Sheet 13

Due date: July 19

Problem 1 (Buildings as graphs)**.** Open Section 4 of the following article:

> L. Kramer, *Some remarks on proper actions, proper metric spaces, and buildings*, Advances in Geometry, vol. 22, no. 4, 2022, pp. 541-559. DOI: 10.1515/advgeom-2022-0018.

If you have difficulty accessing it on the publisher's website, you can also find it on the arXiv.

(a) Explain the connection between the notion of a building given by Kramer (Definition 4.1) and the one seen in class. How does his definition of i-adjacency differ from ours? What does that imply for galleries?

(b) Definition 4.4 introduces **Weyl-transitive** group actions, as well as the familiar strongly transitive actions on a building. Show that strongly transitive actions are Weyl-transitive.

(c) Does every building admit a Weyl-transitive group action?

Problem 2 ((BN2) refined)**.** Recall axiom (BN2) of a BN-pair for a group G: for every $w \in W, s \in S$, we have

$$BwB \cdot BsB \subseteq BwB \cup BwsB.$$

(a) Show that if $l(ws) > l(w)$, then $BwB \cdot BsB = BwsB$.

Hint: Use the fact that $g \in BwB$ if and only if, in the associated building, we have $\delta(C, gC) = w$, where C is the fundamental chamber.

(b) Show that if $l(ws) < l(w)$ and (BN3) is satisfied (so the associated building is thick), then $BwB \cdot BsB = BwB \cup BwsB$.

Hint: Show that if $g \in BwB$, there is a gallery from C to gC, such that the second-to-last chamber has the form g_0C with $g_0 \in BwsB$. Then pick a third chamber hC that is s-adjacent to g_0C and gC, and write $g = (gh^{-1})(hg_0)$.

Problem 3 (Opposition in $\mathrm{GL}_n(\mathbb{K})$)**.** Fix $n \geq 1$, let $\mathbb{K}$ be a field, and recall the building Δ of type A_n associated to the vector space $\mathbb{K}^{n+1}$, from Sheet 10, Problem 2.

(a) What is the top element w_0 for $\mathrm{Sym}(n+1)$, regarded as a Coxeter group of type A_n?

Hint: Use Sheet 3, Problem 2 (d).

(b) Show that the action $\mathrm{GL}_n(\mathbb{K}) \curvearrowright \Delta$ is strongly transitive.

(c) Recall that the Borel subgroup $B := \mathrm{Stab}(C)$ for the resulting BN-pair consists of the upper triangular matrices.

What is the parabolic subgroup opposite to B in the fundamental apartment?

Problem 4 (Your turn!)**.** (a) What is your favorite fact that you have learned in this lecture? Find a colleague who has not attended the course and tell them about it! It is a difficult but invaluable exercise to adapt the level of detail in your discourse to the background of your audience.

(b) Design an exercise as if you were preparing a problem sheet. You will see this forces you to be aware of what parts of the material you are confident about, and where you might have gaps in your knowledge. Here are some guidelines:

- Your problem should be solvable using the results from class and the "pre-requisite" basic knowledge. Other ingredients should be stated as black-boxes, or be part of the exercise. Sometimes it is also a good idea to recall known results your students might have forgotten (possibly via a hint).
- You do not want your students to get stuck and give up on the problem, so when in doubt, it is safer to drop that helpful hint. Besides, you, who are right now thinking intensely about the problem, are probably underestimating its difficulty. Hints are also helpful for when you want to remember your own solution!
- The hardest part of Mathematics is often not proving results, but conjecturing what might be true. To train the mind for treading in uncertainty, it is often a good idea to pose questions of the form "What is…", "Is it true that…", as opposed to "Prove that…".
- For inspiration, you may look at exercises from a book and adapt them to match the conventions and established results from the lecture. But another idea is to find a nice result that is not covered in the lecture, and turn the main steps of its proof into a sequence of exercises.
- When you are entirely out of ideas, one desperate last resort is to pass the task along to your students. If you phrase the assignment in a way that highlights the pedagogical value of stepping into the TA's shoes, nobody will notice your trick!