Cyclotomic Iwasawa theory of motives

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Abstract

We construct Selmer modules for cyclotomic deformations of motives, whose characteristic ideals recover the algebraic p-adic L-functions of Perrin-Riou. These provide an algebraic counterpart to the unbounded p-adic L-functions attached to supersingular modular forms, for example, and we use Kato's Euler system to relate the two via a divisibility. Our main tool is to extend the main results of local Iwasawa theory to all (φ, Γ)-modules over the Robba ring.

Introduction

Main results

In her monograph [20], Perrin-Riou described a cyclotomic Iwasawa theory for motives. This theory had the surprising feature of defining algebraic p-adic L-functions, but *not* as the characteristic ideals of any apparent Selmer modules. Our ultimate goal in this article is to show that the Selmer modules constructed by the methods of [22] are indeed the ones missing from Perrin-Riou's theory. Along the way, we extend to all (φ, Γ)-modules over the Robba ring the main results of local Iwasawa theory, and deduce that they satisfy the key finiteness and duality results needed in [22]. In the special case of motives associated to modular forms, we extend consequences of Kato's Euler system to the nonordinary case.

Let us make our main results precise under the simplifying assumptions of good reduction and base field $\mathbf{Q}$. Fix a prime number p and a finite extension $E/\mathbf{Q}_p$, and write $\Gamma = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})^+/\mathbf{Q})$, $\Lambda = \mathcal{O}_E[\![\Gamma]\!]$, and $\Lambda_\infty = \Gamma(W, \mathcal{O})$, where W is the generic fiber of $\mathrm{Spf}(\Lambda)$. (In the work of Perrin-Riou, Λ_∞ is sometimes denoted $\mathcal{H}(\Gamma)$.) Normalize the Hodge–Tate weight of the cyclotomic character to be -1.

Fix a finite set S of primes containing p, write $G_{\mathbf{Q},S}$ for the Galois group of a maximal extension $\mathbf{Q}_S$ of $\mathbf{Q}$ unramified outside $S \cup \{\infty\}$, and fix a complex conjugation $c \in G_{\mathbf{Q},S}$. Let V be a finite-dimensional E-vector space with a continuous, linear $G_{\mathbf{Q},S}$-action, whose restriction to a decomposition group above p is crystalline. In this paper we naturally attach to each φ-stable subspace $N \subseteq \mathbf{D}_{\mathrm{crys}}(V)$ of dimension $\dim_{\mathbf{Q}_p} V^{c=1}$ a coherent sheaf $\mathcal{S} = \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{\mathbf{Q},S}, V^*(1))$ over W, identified to a Λ_∞-module, satisfying the following theorem.

Theorem. *(1) If V is Panchishkin-ordinary and $N = \mathbf{D}_{\mathrm{crys}}(F^+)$ comes from its filtration, then $\mathcal{S} = \mathrm{Sel}(\mathbf{Q}(\mu_{p^\infty})^+, V/T)^\vee \otimes_\Lambda \Lambda_\infty$ is the base change of the classical ordinary Iwasawa compact dual Selmer group, for $T \subseteq V$ any Galois-stable $\mathbf{Z}_p$-lattice.*

(2) Let E_ε/E be a finite extension, let $\varepsilon\colon \Gamma \to E_\varepsilon^\times$ have finite order, and let $i \in \mathbf{Z}$. Assume that $V(\varepsilon\chi_{\mathrm{cycl}}^i)^{G_{\mathbf{Q},S}} = 0$ and both $X = N^*(1-i), (\mathbf{D}_{\mathrm{crys}}(V)/N)(i)$ satisfy $\mathrm{Fil}^0 X = X$ and if $\varepsilon = 1$ then $X^{\varphi=1} = 0$. Then $\mathcal{S} \otimes_{\Lambda_\infty, \varepsilon\chi_{\mathrm{cycl}}^i} E_\varepsilon$ is canonically dual to the Bloch–Kato Selmer group $\mathrm{H}^1_{\mathrm{f}}(G_{\mathbf{Q}}, V(\varepsilon\chi_{\mathrm{cycl}}^i))$.

(3) Assume that $V^*(1)^{\mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q}(\mu_{p^\infty})^+)} = 0$ and $\mathbf{D}_{\mathrm{crys}}(V|_{G_{\mathbf{Q}_p}})^{\varphi=p^j} = 0$ for all $j \in \mathbf{Z}$. Then Perrin-Riou's algebraic p-adic L-function is given up to units by

$$\mathbb{I}_{\mathrm{arith},\{p,\infty\}}(\det_{\mathbf{Q}_p} N) = (\mathrm{char}_{\Lambda_\infty} \mathcal{S}) \cdot \Gamma^{-1}_{(\mathbf{D}_{\mathrm{crys}}(V)/N)^*(1)},$$

where the final factor depends only on the Hodge–Tate weights of $(\mathbf{D}_{\mathrm{crys}}(V)/N)^*(1)$. If $\mathcal{S}$ is torsion, then the Weak Leopoldt Conjecture (WLC) holds for both $V, V^*(1)$.

Part (3) explains much about Perrin-Riou's theory. For example, her computation of the order of vanishing of her p-adic L-function [20, Chapter 3] is simply a reflection of the control theorem: $\mathrm{length}_{\Lambda_\infty} \mathcal{S}_I \geq \dim_E \mathrm{H}^1_{\mathrm{f}}(G_{\mathbf{Q}}, V)$, where $I \subseteq \Lambda_\infty$ is the augmentation ideal, with equality under a semisimplicity hypothesis. Moreover, her "Gamma factors at infinity" seem designed simply to balance out the factor $\Gamma^{-1}_{(\mathbf{D}_{\mathrm{crys}}(V)/N)^*(1)}$.

In part (2) above, the condition on $V(\varepsilon\chi_{\mathrm{cycl}}^i)^{G_{\mathbf{Q},S}}$ avoids a conjectural pole of the p-adic L-function, the conditions on the filtration correspond to considering only its critical values, and the conditions on Frobenius avoid a finite set of conjectural "exceptional zeroes" (but see [3] for a treatment of the latter). The hypotheses in (3) are similar (but for all χ at once), and show up throughout Perrin-Riou's work; see [20, §2.4.7] for the classical Panchishkin-ordinary case.

Let us specialize to the case where f is a normalized cuspidal elliptic modular newform of weight $k \geq 2$, level N with $p \nmid N$, and coefficients in E, and V is its associated (cohomological) E-valued Galois representation. Suppose that the Frobenius operator on $\mathbf{D}_{\mathrm{crys}}(V)$ is semisimple with distinct eigenvalues. Choose an eigenspace N with eigenvalue α, and assume that N is complementary to the Hodge filtration; in particular, both eigenspaces are allowed except in the locally-split ordinary case, no matter whether f is ordinary or nonordinary. Let (f^c, α^c) be the complex conjugate of (f, α), and let $\mathrm{L}_p(f^c, \alpha^c) \in \Lambda_\infty$ be its p-adic L-function.

Theorem. *One has $\mathrm{L}_p(f^c, \alpha^c) \in \mathrm{char}_{\Lambda_\infty} \mathcal{S}$.*

In short, if we allow the use of coherent analytic sheaves on W in addition to finite type $\Lambda[1/p]$-modules, then we can bring many nonordinary cases of Iwasawa theory (and even some nonclassical ordinary ones!) onto an equal footing with the ordinary case. Since $\Lambda[1/p] \to \Lambda_\infty$ is faithfully flat, this comes at the only expense of ignoring p-torsion information. This shift, from $\Lambda[1/p]$ to W, allows us to give an algebraic counterpart to the p-adic L-functions of nonordinary modular forms, despite the fact that the latter have infinitely many zeroes.

Overview

The first section, which is preliminary, sketches the structure and duality theory of a nice category of modules, called "coadmissible", over the non-Noetherian ring Λ_∞.

In the second section we study the Iwasawa cohomology of general (φ, Γ)-modules over the Robba ring. For a Galois representation V (resp. a (φ, Γ)-module D), we define its cyclotomic deformation to be $\overline{V} = V \otimes_{\mathbf{Q}_p} \widetilde{\Lambda}_\infty^\iota$ (resp. $\overline{D} = D \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p}} \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\Lambda}_\infty^\iota)$), where the $\widetilde{(\cdot)}^{\iota}$ denotes an appropriate Galois action. The use of Λ_∞ instead of the more traditional Λ is necessary for working with (φ,Γ)-modules over the Robba ring, and is relatively harmless: in [22] it is shown that

$$\mathrm{H}^*(G_{\mathbf{Q}_p}, V \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^{\iota}) \otimes_{\Lambda} \Lambda_\infty \xrightarrow{\sim} \mathrm{H}^*(G_{\mathbf{Q}_p}, \overline{V}) \cong \mathrm{H}^*(G_{\mathbf{Q}_p}, \overline{\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)}).$$

(To avoid confusion, we use $\mathrm{H}^*_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p}, V)$ to denote the classical Iwasawa cohomology $\mathrm{H}^*(G_{\mathbf{Q}_p}, V \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^{\iota})$.) We show that $\mathrm{H}^*(G_{\mathbf{Q}_p}, \overline{D})$ is computed by the (co)invariants of ψ on D, which we denote by $\mathrm{H}^*_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D)$, and we show that the latter have the desired finiteness property. Once this is done, it is little more work to prove a duality result for the Iwasawa cohomology of D.

In the third section we provide a theory of Wach modules for crystalline (φ,Γ)-modules. Similarly to in [4] we show that one may compute Iwasawa cohomology as the (co)invariants of ψ on the Wach module; on the other hand, the Wach module is clearly related to the crystalline periods. Thus, we extend the definition and main properties of Perrin-Riou's big logarithm map to (φ,Γ)-modules.

The fouth section shows how to deduce the first theorem above. The subspace N corresponds to a subobject F of $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, and the natural map $\overline{F} \to \overline{D}$ is used to form a (strict) ordinary local condition in $\mathrm{H}^1(G_{\mathbf{Q}_p}, \overline{V})$. The claims (1,2) then follow from the methods of [22], since we know by the second section that the Galois cohomology of $\overline{F}$ obeys the finiteness theorem. Having access to the big logarithm map for D/F, and knowing its determinant, the comparison (3) to Perrin-Riou's algebraic p-adic L-function is reduced to a formal calculation.

Although part (3) implies the second theorem immediately by work of Kato and Perrin-Riou, in the final section we work out the case of modular forms in some detail to get a more general result. The point is that, as long as f as finite slope, the subobject F always becomes crystalline over a finite p-cyclotomic extension.

Relations to other work

In his Ph.D. thesis [21], R. Pollack noticed that when $a_p = 0$, special linear combinations of $\mathrm{L}_p(f,\alpha)$ and $\mathrm{L}_p(f,\beta)$ factor as a purely local term times an element of $\Lambda[1/p]$. Building on work of Perrin-Riou [19] and M. Kurihara [13], on the algebraic side S. Kobayashi [12] found alternate local conditions in weight 2 whose Selmer groups match these new, simplified p-adic L-functions. The case of weight 2 but $a_p \neq 0$ was handled in the masters thesis [25] of F. Sprung. This story has since been generalized to modular forms of all weights (and general motives having good reduction) in a series of works using the Wach module of V (which is a sort of refinement of the (φ,Γ)-module); see [15, 14]. Our characteristic ideals are related to theirs via certain functions coming from p-adic Hodge theory, as we explain at the end of §4. It seems to us that two aspects familiar to the ordinary case get divided up between their Selmer modules and ours: their Selmer modules have simple analytic properties, in fact can be defined integrally over Λ, whereas ours are directly related to motivic invariants. In fact, we have come to view the necessity for both theories, and to play one off the other, as *fundamental* to Iwasawa theory, and the fact that the two theories coalesce in the ordinary case as a convenient yet very misleading *coincidence*. In future work we will explain how the method of [19] can be seen in this picture, in such a way that we think illustrates this philosophy.

Ideas similar to ours have been used by D. Benois [2, 3] to study exceptional zeroes of Perrin-Riou's algebraic p-adic L-functions. We expect that his results can be recovered in our language by generalizing Nekovář's construction of height pairings [17, §11] from the classical case. Moreover, this formalism should lead to precise special value formulas for our p-adic L-functions at classical points resembling the algebraic side of the Bloch–Kato conjecture. (Such formulas are a key ingredient in our forthcoming reformulation of [19].)

Dabrowski and Panchishkin long ago predicted (see [9, 18]) that the analytic p-adic L-function $\mathrm{L}_p(V, N)$ grows toward the boundary of W like $\mathrm{O}(\log^h)$, where h is the difference between the Newton and Hodge degrees of N. It is likely that one can prove the analogous statement on the algebraic side: one must show that, in the notation of this paper, the characteristic ideal of the cokernel of $\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V) \to \mathrm{H}^1_{\mathrm{Iw}}(G_K, D/F)$ grows like $\mathrm{O}(\log^h)$, where $h = \deg(F)$ is the slope of $\det(F)$. See Equation 4.4 for a partial result in this direction when $\dim_{\mathbf{Q}_p} V^{c=-1} = 1$.

Although Perrin-Riou's formulation of Iwasawa theory is sufficient for obtaining upper bounds on algebraic p-adic L-functions via Euler systems, the only known means of giving lower bounds is by actually constructing cocycles, as in Ribet's method using congruences of automorphic forms. Thus our approach seems necessary in order to prove a full main conjecture in the nonordinary case. For example, we challenge the reader to generalize Skinner–Urban's work [24] to the general good reduction, finite-slope case.

