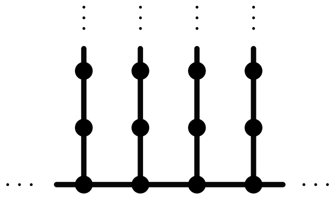
Sheet 9

Due date: June 21

Problem 1 (Infinite trees)**.** Recall that a building of type $\mathrm{I}_2(\infty)$ is the same as an infinite tree without leaves, such as the graph Γ below:



(a) List all subgraphs of Γ that are isomorphic to the Coxeter complex of type $\mathrm{I}_2(\infty)$.

(b) Find two different systems of apartments ("atlases") making Γ into a building.

Problem 2 (p-adic arithmetic)**.** For a fixed prime p, the field of p-adic rationals $\mathbb{Q}_p$ is the Cauchy completion of $\mathbb{Q}$ with respect to the p-adic norm (and operations induced from the usual addition and multiplication). Recall that p-adic rationals can be uniquely expressed in the form

$$\sum_{i=N}^{\infty} a_i p^i, \qquad \text{with } a_i \in \{0,\dots,p-1\},\ N \in \mathbb{Z} \text{ and } a_N \neq 0$$

Inside them, the p-adic integers $\mathbb{Z}_p$ are the sub-ring of p-power series with $N \geq 0$.

(a) Exhibit $-1 \in \mathbb{Z}_p$ as a p-power series of the above form.

(b) Show that for each $m \in \mathbb{N}_{\geq 1}$, its reciprocal $\frac{1}{m}$ lies in $\mathbb{Z}_p$ if and only if $p \nmid m$.

(c) Recall that two elements a, b in a ring R are called **associate** if there is a unit $u \in R$ such that $a = ub$. When are two elements of $\mathbb{Z}_p$ associate?

Problem 3 (Vertex stabilizer)**.** For a fixed prime p and $n \in \mathbb{N}$, recall that two bases of $\mathbb{Q}_p^n$ span the same lattice if and only if they are related by a matrix in $\mathrm{GL}_n(\mathbb{Z}_p)$. Let L be a lattice in $\mathbb{Q}_p^n$, and $A \in \mathrm{SL}_n(\mathbb{Q}_p)$. Show that if AL is homothetic to L, then $A \in \mathrm{SL}_n(\mathbb{Z}_p)$ (and hence $L = L'$).

Problem 4 (Common bases for lattices)**.** Fix a prime p and $n \in \mathbb{N}$.

(a) Show that every matrix $M \in \mathrm{GL}_n(\mathbb{Q}_p)$ can be factored as

$$M = A \begin{bmatrix} p^{a_1} & 0 & \cdots & 0 \\ 0 & p^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p^{a_n} \end{bmatrix} B,$$

where $a_1, \dots, a_n \in \mathbb{Z}$ and $A, B \in \mathrm{GL}_n(\mathbb{Z}_p)$.

Hint: Begin by choosing an entry of M with minimal p-valuation and use row an column operations to move it to the top-left corner. Can you perform Gaussian elimination using only matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$?

(b) Let L and L' be lattices in $\mathbb{Q}_p^n$. Show that there exist a $\mathbb{Z}_p$-basis $(b_1, \ldots, b_n)$ of L and integers $a_1, \ldots, a_n \in \mathbb{Z}$ such that $(p^{a_1} b_1, \ldots, p^{a_n} b_n)$ is a basis of L'.

(c) Prove that the integers $a_1, \ldots, a_n$ from (b) are unique up to reordering.

Hint: Compare L and L' to their maximal common sub-lattice $L \cap L'$.