Geometric group theory, Winter 2024/25
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Sheet 1

Due date: Oct 25

Problem 1 (Outer semidirect products)**.** Let N, Q be groups. Recall that given an action $\alpha\colon Q \to \mathrm{Aut}(N)$ of Q on N by group automorphisms, we define the **outer semidirect product** $N \rtimes_\alpha Q$ (written simply $N \rtimes Q$ if there is no room for confusion) as the group with underlying set $N \times Q$ and group operation given by

$$(n_1, q_1) \cdot (n_2, q_2) := (n_1\, \alpha(q_1)(n_2),\, q_1 q_2).$$

(a) Show that $N \rtimes Q$ is indeed a group, fitting into a short exact sequence of the form

$$\mathbb{1} \to N \to N \rtimes Q \to Q \to \mathbb{1}.$$

(Look up the meaning of "short exact sequence" if you have not encountered them yet.)

What happens if α is the trivial action?

(b) Consider a short exact sequence of groups

$$\mathbb{1} \to N \xrightarrow{f} G \xrightarrow{g} Q \to \mathbb{1},$$

and show the following conditions are equivalent:

1. This short exact sequence **splits**, that is, there is a homomorphism $s\colon Q \to G$ such that $g \circ s = \mathrm{id}_Q$ (called a **section**).
2. There are an action $\alpha\colon Q \to \mathrm{Aut}(N)$ and an isomorphism $\varphi\colon G \to N \rtimes_\alpha Q$ making the following diagram commute:

$$\begin{array}{ccccc} N & \xrightarrow{\;f\;} & G & \xrightarrow{\;g\;} & Q \\ \| & & \downarrow{\scriptstyle\varphi} & & \| \\ N & \longrightarrow & N \rtimes Q & \longrightarrow & Q \end{array}.$$

In this case we also say G is a semidirect product of N and Q.

(c) Give an example of a short exact sequence of groups that does not split.

Problem 2 (Inner semidirect products)**.** Let G be a group and $N, Q \leq G$ be subgroups such that N is normal in G, $N \cap Q = \{1\}$ and $G = NQ := \{nq \mid n \in N, q \in Q\}$. Consider the action $\kappa\colon Q \to \mathrm{Aut}(N)$ given by Q-conjugation on N, that is, for $q \in Q$ and $n \in N$,

$$\kappa(q)(n) := qnq^{-1}.$$

Prove that the inclusion maps $\iota_N : N \hookrightarrow G$ and $\iota_Q : Q \hookrightarrow G$ induce a group isomorphism $N \rtimes_\kappa Q \to G$. In this case, G is said to be the **inner semidirect product** of N and Q.

Problem 3 (The normal core)**.** Let G be a group and H a subgroup. The **normal core** of H in G is the intersection of all its conjugates:

$$\mathrm{Core}_G(H) := \bigcap_{g \in G} gHg^{-1}.$$

(a) Show that $\mathrm{Core}_G(H)$ is the largest normal subgroup of G that is contained in H.

(b) Show that if $S \subseteq G$ contains a representative of each left coset of H in G, then

$$\mathrm{Core}_G(H) = \bigcap_{s \in S} sHs^{-1}.$$

(c) Show that if H and K are subgroups of G with finite index, then $H \cap K$ is also a subgroup of G with finite index.

(d) Conclude that if H has finite index in G, then so does its normal core $\mathrm{Core}_G(H)$.