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1 Coadmissible Λ_∞-modules

Fix a prime p, and let F be a field of characteristic zero with a chosen algebraic closure F^{alg}. Throughout this section we either write $F_n = F(\mu_{p^n})$ for $n \leq \infty$, or we write $F_n = F \cdot \mathbf{Q}(\mu_{p^n})^+$ for $n \leq \infty$, and we put $G = G_F = \mathrm{Gal}(F^{\mathrm{alg}}/F)$, $\Gamma = \Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $H = H_F = \ker(G \twoheadrightarrow \Gamma)$. Assume that F_∞/F is an infinite extension. We write $\Lambda = \Lambda_F = \mathcal{O}_E[\![\Gamma_F]\!]$ for the completed group ring, which is a separated and complete Noetherian semilocal ring with finite residue field. As in [22, §1.4], we write $\Lambda_n = \Lambda_{F,n}$ for the p-adic completion of $\Lambda[\mathfrak{m}^n/p]$, as well as $\Lambda_\infty = \Lambda_{F,\infty} = \varprojlim_n(\Lambda_n[1/p])$. The p-adic analytic space $W = W_F$ arising as the generic fiber of the formal scheme $\mathfrak{W} = \mathfrak{W}_F = \mathrm{Spf}(\Lambda)$ is called the *weight space*, and it is admissibly covered by the collection $\mathcal{U}$ of affinoid subdomains Y_n with $\Gamma(Y_n, \mathcal{O}) = \Lambda_n[1/p]$. Thus $\Gamma(W, \mathcal{O}) = \Lambda_\infty$.

The ring Λ is the product of domains in bijection with the Pontryagin dual $\Gamma^\vee_{\mathrm{tors}}$ of Γ_{tors}. For $\eta \in \Gamma^\vee_{\mathrm{tors}}$, write e_η for the corresponding idempotent, and for any object X on which Λ acts write $X_\eta = e_\eta X$. The rings $\Lambda_{n,\eta}$ are domains for all $n \leq \infty$, and when

$n < \infty$ each $\Lambda_n[1/p]_\eta$ is Euclidean, so $\mathrm{Div}(Y_n) = (\mathrm{Frac}\,\Lambda_n[1/p])^\times/\Lambda_n[1/p]^\times$. We write $\mathrm{Div}(W) := \varprojlim_n \mathrm{Div}(Y_n)$ for the collection of formal sums $\sum_{\mathfrak{p}} n_{\mathfrak{p}}\mathfrak{p}$ of closed points of $\bigcup_n \mathrm{Spec}\,\Lambda_n[1/p]$ such that each $\mathrm{Spec}\,\Lambda_n[1/p]$ contains only finitely many $\mathfrak{p}$ with $n_{\mathfrak{p}} \neq 0$. By work of Lazard, one similarly has $\mathrm{Div}(W) = \mathcal{K}_\infty^\times/\Lambda_\infty^\times$, with $\mathcal{K}_\infty$ the ring of total fractions of Λ_∞.

Recall from [23] the notion of a *coadmissible* Λ_∞*-module*. These are the modules arising as global sections of coherent analytic sheaves on W; alternatively, they are the inverse limits of systems $(M_n)_n$, where each M_n is a finitely generated $\Lambda_n[1/p]$-module and each natural map $M_{n+1} \to M_n$ induces an isomorphism $M_{n+1} \otimes_{\Lambda_{n+1}[1/p]} \Lambda_n[1/p] \xrightarrow{\sim} M_n$. As W is quasi-Stein, the global sections functor induces an equivlaence between coherent analytic sheaves on W and coadmissible Λ_∞-modules. We say that a coadmissible Λ_∞-module M (resp. one of its elements m) is *torsion* if every M_η (resp. m_η) is torsion. The structure of coadmissible Λ_∞-modules is rather simple.

Proposition 1.1. *(1) The torsion submodule M_{tors} of a coadmissible Λ_∞-module M is also coadmissible, and M/M_{tors} restricts to a finitely generated free module over each integral factor of Λ_∞.*

(2) The torsion coadmissible Λ_∞-modules are those isomorphic to $\prod_{\alpha\in I} \Lambda_\infty/\mathfrak{p}_\alpha^{n_\alpha}$ for some collection $\{\mathfrak{p}_\alpha\}_{\alpha\in I}$ of (not necessarily distinct) closed points of $\bigcup_n \mathrm{Spec}\,\Lambda_n[1/p]$ such that each for each n there are only finitely many α with $\mathfrak{p}_\alpha \in \mathrm{Spec}\,\Lambda_n[1/p]$.

In particular, a coadmissible Λ_∞-module arises via base change from a finitely generated $\Lambda[1/p]$-module if and only if its torsion submodule has finite support.

We define the *rank* of a coadmissible Λ_∞-module M to be the tuple $(\mathrm{rank}_{\Lambda_{\infty,\eta}} M_\eta)_\eta$. If it is torsion, its *divisor* $\mathrm{div}(M) = \sum_{\mathfrak{p}}(\mathrm{length}_{\Lambda_{\infty,\mathfrak{p}}} M_{\mathfrak{p}})\mathfrak{p}$ belongs to $\mathrm{Div}(W)$. We define its *characteristic ideal* $\mathrm{char}_{\Lambda_\infty} M$ to be the principal ideal whose η-component is zero if M_η is not torsion, and otherwise generated by $f_\eta \in \Lambda_{\infty,\eta}$ satisfying $\mathrm{div}(M_\eta) = \mathrm{div}(f_\eta)$.

We also have a variant of Grothendieck duality for coadmissible Λ_∞-modules. Given a bounded complex $C^\bullet$ of Λ_∞-modules with coadmissible cohomology, let

$$\mathscr{D}(C^\bullet) = \mathrm{Hom}_{\Lambda_\infty}\left(C^\bullet, \mathrm{Cone}\left[\mathcal{K}_\infty \longrightarrow \frac{\mathcal{K}_\infty}{\Lambda_\infty}\right][-1]\right).$$

It is easy to check by hand that this operation preserves short exact sequences of complexes and quasi-isomorphisms, and thus passes to a contravariant functor

$$\mathscr{D}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coadm}}(\Lambda_\infty) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{coadm}}(\Lambda_\infty)$$

which is an anti-involution; base changing this operation from Λ_∞ to $\Lambda_n[1/p]$ yields the usual Grothendieck duality functor $\mathbf{R}\,\mathrm{Hom}(-, \Lambda_n[1/p])$. If M is coadmissible, then we write for brevity $\mathscr{D}^i(M) = \mathrm{H}^i\mathscr{D}([M])$; one has canonical identifications

$$\mathscr{D}^0(M) = \mathrm{Hom}_{\Lambda_\infty}(M/M_{\mathrm{tors}}, \Lambda_\infty), \qquad \mathscr{D}^1\left(\prod_{\alpha\in I} \Lambda_\infty/\mathfrak{p}_\alpha^{n_\alpha}\right) = \prod_{\alpha\in I} \mathfrak{p}_\alpha^{-n_\alpha}/\Lambda_\infty,$$

and of course all other $\mathscr{D}^i(M) = 0$.

Remark 1.2. We conjecture that the above formalism generalizes from Λ_∞ to any A_∞ as in [22, §1.4] with (A, I) local, by replacing $\mathrm{Cone}[\mathcal{K}_\infty \to \mathcal{K}_\infty/\Lambda_\infty][-1]$ with $\omega_A^\bullet \otimes_A A_\infty$, where $\omega_A^\bullet$ is a bounded complex of injectives representing the Grothendieck dualizing complex for A. The key claim to be proved is that $\omega_A^\bullet \otimes_A A_n[1/p]$ is a Grothendieck dualizing complex for each $A_n[1/p]$.

Finally, we use $\widetilde{\Lambda}$ to denote Λ equipped with the linear Γ-action, and hence also G-action, given by multiplication by $\Gamma \subset \Lambda^\times$. We put $\widetilde{\Lambda}_n = \Lambda_n \otimes_\Lambda \widetilde{\Lambda}$ for $n \leq \infty$. We also write $\iota\colon \Gamma \to \Gamma$ for the inversion map $\gamma \mapsto \gamma^{-1}$, and use the same symbol for the involution it induces on $\Lambda_{(n)}$. Given a $\Lambda_{(n)}$-module M, we write $M^\iota = \Lambda_{(n)}^\iota \otimes_{\Lambda_{(n)}} M$, with $\Lambda_{(n)}$-module structure given by the first factor. Thus, $\widetilde{\Lambda}_{(n)}^\iota$ has G-action through multipliation by the inverses of images of elements in Γ.

2 Iwasawa cohomology

If $K/\mathbf{Q}$ is a finite extension and S is a finite set of primes containing all v dividing p, we write S_n for the set of primes of K_n lying above places of S; we fix a maximal extension K_S of K unramified outside $S \cup \{v | \infty\}$, noting that $K_n \subseteq K_S$ and K_S serves as a K_{n,S_n}; and we write $G = G_{K,S} = \mathrm{Gal}(K_S/K)$ and $G_n = G_{K_n,S_n} = \mathrm{Gal}(K_{n,S_n}/K_n)$. If $K/\mathbf{Q}_\ell$ is a finite extension, we write $G = G_K$ and $G_n = G_{K_n}$. For one of these choices of K (and possibly S), given $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G)$ its *classical Iwasawa cohomology* is defined to be

$$\mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,T) = \mathbf{R}\varprojlim_n \mathbf{R}\Gamma_{\mathrm{cont}}(G_n,T) \text{ and } \mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,T[1/p]) = \mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,T)[1/p]$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda[1/p])$, respectively. By a variant of Shapiro's lemma, for $X = T, T[1/p]$ one has

$$\mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,X) \cong \mathbf{R}\Gamma_{\mathrm{cont}}(G, X \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\iota).$$

If K is finite over $\mathbf{Q}$ and $v \in S$, then v only splits finitely in K_∞/K, hence Λ_K is a finite Λ_{K_v}-algebra, and we get a restriction map

$$\mathrm{res}_v\colon \mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G_{K,S},X) \to \mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G_v,X)$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda_{K_v})$.

For the rest of this section, K is finite over $\mathbf{Q}_p$. We let F_n denote $K(\mu_{p^n})$ rather than $F \cdot \mathbf{Q}(\mu_{p^n})^+$; analogous results for the latter choice are deduced from the stated ones by applying the even idempotent projector. We use the language of families of Galois representations, (φ,Γ)-modules, and their cohomology developed in [22], to which we refer for notations.

Well-known work of Fontaine and Perrin-Riou computes the structure of the Λ-modules $\mathrm{H}^*_{\mathrm{cl.Iw}}(G,T)$: one has a canonical ismomorphism

$$\mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,T) \cong \left[\mathbf{D}^\dagger(T) \xrightarrow{\psi-1} \mathbf{D}^\dagger(T)\right]$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda)$, where the right hand side is concentrated in degrees $1,2$, such that the torsion submodule of $\mathrm{H}^1_{\mathrm{cl.Iw}}(G,T) = \mathbf{D}^\dagger(T)^{\psi=1}$ is identified with $\mathbf{D}^\dagger(T)^{\varphi=1}$. The rank of $\mathrm{H}^i_{\mathrm{cl.Iw}}(G,T)$ is $[K:\mathbf{Q}_p]\,\mathrm{rank}_{\mathbf{Z}_p} T$ if $i=1$ and 0 otherwise; and their torsion submodules are computed by $\mathrm{H}^1_{\mathrm{cl.Iw}}(G,T)_{\mathrm{tors}} \cong T^H$ and $\mathrm{H}^2_{\mathrm{cl.Iw}}(G,T) \cong [T^*(1)^H]^*$ as Λ-modules.

We define, for $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G)$, its *Iwasawa cohomology* to be

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G,V) = \mathbf{R}\Gamma_{\mathrm{cont}}(G, V \otimes_{\mathbf{Q}_p} \widetilde{\Lambda}^\iota_\infty).$$

It follows from [22, Theorem 1.6] that the natural maps

$$\mathbf{R}\Gamma_{\mathrm{cl.Iw}}(G,V)\overset{\mathbf{L}}{\underset{\Lambda}{\otimes}}\Lambda_\infty \to \mathbf{R}\Gamma_{\mathrm{Iw}}(G,V) \quad\text{and}\quad \mathrm{H}^*_{\mathrm{cl.Iw}}(G,V)\otimes_\Lambda \Lambda_\infty \to \mathrm{H}^*_{\mathrm{Iw}}(G,V)$$

are isomorphisms. (A "tempered growth" variant of these isomorphisms appears in [8, Proposition II.3.1].) It is then clear that the Iwasawa cohomology groups are coadmissible Λ_∞-modules. When K is finite over $\mathbf{Q}$ and $v\in S$ we get restriction maps

$$\mathrm{res}_v\colon \mathbf{R}\Gamma_{\mathrm{Iw}}(G_{K,S},V)\to \mathbf{R}\Gamma_{\mathrm{Iw}}(G_v,V)$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda_{K_v,\infty})$. When when K is finite over $\mathbf{Q}_p$ the work of Fontaine and Perrin-Riou carries over, provided one replaces Fontaine's isomorphism by

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(G,V)\cong \left[\mathbf{D}^\dagger_{\mathrm{rig}}(V)\xrightarrow{\psi-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right]$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\Lambda_\infty)$.

By analogy, for a (φ,Γ)-module $D^{(s)}$ over $\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K}$, we define the *Iwasawa cohomology* of $D^{(s)}$ to be complex

$$\mathrm{C}^\bullet_{\mathrm{Iw}}(D^{(s)})=\left[D^{(s)}\xrightarrow{\psi-1}D^{(s)}\right]$$

in $\mathbf{K}^{\mathrm{b}}(\Lambda_\infty)$ concentrated in degrees $1,2$, and write $\mathbf{R}\Gamma_{\mathrm{Iw}}(G,D^{(s)})$ and $\mathrm{H}^*_{\mathrm{Iw}}(G,D^{(s)})$ for the respective objects it determines in $\mathbf{D}^{\mathrm{b}}(\Lambda_\infty)$ and $\mathbf{Gr}^{\mathrm{b}}(\Lambda_\infty)$. The snake lemma shows that it gives rise to a cohomological δ-functor. We go on to generalize to (φ,Γ)-modules the work of Perrin-Riou and Fontaine mentioned in §2.

