# 10 A building associated with a BN-pair

Recall BN-pairs

**10.1 Def** Let $G$ be a group with a BN pair.
For each $i \in I$ define
$$P_i := B \sqcup B s_i B.$$

**10.2 Lemma**: $P_i$ is a subgrp of $G$.

**Proof**: Check grp axioms using (BN2):
$$BwB \cdot Bs_iB = BwBs_iB$$
$$\subseteq BwB \cup Bws_iB$$
with $w = \mathbb{1}$ and $BwB = B$. $\square$

The following theorem provides a method to construct a building from a BN pair. This is the converse to the theorem we've seen that constructed BN-pairs from buildings using strongly transitive actions.

## 10.3 Thm The building of a BN pair

Let $G$ be a group with subgroups $B, N$ s.th. (BN0) – (BN2) hold.
Then there exists a building $\Delta = \Delta(B,N)$ of type $(W,S)$ constructed as follows:

i) Put $\mathcal{C} := \{ gB \mid g \in G \}$ the set of chambers.

ii) $i$-adjacency is given by
$$gB \sim_i hB \iff g^{-1}h \in P_i$$

A $W$-valued distance is given by
$$\delta(gB, hB) := w \quad \text{if} \quad g^{-1}h \in BwB.$$

If in addition (BN3) holds $\Delta$ is thick.

Now let $C_0 = B$, $A_0 = \{ wC_0 \mid w \in W \}$ and define $\mathcal{A} := \{ gA_0 \mid g \in G \}$ then $\mathcal{A}$ is a system of apartments and $G$ acts on the building $\Delta$ transitive on pairs in $\mathcal{C}$ of a same $W$-distance with $B$ the pointwise stabilizer of $C_0$ and $N$ stabilizing the Apartment $A_0$ (*)

**Proof:** The fact that ii) is an equivalence relation is clear from the construction.

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construction.

We need to prove:

a) $\delta$ is indeed a $W$-distance

b) the stament (*) which includes:

- $A_0$ is an apartment
- assertions on the action

<u>Towards a)</u>:

We need to prove the following:

Let $s_{i_1} \dots s_{i_k}$ be a reduced word for $w$.

Then: there exists a gallery of type $(i_1, \dots, i_k)$ from $gB$ to $hB$ $\Longleftrightarrow$ $\delta(gB, hB) = s_{i_1} \cdots s_{i_k} = w$

By construction we may assume that one of the chambers equals $B$ and we only consider the case where

$$c = B \quad \text{and} \quad d = gB.$$

<u>We first prove „$\Leftarrow$"</u>

So $\delta(c,d) = w = \underbrace{s_{i_1} \cdots s_{i_k}}_{\text{reduced!}}$.

We argue by induction on $k$:

<u>$k=1$:</u> in this case $c$ and $d$ are,

by definition, adjacent and the

$k=1$: in this case $c$ and $d$ are, by definition, $i_1$-adjacent and the gallery in question is just $(c,d)$.

$k \to k+1$: suppose $\delta(c,d) = w = s_{i_1} \cdots s_{i_k} s_{i_{k+1}}$

Hence $w = s_{i_1} \cdot w'$ with $w' = s_{i_2} \cdots s_{i_k}$.

We use the $B$ left-action to reduce to the case that $d = wB$ as follows:

$d = gB$ for some $g \in G$.
Bruhat decomp. $\Rightarrow$ $g \in BwB$ for some $w$.
Hence $gB = bwB$ for some $b \in B$.
Acting on $c, d$ by $b^{-1}$ from the left yields $b^{-1}c = b^{-1}B = B$ and
$b^{-1}d = b^{-1}bwB = wB$.

and $b^{-1}c = c$ and $b^{-1}d$ are two chambers at the same $W$-distance.

W.l.o.g. we may thus assume that $c = B$ and $d = wB$ for some $w \in W$.

But then $sd = swB = w'B$.
With $l(w') = k$.
By induction there is thus a gallery $\gamma'$ of type $i_2 \ldots i_k$ from $c$ to $sd$.

of type $i_2 \dots i_k$ from $c$ to $sd$.

Suppose $\gamma' = (c = c_0, c_1, \dots, c_k = sd)$.

The gallery $\gamma'' = (sc, c_0, c_1, \dots, c_k)$ is then from $sc$ to $c_k = sd$. As $w$ is reduced this gallery is minimal.

Hence $s.\gamma'' = (c, sc_0, sc_1, \dots, sc_k)$ is a gallery from $c$ to $d$ of type $w$. $\quad sc_k = s \cdot sd = d$

Now prove „$\Rightarrow$"

Suppose $\gamma = (c = c_0, c_1, \dots, c_k = d)$ is a minimal gallery from $c$ to $d$.

