$_n$ is given by $V = \mathbb{Z}/n\mathbb{Z}$, the orientation $E_+ = \{[i, i+1] | i \in \mathbb{Z}/n\mathbb{Z}\}$ and $o([i, i+1]) = i$ as well as $t([i, i+1]) = i + 1$.*

*ii) A **circuit** (of length n) in a graph is any subgraph isomorphic to Circ$_n$.*

*iii) A graph is called **combinatorial** if it has no circuit of length ≤ 2.*

*iv) A **geometric edge** of a combinatorial graph is a set $\{P, Q\}$ of extremities of a path of length 1.*

Definition 1.5. *Let G be a group and let S be a subset of G. We let $\Gamma := \Gamma(G, S)$ be the oriented graph with $V = G$, $E_+ = G \times S$ and $o(g, s) = g, t(g, s) = gs \ \forall (g, s) \in G \times S$. If S generates G we call $\Gamma(G, S)$ a **Cayley**-graph.*

Proposition 1.6. *Let $\Gamma = \Gamma(G, S)$ be the graph defined by a group G and a subset S of G.*

i) Γ is connected if and only if S generates G.

ii) Γ contains a circuit of length 1 if and only if $1_G \in S$.

iii) Γ is combinatorial if and only if $S \cap S^{-1} = \emptyset$.

2 Trees

Definition 2.1. *i) A **tree** is a connected, non-empty graph without circuits.*

*ii) A **geodesic** in a tree is a path without backtracking.*

*iii) The length of a geodesic from P to Q (2 vertices in a tree Γ) is called the **distance** from P to Q and denoted by $l(P, Q)$.*

Proposition 2.2. *Let P and Q be two vertices in a tree Γ. There is exactly one geodesic from P to Q and it is an injective path.*

Corollary 2.3. *In a tree Γ with $V := V(\Gamma)$ the function $l : V \times V \to \mathbb{N}, (P, Q) \mapsto l(P, Q)$ is well-defined and (V, l) is a metric space.*

Definition 2.4. *Let Γ be a graph, $V := V(\Gamma), E := E(\Gamma)$ and $P \in V$ a vertex.*

*i) Define $E_P := \{e \in E | t(e) = P\}$. The **index** of P is defined as the cardinatlity of E_P. If $|E_P| \leq 1$ one says that P is a **terminal vertex**. If $|E_P| = 0$ we call P **isolated**.*

ii) Let $\Gamma \setminus P$ denote the subgraph of Γ with vertex set $V \setminus \{P\}$ and edge set $E \setminus (E_P \cup \overline{E_P})$.

Proposition 2.5. *Let P be a non-isolated terminal vertex of a graph Γ.*

i) *Γ is connected if and only if $\Gamma \setminus P$ is connected.*

ii) *Every circuit of Γ is contained in $\Gamma \setminus P$.*

iii) *Γ is a tree if and only if $\Gamma \setminus P$ is a tree.*

Proposition 2.6. *Let Γ be a tree of diameter $n < \infty$.*

i) *The set $t(\Gamma)$ of terminal vertices of Γ is non-empty.*

ii) *If $n \geq 2$, $V(\Gamma) \setminus t(\Gamma)$ is the vertex set of a subtree of diameter $n-2$.*

iii) *If $n = 0$ we have $\Gamma \cong Path_0$ and if $n = 1$ we have $\Gamma \cong Path_1$.*

Corollary 2.7. *A tree of even finite diameter (resp. odd finite diameter), has a vertex (resp. geometric edge), which is invariant under all automorphisms.*

3 Subtrees

Definition 3.1. *Let Γ be a non-empty graph. A* ***maximal tree*** *of Γ is a maximal element of the set of subgraphs of Γ which are trees, ordered by inclusion.*

Remark. By Zorn's Lemma every non-empty graph has at least one maximal tree.

Proposition 3.2. *Let Λ be a maximal tree of a connected non-empty graph Γ. Then Λ contains all the vertices of Γ.*

Proposition 3.3. *Let Γ be a connected graph with a finite number of vertices. Put*

$$s(\Gamma) := |V(\Gamma)|, \qquad a(\Gamma) := \frac{1}{2}|E(\Gamma)|$$

Then $a(\Gamma) \geq s(\Gamma) - 1$ and equality holds if and only if Γ is a tree.