The G-representations $\widetilde{\Lambda}^\iota_n[1/p]$ fit into a family of G-representations over W, and therefore determine a family of (φ,Γ)-modules over $\mathcal{O}_W\mathbin{\widehat{\otimes}}_{\mathbf{Q}_p}\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K}$ which is denoted by $\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_\infty)$. It is easy to compute that

$$\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_n[1/p])\cong(\widetilde{\Lambda}^\iota_n[1/p]\mathbin{\widehat{\otimes}}_{\mathbf{Q}_p}\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K})\cdot e,$$

with φ, ψ and Γ acting on the right hand side $(\widetilde{\Lambda}_n[1/p]\mathbin{\widehat{\otimes}}1)$-linearly via $\varphi(e)=e$, $\psi(e)=e$, and $\gamma(e)=(\gamma^{(-1)}\mathbin{\widehat{\otimes}}1)\cdot e$ for $\gamma\in\Gamma$.

For any (φ,Γ)-module $D^{(s)}$ over $E\otimes_{\mathbf{Q}_p}\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K}$, we define its *cyclotomic deformation* to be the family

$$\overline{D}^{(s)}=D^{(s)}\underset{(E\otimes_{\mathbf{Q}_p}\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K})}{\otimes}\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_\infty)$$

of (φ,Γ)-modules over $\mathcal{O}_W\mathbin{\widehat{\otimes}}_{\mathbf{Q}_p}\mathrm{B}^{\dagger(,s)}_{\mathrm{rig},K}$, and put $\overline{D}^{(s)}_n=\Gamma(Y_n,\widetilde{D}^{(s)})$. It follows from the preceding computation that the natural maps $D^{(s)}\mathbin{\widehat{\otimes}}_E\widetilde{\Lambda}^\iota_n[1/p]\to\overline{D}^{(s)}_n$ are isomorphisms, provided φ, ψ and Γ are extended to the left hand sides $\widetilde{\Lambda}_n[1/p]$-linearly with $\varphi(d\mathbin{\widehat{\otimes}}1)=\varphi(d)\mathbin{\widehat{\otimes}}1$, $\psi(d\mathbin{\widehat{\otimes}}1)=\psi(d)\mathbin{\widehat{\otimes}}1$, $\gamma(d\mathbin{\widehat{\otimes}}1)=\gamma(d)\mathbin{\widehat{\otimes}}\gamma^{-1}$ for $\gamma\in\Gamma$. In the case where $D^{(s)}=\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(V)$ with $V\in\mathbf{Rep}_E(G)$, one easily checks that $\overline{\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(V)}\cong\mathbf{D}^{\dagger(,s)}_{\mathrm{rig}}(\overline{V})$. The four results below follow from these facts and well-known explicit computations of D^s/t and $\overline{D}^s/t$ with their actions of φ, ψ, and Γ (cf. [16, §3]).

Proposition 2.1. *Let* $D^* = \mathrm{Hom}_{\mathrm{B}^\dagger_{\mathrm{rig},K}}(D, \mathrm{B}^\dagger_{\mathrm{rig},K})$.

(1) The Λ_∞*-module* $(D/t)^{\psi=1}$ *is coadmissible and torsion.*

(2) The Λ_∞*-module* $(D/t)/(\psi-1)$ *vanishes.*

(3) If $D^* = \mathrm{Hom}_{\mathrm{B}^\dagger_{\mathrm{rig},K}}(D, \mathrm{B}^\dagger_{\mathrm{rig},K})$*, then there is a canonical isomorphism*

$$\mathscr{D}((D/t)^{\psi=1}) \cong \left((t^{-1}D^*(1)/D^*(1))^{\psi=1}\right)^{\iota}[-1].$$

Proposition 2.2. *Let* $\gamma \in \Gamma/\Delta$ *be a topological generator.*

(1) If D *is étale, then the operator* $\gamma - 1$ *is bijective on* $D^{\Delta=1,\psi=0}$ *and* $\overline{D}^{\Delta=1,\psi=0}$.

(2) The operator $\gamma-1$ *is bijective on* $(D/t)^{\Delta=1,\psi=0}$ *and* $(\overline{D}_n/t)^{\Delta=1,\psi=0}$.

Lemma 2.3. *Let* $W \in \mathbf{Rep}_E(\Gamma)$.

(1) One has $(W \otimes_E R)^\Gamma = 0$ *for* $R = \widetilde{\Lambda}^\iota[1/p]$, $\widetilde{\Lambda}^\iota_n[1/p]$, $\widetilde{\Lambda}^\iota_\infty$.

(2) If $n \geq 1$*, then for all* $m \gg 0$ *one has* $\left(W \otimes_E (\ker \mathrm{Tr}_{K_m/K_{m-1}} \otimes_{\mathbf{Q}_p} \widetilde{\Lambda}^\iota_n[1/p])\right)_\Gamma = 0$.

Proposition 2.4. *One has* $\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma, \overline{D}_n) \cong \left[D \,\widehat{\otimes}_{\Lambda_\infty} \Lambda_n[1/p]\right][-1]$*, where the right hand side is concentrated in degree* 1.

We make heavy use of clever dévissage arguments due to Liu, which are centered around the following two facts.

Proposition 2.5. *Let* D *be a* (φ,Γ)*-module over* $\mathrm{B}^\dagger_{\mathrm{rig},K}$ *that is pure of slope* $\lambda \in \mathbf{Q}$.

(1) Then there exists an étale (φ,Γ)*-module* D' *that is a successive extension of various* $t^m D$ *with* $m \in \mathbf{Z}$.

(2) If $\lambda > 0$*, then there exists an extension*

$$0 \to D \to D'' \to t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K} \to 0$$

such that every constituent of the slope filtration of D'' *has nonnegative degree that is strictly less than* $\deg D$.

Proof. The claim (1) may be gleaned from the discussion of [16, §4.1], and the claim (2) follows from the argument of [16, Theorem 4.7 and Remark 4.6]. □

Theorem 2.6. *Let* D *be a* (φ,Γ)*-module over* $E \otimes_{\mathbf{Q}_p} \mathrm{B}^\dagger_{\mathrm{rig},K}$.

(1) The $\mathrm{H}^i_{\mathrm{Iw}}(G,D)$ *are coadmissible* Λ_∞*-modules, vanish for* $i \neq 1,2$*, are torsion for* $i=2$*, and are of rank equal to* $[K:\mathbf{Q}_p]\operatorname{rank} D$ *for* $i=1$.

(2) If $\gamma \in \Gamma/\Delta$ *is a topological generator, then* $\gamma - 1$ *is invertible on* $D^{\Delta=1,\psi=0}$ *and* $\overline{D}^{\Delta=1,\psi=0}$*, and the morphisms of complexes*

$$\begin{array}{lcccccc}
\mathrm{C}^\bullet_{\varphi,\gamma}: & [D^\Delta & \xrightarrow{(\varphi-1,\gamma-1)} & D^\Delta \oplus D^\Delta & \xrightarrow{(1-\gamma,\varphi-1)} & D^\Delta] \\
& \mathrm{id}\downarrow & & -\psi\downarrow\mathrm{id} & & \downarrow -\psi \\
\mathrm{C}^\bullet_{\psi,\gamma}: & [D^\Delta & \xrightarrow{(\psi-1,\gamma-1)} & D^\Delta \oplus D^\Delta & \xrightarrow{(1-\gamma,\psi-1)} & D^\Delta]
\end{array}$$

and

$$\begin{array}{lcccccc}
\overline{\mathrm{C}}^\bullet_{\varphi,\gamma}: & [\overline{D}^\Delta & \xrightarrow{(\varphi-1,\gamma-1)} & \overline{D}^\Delta \oplus \overline{D}^\Delta & \xrightarrow{(1-\gamma,\varphi-1)} & \overline{D}^\Delta] \\
& \mathrm{id}\downarrow & & -\psi\downarrow\mathrm{id} & & \downarrow -\psi \\
\overline{\mathrm{C}}^\bullet_{\psi,\gamma}: & [\overline{D}^\Delta & \xrightarrow{(\psi-1,\gamma-1)} & \overline{D}^\Delta \oplus \overline{D}^\Delta & \xrightarrow{(1-\gamma,\psi-1)} & \overline{D}^\Delta]
\end{array}$$

are quasi-isomorphisms.

(3) One has a canonical isomorphism $\mathbf{R}\Gamma_{\mathrm{Iw}}(G,D) \cong \mathbf{R}\Gamma(G,\overline{D})$.

Remark 2.7. Strictly speaking, the isomorphism of part (3) of the theorem is nonsense, because the two sides do not belong to the same category. It means that the rules

$$Y_n \mapsto \mathbf{R}\Gamma_{\mathrm{Iw}}(G,D) \overset{\mathbf{L}}{\underset{\Lambda_\infty}{\otimes}} \Lambda_n[1/p] \quad \text{and} \quad \mathbf{R}\Gamma(G,\overline{D})$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{ft}}(\mathcal{U})$ agree, or equivalently that $\mathbf{R}\Gamma_{\mathrm{Iw}}(G,D) \cong \mathbf{R}\varprojlim_n \mathbf{R}\Gamma(G,\overline{D}_n)$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coadm}}(\Lambda_\infty)$.

Proof. It suffices to forget the E-action everywhere and assume that $E = \mathbf{Q}_p$.

Suppose that D is filtered by sub-(φ,Γ)-modules $F^* \subseteq D$ that are direct summands. An easy induction, using that Iwasawa cohomology is a δ-functor, reduces us to proving the theorem for the Gr^*_X. In particular, we may assume that D is pure of some slope $\lambda \in \mathbf{Q}$, and let D' be as in Proposition 2.5(1), say $D' = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Since there is an exact sequence $0 \to D'' \to D' \to t^{m''}D \to 0$, by the work of Perrin-Riou and Fontaine, we have that

$$\mathrm{H}^2_{\mathrm{Iw}}(G,V) \cong D'/(\psi-1) \twoheadrightarrow t^{m''}D/(\psi-1)$$

for some $m'' \in \mathbf{Z}$. Since $\mathrm{H}^2_{\mathrm{Iw}}(G,V)$ is coadmissible and torsion, so is $t^{m''}D/(\psi-1)$, and by Proposition 2.1 so is $D/(\psi-1)$.

On the other hand, we have an exact sequence

$$D'^{\psi=1} \to (t^{m''}D)^{\psi=1} \to D''/(\psi-1) \to D'/(\psi-1),$$

and applying the preceding paragraph to D'' and invoking the facts that $D'^{\psi=1} \cong \mathrm{H}^2_{\mathrm{Iw}}(G,V)$ and $D'/(\psi-1) \cong \mathrm{H}^2_{\mathrm{Iw}}(G,V)$, we see that $(t^{m''}D)^{\psi=1}$ is coadmissible. It follows by Proposition 2.1 that $D^{\psi=1}$ is coadmissible and of the same Λ_∞-rank as $(t^{m''}D)^{\psi=1}$. To compute this rank, using dévissage and the fact that all the $\mathrm{H}^2_{\mathrm{Iw}}$ are torsion we calculate that

$$\begin{aligned}\frac{\operatorname{rank} D'}{\operatorname{rank} D}[K:\mathbf{Q}_p]\operatorname{rank} D &= [K:\mathbf{Q}_p]\operatorname{rank} D' = [K:\mathbf{Q}_p]\dim_{\mathbf{Q}_p} V = \operatorname{rank}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G,V)\\ &= \operatorname{rank}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G,D') = \sum_i \operatorname{rank}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G,t^{m_i}D)\\ &= \sum_i \operatorname{rank}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G,D) = \frac{\operatorname{rank} D'}{\operatorname{rank} D}\operatorname{rank}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G,D).\end{aligned}$$

This gives of part (1) of the theorem.

For part (2), to see the bijectiveness of $\gamma-1$, we note that by the surjectivity of ψ and the snake lemma, the functors $F\colon D \mapsto D^{\Delta=1,\psi=0}$ and $\overline{F}\colon D \mapsto \overline{D}^{\Delta=1,\psi=0}$ are exact, and therefore we may perform a dévissage. Thus, we may assume that D is pure of some slope $\lambda \in \mathbf{Q}$, and choose D' as in Proposition 2.5(1).

Since there is a short exact sequence of the form $0 \to D'' \to D' \to t^{m''}D \to 0$ we get from Proposition 2.2(1) that $\gamma-1$ is surjective on $(t^{m''}D)^{\Delta=1,\psi=0}$ and $(t^{m''}\overline{D})^{\Delta=1,\psi=0}$, and since there is a short exact sequence of the form $0 \to t^{m'''}D \to D' \to D''' \to 0$ we get from Proposition 2.2(1) that $\gamma-1$ is injective on $(t^{m'''}D)^{\Delta=1,\psi=0}$ and $(t^{m'''}\overline{D})^{\Delta=1,\psi=0}$. But by Proposition 2.2(2), the injectiveness (resp. surjectiveness) of $\gamma-1$ on $D^{\Delta=1,\psi=0}$ and $\overline{D}^{\Delta=1,\psi=0}$ is equivalent to the injectiveness (resp. surjectiveness) of $\gamma-1$ on $(t^mD)^{\Delta=1,\psi=0}$ and $(t^m\overline{D})^{\Delta=1,\psi=0}$ for any $m \in \mathbf{Z}$.

To see that the morphisms of complexes $\mathrm{C}^\bullet_{\varphi,\gamma} \to \mathrm{C}^\bullet_{\psi,\gamma}$ and $\overline{\mathrm{C}}^\bullet_{\varphi,\gamma} \to \overline{\mathrm{C}}^\bullet_{\psi,\gamma}$ are quasi-isomorphisms, we note the contituent maps are surjective in every degree, so it suffices to show that the kernel complexes are acyclic. The latter claim amounts to asking that $\gamma - 1$ be bijective on $D^{\Delta=1,\psi=0}$ and $\overline{D}^{\Delta=1,\psi=0}$, which we have verified. This shows part (2) of the theorem.

