Sheet 8

Due date: Dec 13

Problem 1 (Normal form for free products)**.** Given groups $(G_i)_{i \in I}$, recall that the elements of the free product $G := *_{i \in I} G_i$ are words in $\bigsqcup_{i \in I} G_i$, up to a certain equivalence relation $\sim$. Such a word $(g_1, \ldots, g_l)$ is called **reduced** if

- no entry g_j is the trivial element of the corresponding group G_{i_j}, and
- no two consecutive entries g_j, g_{j+1} lie in the same free factor, that is, $i_j \neq i_{j+1}$ for all $j \in \{1, \ldots, l-1\}$.

(a) Show that every element of G admits a unique representation as a reduced word.

Hint: Recall that $\sim$ is generated by "elementary moves" of two types, which decrease length by one. Given a sequence $u = w_0 \sim w_1 \sim \ldots \sim w_k = v$ of elementary moves (and their inverses) between reduced words u, v, consider what happens when length increases and then decreases in succession.

(b) Sketch the Cayley graph of $\mathbb{Z}^2 * \mathbb{Z}$ with respect to the generating set $\{(1,0), (0,1), 1\}$.

(c) As you know, the n-regular tree T_n is a Cayley graph whenever n is even (see Sheet 6, Problem 2(c)). What if n is odd?

Problem 2 (Geometric properties)**.** Which of the following properties of finitely generated groups are geometric?

(a) Being finite,

(b) Having a generating set with n elements (for a fixed $n \in \mathbb{N}$),

(c) Being finitely presented,

(d) Being abelian,

(e) Being free,

(f) Having elements of order n (for a fixed $n \in \mathbb{N}_{\geq 1}$),

(g) Being a free product of two non-trivial groups.

Problem 3 (Virtually free groups from free products)**.** Let A, B be groups and consider the canonical map

$$\pi \colon A * B \to A \times B$$

induced by $\pi(a) = (a, 1)$ for $a \in A$ and $\pi(b) = (1, b)$ for $b \in B$.

(a) Show that $\ker(\pi)$ is a free group with free generating set

$$S := \{[a, b] \mid a \in A \setminus \{1_A\}, b \in B \setminus \{1_B\}\}.$$

(b) Deduce that if A and B are nontrivial finite groups and one of them has order at least 3, then $A * B$ is quasi-isometric to F_2.

Problem 4 (Product of hyperbolic spaces)**.** Given two metric spaces $(X, d_X), (Y, d_Y)$, we equip their product $X \times Y$ with the metric

$$d((x, y), (x', y')) := d_X(x, x') + d_Y(y, y').$$

(a) Show that the product of two geodesic metric spaces is geodesic.

(b) Suppose X, Y are hyperbolic spaces (that is, δ-hyperbolic for some $\delta \geq 0$). When is $X \times Y$ hyperbolic?

Hint: You may use the fact, to be shown in class, that being hyperbolic is a geometric property.