Remark. The Betti numbers B_i of the non-empty graph Γ are $B_0 = 1, B_1 = a(\Gamma) - a(\Lambda)$ and $B_i = 0$ for $i \geq 2$, where Λ is a maximal tree of Γ. The formula $a(\Gamma) = s(\Gamma) - 1 + (a(\Gamma) - a(\Lambda))$ can then be written:

$$s(\Gamma) - a(\Gamma) = \sum_i (-1)^i B_i$$

which is a special case of the **Euler-Poincaré formula**.

Definition 3.4. *Let Γ be a graph and $E := E(\Gamma), V := V(\Gamma)$. We form the topological space T which is the disjoint union of V and $E \times [0,1]$, where E and V are provided with the discrete topology. Let R be the finest equivalence relation on T for which $(e,t) \equiv (\bar{e}, 1-t), (e,0) \equiv o(e)$ and $(e,1) \equiv t(e)$ for $e \in E$ and $t \in [0,1]$. The quotient space $real(\Gamma) = T/R$ is called the* ***realization*** *of the graph Γ.*

Remark. Recall the following definitions and results from algebraic topology. Let X, Y be two topological spaces and $I := [0,1]$ the unit interval.

i) A **homotopy** between X and Y is a family of maps $h_t : X \to Y, t \in I$ such that the associated map $H : X \times I \to Y$ given by $H(x,t) = h_t(x)$ is continuous.

ii) Two maps $h_0, h_1 : X \to Y$ are **homotopic** if there exists a homotopy h_t connecting them and one writes $h_0 \simeq h_1$.

iii) A map $h : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $h \circ g \simeq id_Y$ and $g \circ h \simeq id_X$. In this case the spaces X and Y are said to be **homotopic equivalent**.

iv) A space which is homotopic equivalent to a point is called **contractible**.

v) Let $A \subseteq X$, the pair (X, A) is said to have the **homotopy extension property** if, given a homotopy $h_t : A \to Y$ and a map $H_0 : X \to Y$ such that $H_0|_A = h_0$, there exists an extension of H_0 to a homotopy $H_t : X \to Y$ such that $H_t|_A = h_t$.

vi) For any CW-complex X and any subcomplex A the pair (X, A) has the homotopy extension property.

vii) A **bouquet of (n) circles** is the quotient space C/S of a disjoint union of n circles C and a set S which contains one point from each circle.

Proposition 3.5. *The realization of a tree is contractible.*

Definition 3.6. *Let Γ be a connected non-empty graph and let Λ be a subgraph of Γ which is a disjoint union of a family $\Lambda_i (i \in I)$ of trees. We define a graph Γ/Λ such that $real(\Gamma/\Lambda)$ is the quotient space of $real(\Gamma)$ obtained by identification of each subspace $real(\Lambda_i)$ to a point. More precisely set $V(\Gamma/\Lambda) := V(\Gamma)/R$ where the classes of the equivalence relation R are the sets $V(\Lambda_i)$ and the elements of $V(\Gamma) \setminus V(\Lambda)$. Further let $E(\Gamma/\Lambda) := E(\Gamma) \setminus E(\Lambda)$ with the involution $e \mapsto \bar{e}$ induced by that on $E(\Gamma)$. Finally,*

$$E(\Gamma/\Lambda) \to V(\Gamma/\Lambda) \times V(\Gamma/\Lambda)$$

is induced by

$$E(\Gamma) \to V(\Gamma) \times V(\Gamma)$$

by passing to quotients.

Proposition 3.7. *Let Γ be a graph and Λ as in 3.6. The canonical projection $real(\Gamma) \to real(\Gamma/\Lambda)$ is a homotopy equivalence.*

Corollary 3.8. *Let Γ be a connected non-empty graph. Then $real(\Gamma)$ is homotopic equivalent to a bouqet of circles. Furthermore, Γ is a tree if and only if $real(\Gamma)$ is contractible.*

Corollary 3.9. *Let Γ be a graph and Λ as in 3.6. Γ is a tree if and only if Γ/Λ is one.*