Part (3) follows from assembling the functorial isomorphisms

$$\begin{aligned}
\mathbf{R}\Gamma(G,\overline{D}_n) &\cong [\overline{\mathrm{C}}^\bullet_{\varphi,\gamma}] \cong [\overline{\mathrm{C}}^\bullet_{\psi,\gamma}] \cong \mathrm{Cone}\left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma,\overline{D}_n) \xrightarrow{\psi-1} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma,\overline{D}_n)\right][-1] \\
&\overset{(\star)}{\cong} \mathrm{Cone}\left[D \mathbin{\widehat{\otimes}_{\Lambda_\infty}} \Lambda_n[1/p] \xrightarrow{\psi-1} D \mathbin{\widehat{\otimes}_{\Lambda_\infty}} \Lambda_n[1/p]\right][-2] \\
&\overset{(*)}{\xleftarrow{\sim}} \mathrm{Cone}\left[D \otimes_{\Lambda_\infty} \Lambda_n[1/p] \xrightarrow{\psi-1} D \otimes_{\Lambda_\infty} \Lambda_n[1/p]\right][-2] \\
&= \mathbf{R}\Gamma^\bullet_{\mathrm{Iw}}(D) \overset{\mathbf{L}}{\underset{\Lambda_\infty}{\otimes}} \Lambda_n[1/p].
\end{aligned}$$

The isomorphism $(\star)$ is the content of Proposition 2.4, and the natural map $(*)$ is a quasi-isomorphism because the right hand side has cohomology that is finitely generated over $\Lambda_n[1/p]$ by part (1) of the theorem. □

We remark in passing that if D is trianguline, for example if it becomes semistable over an abelian extension and φ on D_{crys} is semisimple, then the preceding result can be proved by reducing to the rank one case and then making a direct computation.

Theorem 2.8. *Let D be a (φ,Γ)-module over $E \otimes_{\mathbf{Q}_p} \mathrm{B}^\dagger_{\mathrm{rig},K}$, with dual*

$$D^* = \mathrm{Hom}_{E\otimes_{\mathbf{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}}(D, E \otimes_{\mathbf{Q}_p} \mathrm{B}^\dagger_{\mathrm{rig},K}).$$

Then one has a canonical isomorphism $\mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(G,D) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}(G,D^*(1))^\iota[2]$.

Proof. This argument is just an adaptation of [16, Theorem 4.7]. It is routine to check the various compatibilities, so we omit them. By forgetting the E-action everywhere, it suffices to assume that $E = \mathbf{Q}_p$, which we do for simplicity.

The desired morphisms over the respective $\Lambda_n[1/p]$ are adjoint to the pairing

$$\begin{aligned}
&\mathbf{R}\Gamma(G, D \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_n)) \overset{\mathbf{L}}{\underset{\Lambda_n}{\otimes}} \mathbf{R}\Gamma(G, D^*(1) \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_n))^\iota \\
&= \mathbf{R}\Gamma(G, D \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\Lambda}^\iota_n)) \overset{\mathbf{L}}{\underset{\Lambda_n}{\otimes}} \mathbf{R}\Gamma(G, D^*(1) \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(\widetilde{\Lambda}_n)) \\
&\overset{\cup}{\to} \mathbf{R}\Gamma(G, (D \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} D^*(1)) \mathbin{\widehat{\otimes}_{\mathbf{Q}_p}} \Lambda_n) \\
&= \mathbf{R}\Gamma(G, D \otimes_{\mathrm{B}^\dagger_{\mathrm{rig},K}} D^*(1)) \otimes_{\mathbf{Q}_p} \Lambda_n \\
&\overset{\mathrm{ev}}{\to} \mathbf{R}\Gamma(G, \mathrm{B}^\dagger_{\mathrm{rig},K}(1)) \otimes_{\mathbf{Q}_p} \Lambda_n \cong \mathbf{R}\Gamma_{\mathrm{cont}}(G, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} \Lambda_n \\
&\to \tau_{\geq 2}\mathbf{R}\Gamma_{\mathrm{cont}}(G, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} \Lambda_n \overset{\mathrm{inv}\otimes 1}{\cong} \Lambda_n[-2].
\end{aligned}$$

They exist even on the level of cochains, compatibly for varying n, so they compile to a map on cochains over Λ_∞, and the duality operations compile to give $\mathscr{D}$. We need to check that what we get is a perfect pairing.

Because both $\mathscr{D} \circ \mathbf{R}\Gamma_{\mathrm{Iw}}(G,-)$ and $\mathbf{R}\Gamma_{\mathrm{Iw}}(G,-^*(1))^\iota[2]$ are exact functors, given any exact triangle it suffices to know the result for any two members. Therefore, we may assume that D has pure slope. Since $\mathscr{D}$ is an (anti-)involution, by replacing D with $D^*(1)$ if necessary, it suffices to treat the case where D has nonnegative slope. Invoking Proposition 2.5(2), exactness and induction on $\deg D$ reduce us to considering only the cases where D is étale, and where $D = t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K}$.

When $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is étale the analogous result for the Iwasawa cohomology of V is well-known (see for example [17, 5.2.6] applied to $V \otimes_{\mathbf{Q}_p} \widetilde{\Lambda}^\iota$), and the result for the Iwasawa cohomology of D follows. To treat $D = t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K}$, we apply the following general argument with $E = t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K}$, which shows that the case of E is equivalent to the case of tE, and then note that $\mathrm{B}^\dagger_{\mathrm{rig},K}$ is étale.

For a general (φ,Γ)-module E over $\mathrm{B}^\dagger_{\mathrm{rig},K}$, it follows from Theorem 2.6 and the short exact sequence $0 \to tE \to E \to E/t \to 0$ that the analogue of that theorem holds for E/t, for any E as in the theorem; in fact, by Proposition 2.1(1,2), $\mathrm{H}^1_{\mathrm{Iw}}(G, E/t)$ is coadmissible and torsion, and $\mathrm{H}^2_{\mathrm{Iw}}(G, E/t) = 0$. Examining the diagram

$$\begin{array}{ccccc} \mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(E/t) \to & \mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(E) & \to & \mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(Et) & \\ & \downarrow & & \downarrow & \\ & \mathbf{R}\Gamma_{\mathrm{Iw}}(E^*(1))^\iota[2] & \to & \mathbf{R}\Gamma_{\mathrm{Iw}}(t^{-1}E^*(1))^\iota[2] & \to \mathbf{R}\Gamma_{\mathrm{Iw}}(t^{-1}E^*(1)/E^*(1))^\iota[2] \end{array}$$

with exact triangles for rows, it suffices to complete it via an isomorphism

$$\mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(G, E/t) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}(G, t^{-1}E^*(1)/E^*(1))^\iota[1].$$

Such an isomorphism is obtained by combining the three parts of Proposition 2.1, and we leave it to the reader to check that it fits into the above diagram. □

In fact, one has much finer information on the torsion in Iwasawa cohomology, as the following result shows.

Proposition 2.9. *Let D be a (φ,Γ)-module over $E \otimes_{\mathbf{Q}_p} \mathrm{B}^\dagger_{\mathrm{rig},K}$. The vector spaces $\mathrm{H}^i_{\mathrm{Iw}}(G, D)_{\mathrm{tors}}$ are finite-dimensional over E.*

Proof. By the preceding duality theorem, it suffices to show the claim for $\mathrm{H}^2_{\mathrm{Iw}}(G, D)$. For each character $\chi\colon \Gamma \to E^\times$ the machinery of [22] gives an isomorphism

$$\mathrm{H}^2_{\mathrm{Iw}}(G, D) \otimes_{\Lambda_\infty,\chi} E \cong \mathrm{H}^2(G, D(\chi^{-1})),$$

and it suffices to show that the right hand side is nonzero for only finitely many χ (as one ranges over all E). By Tate local duality as in [16], it suffices to show that $\mathrm{H}^0(G, D(\chi))$ is nonzero only for finitely many χ. In fact, let $s_1, \ldots, s_2$ (resp. s) be the Hodge–Tate–Sen weights of D (resp. χ). If $\mathrm{H}^0(G, D(\chi)) \neq 0$ then some $s_i + s \in \mathbf{Z}$; thus, replacing D by $D(\chi)$ if necessary, it suffices to show that only finitely many χ with $s \in \mathbf{Z}$ have $\mathrm{H}^0(G, D(\chi)) \neq 0$. But in fact such χ have the form $\epsilon\chi_{\mathrm{cycl}}^{-s}$ with ϵ of finite order, and $\mathrm{H}^0(D(\chi)) = D(\chi)^{+,\varphi=1}_{\mathrm{crys}} \cong D(\epsilon)^{+,\varphi=p^{-s}}_{\mathrm{crys}}$. It is then easy to see that $D(\epsilon)_{\mathrm{crys}} = 0$ if ϵ has sufficiently large conductor, and when it is nonzero there are only finitely many eigenvalues of φ, so the claim folows. □

Since the $\Lambda_n[1/p]$ for $n \leq \infty$ are rings of uniformly bounded Tor-dimension (equal to 1), and the preceding proposition allows us to bound the number of generators of $\mathrm{H}^*_{\mathrm{Iw}}(G, D)$, we obtain the following strengthening of the preceding finiteness results.

Corollary 2.10. *The object* $\mathbf{R}\Gamma_{\mathrm{Iw}}(G, D)$ *can be represented in the derived category of* Λ_∞*-modules by a perfect complex in degrees* $[0, 2]$*. In particular, cyclotomic deformations satisfy the Finiteness Theorem of [22, Theorem 2.5].*

3 Wach modules and the big logarithm

Continuing the preceding section, we assume $K/\mathbf{Q}_p$ is a finite *unramified* extension. Write $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$ and $D_{\mathrm{pcrys}} = \bigcup_n D[1/t]^{\Gamma_n}$. Recall that π is a formal variable, and one has the rings $\mathrm{A}_K^+ = \mathcal{O}_K[\![\pi]\!]$, $\mathrm{B}_K^+ = \mathrm{A}_K^+[1/p]$, and $\mathrm{B}_{\mathrm{rig},K}^+ = \varprojlim_n A_K^+[\pi^n/p]^\frown[1/p]$ (where the completion is for the p-adic topology). They are all embedded into each $\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}$ and stable under the actions of φ and Γ. Let $q = \varphi(\pi)/\pi \in \mathrm{A}_K^+$. A simple refinement of the method of [7, §3.3], [6, §II] and [5, §II–III] gives the following result.

Theorem 3.1. *(1) Assume given a* (φ, Γ)*-module* D *of rank* d *that becomes crystalline over* K_n *of Hodge–Tate weights* $h_1 \le \cdots \le h_d$*. There exists a unique free* $\mathrm{B}_{\mathrm{rig},K}^+$*-submodule* $D_{\mathrm{W}} \subseteq D$ *of rank* d *with the following properties:*

- $D_{\mathrm{W}} \otimes_{\mathrm{B}_{\mathrm{rig},K}^+} \mathrm{B}_{\mathrm{rig},K}^\dagger = D$.
- $[D_{\mathrm{W}} : D_{\mathrm{pcrys}} \otimes_K \mathrm{B}_{\mathrm{rig},K}^+] = [(t/\pi)^{h_1}; \ldots ; (t/\pi)^{h_d}]$.
- $[D_{\mathrm{W}} \colon \varphi^* D_{\mathrm{W}}] = [q^{h_1}; \ldots ; q^{h_d}]$*, where* $\varphi^* D_{\mathrm{W}} = \varphi(D_{\mathrm{W}}) \otimes_{\varphi(\mathrm{B}_{\mathrm{rig},K}^+)} \mathrm{B}_{\mathrm{rig},K}^+$.
- Γ *leaves* D_{W} *stable and* Γ_n *acts trivially on* D_{W}/π.

(2) The rule $D \mapsto D_{\mathrm{W}}$ *is an exact equivalence between the category of* (φ, Γ)*-modules becoming crystalline over* K_n *and the category of finitely generated free* $\mathrm{B}_{\mathrm{rig},K}^+$*-modules* N *equipped with a* φ*-linear map* $\varphi \colon N \to N[q^{-1}]$ *such* $(\varphi^* N)[q^{-1}] = N[q^{-1}]$ *and a commuting semilinear action of* Γ *with* Γ_n *acting trivially modulo* π*.*

(3) One has canonical identifications

$$D_{\mathrm{W}}/\pi = D_{\mathrm{pcrys}}, \quad D_{\mathrm{W}}[(t/\pi)^{-1}] = D_{\mathrm{pcrys}} \otimes_K \mathrm{B}_{\mathrm{rig},K}^+[(t/\pi)^{-1}],$$
$$q^{h_d} D_{\mathrm{W}} \subseteq \varphi^* D_{\mathrm{W}} \subseteq q^{h_1} D_{\mathrm{W}},$$
$$(t^k D)_{\mathrm{W}} = (t/\pi)^k D_{\mathrm{W}}, \quad D(k)_{\mathrm{W}} = \pi^{-k} D_{\mathrm{W}}(k), \text{ and}$$
$$\mathbf{D}_{\mathrm{rig}}^\dagger(V)_{\mathrm{W}} = \mathrm{N}(V) \otimes_{\mathrm{B}_K^+} \mathrm{B}_{\mathrm{rig},K}^+,$$

where $\mathrm{N}(V)$ *is the usual Wach module of a crystalline* G_K*-representation* V*.*

Remark 3.2. Although most references refer only to the crystalline case, it is well-known that their arguments carry over with little modification to the crystalline-over-some-K_n case (see, for example, [7]), so we suppress any further discussion of this gap.

One easily sees that $\pi^{-a} D_{\mathrm{W}}$ is stable under ψ for all $a \ge h_d$, so we often consider $\psi - 1$ as an endomorphism.

The following result extends [4, Theorem A.3] to crystalline (φ, Γ)-modules. Write $\lambda(D)$ for the largest integer n for which $\varphi - p^n$ is not bijective on D_{pcrys} (or $-\infty$ if no such n exists), and $a(D) = \max\{h_d, \lambda(D) + 1\}$.

Theorem 3.3. *If* $a \ge a(D)$ *then the natural map* $[\pi^{-a} D_{\mathrm{W}} \xrightarrow{\psi-1} \pi^{-a} D_{\mathrm{W}}] \to [D \xrightarrow{\psi-1} D]$ *is a quasi-isomorphism (both complexes considered as concentrated in degrees* $1, 2$*).*

Proof. The proof begins by establishing some simplifications.

