UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Sheet 12

Due date: July 12

Our goal for this exercise sheet is to prove the Theorem of Tits. Fix a building Δ of type (W, S) with a strongly transitive label-preserving action of a group G. Choose also an apartment Σ and a chamber $C \subset \Sigma$, and recall the subgroups

$$B := \mathrm{Stab}_G(C), \qquad N := \mathrm{Stab}_G(\Sigma), \qquad T := B \cap N.$$

Problem 1 (The torus)**.** Since the G-action is color-preserving, B fixes C pointwise. On the other hand, N does not fix Σ pointwise.

(a) What is the maximal subgroup of G that does?

Hint: Recall the standard uniqueness argument from Sheet 11, Problem 1.

(b) Conclude axiom (BN1), that T is normal in N, with $N/T \cong W$ (BN1).

For each $w \in W$, fix from now on a pre-image $\tilde{w}$ under the quotient $N \twoheadrightarrow W$.

Problem 2 (Stabilizers)**.** Given a special subgroup $W_0 \subseteq W$, let σ be the corresponding face of C.

(a) Show that

$$\mathrm{Stab}_G(\sigma) = \bigcup_{w \in W_0} B\tilde{w}B. \qquad (\star)$$

Hint: Given $g \in \mathrm{Stab}_G(\sigma)$, there is an apartment containing C and gC...

(b) Deduce $G = BNB$, so in particular $B \cup N$ generates G and (BN0) holds.

Problem 3 (Retractions, algebrically)**.** Recall the retraction $\rho_{\Sigma,C} \colon \Delta \to \Sigma$, which restricts to an isomorphism on each apartment containing C. Define $\rho\colon G \to W$ by setting $\rho(g)$ to be the element of W such that $\rho_{\Sigma,C}(gC) = \rho(g)C$.

(a) Prove that ρ maps each B-double coset $B\tilde{w}B$ to w.

Hint: Show first that for every $b \in B$, the restriction of $\rho_{\Sigma,C}$ to the apartment $b\Sigma$ is given by acting with b^{-1}.

(b) Conclude that $(\star)$ may be written as a disjoint union

$$\mathrm{Stab}_G(\sigma) = \bigsqcup_{w \in W_0} B\tilde{w}B,$$

and deduce in particular the Bruhat decomposition.

Problem 4 (s-adjacent double cosets)**.** Use the function ρ to show axiom (BN2): for every $w \in W$ and $s \in S$, we have

$$B\tilde{w}B \cdot B\tilde{s}B \subseteq B\tilde{w}B \cup B\widetilde{ws}B.$$

Hint: Given $g \in B\tilde{w}B$ and $h \in B\tilde{s}B$, use the fact that $\rho_{\Sigma,C}$ preserves s-adjacency to show that $\rho(gh)$ equals w or ws.

Problem 5 (The thick case)**.** Let $s \in S$. Show that if C has two s-adjacent chambers (other than C itself), then the condition in axiom (BN3) is satisfied:

$$sBs \nsubseteq B.$$

Hint: Show that if $g \in G$ moves C to an s-adjacent chamber that is not in Σ, then $g \in BsB$ but $g \notin sB$.