José Pedro Quintanilha
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→ aka Σ-invariants, BNSR-invariants (for Bieri-Neumann-Strebel-Renz)

Geometric invariants for locally compact groups

joint w/ Kai-Uwe Bux and Elisa Hartmann

I will focus on this one today →
- the homotopical perspective: arXiv: 2410.19501 [math.GR]
- the homological perspective: arXiv: 2411.08775 [math.AT]

Finiteness properties of groups

Let $n \geq 1$, G a (discrete) group

Def. G is of type F_n if there is a $K(G,1)$ with finite n-skeleton.

Equivalently:

G is of type F_n $\Leftrightarrow$ There is a CW-complex Y with a free G-action such that

1. $Y^{(0)} = G$
2. $Y^{(n)}$ has only finitely many G-cells (G-orbits of cells)
3. Y is $(n-1)$-connected ($\pi_i(Y)$ trivial for all $i \leq n-1$)

Obs.
- $F_1 \Leftarrow F_2 \Leftarrow \ldots$
- May assume $\dim(Y) \leq n$.

For $n=1$: Assuming Y is a 1-complex, get:
1. $Y = \mathrm{Cay}(G,S)$ for some $S \subseteq G$ (maybe not gen set)
2. S is finite
3. $\mathrm{Cay}(G,S)$ is connected, i.e. S is a gen. set.

So: G is of type F_1 $\Leftrightarrow$ G is finitely generated

Similarly: G is of type F_2 $\Leftrightarrow$ G is finitely presented
(Y is a G-finite Cayley complex)

The geometric invariants

We will define the geometric invariants $\Sigma^n(G)$ as certain subsets of $\mathrm{Hom}(G,\mathbb{R})$.

Notation. Given $\chi: G \longrightarrow \mathbb{R}$:

- $G_\chi := \{ g \in G \mid \chi(g) \geq 0 \}$
- If Y is a CW-complex with $Y^{(0)} = G$,
  $Y_\chi :=$ the maximal subcomplex of Y with $Y_\chi^{(0)} = G_\chi$

Def. of $\Sigma^n(G) \subseteq \mathrm{Hom}(G,\mathbb{R})$: [BNSR, 80s]

$\chi \in \Sigma^n(G) \Leftrightarrow$ There is a CW-complex Y with a free G-action such that

1. $Y^{(0)} = G$
2. $Y^{(n)}$ has only finitely many G-cells
3. Y_χ is $(n-1)$-connected

Obs:
- $\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \dots$
- G is of type $F_n \Leftrightarrow 0 \in \Sigma^n(G)$
- Little exercise: $\Sigma^n(G) \neq \emptyset \Rightarrow 0 \in \Sigma^n(G)$
- For $\lambda \in]0,\infty[$: $\chi \in \Sigma^n(G) \Rightarrow \lambda\chi \in \Sigma^n(G)$

If $\Sigma^n(G) \neq \emptyset$, then it is a cone in $\mathrm{Hom}(G,\mathbb{R})$

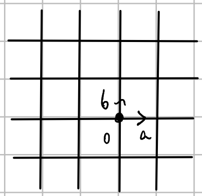
Thm [BNSR]. In fact, it is the cone over an open subset of $\mathrm{Hom}(G,\mathbb{R}) \setminus \{0\}$.

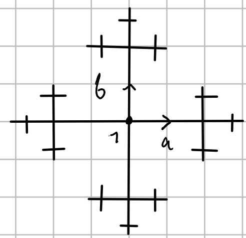
Ex. $n = 1$

$\chi \in \Sigma^1(G) \iff$ There is a finite subset $S \subseteq G$ s.t. $\mathrm{Cay}(G, S)_\chi$ is connected.

fact / exercise (specific to $n=1$) $\iff$ For every generating set $S \subseteq G$, $\mathrm{Cay}(G,S)_\chi$ is connected

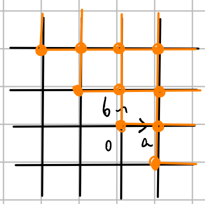
$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ $\qquad$ $F_2 = \langle a, b \rangle$

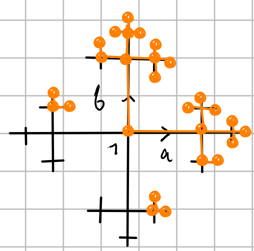




Consider the character χ: $a \mapsto 1$, $b \mapsto 1$

$\mathrm{Cay}(G, \{a,b\})_\chi$:





connected $\qquad$ not connected

$\chi \in \Sigma^1(\mathbb{Z}^2)$ $\qquad$ $\chi \notin \Sigma^1(F_2)$

In fact, we have: $\Sigma^1(\mathbb{Z}^2) = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{R})$
$\Sigma^1(F_2) = \{0\}$

Motivation: It is usually difficult to transfer finiteness properties to subgroups. But:

Thm. [BNRS] Suppose $1 \to N \to G \to Q \to 1$ is a s.e.s. with Q abelian. If $\Sigma^n(G)$ contains all characters that vanish on N (in particular, G is of type F_n), then N is of type F_n.

