Sheet 6

Due date: Nov 29

Problem 1 (Properly discontinuous actions)**.** Consider a properly discontinuous action of a group G on a metric space (X, d).

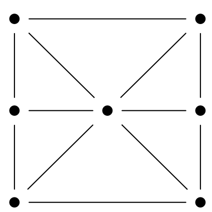
(a) Prove that, for all $x \in X$, the stabilizer $G_x := \{g \in G \mid g \cdot x = x\}$ of x in G is finite.

(b) Show that, for every $x \in X$, there is a ball B_x in X centered at x such that $gB_x \cap B_x = \emptyset$ for every $g \in G \setminus G_x$.

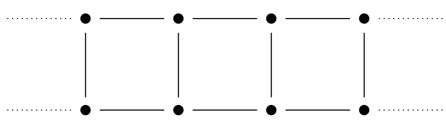
(c) Prove that no G-orbit has an accumulation point in X.

Problem 2 (Cayley or not Cayley?)**.** Among the following graphs, which of them are Cayley graphs?

(a)

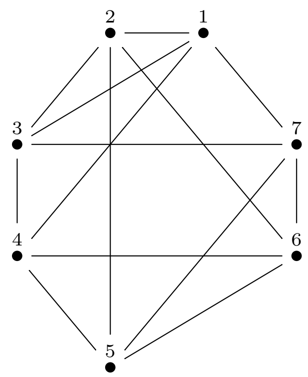


(b)



(c) The **$2n$-regular tree** T_{2n} with $n \geq 0$, i.e., the tree in which every vertex has exactly $2n$ neighbors.

(d)



Convention: in the following problem, every generating set of a group is supposed to not have the unit as element.

Problem 3 (Quasi-isometries on groups)**.** Let G be a group with a generating set S. Consider the **word metric** d_S of (G, S), i.e., for all $g, h \in G$ define

$$d_S(g, h) := \min\{n \in \mathbb{Z}_{\geq 0} \mid \exists\, s_1, \ldots, s_n \in S \cup S^{-1} \text{ such that } h = gs_1 \cdots s_n\}.$$

For $g \in G$ and $r \in \mathbb{Z}_{\geq 0}$, let

$$B_r^{G,S}(g) := \{h \in G \mid d_S(g, h) \leq r\}.$$

Two finitely generated groups G and H are said to be **quasi-isometric** if there are finite generating sets S and T of G and H, respectively, such that (G, d_S) is quasi-isometric to (H, d_T).

(a) Let G be a finitely generated group. By Satz 6.5 of the lecture notes, for every two *finite* generating sets S and T of G, the metric spaces (G, d_S) and (G, d_T) are quasi-isometric. Does this statement remain true if we drop the hypothesis that S and T are both finite?

(b) Let G and H be groups with finite generating sets S and T, respectively. Consider a quasi-isometric embedding $f\colon (G, d_S) \to (H, d_T)$. Show that there is $c \in \mathbb{Z}_{\geq 1}$ such that, for every $r \in \mathbb{Z}_{\geq 0}$,

$$f(B_r^{G,S}(1_G)) \subseteq B_{cr+c}^{H,T}(f(1_G))$$

and

$$|B_r^{G,S}(1_G)| \leq |B_{c^2}^{G,S}(1_G)| \cdot |B_{cr+c}^{H,T}(1_H)|.$$

(c) Fix $n \in \mathbb{Z}_{\geq 1}$. Let G be the additive group $\mathbb{Z}^n$ and $S = \{e_i \mid 1 \leq i \leq n\}$ be its canonical $\mathbb{Z}$-basis. Prove that there are $k, K \in \mathbb{R}_{>0}$ such that for every $r \gg 1$ one has

$$kr^n \leq |B_r^{G,S}(0)| \leq Kr^n.$$

(d) Deduce that, for all $m, n \in \mathbb{Z}_{\geq 1}$ with $m \neq n$, the groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not quasi-isometric.