We first point out that for $a \geq a(D)$ the natural map

$$[\pi^{-a} D_{\mathrm{W}} \xrightarrow{\psi-1} \pi^{-a} D_{\mathrm{W}}] \to [\pi^{-a-1} D_{\mathrm{W}} \xrightarrow{\psi-1} \psi^{-a-1} D_{\mathrm{W}}]$$

is a quasi-isomorphism. This is because, applying the snake lemma to the operator $\psi-1$ on the short exact sequence $0 \to \pi^{-a} D_{\mathrm{W}} \to \pi^{-a-1} D_{\mathrm{W}} \to \pi^{-a-1} D_{\mathrm{W}}/\pi^{-a} D_{\mathrm{W}} \to 0$, we only need that $\psi - 1$ be bijective on $\pi^{-a-1} D_{\mathrm{W}}/\pi^{-a} D_{\mathrm{W}}$. Since the latter operator can be identified with $p^a\varphi^{-1} - 1$ on D_{pcrys} (compare [4, Lemma A.4]), this is indeed the case, as $a > \lambda(D)$. Thus, we may replace $\pi^{-a} D_{\mathrm{W}}$ by $D_{\mathrm{W}}[\pi^{-1}]$ throughout the proof.

The claim for D is clearly invariant under replacing D by its Tate twists; it is also invariant under replacing D by $t^k D$ for $k \in \mathbf{Z}$. Indeed, to see that the claims for D and tD are equivalent, applying the snake lemma to the operator $\psi - 1$ on the short exact sequences $0 \to tD_{\mathrm{W}}[\pi^{-1}] \to D_{\mathrm{W}}[\pi^{-1}] \to D_{\mathrm{W}}[\pi^{-1}]/t \to 0$ and $0 \to tD \to D \to D/t \to 0$, it suffices to compare the kernel and cokernel of $\psi - 1$ on $D_{\mathrm{W}}[\pi^{-1}]/t$ and on D/t. Using the methods of [16, §3] and §2, both kernels are computed to be $\varprojlim_{n,\psi} D^n_{\mathrm{Sen}}$ and both cokernels are computed to be zero.

Next, if $0 \to D' \to D \to D'' \to 0$ is a short strict exact sequence, then so is $0 \to D'_{\mathrm{W}}[\pi^{-1}] \to D_{\mathrm{W}}[\pi^{-1}] \to D''_{\mathrm{W}}[\pi^{-1}] \to 0$, and applying the snake lemma to the operator $\psi - 1$ on these, followed by the five lemma, shows that if the claim of the theorem holds for two of D, D', D'', then it holds for the third as well.

Let any D be given. By dévissage, it suffices to treat its pure pieces, hence to assume that D is pure of some slope $\lambda \in \mathbf{Q}$. Replacing the given D by suitable $t^k D$, we may assume that $\lambda \geq 0$. We now induct on $\deg(D) = \lambda \cdot d \in \mathbf{Z}_{\geq 0}$. In the base case, $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is étale, and the result follows from applying $\otimes_\Lambda \Lambda_\infty$ to [4, Theorem A.3] (for H^1, the case of H^2 following from the same method). Assuming the result for all degrees strictly less than $\deg(D)$, we replace D by a Tate twist $D(k)$ (which does not change its degree) such that all extensions E of $t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K}$ by $D(k)$ become crystalline over K_n. It is shown in [16, Remark 4.6 and Theorem 4.7] that nontrivial such extensions E exist, and that all the pure slope pieces of E have nonnegative degree strictly less than $\deg(D)$. By inductive hypothesis and dévissage the result holds for E and $t^{-1}\mathrm{B}^\dagger_{\mathrm{rig},K}$, hence also for D. $\square$

Take an element $\gamma \in \Gamma$ of infinite order, and for $i \in \mathbf{Z}$ write

$$\ell_i = \frac{\log \gamma}{\log \chi_{\mathrm{cycl}}(\gamma)} - i = \frac{\log(\gamma \chi_{\mathrm{cycl}}(\gamma)^{-i})}{\log \chi_{\mathrm{cycl}}(\gamma)}, \quad \Gamma_i = \begin{cases} \ell_1^{-1}\ell_2^{-1}\cdots\ell_i^{-1} & \text{if } i \geq 0, \\ \ell_0\ell_{-1}\cdots\ell_{i+1} & \text{if } i < 0, \end{cases}$$
$$\text{and} \qquad \Gamma_D = \Gamma_{h_1}\Gamma_{h_2}\cdots\Gamma_{h_2}.$$

The ℓ_i, Γ_i and Γ_D are independent of γ. Recall that the Mellin transform shows $\mathrm{B}^{+,\psi=0}_{\mathrm{rig},K}$ to be free of rank one over $K \otimes_{\mathbf{Q}_p} \Lambda_\infty$. By [14, Proposition 1.5], one has

$$\Gamma_{h_1} \cdot D_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+,\psi=0}_{\mathrm{rig},K} \subseteq D^{\psi=0}_{\mathrm{W}} \subseteq \Gamma_{h_d} \cdot D_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+,\psi=0}_{\mathrm{rig},K}.$$

Since $\varphi^*(\pi^{-a} D_{\mathrm{W}}) \subseteq q^{h_1 - a} \cdot \pi^{-a} D_{\mathrm{W}}$ for any $a \in \mathbf{Z}$, the operator $1 - \varphi$ on D takes a finite-colength Λ_∞-submodule of $(\pi^{-a} D_{\mathrm{W}})^{\psi=1}$ into $D^{\psi=0}_{\mathrm{W}}$. We deduce from these facts a "big logarithm" map à la Perrin-Riou

$$\mathrm{Log}_D\colon \mathrm{H}^1_{\mathrm{Iw}}(G_K,D) \cong D^{\psi=1} = (\pi^{-a(D)}D_{\mathrm{W}})^{\psi=1} \xrightarrow{1-\varphi} D_{\mathrm{W}}^{\psi=0} \otimes_{\Lambda_\infty} \mathcal{K}_\infty = D_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+,\psi=0}_{\mathrm{rig},K} \otimes_{\Lambda_\infty} \mathcal{K}_\infty \approx D_{\mathrm{pcrys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty,$$

whose target we consider equipped with the Λ_∞-lattice $D_{\mathrm{pcrys}} \otimes_{\mathbf{Q}_p} \Lambda_\infty$.

Theorem 3.4 ("$\delta(D)$")**.** *One has* $\det_{\Lambda_\infty} \mathrm{Log}_D \cdot \mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_K,D) = \Gamma_D \cdot \Lambda_\infty \subset \mathcal{K}_\infty$.

We remind the reader that the right way to interpret the above determinant is $(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^1_{\mathrm{Iw}}(G_K,D)_{\mathrm{tors}})^{-1}$ times the determinant of the map induced by Log_D on the free quotient $\mathrm{H}^1_{\mathrm{Iw}}(G_K,D)/\mathrm{tors}$. Thus one has the slightly more appealing restatement that $\det_{\Lambda_\infty} \mathrm{Log}_D\colon \det_{\Lambda_\infty} \mathbf{R}\Gamma_{\mathrm{Iw}}(G_K,D)^{-1} \xrightarrow{\sim} \Gamma_D \cdot \Lambda_\infty \subset \mathcal{K}_\infty$. The nondegeneracy of Log_D shows, by the way, that $\mathrm{H}^1_{\mathrm{Iw}}(G_K,D)_{\mathrm{tors}} \cong D^{\varphi=1} = (\pi^{-a(D)}D_{\mathrm{W}})^{\varphi=1}$.

Proof. Following the strategy of the proof of the preceding theorem, it suffices to show that the claim for D is invariant under replacing D by $D(k)$ and t^kD for $k \in \mathbf{Z}$, is preserved by short strict exact sequences (if it holds for two, then it holds for the third), and holds in the étale case. Invariance under Tate twisting follows from the computation that $\Gamma_{D(k)} = \Gamma^d_{-k} \cdot \mathrm{Tw}_k \Gamma_D$, plus the canonical identifications $\mathrm{H}^n_{\mathrm{Iw}}(G_K,D(k)) = \mathrm{H}^n_{\mathrm{Iw}}(G_K,D)(k)$ and

$$D(k)_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+,\psi=0}_{\mathrm{rig},K} = (D(k)_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+}_{\mathrm{rig},K})^{\psi=0} = (t^{-k}D_{\mathrm{pcrys}}(k) \otimes_K \mathrm{B}^{+}_{\mathrm{rig},K})^{\psi=0} = \Gamma^{-1}_{-k} \cdot D_{\mathrm{pcrys}} \otimes_K \mathrm{B}^{+,\psi=0}_{\mathrm{rig},K}(k),$$

which give rise to the computations $\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_K,D(k)) = \mathrm{Tw}_k \,\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_K,D)$ and $\det_{\Lambda_\infty} \mathrm{Log}_{D(k)} = \Gamma^d_{-k} \cdot \mathrm{Tw}_k \det_{\Lambda_\infty} \mathrm{Log}_D$. The claims for D and tD are equivalent by similar reasoning, this time using the identity $\Gamma_{tD} = \Gamma_{D(1)} = \ell_{h_1} \cdots \ell_{h_d} \cdot \Gamma_D$ and considering the long exact sequence for Iwasawa cohomology assoicated to $0 \to tD \to D \to D/t \to 0$, noting the computations

$$(D/t)^{\psi=1} = \varprojlim_{n,\psi} D^n_{\mathrm{Sen}} \approx \varprojlim_{n,\frac{1}{p}\mathrm{Tr}} \bigoplus_{i=1}^d t^{h_i} K_n \cong \bigoplus_{i=1}^d \Lambda_\infty/\ell_{h_i},$$

$(D/t)/(\psi-1) = 0$, and $(tD)_{\mathrm{pcrys}} = D_{\mathrm{pcrys}}$. That both sides of the desired identity are multiplicative over short strict exact sequences is easy. The étale case follows from applying $\otimes_{\Lambda[1/p]}\Lambda_\infty$ to the "$\delta(V)$ conjecture", now a well-known theorem, casting it in the language of (φ,Γ)-modules as in [4]. □

Similarly to the finiteness of Iwaswa cohomology, one can prove the above result in the trianguline case by an explicit computation for rank one modules.

Remark 3.5. The ring Λ_∞, consisting of locally analytic distributions on Γ, is sometimes referred to as $\mathcal{H}(\Gamma)$ in the literature. Other times, $\mathcal{H}(\Gamma)$ is used for the subring $\Lambda_{\mathrm{temp}} \subset \Lambda_\infty$ of *tempered distributions*. The methods of this section can easily be refined to work with this ring instead. One easily constructs D_{W} as a $\mathrm{B}^+_{\mathrm{temp},K}$-lattice in D with all the analogous properties, where $\mathrm{B}^+_{\mathrm{temp},K} \subset \mathrm{B}^+_{\mathrm{rig},K}$ consists of the power series of tempered growth. The proof of Theorem 3.3 shows (in this new notation)

$$[\pi^{-a}D_{\mathrm{W}} \xrightarrow{\psi-1} \pi^{-a}D_{\mathrm{W}}] \otimes_{\Lambda_{\mathrm{temp}}} \Lambda_\infty \to [D \xrightarrow{\psi-1} D]$$

to be a quasi-isomorphism, and the proof of Theorem 3.4 goes through without change, using Λ_{temp} and its ring of total fractions in place of Λ_∞ and $\mathcal{K}_\infty$.

4 Global results

We summarize how the preceding sections give a cyclotomic Iwasawa theory for the Selmer groups of Galois representations, and how their characteristic ideals compare to Perrin-Riou's algebraic p-adic L-functions. *From now until the end of the article* we let F_n denote $F \cdot \mathbf{Q}(\mu_{p^n})^+$ instead of $F(\mu_{p^n})$, thus we redefine Γ_F and all derivative objects, and whenever we encounter an object constructed with the old convention we tacitly apply the even idempotent projector to it, bringing it into the present situation.

In this section we suppose that $K/\mathbf{Q}$ is a finite extension and S is a finite set of places containing all v dividing p as at the beginning of §2. For K' a finite subextension of K_∞/K with S' its set of places lying over S, and $\chi\colon \Gamma_K \to \mathcal{O}_E^\times$ a continuous character, we write $I_{\chi,K'} \subset \Lambda_K$ for the ideal corresponding to the kernel of the natural map of $G_{K,S}$-modules $\widetilde{\Lambda}_K \twoheadrightarrow \mathcal{O}_E[\mathrm{Gal}(K'/K)]^\sim(\chi)$, where $\sim$ still denotes the canonical $G_{K,S}$-action. The natural maps $\Lambda_K/I_{\chi,K'}[1/p] \to \Lambda_{K,n}/I_{\chi,K'}[1/p]$ are isomorphisms for $n(\chi,K') \leq n \leq \infty$.

Let $V \in \mathbf{Rep}_E(G_{K,S})$ and put $V^* = \mathrm{Hom}_E(V,E)$ and $D_v = \mathbf{D}^\dagger_{\mathrm{rig}}(V|_{G_v})$ for each v dividing p. Assume that that $V|_{G_v}$ is ordinary in the sense of [22] with distinguished subobject $F_v \subseteq D_v$ for each such v. The distinguished subobjects induce subobjects upon passage to twists by χ, duals, restrictions to K', and cyclotomic deformations. Construct the Selmer complexes $\mathbf{R}\widetilde{\Gamma}_{\mathrm{str}}(G_K,-)$ using the unramified local condition at $v \in S$ not dividing p and the strict ordinary local condition at $v \in S$ dividing p, and similarly the Selmer complexes $\mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G_K,-)$ of the cyclotomic deformations.