Again $w = s_{i_1} \dots s_{i_k}$ and put

$w' := s_{i_2} \dots s_{i_k}$ with $s_{i_1} w' = w$.

W.l.o.g. $c_1 = s_{i_1} \cdot c_0 = s_{i_1} c$

Hence $s\gamma = (sc, sc_1, \dots, sc_k)$

$= (sc, \underbrace{c, \dots, sd}_{})$

And we have the sub-gallery from $c$ to $d$ of type $(i_2, \dots, i_k)$.

Again by induction on $k = \ell(w)$

$sd \subset Bw'B$.

$\underset{\cdot s}{\Rightarrow} d \subset s \cdot Bw'B \subset BswB$.

$\underset{\cdot s}{\Rightarrow}$ $d \subset s \cdot Bw'B \subset BswB$.

uses: $\ell(sw) > \ell(w)$

$\Rightarrow BsBwB = BswB$

$\Rightarrow \delta(c, d) = s \cdot w' = w$. Hence a).

<u>Towards b)</u>

Let $\Sigma = \{nC_0 \mid n \in N\}$. ($gB$)

By (BN1) the set $\Sigma$ equals the set $A_0$.

$T = B \cap N \triangleleft N$ $\quad A_0 = \{wC_0 \mid w \in W\}$

$W = N/T$

By definition $\delta(gB, hB) = g^{-1}h$

and thus $\Sigma$ is isometric to $W$ via

$wC_0 \longmapsto w$.

Hence $\Sigma = A_0$ is isomorphic to the Cayleygraph of $(W, S)$ and hence an apartment.

Multiplication on the left induces $W$ on $\Sigma = A_0$.

The fact that it is an atlas is clear from the construction.

Moreover, if $\delta(B, gB) = w$, then $gB = bwB$ for some $b \in B$

So $b$ sends $(B, wB)$ to $(B, gB)$.

to send $(hB, gB)$ at distance $w$ to any other pair $(h'B, g'B)$ at the same distance proceed as follows:

$$(hB, gB) \overset{\cdot h^{-1}}{\rightsquigarrow} (B, h^{-1}gB) \underset{\text{as above}}{\overset{\text{some}}{\underset{b}{\rightsquigarrow}}} (B, wB)$$

$$\|$$

$$(h', g'B) \overset{\cdot (h')^{-1}}{\rightsquigarrow} (B, h'^{-1}g'B) \underset{\text{some } b'}{\rightsquigarrow} (B, wB)$$

Transitivity follows. Hence b) $\square$

<u>10.4 Rmk:</u> $N$ might not be the full stabilizer of $A_0$.

If in addition $T = \bigcap_{w \in W} wBw^{-1}$ then the BN-pair is called <u>saturated</u> and $N = \mathrm{Stab}_G(A_0)$.

## <u>10.5 Geometric interpretation of axiom (BN2)</u>

recall: $gB$, $g \in G$ are the chambers

$BwB$ = chambers in the $B$-orbit of $wB$

represent the chambers in $BwB$ by

represent the chambers in $BwB$ by galleries from $1 \cdot B$ to $bwB$.

(BN2) means:

if a gallery from $B$ to a $c \in BwB$ is followed by a gallery of type $i$, then the final resulting chamber of the elongated gallery lies in $BwB$ or in $Bws_iB$.

We will now construct a simplicial realization of the building in this theorem using parabolic subgroups:

**Def 10.6** standard parabolic subgroups:

Let $G$ be a group with a BN-pair. For each $J \subseteq I$ let $W_J = \langle s_j \mid j \in J \rangle$.

$(W_J, J)$ is a Coxeter system

Put

$$P_J := \bigsqcup_{w \in W_J} BwB.$$

We call such $P_J$ <u>standard parabolic subgroups</u> of $G$.

A <u>parabolic subgrp</u> is a conjugate of

A *parabolic subgrp* is a conjugate of some $P_J$ for $J \subseteq I$.

Def 16.7 $P_\emptyset = B$ called *standard Borel subgroup*

One can show:

Thm 16.8

$G$ a group with a BN-pair, then

(1) If $B \leq P \leq G$, then $P = P_J$ for some $J \subseteq I$.

(2) $\forall J, K \subseteq I$, $P_J \cap P_K = P_{J \cap K}$ and

$$\langle P_J, P_K \rangle = P_{J \cup K}$$

(3) $\forall J \subseteq I$ the group $P_J$ is the stabilizer of the $J$-residue of $\Delta = \Delta(B, N)$ containing $B$.
In particular, for all $i \in I$ the group $P_i$ is the stabilizer of the $i$-panel of $\Delta$ containing $B$. o.Bew

represents a chamber in this chamber complex.

