UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Sheet 4

Due date: Nov 15

Problem 1 (A presentation for the trivial group)**.** Show that the following is a presentation of the trivial group:

$$\langle x, y \mid xyx^{-1}y^{-2}, yxy^{-1}x^{-2} \rangle.$$

Problem 2 (Normal subgroups of F_n)**.** Let $n \in \mathbb{Z}_{\geq 1}$, and denote by F_n the free group of rank n.

(a) Prove that F_n has a normal free subgroup of index d, for every $d \in \mathbb{Z}_{\geq 1}$.

(b) Prove that F_2 has a normal free subgroup of finite index and rank k, for every $k \in \mathbb{Z}_{\geq 2}$.

(c) Assume that $n \geq 2$ and let p be a prime number. Prove that F_n has exactly $\frac{p^n-1}{p-1}$ normal free subgroups of index p.

Hint: Prove that there are exactly $p^n - 1$ epimorphisms $\mathrm{F}_n \to \mathbb{Z}/p$.

A group is said to be **virtually free** if it has a free subgroup of finite index.

Problem 3 ($\mathrm{SL}_2(\mathbb{Z})$ is virtually free)**.** Let F be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices

$$a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

According to Beispiel 3.20 of the lecture notes, the group F is freely generated by $\{a, b\}$.

Define the subgroups:

$$\Gamma_n := \mathrm{SL}_2(\mathbb{Z}) \cap \begin{bmatrix} 1+n\mathbb{Z} & n\mathbb{Z} \\ n\mathbb{Z} & 1+n\mathbb{Z} \end{bmatrix}, \, n \in \mathbb{Z}_{\geq 0}; \quad \text{and} \quad G := \mathrm{SL}_2(\mathbb{Z}) \cap \begin{bmatrix} 1+4\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 1+4\mathbb{Z} \end{bmatrix}.$$

(a) For $k \geq 1$, consider the set

$$G(k) := \big\{ x \in G \,\big|\, \max\{|x_{11}|, |x_{21}|\} = k \big\}.$$

Let $k \geq 2$. Prove that, for every $x \in G(k)$, there are $f \in F$ and $h \in \mathbb{Z}_{\geq 0}$ with $h < k$ such that $fx \in G(h)$.

Hint: For example, observe that if $|x_{11}| \geq |x_{21}|$, then one has one of the strict inequalities

$$|x_{11} - 2x_{21}| < |x_{11}| \qquad \text{or} \qquad |x_{11} + 2x_{21}| < |x_{11}|.$$

(b) Prove that $F = G$.

(c) Prove that $\mathrm{SL}_2(\mathbb{Z})/\Gamma_2 \cong \mathrm{SL}_2(\mathbb{Z}/2)$.

(d) Prove that $|\mathrm{SL}_2(\mathbb{Z}) : \Gamma_2| = 6$ and $|\Gamma_2 : G| = 2$.

Hint: Show that

$$\Gamma_2 = G \sqcup \left(\mathrm{SL}_2(\mathbb{Z}) \cap \begin{bmatrix} -1+4\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & -1+4\mathbb{Z} \end{bmatrix} \right).$$

(e) Conclude that $\mathrm{SL}_2(\mathbb{Z})$ is virtually free. Is $\mathrm{SL}_2(\mathbb{Z})$ free?

(f) Prove that Γ_n is free whenever n is a multiple of 4.