Theorem 4.1. *(1) The complex $\mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G_K,V)$ has cohomology concentrated in degrees $[0,3]$ consisting of coadmissible Λ_∞-modules.*

(2) The natural map

$$\mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G_K,V) \overset{\mathbf{L}}{\underset{\Lambda_{K,\infty}}{\otimes}} \Lambda_{K,\infty}/I_{\chi,K'} \cong \mathbf{R}\widetilde{\Gamma}_{\mathrm{str}}(G_{K'},V(\chi^{-1}))$$

is an isomorphism. In particular, we have canonical short exact sequences

$$\begin{aligned} 0 \to \widetilde{\mathrm{H}}^i_{\mathrm{str,Iw}}(G_K,V) \underset{\Lambda_{K,\infty}}{\otimes} \Lambda_{K,\infty}/I_{\chi,K'} &\to \widetilde{\mathrm{H}}^i_{\mathrm{str}}(G_{K'},V(\chi^{-1})) \\ &\to \mathrm{Tor}_1^{\Lambda_{K,\infty}}(\widetilde{\mathrm{H}}^{i+1}_{\mathrm{str,Iw}}(G_K,V),\Lambda_{K,\infty}/I_{\chi,K'}) \to 0. \end{aligned}$$

(3) There is a canonical isomorphism

$$\mathscr{D}\mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G_K,V) \cong \mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G,V^*(1))^\iota[3].$$

In particular, we have canonical short exact sequences

$$0 \to \mathscr{D}^1\widetilde{\mathrm{H}}^{4-i}_{\mathrm{str,Iw}}(G_K,V) \to \widetilde{\mathrm{H}}^i_{\mathrm{str,Iw}}(G_K,V^*(1))^\iota \to \mathscr{D}^0\widetilde{\mathrm{H}}^{3-i}_{\mathrm{str,Iw}}(G_K,V) \to 0$$

identifying the first term with the torsion submodule of the second.

(4) Assume that χ is de Rham at p, $V^*(1)(\chi)^{G_{K',S'}} = 0$, and for all places v' of K' dividing p both $X = F_{v'}(\chi)^*(1), (D_{v'}/F_{v'})(\chi)$ satisfy $X^+_{\mathrm{dR}} = X_{\mathrm{dR}}$ and $X^{\varphi=1}_{\mathrm{crys}} = 0$. Then there is a canonical isomorphism

$$\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V) \otimes_{\Lambda_{K,\infty}} \Lambda_{K,\infty}/I_{\chi,K'} \cong \mathrm{H}^1_{\mathrm{f}}(G_{K'},V^*(1)(\chi))^*.$$

Proof. Claims (1) and (2) are clear.

For part (3) By the definition of the Selmer complex as a mapping cone, it suffices to justify that duality holds for each of $\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S},\overline{V})$ and the U_v^+ for $v \in S$. The first claim follows by applying $\overset{\mathbf{L}}{\otimes}_{\Lambda_K} \Lambda_{K,\infty}$ to [17, 8.5.6]. Similarly, if $v \in S$ does not divide p, the second claim follows by applying $\overset{\mathbf{L}}{\otimes}_{\Lambda_K} \Lambda_{K,\infty}$ to [17, 8.9.7.3], noting that the local error term for $V|_{G_v}$ vanishes because p is inverted. Finally, if $v \in S$ divides p, the second claim follows from Theorem 2.8.

Employing the discussion of [22, §3.4], to prove (4) it suffices avoid the $I_{\chi,K'}$-torsion submodule of $\widetilde{\mathrm{H}}^3_{\mathrm{str,Iw}}(G_K,V)$. By part (3), this module is $\mathscr{D}^1$-dual to the $I_{\chi^{-1},K'}$-torsion submodule of $\widetilde{\mathrm{H}}^1_{\mathrm{str,Iw}}(G_K,V^*(1))$. On the other hand, by part (2) this torsion vanishes provided $\widetilde{\mathrm{H}}^0_{\mathrm{str}}(G_{K'},V^*(1)(\chi)) = 0$, and the mapping cone defining the Selmer complex gives $\widetilde{\mathrm{H}}^0_{\mathrm{str}}(G_{K'},V^*(1)(\chi)) \subseteq V^*(1)(\chi)^{G_{K,S'}}$. □

Corollary 4.2. *Put* $\mathcal{S}^i = \widetilde{\mathrm{H}}^i_{\mathrm{str,Iw}}(G_K,V)$.

(1) One has $\mathcal{S}^i = 0$ *for* $i \neq 1,2,3$, $\mathrm{rank}_{\Lambda_\infty} \mathcal{S}^3 = 0$, *and* $\mathrm{rank}_{\Lambda_\infty} \mathcal{S}^1 = \mathrm{rank}_{\Lambda_\infty} \mathcal{S}^2$. *One has* $\mathcal{S}^1[I_{\chi,K'}] \hookrightarrow V(\chi^{-1})^{G_{K',S}}$ *and* $[V(\chi^{-1})^*(1)^{G_{K',S'}}]^* \twoheadrightarrow \mathcal{S}^3[I_{\chi,K'}]$.

(2) The common rank of $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *is invariant under replacing* V *with* $V^*(1)$, *and bounds above the rank of* $\mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V)$.

Proof. (1) The first claim for $i \notin [0,3]$ is clear by construction, and for $i = 0$ it follows from Lemma 2.3(1). The computation of the rank of $\mathcal{S}^3$ follows from the vanishing of $\mathcal{S}^0$ by part (3) of the theorem. To see the equality of ranks in degrees $i = 1,2$, one applies the Iwasawa-theoretic Euler–Poincaré formula to the diagram

$$0 \to \widetilde{\mathrm{H}}^1_{\mathrm{str,Iw}}(G_K,V) \to \mathrm{H}^1_{\mathrm{Iw}}(G_{K,S},V) \to \mathrm{H}^1_{\mathrm{Iw}}(G_K,D/F) \to \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V) \to \mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V) \to \bigoplus_{v\in S} \mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V),$$

noting that all local Iwasawa cohomology in degree 2 is torsion. The computation of torsion in $\mathcal{S}^1$ follows from part (2) of the theorem, and the computation of torsion in $\mathcal{S}^3$ follows from this and part (3) of the theorem.

(2) The first claim follows from part (3) of the theorem, and the second claim follows from the preceding exact sequence. □

We define the *algebraic p-adic* L*-function* for $(V,(F_v)_{v|p})$ to be the principal fractional Λ_∞-ideal $\left(\det_{\Lambda_\infty} \mathbf{R}\widetilde{\Gamma}_{\mathrm{str,Iw}}(G_K,V)\right)^{-1} \subseteq \mathcal{K}_\infty$, or a generator thereof, uniquely determined up to $\Lambda_\infty^\times$, which is equal to $\Lambda[1/p]^\times$ by a theorem of Lazard. When $V^{H_K} = V^*(1)^{H_K} = 0$, the preceding corollary shows that the algebraic p-adic L-function is simply $\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V)$. We assert a *weak Leopoldt conjecture (WLC) for* $(V,(F_v)_{v|p})$, that $\mathrm{rank}_{\Lambda_\infty} \widetilde{\mathrm{H}}^i_{\mathrm{str,Iw}}(G_K,V) = 0$ for $i = 1,2$, or equivalently that the algebraic p-adic L-function is, on each integral factor of $\mathcal{K}_\infty$, not identically zero. By the preceding corollary, the conjecture is invariant under replacing $(V,(F_v)_{v|p})$ by $V^*(1)$ equiped with its dual ordinary local conditions, and it implies the more traditional WLC for V (and $V^*(1)$), which is that $\mathrm{rank}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V) = 0$.

For the remainder of this section, we assume that $K/\mathbf{Q}$ *is unramified above* p. In particular, the groups $\Gamma_K, \Gamma_{K_v}, \Gamma_{\mathbf{Q}}, \Gamma_{\mathbf{Q}_p}$ coincide for all $v|p$. For any complex conjugation c, we let $d^\pm = \dim_{\mathbf{Q}_p}(\mathrm{Ind}_{\mathbf{Q}}^K V)^{c=\pm 1}$, so that $d^+ + d^- = [K:\mathbf{Q}] \dim_{\mathbf{Q}_p} V$. We also assume henceforth that $\dim_{K_v} F_v = d^+$ for all $v|p$. It will be convenient to use semilocal notations, when they are meaningful: $D_* = \bigoplus_{v|p}(D_v)_*$ for $* = \varnothing, \mathrm{W}, \mathrm{crys}, \mathrm{pcrys}$; $\mathrm{H}^*_{\mathrm{Iw}}(G_K, D) = \bigoplus_{v|p} \mathrm{H}^*_{\mathrm{Iw}}(G_{K_v}, D_v)$;

$$\mathrm{Log}_D = \bigoplus_{v|p} \mathrm{Log}_{D_v} : \mathrm{H}^1_{\mathrm{Iw}}(G_K, D) \to D_{\mathrm{pcrys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty;$$

$\Gamma_D = \prod_{v|p} \Gamma_{D_v}$; and similarly with $V^*(1)$ in place of V, F or F/D in place of D, etc.

When V is crystalline at all places v of K lying over p, in [20, Remark 1.4.9] Perrin-Riou's *module of p-adic* L*-functions* for V is given as

$$\begin{aligned}\mathbb{I}_{\mathrm{arith},\{p,\infty\}} = &\prod_{\substack{v\in S\\ v\nmid p}} \left(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_{K_v}, V)\right)^{-1} \cdot \Gamma_D^{-1}\\ &\cdot \left(\bigwedge\nolimits^{d^-}_{\Lambda_\infty} \mathrm{Log}_D\right) \left(\det{}^{-1}_{\Lambda_\infty} \mathbf{R}\Gamma_{\mathrm{Iw}}(G_{K,S}, V)\right)\\ &\subset \bigwedge\nolimits^{d^-}_{\Lambda_\infty} \left(D_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty\right) = \left(\bigwedge\nolimits^{d^-}_{\mathbf{Q}_p} D_{\mathrm{crys}}\right) \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty.\end{aligned}$$

Assume for the moment that WLC holds for V, so that $r_i := \mathrm{rank}_{\Lambda_\infty} \mathrm{H}^i_{\mathrm{Iw}}(G_{K,S}, V)$ satisfies $r_1 = d^-$ and $r_2 = 0$. Then, since $\mathrm{H}^0_{\mathrm{Iw}}(G_{K,S}, V) = 0$, we may rewrite

$$\begin{aligned}\mathbb{I}_{\mathrm{arith},\{p,\infty\}} = &\prod_{\substack{v\in S\\ v\nmid p}} \left(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_{K_v}, V)\right)^{-1} \cdot \Gamma_D^{-1}\\ &\cdot \prod_{i=1}^{2} \left(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^i_{\mathrm{Iw}}(G_{K,S}, V)_{\mathrm{tors}}\right)^{(-1)^i}\\ &\cdot \bigwedge\nolimits^{d^-}_{\Lambda_\infty} \mathrm{Log}_D \left(\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V)\right).\end{aligned}$$

Considering D_{crys} as functionals on $D^*(1)_{\mathrm{crys}}$ and evaluating these functionals on $(D/F)^*(1)_{\mathrm{crys}} = (F_{\mathrm{crys}})^\perp$, one gets a fractional Λ_∞-ideal

$$\mathbb{I}_{\mathrm{arith},\{p,\infty\}}(\det_{\mathbf{Q}_p}(D/F)^*(1)_{\mathrm{crys}}) \subseteq \mathcal{K}_\infty.$$

A generator of this fractional ideal is *Perrin-Riou's algebraic p-adic* L*-function for* $(V, (F_v)_{v|p})$, well-defined up to $\Lambda[1/p]^\times$. The factor $\bigwedge^{d^-}_{\Lambda_\infty} \mathrm{Log}_D(\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V))$, evaluated on $\det_{\mathbf{Q}_p}(D/F)^*(1)_{\mathrm{crys}}$, is of course recomputed as follows: map $\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V)$ along the top row and right column of the diagram

$$\begin{array}{ccccc}
\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V) & \longrightarrow & \mathrm{H}^1_{\mathrm{Iw}}(G_K, D) & \xrightarrow{\mathrm{Log}_D} & D_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty\\
& \searrow^{\mathrm{loc}_{D/F}} & \downarrow & & \downarrow\\
& & \mathrm{H}^1_{\mathrm{Iw}}(G_K, D/F) & \xrightarrow{\mathrm{Log}_{D/F}} & (D/F)_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty,
\end{array}$$

apply $\bigwedge^{d^-}_{\Lambda_\infty}$ to its image, and identify $\det_{\mathbf{Q}_p}(D/F)_{\mathrm{crys}}$ to $\mathbf{Q}_p$ at whim. (We remark that $\mathrm{loc}_{D/F}$ or $\mathrm{Log}_{D/F} \circ \mathrm{loc}_{D/F}$ is the analogue in this situation of a *Coleman map*.) But factoring the map $\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V) \to (D/F)_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \mathcal{K}_\infty$ as $\mathrm{Log}_{D/F} \circ \mathrm{loc}_{D/F}$, and noting that all three of

$$\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S}, V), \quad \mathrm{H}^1_{\mathrm{Iw}}(G_K, D/F), \quad \text{and} \quad (D/F)_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \Lambda_\infty$$

have rank d^-, gives

$$\begin{aligned}\bigwedge\nolimits_{\Lambda_\infty}^{d^-} \mathrm{Log}_D\left(\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S},V)\right)(\det_{\mathbf{Q}_p}(D/F)^*(1)_{\mathrm{crys}}) \\ = \left(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S},V)_{\mathrm{tors}}\right)\cdot(\det\nolimits_{\Lambda_\infty}\mathrm{loc}_{D/F}) \\ \cdot\,\Gamma_{D/F}\cdot\left(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p},D/F)\right)^{-1}.\end{aligned}$$

Rewriting Perrin-Riou's definition using this equation, it now makes sense under the weaker assumption that for each D_v/F_v becomes crystalline over some $K_{v,n}$, using $(D_v/F_v)_{\mathrm{pcrys}}$ in place of $(D_v/F_v)_{\mathrm{crys}}$.

