Sheet 3

Due date: Nov 8

Problem 1 (Undirected trees)**.** Give an example of a group G with a generating system S such that the (undirected) Cayley graph $\mathrm{Cay}(G, S)$ is a tree, but G is not free.

Problem 2 (Spanning trees)**.** A **spanning tree** for a connected graph Γ is a subgraph that is a tree and contains all the vertices of Γ.

(a) Show that every connected graph has a spanning tree.

Hint: Take inspiration from Satz 3.13 of the lecture notes.

(b) Let T be a tree and X a set of vertices of T. Show that there is a smallest sub-tree $T_X \subseteq T$ containing X, and that if X is finite, then so is T_X.

Problem 3 (Rank of finitely generated free groups)**.** Given $n \in \mathbb{N}$, denote by F_n be the free group on n generators.

(a) If G is a finite group, how many group homomorphisms $\mathrm{F}_n \to G$ do there exist?

(b) Deduce that for every $m, n \in \mathbb{N}$, we have $\mathrm{F}_n \cong \mathrm{F}_m$ if and only if $n = m$.

The number of free generators for a finitely generated free group is therefore well-defined, and we call it the **rank** of the free group. This is in fact also true of the cardinality of infinite free generating sets.

(c) Let $\mathrm{F}_{\mathbb{N}} = \langle x_1, x_2, \ldots \rangle$ be the free group on a countably infinite set of generators, and $\mathrm{F}_2 = \langle a, b \rangle$ the free group of rank 2. Show that the homomorphism

$$\begin{aligned} \mathrm{F}_{\mathbb{N}} &\to \mathrm{F}_2 \\ x_i &\mapsto a^i b a^{-i} \end{aligned}$$

is injective. This implies that F_2 contains a copy of $\mathrm{F}_{\mathbb{N}}$ as a subgroup, and thus also of every free group of finite rank.

Hint: We have seen in class how to detect whether words in free groups represent the trivial element…

Problem 4 (Finite groups acting on trees)**.** The goal of this exercise is to show that finite groups acting on nonempty trees always have globally fixed points.

(a) Suppose that T is a finite tree that has at least one edge. Show that T has at least two leaves (a **leaf** is a vertex incident to only one edge).

Hint: Argue by induction on the number of edges, and remember that removing an edge from a tree disconnects it.

(b) Show that every action of a group G on a finite nonempty tree T has a globally fixed vertex or edge (that is, there is a vertex or edge x of T such that for every $g \in G$ we have $g \cdot x = x$).

Hint: Observe that the G-action restricts to the tree T' obtained from T by removing all leaves (along with their incident edges).

(c) Show that every action of a finite group G on a (possibly infinite) nonempty tree T has a globally fixed vertex or edge.

Hint: Choose any vertex v of T and consider the tree $T_{G \cdot x}$, as in Problem 2 (b).