



Sheet 5

Due date: Nov 22

Problem 1. (Count the uncountables) Which of the following classes of groups contain uncountably many isomorphism classes?

- (a) Finite groups
- (b) Countable groups
- (c) Finitely generated groups
- (d) Finitely generated abelian groups
- (e) Finitely presented groups
- (f) Countably presented groups (that is, having a presentation with (at most) countably-many generators and countably-many relators)
- (g) Free groups

Problem 2 (A 2-generated group with uncountably many quotients). In this exercise we will construct a 2-generated group G whose center contains an isomorphic copy of $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Define:

- the free abelian groups A and B as having \mathbb{Z} -bases $(a_n)_{n \in \mathbb{N}_{>0}}$ and $(b_n)_{n \in \mathbb{Z}}$, respectively.
- the \mathbb{Z} -bilinear map $f: B \times B \rightarrow A$ given on basis elements by

$$f(b_m, b_n) := \begin{cases} a_{n-m} & \text{if } n > m, \\ 0 & \text{if } n \leq m, \end{cases}$$

- the binary operation on the set $H := A \times B$, given by

$$(a, b) \cdot (a', b') := (a + a' + f(b, b'), b + b'),$$

- the map

$$T: H \rightarrow H$$

$$\left(a, \sum_{n \in \mathbb{Z}} \beta_n b_n \right) \mapsto \left(a, \sum_{n \in \mathbb{Z}} \beta_n b_{n+1} \right).$$

Show the following statements:

- (a) The operation \cdot makes H into a group.
- (b) The set $A \times \{0\}$ is a central subgroup of H .
- (c) H is generated by the set $\{(0, b_n) \mid n \in \mathbb{N}\}$.
Hint: For each $n \in \mathbb{N}_{>0}$, compute the commutator $[(0, b_0), (0, b_n)]$.
- (d) T is a group automorphism of H .

We now let $G := H \rtimes \mathbb{Z}$, with action $\mathbb{Z} \rightarrow \text{Aut}(H)$ given by $1 \mapsto T$. Prove that G satisfies the desired properties:

- (e) G is generated by $\sigma := ((0, b_0), 0)$ and $\tau := ((0, 0), 1)$.
- (f) G contains a central subgroup isomorphic to $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$, and thus uncountably many normal subgroups.

By Proposition 4.7 from class, it follows that G has uncountably-many pairwise nonisomorphic quotients! We can use G to show that the same is true of the group G_0 from class, presented by two generators s, t and relations

$$\{[s, t^n s t^{-n}], s \mid n \in \mathbb{Z}\} \cup \{[s, t^n s t^{-n}], t \mid n \in \mathbb{Z}\}.$$

- (g) Show that the assignment $s \rightarrow \sigma, t \rightarrow \tau$ induces an epimorphism $G_0 \rightarrow G$.
- (h) Show that the subgroup of G_0 generated by $\{[s, t^n s t^{-n}] \mid n \in \mathbb{Z}\}$ is central and contains a subgroup isomorphic to $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$.

Problem 3 (Word metrics). Fix a group G with generating set S , and assume S is **symmetric** (that is, $S = S^{-1}$).

- (a) Prove that the function

$$d_S: G \times G \rightarrow \mathbb{R}_{\geq 0} \\ (g, h) \mapsto l_S(g^{-1}h),$$

where l_S is the word length relative to S , is a metric on G .

- (b) Let H be a second group, with symmetric generating set T . Show that if T is finite, then every group homomorphism $f: H \rightarrow G$ is Lipschitz with respect to the metrics d_T and d_S .

Reminder: A function $f: X \rightarrow Y$ between metric spaces $(X, d_X), (Y, d_Y)$ is called **Lipschitz** if there is a constant $L \in \mathbb{R}_{\geq 0}$ such that

$$\forall x, x' \in X \quad d_Y(f(x), f(x')) \leq L d_X(x, x').$$

- (c) Conclude that if S and S' are finite symmetric generating sets for G , then the metrics $d_S, d_{S'}$ on G are **bilipschitz equivalent**, that is, there are constants $\alpha, \beta > 0$ such that

$$\forall g, h \in G \quad \alpha d_S(g, h) \leq d_{S'}(g, h) \leq \beta d_S(g, h).$$

Remark: This implies that any properties of the metric space (G, d_S) that are preserved by bilipschitz equivalence are therefore intrinsic to G (that is, independent of S)!