Construction of the poset which gives the simpl. cplx:

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<u>Theorem 10.9</u>

Consider the poset of cosets of proper standard parabolic subgroups of $G$, ordered by inclusion:

$$\{gP_J \mid g \in G,\ J \subseteq I\}.$$

The simplicial complex which is the geometric realization of this poset "is" the building $\Delta = \Delta(B,N)$.

<u>Proof</u>: This is a consequence of the orbit stabilizer theorem combined with the Theorem on parabolic subgroups above. $\square$

## Orbit-stabilizer theorem and Burnside's lemma [ edit ]

Orbits and stabilizers are closely related. For a fixed *x* in *X*, consider the map $f : G \to X$ given by $g \mapsto g \cdot x$. By definition the image $f(G)$ of this map is the orbit $G \cdot x$. The condition for two elements to have the same image is

$$f(g) = f(h) \iff g \cdot x = h \cdot x \iff g^{-1}h \cdot x = x \iff g^{-1}h \in G_x \iff h \in gG_x.$$

In other words, $f(g) = f(h)$ *if and only if* *g* and $h$ lie in the same coset for the stabilizer subgroup $G_x$. Thus, the fiber $f^{-1}(\{y\})$ of *f* over any *y* in *G*·*x* is contained in such a coset, and every such coset also occurs as a fiber. Therefore *f* induces a *bijection* between the set $G/G_x$ of cosets for the stabilizer subgroup and the orbit $G \cdot x$, which sends $gG_x \mapsto g \cdot x$.[11] This result is known as the *orbit-stabilizer theorem*.

If *G* is finite then the orbit-stabilizer theorem, together with Lagrange's theorem, gives

$$|G \cdot x| = [G : G_x] = |G|/|G_x|,$$

in other words the length of the orbit of *x* times the order of its stabilizer is the order of the group. In particular that implies that the orbit length is a divisor of the group order.

Remark 10.10:

1) $\left\{\begin{array}{l}\text{chambers in}\\ \text{the building}\end{array}\right\} \mathrel{\hat{=}} \{g\overset{=P_\emptyset}{B} \mid g \in G\}$

$\{\text{faces of chambers}\} \mathrel{\hat{=}} \{gP_J \mid \emptyset \neq J \subseteq I,\ g \in G\}$

$\left\{\begin{array}{c}\text{codim 1-faces of}\\ \text{chamber } gB\end{array}\right\} \mathrel{\hat{=}} \{gP_i \mid i \in I\}$

2) Each apartment in the building is a Coxeter complex and is hence naturally isomorphic to the simplicial realization of the poset
$$\{wW_J \mid w \in W,\ J \subseteq I\}$$

Remark 10.11: Many buildings for a grp

In case $G = Gl_n(\mathbb{K})$ or $Sl_n(\mathbb{K})$ over a field with valuation $\nu : \mathbb{K} \to \mathbb{Z}$ we are hence in the following situation:

(i) $G$ has a BN-pair
($B = \begin{pmatrix} * & * \\ & \ddots \\ & & * \end{pmatrix}$ and $N$ = monomials)
from which we can construct a (spherical) building $\Delta$ with $W \cong Sym(n)$ i.e. the type of $\Delta$ is $A_{n-1}$

(ii) the $l_2$-tree construction generalizes to this setting and one can construct (via adjacency of homothety classes of lattices again) an affine building $X$ of type $\tilde{A}_{n-1}$. Associated with this affine building is another (different) BN-pair for $Gl_n(\mathbb{K})$ resp. $Sl_n(\mathbb{K})$.

(ii) since $\mathbb{K}$ is a valued field we also have the residue field $k$ wrt this valuation $\nu$ and thus obtain valuation $\nu$ and thus obtain another spherical building $\Delta^0$ of type $A_{n-1}$ and $W \cong Sym(n)$ via the BN-pair $B = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}$, $N$ = monomials for $G$ over $k$, i.e. $GL_n(k)$, resp $SL_n(k)$.

<u>The geometric situation</u> is as follows:

$\Delta$ appears as the boundary at infinity of the building $X$ while $\Delta^0$ is isomorphic to links (at special vertices) inside the bldg $X$, e.g. the vertex corresponding to the lattice class $[\langle e_1, e_2, \dots, e_n \rangle_{\mathcal{O}}]$.

We have only seen glimpses of why this is true and how all that stuff works and there is much more to explore and discover.

Thanks for a
fun semester!