Sheet 10

Due date: Jan 10

Problem 1 (Undistorted subgroups)**.** Let $H \leq G$ be finitely generated groups. We say that H is **undistorted** in G if for some finite generating sets $S \subseteq G$ and $T \subseteq H$, the inclusion $(H, d_T) \hookrightarrow (G, d_S)$ is a quasi-isometric embedding.

(a) Show that this notion does not depend on the choice of S and T, that is, if $(H, d_T) \hookrightarrow (G, d_S)$ is a quasi-isometric embedding, then so is $(H, d_{T'}) \hookrightarrow (G, d_{S'})$ for any finite generating sets S', T'.

(b) Show that if G is hyperbolic, then so is every undistorted subgroup H of G.

Problem 2 (Conjugacy classes in free groups)**.** Let $F = \mathrm{F}(S)$ be the free group on a set S, and let $w = y_1 \cdots y_k$ be a word in $S^{\pm}$. A second such word w' is said to be **cyclically equivalent** to w if it differs by a cyclic shift in the indices:

$$\exists l \in \mathbb{N} : w' = y_{1+l} \cdots y_{k+l} \qquad \text{(indices modulo } k\text{)}.$$

We say w is **cyclically reduced** if it is reduced and, additionally, y_1 is not the inverse of y_k.

(a) Show that two elements of F are conjugate if and only if they can be represented by cyclically equivalent words.

(b) Show that each conjugacy class in F is represented by a cyclically reduced word, which is unique up to cyclic equivalence.

Problem 3 (Taming c-local geodesics)**.** Recall that given $c > 0$, a path $\gamma\colon [0, L] \to X$ in a metric space (X, d) is called a **c-local geodesic** if for every $a, b \in [0, L]$ with $|b - a| \leq c$, we have $d(\gamma(a), \gamma(b)) = |b - a|$.

Let $\delta > 0$ and suppose X is δ-hyperbolic.

(a) Show that for every geodesic quadrilateral in X, each side is contained in the 2δ-neighborhood of the union of the other three.

(b) Let $\gamma'\colon [0, L'] \to X$ be a geodesic and $\gamma\colon [0, L] \to X$ an 8δ-local geodesic with the same endpoints. Show that $\mathrm{im}(\gamma)$ is contained in the 2δ-neighborhood of $\mathrm{im}(\gamma')$.

Hint: Choose $t \in [0, L]$ such that the distance from $\gamma(t)$ to $\mathrm{im}(\gamma')$ is maximal, and define $t_0 := \max\{t - 4\delta, 0\}$ and $t_1 := \min\{t + 4\delta, L\}$. Analyze the quadrilateral formed by $p := \gamma(t_0), q := \gamma(t_1)$, and points p', q' of $\mathrm{im}(\gamma')$ that are closest to p and q, respectively.