Knot theory

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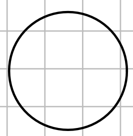
1. Introduction

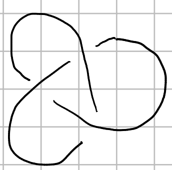
A <u>knot</u> is a smoothly embedded closed curve (i.e. circle)
$K \subset \mathbb{S}^3$ — often regarded as $\mathbb{R}^3 \cup \{\infty\}$ via the stereographic projection

<u>Ex.</u>

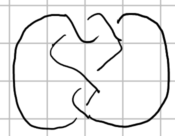


$\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1, \; z=0\}$

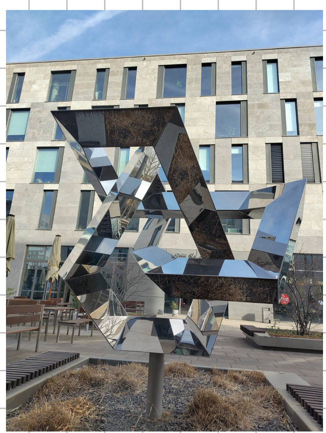
A <u>trivial knot</u>, or <u>unknot</u>

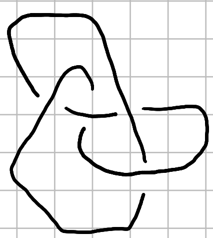


A <u>right-handed trefoil</u>



A <u>figure-eight knot</u> (a.k.a. 4_1)



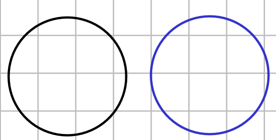


A sculpture outside the Mathematikon

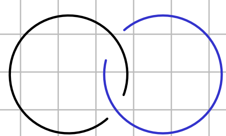
More generally, for $m \in \mathbb{N}$, an <u>m-component link</u> is a compact 1-dimensional smooth submanifold of $\mathbb{S}^3$ with m components $L \subset \mathbb{S}^3$

So $L \underset{\text{diffeo}}{\cong} \bigsqcup_{i=1}^{m} \mathbb{S}^1$

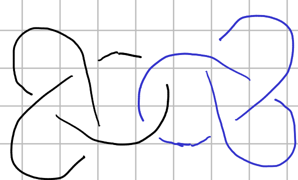
<u>Ex.</u>



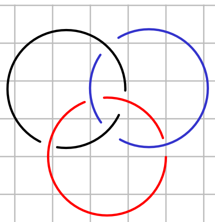
An <u>unlink</u> of two unknots



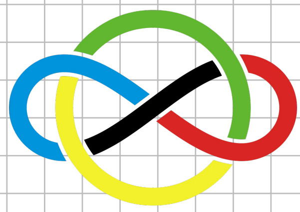
A <u>Hopf link</u>



Two linked trefoils

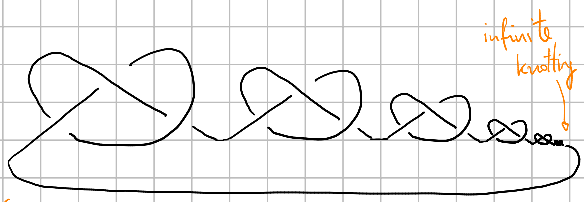


The Borromean rings



The logo of the International Mathematical Olympiad depicts a Whitehead link.

Rmk. • The assumption that knots/links are smoothly embedded (and not merely a subspace homeomorphic to S^1) excludes "infinite knotting":

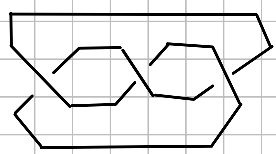


This subspace is homeomorphic to S^1.

This is an example of a "wild knot".

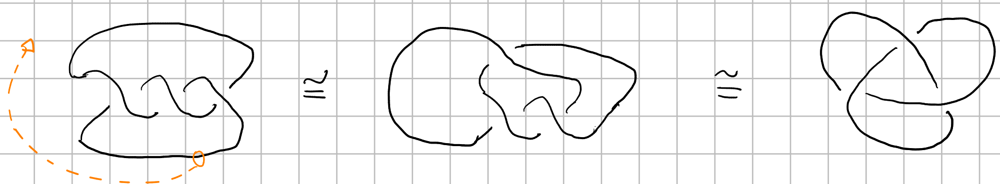
We exclude these from our theory.

• In the literature, the smoothness condition is often replaced by knots being "polygonal curves" with finitely many segments:



This leads to equivalent theories.

