Sheet 8

Due date: June 14

Problem 1 (The strong exchange property)**.** Fix a Coxeter system (W, S). Recall that the Cayley graph $\Gamma = \mathrm{Cay}(W, S)$, together with the set of reflections $T := \bigcup_{w \in W} wSw^{-1}$, is a reflection system – in particular, for each $t \in T$, the wall H_t separates Γ into precisely two connected components. Our goal is to prove:

> **Strong exchange property.** *Let $t \in T$ and let $s_1 \dots s_k$ be an S-word for some $w \in W$. If $l(tw) < l(w)$, then there is an index i such that $tw = s_1 \dots \hat{s}_i \dots s_k$. In other words, $t = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1$.*

(a) Let $t \in T$ and let $w = s_1 \dots s_k$ be an element of W. Show that if the vertices w and 1 of Γ lie on opposite sides of H_t, then for some index i we have

$$t = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1.$$

(b) Show that for every $t \in T$, the vertices 1 and t lie on opposite sides of H_t.

Hint: Write $t = s_1 \dots s_{m-1} s_m s_{m-1} \dots s_1$ and prove that if m is minimal, then the path in Γ given by this word crosses H_t exactly once.

(c) Given $w \in W$ and $t \in T$, show that 1 and w are on the same side of H_t if and only if t and tw are on the same side of H_t.

(d) Show that for every $w \in W$ and $t \in T$, we have $l(tw) < l(w)$ if and only if w and 1 lie on opposite sides of H_t. Deduce the strong exchange property.

Hint: Show first "$\Leftarrow$", then apply it with tw in place of w to prove "$\Rightarrow$".

Problem 2 (Putting order on the orders)**.** Given a Coxeer system (W, S), we defined the Bruhat order $\leq$, the left weak order $\leq_{\mathrm{L}}$, and the right weak order $\leq_{\mathrm{R}}$ on W.

(a) Draw Hasse diagrams for the Coxeter group of type $\mathrm{I}_2(4)$ with respect to each of these orders.

(b) Explain why we did not need to define a "left Bruhat order".

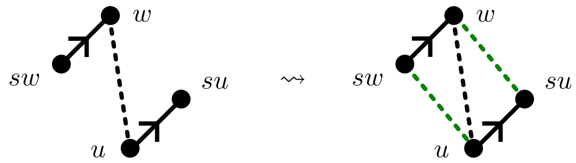
Hint: Do you know a description of the Bruhat order that does not distinguish left and right multiplication?

Problem 3 (Reach the top)**.** A poset $(P, \leq)$ is **directed** if every two elements have a common upper bound:

$$\forall x, y \in P \quad \exists z \in P \quad x \leq z \land y \leq z.$$

(a) For a Coxeter system (W, S), prove that the Bruhat order satisfies:

> **Lifting Property:** If $u < w$ are elements of W and $s \in S$ satisfies $sw < w$ and $u < su$, then we have $u \leq sw$ and $su \leq w$. In the Bruhat graph:



Hint: Start with a reduced word for sw and apply the subword property to the inequality $u < w$.

(b) Use the lifting property to show that $(W, \leq)$ is a directed poset.

Hint: To find a common upper bound for w, u, argue by induction on $l(w) + l(u)$. Reduce to a lower case by choosing $s \in S$ such that $su < u$. Having an upper bound z for su, w, consider the two cases $sz < z$ and $sz > z$.

(c) Conclude that if W is finite, then it has a unique maximal element w_0.

Problem 4 (The top element)**.** Let (W, S) be a finite Coxeter system and $w_0 \in W$ the maximal element.

(a) Show that $w_0^2 = 1$.

(b) Prove that for every $w \in W$, we have

$$l(w_0 w) = l(w_0) - l(w)$$

Hint: For "$\leq$", argue by induction on $l(w_0) - l(w)$.

(c) Use the strong exchange property to show that $l(w_0) = |T|$, where T is the set of reflections.