Sheet 10

Due date: June 28

Problem 1 (Coxeter complex of type A_n)**.** Fix $n \geq 1$ and let $X := (\partial\Delta^n)'$ be the barycentric subdivision of the boundary of the n-simplex. In this exercise, we will make precise and generalize the statement hinted at in Problem 3 (b) of Sheet 6.

(a) Denoting the vertices of Δ^n by $1, \dots, n+1$, the vertices of X are canonically identified with the strict non-empty subsets of $\{1, \dots, n+1\}$. Show that X is an $(n-1)$-dimensional chamber complex, and the assignment

$$\begin{aligned} \lambda\colon \mathrm{V}(X) &\to \{1, \dots, n\} \\ v &\mapsto |v| \end{aligned}$$

is a labeling of X.

(b) Exhibit a canonical bijection between the chambers of X and the elements of $\mathrm{Sym}(n+1)$. The chamber corresponding to id shall be denoted by C.

(c) Consider the action $\mathrm{Sym}(n+1) \curvearrowright X$ induced by vertex permutation on $\partial\Delta^n$. Describe the stabilizers of simplices contained in C.

Hint: Start with the stabilizers of vertices.

(d) Give an isomorphism of labeled chamber complexes $\Sigma \xrightarrow{\cong} X$, where Σ is the Coxeter complex of type A_n.

Problem 2 (A building from a vector space)**.** Let $n \geq 1$ and let V be an $(n+1)$-dimensional vector space over some field $\mathbb{K}$. Consider the simplicial complex Δ given as follows:

- vertices are strict nonzero subspaces of V, and
- k-simplices are sets $\{U_0, \dots, U_k\}$ where $U_0 \subset U_1 \subset \ldots \subset U_k$, also called **flags** of V.

In other words, Δ encodes the projective geometry $\mathbb{K}\mathrm{P}^n$.

(a) Show that Δ is a labelable chamber complex of dimension $n-1$. A chamber of Δ is called a **complete flag**.

(b) A **frame** for V is a set $\mathcal{L} = \{L_1, \dots, L_{n+1}\}$ of 1-dimensional subspaces whose sum is V. We denote by $A_{\mathcal{L}}$ the subcomplex of Δ spanned by the vertices that are sums of elements in $\mathcal{L}$. Show that $A_{\mathcal{L}}$ is a Coxeter complex of type A_n.

Hint: Use the description given by Problem 1.

We wish to show that the collection

$$\mathcal{A} := \{A_{\mathcal{L}} \mid \mathcal{L} \text{ is a frame for } V\}$$

is an atlas making Δ into a building of type A_n.

(c) Prove building axiom (B1), that is, every two chambers of Δ are contained in a common $A_{\mathcal{L}}$.

Hint: Argue by induction on n. Given complete flags of V

$$\mathcal{U}\colon U_1 \subset \ldots \subset U_n, \qquad \mathcal{W}\colon W_1 \subset \ldots \subset W_n,$$

note that U_1 must be in the desired $\mathcal{L}$. If W_i is the first subspace in $\mathcal{W}$ containing U_1, consider the projection $\pi\colon V \twoheadrightarrow V/U_1$ and the complete flags for V/U_1

$$\pi(U_2) \subset \ldots \subset \pi(U_n), \qquad \pi(W_1) \subset \ldots \subset \pi(W_{i-1}) = \pi(W_i) \subset \ldots \subset \pi(W_n).$$

Given a frame $\mathcal{L}_0$ for V/U_1 spanning these flags, how can we produce $\mathcal{L}$?

Recall the following variant of building axiom (B2):

(B2″) Given apartments $A, A' \in \mathcal{A}$ whose intersection contains a chamber C, there is an isomorphism $A \to A'$ fixing $A \cap A'$ pointwise.

(d) Let $\mathcal{L}, \mathcal{L}'$ be frames for V and let C be a chamber in $A_{\mathcal{L}} \cap A_{\mathcal{L}'}$. Prove that there is a unique isomorphism $\varphi\colon A_{\mathcal{L}} \to A_{\mathcal{L}'}$ fixing C pointwise.

Hint: There is a canonical one-to-one correspondence between isomorphisms $A_{\mathcal{L}} \xrightarrow{\cong} A_{\mathcal{L}'}$ and bijections $\mathcal{L} \xrightarrow{\cong} \mathcal{L}'$.

(e) Show that the above φ fixes $A_{\mathcal{L}} \cap A_{\mathcal{L}'}$ pointwise, proving (B2″).

Hint: Let C be given by the complete flag

$$(0 = U_0 \subset)\, U_1 \subset \ldots \subset U_n \,(\subset U_{n+1} = V),$$

and denote by L_i the element of $\mathcal{L}$ for which $U_i = U_{i-1} \oplus L_i$. Given a vertex U of $A_{\mathcal{L}} \cap A_{\mathcal{L}'}$, observe that, for every i, the following equivalences hold:

$$L_i \subseteq U \quad \iff \quad \dim\left(\frac{U \cap U_i}{U \cap U_{i-1}}\right) = 1 \quad \iff \quad \varphi(L_i) \subseteq U.$$

Problem 3 (The first BN-pair)**.** Fix a field $\mathbb{K}$, and let Δ be the building of type A_{n-1} associated to the vector space $\mathbb{K}^n$, as in Problem 2.

(a) Show that the assignment $M \cdot U := M(U)$ defines a simplicial label-preserving action of $G := \mathrm{GL}_n(\mathbb{K})$ on Δ.

(b) The **fundamental chamber** C of Δ is the one given by the complete flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_{n-1} \rangle,$$

where the e_i are the standard basis vectors. Describe the subgroup $B := \mathrm{Stab}_G(C)$.

This subgroup is usually designated by B because it is a Borel subgroup of G. Another mnemonic is that it stabilizes the fundamental chamBer.

(c) The **fundamental apartment** A is given by the frame $\{\langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_n \rangle\}$. Describe the subgroup $N := \mathrm{Stab}_G(A)$.

The notation N is due to the fact that it Normalizes the intersection $B \cap N$, but another mnemonic is that it stabilizes the fundamental apartmeNt.

Later in the lecture, we will see that the pair (B, N) contains a lot of structural information about G.