UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Sheet 9

Due date: Dec 20

Problem 1 (Minimal words in free products)**.** Let $G := *_{i\in I} G_i$ be a free product of groups, and recall that each $g \in G$ is represented by a unique reduced word

$$g_1 * \dots * g_k,$$

with each g_j in one of the groups G_{i_j}. Given generating sets S_i for the G_i, consider also the generating set $S := \bigsqcup_{i\in I} S_i$ of G.

Show that a word w in $S^{\pm}$ is a minimal length representative of g if and only if it is of the form

$$w = w_1 \cdots w_k,$$

where each w_j is a minimal length word in $S^{\pm}_{i_j}$ representing g_j.

Problem 2 (Hyperbolicity and free products)**.** Let G and H be finitely generated groups. Show that their free product $G * H$ is hyperbolic if and only if both G and H are hyperbolic.

Problem 3 ($\mathbb{R}$-trees)**.** An **$\mathbb{R}$-tree** is a metric space (X, d) such that:

- X is **uniquely geodesic**, that is, for every $x, y \in X$ there is a unique geodesic from x to y, whose image we denote by $[x, y]$,
- for all $x, y, z \in X$, we have

$$[x, y] \cap [y, z] = \{y\} \quad \Longrightarrow \quad [x, z] = [x, y] \cup [y, z].$$

(a) Show that if T is a tree, then its geometric realization $|T|$ is an $\mathbb{R}$-tree.

(b) Give an example of an $\mathbb{R}$-tree that is not the geometric realization of a tree.

(c) Show that $\mathbb{R}$-trees are precisely the 0-hyperbolic metric spaces.

Problem 4 (Möbius transformations)**.** The **complex projective plane**, also known as the **Riemann sphere**, is defined as

$$\mathbb{C}\mathrm{P}^1 := (\mathbb{C}^2 \setminus \{0\})/\sim,$$

where $\sim$ identifies $\mathbb{C}$-collinear vectors:

$$(z, w) \sim (z', w') :\Longleftrightarrow \exists \lambda \in \mathbb{C}^\times : (z', w') = \lambda(z, w).$$

The equivalence class of (z, w) is denoted by $[z : w]$. It is common to identify $\mathbb{C}\mathrm{P}^1$ with the set $\mathbb{C} \cup \{\infty\}$ via the bijection $[z : 1] \mapsto z \in \mathbb{C}, [1 : 0] \mapsto \infty$ (if this is new to you, you should convince yourself that this is indeed a bijection).

(a) Show that matrix multiplication yields a well defined action $\alpha\colon \mathrm{PGL}(2, \mathbb{C}) \curvearrowright \mathbb{C}\mathrm{P}^1$, and express α in terms of the description of $\mathbb{C}\mathrm{P}^1$ as $\mathbb{C} \cup \{\infty\}$.

(b) Show that α is **triply transitive**, that is, it is transitive on ordered triples of distinct points in $\mathbb{C}\mathrm{P}^1$.

(c) Show that $\mathrm{PGL}(2,\mathbb{C})$ is generated by the union of the following sets of matrices:

- homotheties: $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, where $a \in \mathbb{C}^\times$,
- translations: $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where $b \in \mathbb{C}$,
- the inversion $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Hint: Can you express the formula from part (a) as a composition of simpler functions?

(d) Find a similar generating set for the subgroup $\mathrm{PSL}(2,\mathbb{R}) < \mathrm{PGL}(2,\mathbb{C})$, and deduce that α restricts to an action of $\mathrm{PSL}(2,\mathbb{R})$ on the upper half-plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

(e) Prove that this action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ is faithful and transitive. What is the stabilizer of the point $i \in \mathbb{H}^2$?