In Knot Theory, we are interested in knots / links up to smooth isotopy:

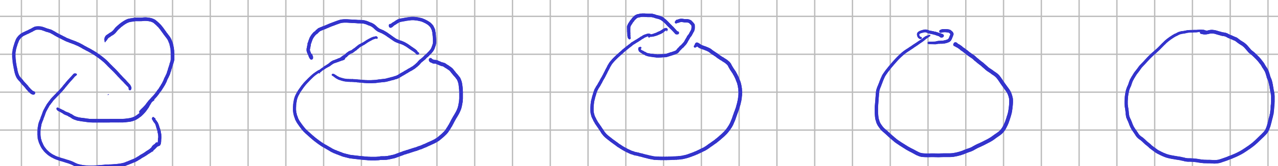


Def. Two m-component links L, L' are <u>isotopic</u> if there are smooth embeddings $f, f' : \bigsqcup_{i=1}^{m} S^1 \hookrightarrow S^3$ with $f(\bigsqcup_{i=1}^{m} S^1) = L$, $f'(\bigsqcup_{i=1}^{m} S^1) = L'$, and a <u>smooth isotopy from f to f'</u>.

that is, a smooth map $F : \bigsqcup_{i=1}^{m} S^1 \times [0,1] \longrightarrow S^3$
such that:
- for each $t \in [0,1]$, the map $F_t : \bigsqcup_{i=1}^{m} S^1 \to S^3$ is an embedding
- $F_0 = f$, $F_1 = f'$.

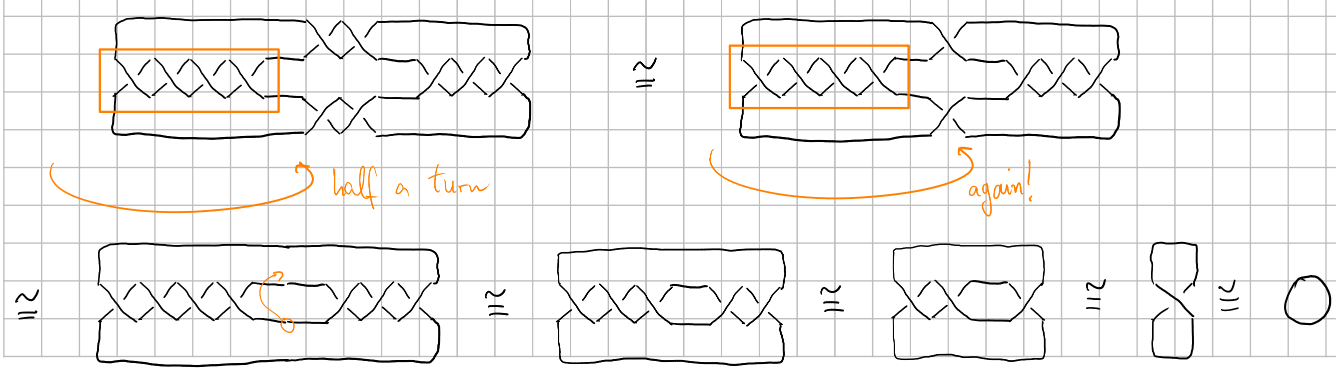
For each $m \in \mathbb{N}$, this defines an equivalence relation on the set of m-component links.

Remark. It is crucial for the definition that
$$F : \bigsqcup_{i=1}^{m} S^1 \times [0,1] \xrightarrow{m} S^3$$
be smooth! If F were allowed to be only a homotopy through smooth embeddings, then every two knots would be isotopic:



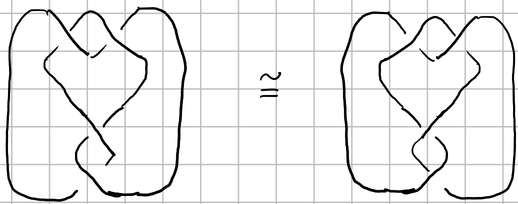
Isotopic knots can look very different!

Ex. The <u>Goeritz unknot</u>



A drawing of an unknot, by Ana Chávez Cáliz.

<u>Exercise</u>: Show that the figure-8 knot is isotopic to its <u>mirror image</u>:



Here is an equivalent characterization of link isotopy:

<u>Prop.</u> Two links L, L' are isotopic iff there is a smooth isotopy $G : S^3 \times [0,1] \longrightarrow S^3$ "ambient isotopy" such that $G_0 = id_{S^3}$ and $G(L, 1) = L'$.

<u>Pf.</u> $(\Leftarrow)$ Choose a smooth parametrization $f : \coprod_{i=1}^{m} S^1 \hookrightarrow S^3$ of L.

The composition

$$f' : \coprod_{i=1}^{m} S^1 \xhookrightarrow{f} S^3 \xrightarrow{G_1} S^3$$

is a parametrization of L', and

$$F : \coprod_{i=1}^{m} S^1 \times [0,1] \xrightarrow{f \times id} S^3 \times [0,1] \xrightarrow{G} S^3$$

is a smooth isotopy from f to f'.

($\Rightarrow$) Follows directly from the following theorem:

Isotopy Extension Theorem (simplified version, see [Friedl, Topology] for a much more general statement.)

Let M, N be smooth manifolds with empty boundary, M compact, and

$$F: \quad M \times [0,1] \longrightarrow N$$

a smooth isotopy. Then there is a smooth isotopy

$$G: \quad N \times [0,1] \longrightarrow N$$

such that $G_0 : N \to N$ is the identity, and F is the composition

$$M \times [0,1] \xrightarrow{F_0 \times id} N \times [0,1] \xrightarrow{G} N. \qquad \square$$

In particular, the oriented diffeomorphism type of the space $S^3 \setminus L$ is preserved under link isotopy!