Compactness properties – bringing in the topology

For locally compact Hausdorff topological groups, [Abels, Tiemeyer ; 1997] define compactness properties

$$C_1 \Leftarrow C_2 \Leftarrow \dots$$

"compactly generated" "compactly presented" presentation by compact generating set and relations of bounded length

- For discrete groups: type $F_n \Leftrightarrow$ type C_n
  (via Brown's criterion)

Thm [AT]: $H \leq G$ closed cocompact subgroup.
(e.g. H uniform lattice)
Then H and G have the same compacness properties.

[Primary use case: Totally Disconnected Locally Compact groups. (TDLC)
(e.g. discrete groups, profinite groups, $SL_n(\mathbb{Q}_p)$, Aut(locally finite graph))
If G is locally cpt Hausdorff, then it fits into a s.e.s

$$1 \to \underbrace{G_0}_{\text{identity component}} \to G \to \underbrace{G/G_0}_{\text{TDLC}} \to 1$$

has all compacness properties [Chaufi, Witzel] have the same compactness properties]

Our contribution:

$$\begin{array}{ccc} F_n & \overset{\text{BNSR}}{\rightsquigarrow} & \Sigma^n \\ \text{AT} \updownarrow & & \downarrow \\ C_n & \dashrightarrow & \boxed{\Sigma^n_{top}} \subseteq \mathrm{Hom}_{top}(G,\mathbb{R}) \end{array}$$

defined geometric invariants for locally compact groups

The definition of $\Sigma^n_{top}(G)$

G : locally compact Hausdorff group

EG : free simplicial set on G, i.e :

↳ k-simplices are tuples $(g_0, \ldots, g_k) \in G^{k+1}$

↳ i-th face / degeneracy map given by suppressing / doubling i-th entry

(gives rise to the bar resolution)

"An infinite-dimensional simplex with one vertex per element of G."

$\mathcal{C}$: The poset of compact subsets $C \subseteq G$

$\chi : G \to \mathbb{R}$ continuous homomorphism

We consider the following filtration of EG :

$$\left(G_\chi \cdot EC \right)_{C \in \mathcal{C}}$$

Def: $\Sigma^n_{top}(G) := \{ \chi : G \to \mathbb{R} \mid (G_\chi \cdot EC) \text{ is } \underbrace{\text{essentially } (n-1)\text{-connected}} \}$

For each $C \in \mathcal{C}$ there is $D \in \mathcal{C}_{\supseteq C}$ st
$G_\chi \cdot EC \hookrightarrow G_\chi \cdot ED$ is π_k trivial on $k \leq n-1$

- Setting $\chi = 0$ recovers Abels-Tiemeyer's definition of property C_n.
- $\Sigma^n_{top}(G) = \Sigma^n(G)$ for G discrete
  (Uses a modified version of Brown's criterion)
- We recover the definition of Σ^1_{top} and Σ^2_{top} given by Kochloukova (2009) in terms of generating sets and presentations.

Some classical results we extended to this context:

- $\Sigma^n_{top}(G) \subseteq Hom_{top}(G, \mathbb{R})$ is the cone over an open subset of $Hom_{top}(G,\mathbb{R}) \setminus \{0\}$

- For $H \leq G$ closed cocompact and $\chi : G \to \mathbb{R}$:
$$\chi \in \Sigma^n_{top}(G) \Leftrightarrow \chi|_H \in \Sigma^n_{top}(H)$$

- For a s.e.s. $1 \to N \to G \xrightarrow{p} Q \to 1$ with N of type C_n and $\chi : Q \to \mathbb{R}$:
  - $\chi \in \Sigma^n_{top}(Q) \Rightarrow \chi \circ p \in \Sigma^n_{top}(G)$
  - $\chi \circ p \in \Sigma^{n+1}_{top}(G) \Rightarrow \chi \in \Sigma^{n+1}_{top}(Q)$

[New even for $\chi = 0$!]

- For $1 \to N \to G \xrightarrow{p} Q \to 1$ is a s.e.s. with Q abelian, if $\Sigma^n_{top}(G)$ contains all characters that vanish in N, then N is of type C_n.

The homological story

[Abels-Tiemeyer]

- There are homological "finiteness compactness properties" FP_n / CP_n
- FP_n has been refined to "homological Sigma-sets" $\Sigma^n(G;\mathbb{Z})$ [Bieri, Renz]
- Our second paper unifies them into
$$\Sigma^n_{top}(G;\mathbb{Z})$$
- We relate the two theories by a Hurewicz-like theorem:
$$\Sigma^1_{top}(G) = \Sigma^1_{top}(G;\mathbb{Z})$$

for $n \geq 2$: $\quad \Sigma^n_{top}(G) = \Sigma^n_{top}(G;\mathbb{Z}) \cap \Sigma^2(G)$