Assume henceforth only that each D_v/F_v becomes crystalline over some $K_{v,n}$, and define *Perrin-Riou's algebraic p-adic* L*-function* for V and $(F_v)_{v|p}$, well-defined up to $\Lambda[1/p]^\times$, to be a generator of the fractional Λ_∞-ideal

$$\begin{aligned}\mathbb{I}_{\mathrm{arith},\{p,\infty\}}(\det\nolimits_{\mathbf{Q}_p}(D/F)^*(1)_{\mathrm{pcrys}}) := \prod_{\substack{v\in S\\ v\nmid p}}\left(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V)\right)^{-1} \\ \cdot\,\Gamma_F^{-1}\cdot\left(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p},D/F)\right)^{-1} \\ \cdot\left(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V)\right)\cdot(\det\nolimits_{\Lambda_\infty}\mathrm{loc}_{D/F})\end{aligned}$$

(which *a priori* might be zero). We compare it to $\mathrm{char}_{\Lambda_\infty}\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V)$.

Consider the local conditions $U^+_{v,\mathrm{Iw}}$ for $v\in S$ arising in the definition of the Selmer complex, or rather the complexes $U^-_{v,\mathrm{Iw}} = \mathrm{Cone}\left[U^+_{v,\mathrm{Iw}}\to\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K_v},V)\right]$. If $v\nmid p$ one easily computes from the definition of the unramified local condition that

$$\mathrm{H}^0U^-_{v,\mathrm{Iw}}=0,\quad \mathrm{H}^1U^-_{v,\mathrm{Iw}}=(V_{I_v}\otimes_{\mathbf{Q}_p}\Lambda_\infty)^{G_{\mathbf{F}_v}}=0,\quad\text{and}\quad \mathrm{H}^2U^-_{v,\mathrm{Iw}}=\mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V),$$

where $1\to I_v\to G_{K_v}\to G_{\mathbf{F}_v}\to 1$ exhibits the inertia and residual Galois groups of K_v. If $v|p$ one has $\mathrm{H}^*U^-_{v,\mathrm{Iw}}=\mathrm{H}^*_{\mathrm{Iw}}(G_{K_v},D_v/F_v)$. *Assume that the natural surjection* $\mathrm{H}^2_{\mathrm{Iw}}(G_K,D)\to\mathrm{H}^2_{\mathrm{Iw}}(G_K,D/F)$ *is an isomorphism*; for example, if $V|_{G_{K_v}}$ is crystalline for all $v|p$ then

$$\mathrm{H}^2_{\mathrm{Iw}}(G_K,D)^* = \bigoplus_{v|p}\mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V)^* = \bigoplus_{v|p}V^*(1)^{H_{K_v}} = \bigoplus_{j\in\mathbf{Z}}(D^*(1)^{\varphi=p^{-j}}_{\mathrm{crys}}\otimes_{\mathbf{Q}_p}\chi^j_{\mathrm{cycl}}),$$

and the right hand side vanishes for generic V. By the preceding computations, the long exact sequence arising from the definition of the Selmer complex as a mapping cone gives:

$$\begin{aligned}0\to\widetilde{\mathrm{H}}^1_{\mathrm{str,Iw}}(G_K,V)\to\mathrm{H}^1_{\mathrm{Iw}}(G_{K,S},V)\xrightarrow{\mathrm{loc}_{D/F}}\mathrm{H}^1_{\mathrm{Iw}}(G_K,D/F)\to\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V) \\ \to\mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V)\to\bigoplus_{v\in S}\mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V)\to\widetilde{\mathrm{H}}^3_{\mathrm{str,Iw}}(G_K,V)\to 0.\end{aligned}\tag{4.1}$$

Note that WLC holds for $(V,(F_v)_{v|p})$ if and only if the map $\mathrm{loc}_{D/F}$ is injective modulo torsion. We compute

$$\mathrm{char}_{\Lambda_\infty}\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V)=(\mathrm{char}_{\Lambda_\infty}\mathrm{H}^2_{\mathrm{p,Iw}}(G_{K,S},V))\cdot(\det\nolimits_{\Lambda_\infty}\mathrm{loc}_{D/F}),\tag{4.2}$$

where

$$\mathrm{H}^*_{\mathrm{p,(cl.)Iw}}(G_{K,S},V) = \ker\left[\mathrm{H}^*_{\mathrm{(cl.)Iw}}(G_{K,S},V) \to \bigoplus_{v\in S} \mathrm{H}^*_{\mathrm{(cl.)Iw}}(G_{K_v},V)\right],$$

$$\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{p,Iw}}(G_{K,S},V) = \big(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_{K,S},V)\big)\cdot \prod_{v\in S} \big(\mathrm{char}_{\Lambda_\infty} \mathrm{H}^2_{\mathrm{Iw}}(G_{K_v},V)\big)^{-1} \cdot \mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^3_{\mathrm{str,Iw}}(G_K,V). \tag{4.3}$$

In conclusion, we have the following theorem.

Theorem 4.3. *Let K,S,V,D_v,F_v be as at the beginning of this section, with $K/\mathbf{Q}$ unramified at p. Assume that each D_v/F_v becomes crystalline over some $K_{v,n}$, and that the natural surjection $\mathrm{H}^2_{\mathrm{Iw}}(G_K,D) \to \mathrm{H}^2_{\mathrm{Iw}}(G_K,D/F)$ is an isomorphism. Then Perrin-Riou's algebraic p-adic* L*-function is given by*

$$\mathbb{I}_{\mathrm{arith},\{p,\infty\}}(\det_{\mathbf{Q}_p}(D/F)^*(1)_{\mathrm{pcrys}}) = \Big(\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_K,V)\Big)\cdot \Gamma_F^{-1} \cdot \Big(\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^3_{\mathrm{str,Iw}}(G_K,V)\Big)^{-1},$$

where the last factor disappears if $V^*(1)^{H_K}=0$. WLC holds for $(V,(F_v)_{v|p})$ if and only if the common quantity is nonzero on each component of $\operatorname{Spec}\Lambda_\infty$*.*

Remark 4.4. Assuming the hypotheses of the theorem also when the roles of $V, V^*(1)$ are reversed, Theorems 4.1 and 4.3 give a new proof of the functional equation of Perrin-Riou's algebraic p-adic L-function, and partially recover her results on its behavior at $s=0$, namely on the order of vanishing.

We now compare the characteristic ideals constructed in [15, 14] to ours. In keeping with *op. cit.*, we assume for simplicity that $E=\mathbf{Q}_p$, $K=\mathbf{Q}$, and the restriction $V|_{G_{\mathbf{Q}_p}}$ is crystalline, negative (in our normalizations, all its Hodge–Tate weights are nonpositive), and without quotient isomorphic to the trivial representation. In particular, the Wach module D_{W} satisfies $D_{\mathrm{W}} \subseteq \varphi^* D_{\mathrm{W}}$, $\mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p},V) = D_{\mathrm{W}}^{\psi=1}$, and $D_{\mathrm{W}}^{\psi=0} \subseteq D_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} \mathrm{B}^{+,\psi=0}_{\mathrm{rig},\mathbf{Q}_p}$. The same properties are satisfied by the usual Wach module $\mathrm{N}(T)$ over Λ, with $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p},T)$ in place of $\mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p},T)$, where $T\subseteq V$ is a $G_{\mathbf{Q},S}$-stable $\mathbf{Z}_p$-lattice. One chooses a basis $n_1,\dots,n_d$ of $\mathrm{N}(T)$ with the property that $(1+\pi)\varphi(n_1),\dots,(1+\pi)\varphi(n_d)$ is a Λ-basis of $(\varphi^*\mathrm{N}(T))^{\psi=0}$, and defines the *Coleman maps*

$$\mathrm{Col}\colon \mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p},T) = \mathrm{N}(T)^{\psi=1} \xrightarrow{1-\varphi} (\varphi^*\mathrm{N}(T))^{\psi=0} \approx \Lambda^{\oplus d} \quad \text{and} \quad \mathrm{Col}_I = \mathrm{proj}_I \circ \mathrm{Col}$$

with respect to this basis, for $I\subseteq\{1,\dots,d\}$. We denote by Col'_I its restriction to $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S},T)$. We assume $D_{\mathrm{crys}}^{\varphi=p^n}=0$ for all $n\in\mathbf{Z}$, so that the map $1-\varphi$ above is injective and the Λ- and Λ_∞-modules $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S},T)$, $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p},T)$ and $\mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p},D/F)$ are free. Selmer groups were defined in *op. cit.* for modular forms, and the definition generalizes as follows. In a standard way one forms a local condition $U_p^{+,\bullet}$ for $T\otimes_{\mathbf{Z}_p}\widetilde{\Lambda}^\iota$ with $\mathrm{H}^0(U_p^+) = \mathrm{H}^0_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p},T)$, $\mathrm{H}^1(U_p^+) = \ker\mathrm{Col}_I$, and all other $\mathrm{H}^i(U_p^+)=0$; one has $\mathrm{H}^1(U_p^-) = \operatorname{img}\mathrm{Col}_I$, $\mathrm{H}^2(U_p^-) = \mathrm{H}^2_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p},T)$, and all other $\mathrm{H}^i(U_p^-)=0$. Using this condition at p and the unramified local condition at each $v\nmid p$

gives rise to a Selmer complex whose cohomology will be denoted $\widetilde{\mathrm{H}}^*_{I,\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V)$. There is a mapping cone exact sequence

$$0 \to \widetilde{\mathrm{H}}^1_{I,\mathrm{cl.Iw}}(G_{\mathbf{Q}}, T) \to \mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, T) \xrightarrow{\mathrm{Col}'_I} \mathrm{img}\,\mathrm{Col}_I \to \widetilde{\mathrm{H}}^2_{I,\mathrm{cl.Iw}}(G_{\mathbf{Q}}, T)$$
$$\to \mathrm{H}^2_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, T) \to \bigoplus_{v \in S} \mathrm{H}^2_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_v}, T) \to \widetilde{\mathrm{H}}^3_{I,\mathrm{cl.Iw}}(G_{\mathbf{Q}}, T) \to 0,$$

just like Sequence (4.1); cf. [14, Sequence (60)]. We can expect a reasonable numerology of Λ-ranks only when $\#I = d^-$, so assume the subset I satisfies this requirement. One deduces an identity

$$\mathrm{char}_\Lambda \widetilde{\mathrm{H}}^2_{I,\mathrm{cl.Iw}}(G_{\mathbf{Q}}, T) = (\mathrm{char}_\Lambda \mathrm{H}^2_{\mathrm{p,cl.Iw}}(G_{\mathbf{Q},S}, T)) \cdot (\det_\Lambda \mathrm{Col}'_I),$$

just like Equation (4.2), and we are reduced to comparing Col'_I with $\mathrm{loc}_{D/F}$. Such a comparison is highly dependent on the original choice of basis $n_1, \ldots, n_d$ of $\mathrm{N}(T)$, but there is still a tautological relationship coming from the linear algebra of the situation; we make it precise when $d^- = 1$. Fixing a basis m of $\mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F)$, we may write the natural map $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q}_p}, T) \to \mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F)$ uniquely in the form

$$x \mapsto (\log_1 \;\cdots\; \log_d) \begin{pmatrix} \mathrm{Col}_{\{1\}}(x) \\ \vdots \\ \mathrm{Col}_{\{d\}}(x) \end{pmatrix} \cdot m$$

with $\log_i \in \mathcal{K}_\infty$. (By [14, Theorem B], the $\log_i$ have finitely many poles, occurring at precisely known points.) It follows formally that

$$\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{K,S}, V) = \sum_{i=1}^{d} \log_i \cdot \mathrm{char}_\Lambda \widetilde{\mathrm{H}}^2_{\{i\},\mathrm{cl.Iw}}(G_{K,S}, V), \tag{4.4}$$

abusively confusing principal ideals with choices of generators. This exhibits our p-adic L-function as a linear combination of various bounded functions on W of global origin, with unbounded coefficients of purely local origin.

5 Modular forms

Given a normalized elliptic modular cuspidal new eigenform f and a prime p, Kato proved one divisibility in his so-called "Iwasawa main conjecture without p-adic L-functions", which is purely a statement relating naked Iwasawa cohomology to his Euler system. In the *classically ordinary* case, he deduced from this one divisibility in his so-called "Iwasawa main conjecture with p-adic L-functions", which relates Greenberg's ordinary Iwasawa Selmer group to one of the p-adic L-functions. Here we extend this latter deduction to all *finite slope* forms, and *both* p-adic L-functions for each form that admits two (in most cases).

We continue the notations of the preceeding section, taking $p > 2$, $K = \mathbf{Q}$, and S to be the set of places dividing Mp, where $M \geq 1$. Note that $\Gamma_{\mathbf{Q}} = \Gamma_{\mathbf{Q}_p}$; we write Γ, etc. without regard for the base field.

Let $f \in \mathrm{S}_k(\Gamma_1(M), \psi; E)$ be a normalized elliptic modular cuspidal new eigenform with $k \geq 2$, having q-expansion $\sum_{n \geq 1} a_n q^n$. Inside the cohomology of the appropriate

Kuga–Sato variety, there is associated to f an irreducible 2-dimensional E-valued representation V of $G_{\mathbf{Q},S}$ which is de Rham at p, characterized up to scalar multiple by the relations

$$\operatorname{trace}(\mathrm{Frob}_\ell|V) = a_\ell \quad \text{and} \quad \det(\mathrm{Frob}_\ell|V) = \ell^{k-1}\psi(\ell)$$

for all $\ell \nmid Mp$, where Frob_ℓ denotes the geometric Frobenius. We put $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V|_{G_{\mathbf{Q}_p}})$.