A third equivalent description:

Prop. Two links L, L' are smoothly isotopic if and only if there is an orientation-preserving diffeomorphism

$$\varphi : S^3 \longrightarrow S^3$$

with $\varphi(L) = L'$

Pf. ($\Rightarrow$) If $L \cong L'$, then the previous proposition gives a smooth isotopy $G: S^3 \times [0,1] \to S^3$ with $G_0 = id_{S^3}$ and $G_1(L) = L'$.

Since G_0 is orientation-preserving and G_1 is homotopic to G_0, also G_1 is orientation-preserving.

So we take $\varphi := G_1$.

($\Leftarrow$) This is a direct consequence of the following (hard) fact:

Theorem (Cerf). Every orientation-preserving diffeomorphism $\varphi : S^3 \longrightarrow S^3$ is smoothly isotopic to the identity. $\square$

Motivating questions:

- Can we decide if a given knot is isotopic to the unknot?
- Given two knots/links, are they isotopic?
  If not, how can one prove it?

$$\text{Links}/\text{isotopy} \rightsquigarrow \text{Invariants}$$

(e.g. integers, polynomials, properties, groups ...)

Typically, knot/link invariants are defined

- topologically (e.g. topological features of $S^3 \setminus L$)

or

- diagramatically
  (requires showing that the value is independent on the choice of diagram.)

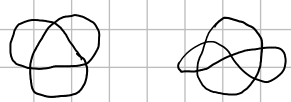
Link diagrams

Def. An m-component <u>link diagram</u> consists of

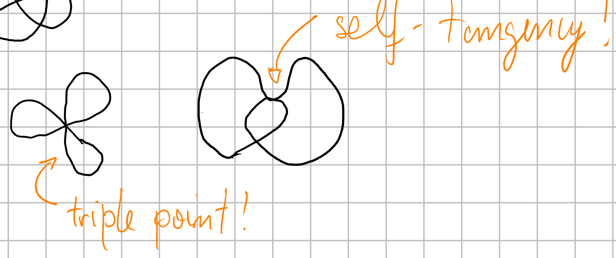
- the image of an immersion $\varphi : \bigsqcup_{i=1}^{m} S^1 \hookrightarrow \mathbb{R}^2$ such that every $p \in \mathbb{R}^2$ has at most 2 pre-images, and whenever $\varphi(x) = \varphi(y)$ with $x \neq y$, this intersection is transverse (that is, $\operatorname{im}(D_x\varphi) \oplus \operatorname{im}(D_y\varphi) = \mathbb{R}^2$).

  The point $\varphi(x) = \varphi(y)$ is a <u>crossing</u> of the diagram.

  <u>ex:</u>

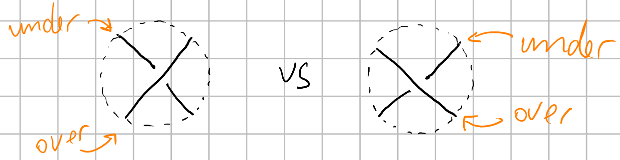
  

  <u>not allowed:</u>

- At each crossing, the datum of which strand is "over-crossing", and which is "under-crossing"

  locally:

<u>Fact:</u>
- A link diagram has only finitely many crossings.
- Every m-component diagram determines a unique m-component link (up to isotopy).
- Every link $L \subset S^3$ can be isotoped to an $L' \subset \mathbb{R}^3 \subset S^3$ such that the orthogonal projection to the xy-plane yields a diagram.

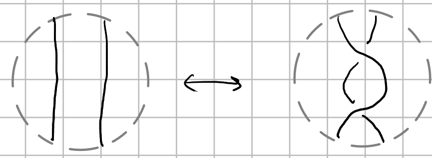
To define a link invariant from a diagram, one needs to show it is unchanged upon passing to a different diagram of the same link.

Reidemeister's Theorem

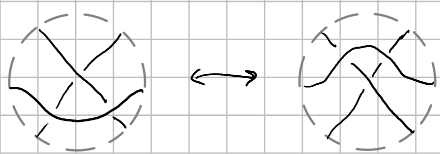
Every two diagrams for a link differ by a sequence of the following moves (plus isotopies of $\mathbb{R}^2$):

R1 :

R2 :



R3 :

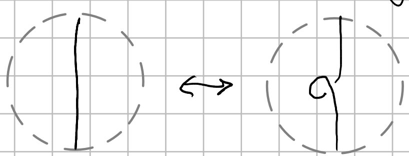


(Each move is numbered according to how many strands are involved.)

So. To prove that a feature of a diagram is a link invariant, one only needs to show it is unchanged under all three Reidemeister moves.

Exercise. Show that the following variants of the Reidemeister moves follow from the given ones:

R1′ :



R3′ :