Sheet 12

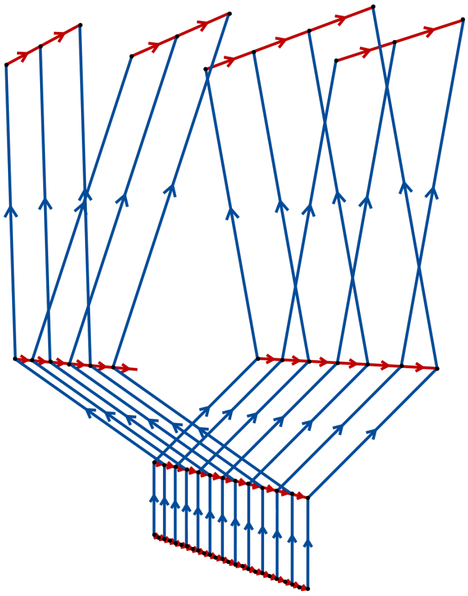
Due date: Jan 24

Problem 1 (Ends in small free products)**.** Given two finite groups G, H, express the number of ends of their free product $G * H$ as a function of $|G|$ and $|H|$.

Problem 2 (The geometry of $BS(1,2)$)**.** Recall the Baumslag-Solitar group

$$BS(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$$

from Sheet 11. Its Cayley graph with respect to $\{a, b\}$ looks as follows:



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(a) Explain this picture of the Cayley graph. More explicitly: indicate which edges correspond to a and to b, and label some of the vertices. How is this picture compatible with the unique words established in Sheet 11?

(b) How many ends does $BS(1,2)$ have?

Problem 3 (Azrelà-Ascoli)**.** The goal of this exercise is to prove the following formulation of Azrelà-Ascoli's Theorem:

> Let X, Y be metric spaces, with X separable and Y proper. Let $(f_n : X \to Y)_{n \in \mathbb{N}}$ be a sequence of functions that are:
>
> - **pointwise bounded**, that is, for every $x \in X$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is bounded,
> - **uniformly equicontinuous**, that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $n \in \mathbb{N}$ and all $x, x' \in X$, we have
>
> $$d(x, x') < \delta \implies d(f_n(x), f_n(x')) < \varepsilon.$$

Then $(f_n)_{n\in\mathbb{N}}$ has a subsequence that converges to a continuous function $f\colon X \to Y$ uniformly on compact sets.

(a) Given a functions f_n as in the statement and a countable subset $Q \subseteq X$, show that there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ such that for every $q \in Q$, the sequence of points $f_{n_m}(q)$ is convergent.

Hint: If $Q = \{q_0, q_1, \ldots\}$, start by finding a subsequence that converges on q_0, then one converging on q_0 and q_1, etc.

(b) Conclude that there is a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ that converges pointwise on a dense subset of X to a continuous function $f\colon X \to Y$.

(c) Show that the family $\{f_n\}_{n\in\mathbb{N}} \cup \{f\}$ is uniformly equicontinuous.

Hint: Let $Q \subseteq X$ be a dense set as before. For every $\varepsilon > 0$ and respective $\delta > 0$ as given by uniform equicontinuity of the f_n, observe first that for all $q, q' \in Q$

$$d(q, q') < \delta \implies d(f(q), f(q')) \leq \varepsilon.$$

Then extend this estimate to all $x, x' \in X$.

(d) Show that for every compact set $C \subseteq X$, the sequence of restricted functions $f_{n_m}|_C$ converges uniformly to $f|_C$.

Hint: The role of the compactness assumption on C is to, given $\delta > 0$, produce a finite subset $F \subseteq Q$ such that every $x \in C$ is within distance δ from some $q \in F$.