Sheet 2

Due date: Nov 1

Let $n \geq 3$. Recall that the **dihedral group of order** $2n$, written D_n, is the group of all isometries of a regular n-gon $\mathcal{P}_n$ in $\mathbb{R}^2$. Denote by m the midpoint of $\mathcal{P}_n$ and, for every vertex v of $\mathcal{P}_n$, denote by $s_v \in D_n$ the reflection of $\mathcal{P}_n$ whose axis is the line through m and v. Lemma 1.6 of the lecture notes asserts that, for every vertex v of $\mathcal{P}_n$ and for every rotation t of $\mathcal{P}_n$ with center m and angle $\frac{2\pi}{n}$, the group D_n is generated by s_v and t. Moreover,

$$D_n = \{t^k s_v^\epsilon \mid k \in \{0, 1, \dots, n-1\}, \epsilon \in \{0, 1\}\}.$$

Problem 1 (Finite dihedral groups)**.** From now on, fix an arbitrary reflection s and an arbitrary rotation t of $\mathcal{P}_n$ of angle $\frac{2\pi}{n}$.

(a) Prove that $sts = t^{-1}$.

(b) Let $g \in D_n$. Prove that g is a rotation of $\mathcal{P}_n$ if and only if $g = t^k$ for some $0 \leq k < n$. Moreover, show that g is a reflection of $\mathcal{P}_n$ if and only if $g = t^k s$ for some $0 \leq k < n$.

(c) Let $T \subseteq D_n$ be the set of all rotations of $\mathcal{P}_n$. Prove that $T \trianglelefteq D_n$ and that $T \cong \mathbb{Z}/n$ via $k \mapsto t^k$.

(d) Consider the action $\iota \colon \mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z}/n)$ given by $\iota(1)(x) = -x$ for every $x \in \mathbb{Z}/n$. Show that

$$\mathbb{Z}/n \rtimes_\iota \mathbb{Z}/2 \cong D_n.$$

Problem 2 (The dihedral group D_5)**.** For this problem we focus on the dihedral group D_5.

(a) Let s be a reflection of $\mathcal{P}_5$ and t a rotation of $\mathcal{P}_5$ of angle $\frac{2\pi}{5}$. Draw the Cayley graphs of D_5 with respect to the generating sets

$$\{s, t\}, \quad \{s, t, t^{-1}\}, \quad \text{and} \quad \{s, st\}.$$

(b) Prove that, for every $n \geq 1$, the product of $2n$ reflections of $\mathcal{P}_5$ is a rotation of $\mathcal{P}_5$, and the product of $2n+1$ reflections of $\mathcal{P}_5$ is a reflection of $\mathcal{P}_5$.

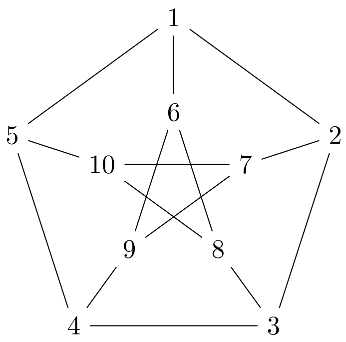
Hint: Use Problem 1(b).

(c) Let S be a generating set of D_5 consisting of elements of order 2. Prove that the undirected Cayley graph $\Gamma(D_5, S)$ of D_5 with respect to S does not have cycles of length 5.

(d) Let S be a generating set of D_5 with $|S| = 3$, such that the undirected Cayley graph $\Gamma(D_5, S)$ has *valency* 3, i.e., every vertex of $\Gamma(D_5, S)$ is in exactly three edges. Prove that S either consists of three distinct elements of order 2, or it equals $\{a, b, b^{-1}\}$ where a has order 2 and b has order 5.

Hint: Observe that every element $a \in S$ of order 5 determines a cycle of length 5 in $\Gamma(D_5, S)$.

(e) Let $\mathcal{P}$ be the Petersen graph:



Prove that $\mathcal{P}$ is not isomorphic to any Cayley graph of D_5.

Problem 3 (Groups of order 10)**.** The goal of this exercise is to classify the groups of order 10.

(a) Show that every group G of order 10 is isomorphic to a semidirect product $\mathbb{Z}/5 \rtimes_\alpha \mathbb{Z}/2$.

Hint: You may use the fact that G has a subgroup of order 5. This is a consequence of the first Sylow theorem.

(b) Show that there are precisely two actions $\alpha\colon \mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z}/5)$.

Hint: Observe that every automorphism of $\mathbb{Z}/5$ is given by multiplication by some element of $(\mathbb{Z}/5)^\times$. When does such an automorphism have order dividing 2?

(c) Conclude that the only groups of order 10 up to isomorphism are $\mathbb{Z}/10$ and D_5.

(d) Show that the Petersen graph is not isomorphic to any Cayley graph.

Hint: Cayley graphs of abelian groups often like to have cycles of length 4 ...