Geometric group theory, Winter 2024/25
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Sheet 7

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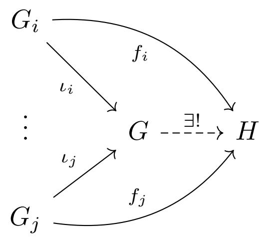
Problem 1 (Free products)**.** Let $(G_i)_{i\in I}$ be a set of groups. A **free product** of the G_i is a group G together with homomorphisms $(\iota_i\colon G_i \to G)_{i\in I}$ satisfying the following universal property: For every group H and homomorphisms $(f_i\colon G_i \to H)_{i\in I}$, there is a unique homomorphism $f\colon G \to H$ such that $f \circ \iota_i = f_i$ for all $i \in I$. Diagrammatically:



Note: If you have encountered coproducts before, you should recognize this as the definition of a coproduct in the category of groups.

(a) Show for every two free products G, G', with structure maps $(\iota_i), (\iota'_i)$ respectively, there is a unique isomorphism $\varphi\colon G \to G'$ such that for every $i \in I$ we have $\varphi \circ \iota_i = \iota'_i$.

Let $\mathcal{G}$ be the set of all finite words with entries in the disjoint union $\bigsqcup_{i\in I} G_i$, and let $\sim$ be the equivalence relation on $\mathcal{G}$ generated by the following moves:

- suppressing occurrences of trivial elements:

$$(g_1, \dots, g_{m-1}, 1, g_{m+1}, \dots, g_k) \sim (g_1, \dots, g_{m-1}, g_{m+1}, \dots, g_k),$$

- multiplying consecutive entries g_m, g_{m+1} that are elements of the same group G_i:

$$(g_1, \dots, g_m, g_{m+1}, \dots, g_k) \sim (g_1, \dots, g_m g_{m+1}, \dots, g_k).$$

We define

$$\mathop{\ast}_{i\in I} G_i := \mathcal{G}/\sim,$$

and denote the equivalence class of a word $(g_1, g_2, \dots, g_k) \in \mathcal{G}$ by $g_1 * g_2 * \dots * g_k$.

(b) Show that the operation on $\mathcal{G}$ of concatenating words descends to an operation on $*_{i\in I} G_i$ making it into a group.

(c) Supply structure maps $(\iota_i\colon G_i \to \mathop{\ast}_{i\in I} G_i)$ that make $*_{i\in I} G_i$ a free product of the G_i.

In light of (a), one often refers to $\mathop{\ast}_{i\in I} G_i$ as *the* free product of the G_i.

(d) Given presentations $\langle S_i \mid R_i \rangle$ of the G_i, write down a presentation of $*_{i\in I} G_i$.

Problem 2 (Free groups and quasi-isometry)**.** Recall that every finitely generated group G has a canonical quasi-isometry type, represented by the Cayley graph $\mathrm{Cay}(G, S)$ with respect to any finite generating set S.

For which pairs of numbers $n, m \in \mathbb{N}$ is F_n quasi-isometric to F_m? Prove your assertion.

Problem 3 (Metrics on quotients)**.** Consider an action of a group G on a metric space (X, d), and denote the orbit of a point $x \in X$ by $\bar{x}$.

(a) A **pseudo-metric space** is a set Y with a function $\delta\colon Y \to \mathbb{R}_{\geq 0}$ satisfying the usual properties of a distance, except that we do not impose $\delta(x, y) \neq 0$ for $x \neq y$. Such δ is called a pseudo-metric.

Show that
$$\bar{d}(\bar{x}, \bar{y}) := \inf\{d(x, y) \mid x \in \bar{x}, y \in \bar{y}\}$$
defines a pseudo-metric $\bar{d}$ on the orbit space $G\backslash X$.

(b) Show that if, in addition, the G-action is properly discontinuous, then the above infimum is attained, and $\bar{d}$ is a metric making the projection map $p\colon X \to G\backslash X$ continuous.

(c) Give an example of a group G with an action by isometries on a metric space (X, d) such that $\bar{d}$ is not a metric on $G\backslash X$.

Problem 4 (Topological Milnor-Švarc)**.** Prove the following variant of the Milnor-Švarc Lemma by reducing it to the "geometric version" seen in class. (Recall that a metric space X is **proper** if its closed balls are compact.)

> **Topological Milnor-Švarc (TMŠ).** Let G be a group acting properly and cocompactly by isometries on a proper geodesic metric space (X, d). Then G is finitely generated, and for every $x \in X$ the map
> $$\begin{aligned} G &\to X \\ g &\mapsto g \cdot x \end{aligned}$$
> is a quasi-isometry.