Fix a finite Galois extension $L/\mathbf{Q}_p$ over which V becomes semistable, write F_L for the maximal aboslutely unramified subfield of L, and let $m = [F_L : \mathbf{Q}_p]$. Enlarge E, if necessary, so that L can be embedded into E as a $\mathbf{Q}_p$-algebra. Then we may view D_{pst} as a free $(F_L \otimes_{\mathbf{Q}_p} E)$-module of rank 2, and φ as a $(\varphi \otimes 1)$-linear operator. On a suitable basis, the matrix entries of φ can be taken to lie in $1 \otimes E$. We assume that *f has finite slope at p*, by which we mean that the matrix of φ is nonscalar: even after enlarging E, φ cannot be taken to have the form $I_2 \otimes \alpha$ with $\alpha \in E^\times$ and where $I_2 \in \mathrm{GL}_2(F_L)$ is the identity matrix. Under our assumption, after perhaps enlarging E, the filtered $(\varphi, N, G_{\mathbf{Q}_p})$-module D_{pst} has the following structure. (For details on the classification of rank two filtered (φ, N, G)-modules with coefficients, we refer the reader to [10, §6].) There exists an $(F_L \otimes_{\mathbf{Q}_p} E)$-basis $\{e_1, e_2\}$ and finite order characters $\psi_i \colon G_{\mathbf{Q}_p} \to E^\times$ such that $G_{\mathbf{Q}_p}$ acts on e_i through $1 \otimes \psi_i$, as well as nonzero $\alpha_1, \alpha_2 \in \mathcal{O}_E$ with $\mathrm{ord}_p\,\alpha_1 \le \mathrm{ord}_p\,\alpha_2$ such that exactly one of the following situations applies.

- φ is semisimple. Then $\alpha_1^m \ne \alpha_2^m$, and $\varphi(e_i) = (1 \otimes \alpha_i)e_i$ for $i = 1, 2$. The vectors e_1, e_2 realize the unique E-linear $(\varphi, G_{\mathbf{Q}_p})$-stable decomposition. If $N \ne 0$, then $N(e_2) = e_1$ and $N(e_1) = 0$, so only the factor containing e_1 is $(\varphi, N, G_{\mathbf{Q}_p})$-stable, and $\mathrm{ord}_p\,\alpha_2 = \mathrm{ord}_p\,\alpha_1 + 1$.
- φ is not semisimple. Then $\psi_1 = \psi_2$, $\alpha_1 = \alpha_2$, and $N = 0$. One has $\varphi(e_1) = (1 \otimes \alpha_1)e_1$, e_1 spans the unique nontrivial $(\varphi, G_{\mathbf{Q}_p})$-stable E-subspace, and $\varphi(e_2) = (1 \otimes \alpha_2)e_2 \pmod{e_1}$.

(According to common conjecture, the second case never occurs.) In particular, we may take L to be some $\mathbf{Q}_{p,n}$, so $F_L = \mathbf{Q}_p$. In either case, only the α_i^m, and not the α_i, are uniquely determined. One has $\mathrm{ord}_p\,\alpha_1 + \mathrm{ord}_p\,\alpha_2 = k - 1$, with $\mathrm{ord}_p\,\alpha_1 = 0$ if and only if f is classically ordinary at p. Also, D_{dR} is 2-dimensional over E, and the Hodge filtration satisfies $D_{\mathrm{dR}} = H^0 \supsetneq H^1 = H^{k-1} \supsetneq H^k = 0$; put $H = H^1$. Weak admissibility means that $e_1 \notin L \otimes_{\mathbf{Q}_p} H$, and that $e_2 \notin L \otimes_{\mathbf{Q}_p} H$ unless possibly if $N \ne 0$, or φ is not semisimple, or V is deomposible (in which case $\mathrm{ord}_p\,\alpha_1 = 0$, and we call f "split ordinary" at p).

Thus the "finite slope" hypothesis is equivalent to V becoming semistable over an abelian extension, or some twist $f \otimes \varepsilon$ by a Dirichlet character ε having an associated U_p-eigenform with nonzero U_p-eigenvalue (in the above, take $\varepsilon = \psi_1^{-1}$), justifying the name. The cases excluded by this assumption consist of a small number of cases where f is principal series at p (those where f becomes semistable only over a nonabelian extension of $\mathbf{Q}_p$), and all cases where f is supercuspical at p.

The nontrivial nearly ordinary filtrations are rank-one $(\varphi, N, G_{\mathbf{Q}_p})$-stable $(1 \otimes E)$-subspaces of D_{pst}; these are determined by e_i, with necessarily $i = 1$ if either $N \ne 0$ or φ is not semisimple. The ordinary hypothesis requires that e_i not lie in the Hodge filtration $L \otimes_{\mathbf{Q}_p} H \subset L \otimes_{\mathbf{Q}_p} D_{\mathrm{dR}}$; thus e_i spans an ordinary filtration except when $i = 2$ in the $N \ne 0$, nonsemisimple φ, and split ordinary cases. We choose such an i, and take $F \subseteq D$ such that F_{pst} is the span of e_i. Since the theory behaves well under twisting, for simplicity we replace f by $f \otimes \psi_1^{-1}$ (identifying ψ_1 to a Dirichlet character of p-power conductor), and assume that F is crystalline and $\psi_2 = \psi$. Let $\alpha f = U_p f$ (so we have $\alpha_i = \alpha$), and put $i' = 3 - i$.

We determine when Theorem 4.1(4) applies to $K' = \mathbf{Q}$ and $\chi = \varepsilon \chi_{\mathrm{cycl}}^j$, for ε of finite order factoring through Γ and $j \in \mathbf{Z}$. The Hodge–Tate weight of $F(\chi)^*(1)$ (resp. $(D/F)(\chi)$) is equal to $j-1$ (resp. $k-1-j$), so the condition on Hodge–Tate weights requires that $0 < j < k$. For the condition on Frobenius, we first see that $F(\chi)^*(1)^{\varphi=1}_{\mathrm{crys}}$ is nonzero if and only if $\varepsilon = 1$ and $\alpha^m = p^{(j-1)m}$. Considering that the determinant of V gives $F \otimes (D/F) \cong \mathbf{D}^\dagger_{\mathrm{rig}}(\chi_{\mathrm{cycl}}^{1-k}\psi)$, we have $(D/F)(\chi) \cong F^*(\chi_{\mathrm{cycl}}^{j+1-k}\psi\varepsilon)$. Therefore, $(D/F)(\chi)^{\varphi=1}_{\mathrm{crys}}$ is nonzero if and only if $\varepsilon = \psi^{-1}$ and $\alpha^m = p^{(k-j-1)m}$. Finally, $V^*(1)(\chi)^{G_{\mathbf{Q},S}} = 0$ always because V is irreducible. Thus the theorem applies exactly to the χ corresponding to critical values of f where the p-adic L-function $\mathrm{L}_p(f,\alpha)$ does not have a so-called exceptional zero.

We now treat the characteristic ideal, and assume henceforth that *φ on $\mathbf{D}_{\mathrm{crys}}$ is semsimple.* Note that $V|_{H_{\mathbf{Q}}}$ remains irreducible, and in particular $V^{H_{\mathbf{Q}}} = 0$ and $V^*(1)^{H_{\mathbf{Q}}} = 0$. We also claim the surjection $\mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, V) \to \mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, F)$ is injective. In the case where f is nonordinary (resp. potentially crystalline) at p, then the irreducibility of $V|_{H_{\mathbf{Q}_p}}$ (resp. purity of étale cohomology) forces $\mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, V)^* \cong V^*(1)^{H_{\mathbf{Q}_p}} = 0$. In the case where f is ordinary with nonzero monodromy at p, hence of weight two, then the fact that the span of e_1 is the *unique* $(\varphi, N, G_{\mathbf{Q}_p})$-stable subspace of D_{pst}, combined with Tate local duality and base change, allows one to show the desired map to be an isomorphism. In particular, if $\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{\mathbf{Q}}, V)$ is torsion, Theorem 4.3 applies.

We next recall Kato's divisibility "without p-adic L-functions" for f. Let $\kappa \in \mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V)$ denote the restriction to the p-cyclotomic tower of Kato's Euler system.

Theorem 5.1 ([11, Theorems 12.4(1,2), 12.5(2,3)]). *The module $\mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V)$ is free over $\Lambda[1/p]$ of rank one, the $\Lambda[1/p]$-modules*

$$\mathrm{H}^2_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V) \quad \textit{and} \quad \mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V)/\Lambda[1/p]\kappa$$

are torsion, and

$$\mathrm{char}_{\Lambda[1/p]} \mathrm{H}^2_{\mathrm{p,cl.Iw}}(G_{\mathbf{Q},S}, V) \quad \textit{divides} \quad \mathrm{char}_{\Lambda[1/p]} \mathrm{H}^1_{\mathrm{cl.Iw}}(G_{\mathbf{Q},S}, V)/\Lambda[1/p]\kappa.$$

(Kato's formulation of the theorem actually uses with a modified version of Galois cohomology, with local conditions at all $v \neq p$; cf. [11, §8.2] and the proof of [11, Lemma 8.5], and employ Equation (4.3) to compare it with $\mathrm{H}^2_{\mathrm{p,cl.Iw}}(G_{\mathbf{Q},S}, V)$.)

Applying $\otimes_{\Lambda[1/p]}\Lambda_\infty$ then $\mathrm{loc}_{D/F}$ to the above theorem, Equation (4.2) and the torsion-freeness of $\mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q},S}, V)$ give the following claim.

Corollary 5.2. *The characteristic ideal $\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{\mathbf{Q}}, V)$ divides*

$$\mathrm{char}_{\Lambda_\infty} \mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F)/\Lambda_\infty \, \mathrm{loc}_{D/F}\, \kappa,$$

with one nonzero if and only if the other is.

In order to compute the index of $\mathrm{loc}_{D/F}\,\kappa$ above, we apply $\mathrm{Log}_{D/F}$ and make use of Kato's explicity reciprocity law.

To apply $\mathrm{Log}_{D/F}$, put $\delta_i = \mathrm{char}_{\Lambda_\infty} \mathrm{H}^i_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F)_{\mathrm{tors}}$ and $\delta = \prod_{i=1}^2 \delta_i^{(-1)^i}$. An argument similar to the surjectivity of $\mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D) \to \mathrm{H}^2_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F)$ shows that the δ_i are nonidentity precisely in the case of semistable reduction, weight 2, and $\psi|_{G_{\mathbf{Q}_p}}$ factors through Γ (the condition "(C)" of [11, Theorem 12.5(3)]), in which case $D/F = \mathbf{D}^\dagger_{\mathrm{rig}}(\mathbf{Q}_p(-1)(\psi))$ and $\delta_i = \mathrm{char}_{\Lambda_\infty} \mathbf{Q}_p(-i)(\psi)$. Since D/F is potentially crystalline of rank one with Hodge–Tate weight $k-1$, we have that $(D/F)_{\mathrm{pcrys}}$ is free over E with basis given by the image $\overline{e}_{i'}$ of $e_{i'}$. Theorem 3.4 then shows that $\overline{e}^*_{i'} \circ \mathrm{Log}_{D/V}$ has torsion kernel with characteristic ideal δ_1, and has image equal to the fractional ideal $\Gamma_{k-1} \cdot \delta_2^{-1} \subset \mathcal{K}_\infty$. Therefore, $\mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}_{\mathrm{str,Iw}}(G_{\mathbf{Q}}, V)$ divides $\Gamma^{-1}_{k-1}\delta \cdot \overline{e}^*_{i'} \, \mathrm{Log}_{D/F}\, \kappa$.

To state Kato's explicit reciprocity law, write $e_i^*, e_{i'}^*$ for the basis of D^*_{pst} dual to $e_i, e_{i'}$; note that $e^*_{i'}$ is an E-basis for $(D^*)_{\mathrm{pcrys}}$.

Theorem 5.3 ([11, Theorem 16.6(2)])**.** *One can make sense of* $\mathrm{Log}_V : \mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, V) \to ((D^*)_{\mathrm{pcrys}})^* \otimes_{\mathbf{Q}_p} \Lambda_\infty$ *(V only being de Rham) and* $e^*_{i'} \mathrm{Log}_V(\kappa) = \mathrm{L}_p(f^c, \alpha^c)$*, where the superscript c denotes complex conjugation and* L_p *denotes the associated p-adic* L*-function.*

We expect that the diagram

$$\begin{array}{ccc} \mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, V) & \xrightarrow{e^*_{i'} \circ \mathrm{Log}_V} & \Lambda_\infty \\ \downarrow & & \downarrow \Gamma_{k-1}\delta^{-1} \\ \mathrm{H}^1_{\mathrm{Iw}}(G_{\mathbf{Q}_p}, D/F) & \xrightarrow{\overline{e}^*_{i'} \circ \mathrm{Log}_{D/F}} & \Lambda_\infty \end{array} \tag{5.1}$$

commutes up to $\Lambda_\infty^\times$. In the case where f has potentially good reduction, the desired commutativity is a simple verification using the constructions in §3, but otherwise the existence of Log_V relies on unpublished work of Kato–Kurihara–Tsuji and we are presently unable to check the claim. Assuming the commutativity, then the explicit reciprocity law amounts to the identity

$$\overline{e}^*_{i'} \, \mathrm{Log}_{D/F}\, \kappa = \Gamma_{k-1}\delta^{-1} \mathrm{L}_p(f^c, \alpha^c),$$

and we arrive at our sought divisibility "with p-adic L-functions" for f, as in the following theorem.

Theorem 5.4. *Let f have semisimple crystalline Frobenius and nonzero U_p-eigenvalue α. Assume the diagram (5.1) commutes up to $\Lambda_\infty^\times$, for example f has potentially good reduction. Then*

$$\mathrm{L}_p(f^c, \alpha^c) \in \mathrm{char}_{\Lambda_\infty} \widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{\mathbf{Q}}, V).$$

In particular, $\widetilde{\mathrm{H}}^2_{\mathrm{str,Iw}}(G_{\mathbf{Q}}, V)$ *is torsion, and Theorem 4.3 applies.*

Remark 5.5. One can proceed with the constructions of this section in the case where f has split ordinary reduction, taking the nonordinary nearly ordinary filtration. Of course Theorem 4.1(4) does not apply as stated, because we cannot hope to obtain the Bloch–Kato Selmer groups in this way, but one still obtains a control theorem for strict ordinary Selmer groups (there are no changes to the argument), which may have some strange p-adic meaning here. Similarly, one will obtain a torsion characteristic ideal once one knows that the image of Kato's Euler system is nonzero. Recently Bellaiche [1] has constructed a nonzero p-adic L-function for this case by analytic means; we hope it can be shown to equal the image of Kato's Euler system.

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