Sheet 0

Classroom exercises, Oct 21

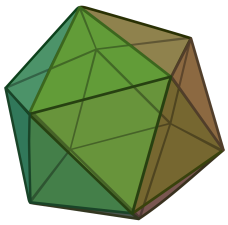
Problem 1 (Group actions)**.** Stare at a face of a cube. There are 4 rotations of the cube (including the identity) that preserve this face, and the cube has 6 faces. Note that $4 \times 6 = 24$. Stare at an edge of the cube. There are 2 rotations of the cube that preserve this edge, and the cube has 12 edges. Note that $2 \times 12 = 24$. Stare at a vertex of the cube. There are 3 rotations of the cube that preserve this vertex, and the cube has 8 vertices. Note that $3 \times 8 = 24$.

(a) Write down the definition of an **action** of a group G on a set X.

(b) Given an action $G \curvearrowright X$, the **stabilizer** of a point $x \in X$ is the subgroup $\operatorname{Stab}_G(x) := \{g \in G \mid gx = x\}$, and the **orbit** of x is $Gx := \{gx \in X \mid g \in G\}$. Show that if G is finite, then for each $x \in X$, we have $|G| = |Gx| \times |\operatorname{Stab}_G(x)|$.

(c) Why does the cube seem to like the number 24?

(d) What is the order of the group of orientation-preserving symmetries of the icosahedron?



Problem 2 (Dihedral groups and cyclic groups)**.** Recall that for every $n \geq 3$, the **dihedral group** D_n is defined as the isometry group of a regular n-gon in $\mathbb{R}^2$.

(a) Can we extend this definition to allow $n = 1$ and $n = 2$?

(b) Look up the definition of the **infinite dihedral group** D_∞. How does it generalize the finite dihedral groups?

(c) Given $n \geq 1$, is there a subset of $\mathbb{R}^2$ whose isometry group is isomorphic to the cyclic group $\mathbb{Z}/n$? Is there one whose isometry group is isomorphic to $\mathbb{Z}$?

Problem 3 (The commutator subgroup)**.** The **commutator** of two elements g, h of a group G is defined as

$$[g, h] := ghg^{-1}h^{-1}.$$

The **commutator subgroup** of G, denoted by $[G, G]$, is the subgroup spanned by all commutators.

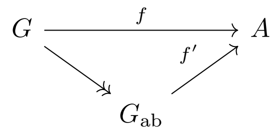
(a) Is the product $[g, h] \cdot [g', h']$ of two commutators again a commutator?

(b) Show that $[G, G]$ is a normal subgroup of G.

(c) The **abelianization** of G is the quotient

$$G_{\text{ab}} := G/[G, G].$$

Show that G_{ab} satisfies the following universal property: For every homomorphism $f\colon G \to A$ with A an abelian group, there is a unique homomorphism $f'\colon G_{\text{ab}} \to A$ such that the following triangle commutes



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where the map $G \twoheadrightarrow G_{\text{ab}}$ is the quotient map.

Problem 4 (Restricted direct products)**.** Given a family of groups $(G_i)_{i\in I}$, its **restricted direct product** is the group

$$\bigoplus_{i\in I} G_i := \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i \,\middle|\, g_i = 1 \text{ for all but finitely many } i \right\}.$$

(a) Prove that for $\mathbb{Q}_+$ the group of positive rationals with the usual multiplication, we have

$$\mathbb{Q}_+ \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}.$$

Hint: Rational numbers also like being decomposed into primes.

(b) Suppose I is countably infinite and every G_i is nontrivial and (at most) countable. Show that $\prod_{i\in I} G_i$ is uncountable, but $\bigoplus_{i\in I} G_i$ is countable.