9 Buildings as chamber complexes

First: chamber complexes:

9.1 Def. Let I be a finite set and $\mathcal{C}$ a possibly infinite set. Elements of $\mathcal{C}$ are called <u>chambers</u> and <u>$\mathcal{C}$ is a chamber system over I</u> if the following holds:

each $i \in I$ determines an equivalence relation on $\mathcal{C}$, denoted $\sim_i$.

We call $x, y \in C$ <u>i-adjacent</u> if $x \sim_i y$.
We say they are <u>adjacent</u> if $\exists\, i \in I$ s.th. $x \sim_i y$.

You can think of a chamber system as an edge colored graph with vertex set $\mathcal{C}$ and colors $i \in I$ of the edges. Note that an edge may have multiple colors.

9.2 Example 1

(W,S) Coxeter system, $S = \{s_i \mid i \in I\}$.
Put $\mathcal{C} = W$, I as above.
Define $x \sim_i y$ for all $x, y \in W$ if

$$x^{-1}y \in W_{\{s_i\}} = \langle s_i \rangle$$

i.e. $x \sim_i y$ iff $x = y$ or $x = ys_i$.

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This yields a chamber system.

9.3 Example 2

Let G be a group, $B \lneq G$ a proper subgrp.
For each $i \in I$ ← some index set, let P_i be
a group $B \lneq P_i \lneq G$.

See e.g. the parabolics we had considered in the context of the Fano plane () and ()

Put $\mathcal{C} := \{gB \mid g \in G\}$.
$= G/B$

Define $gB \sim_i hB :\Leftrightarrow gP_i \sim hP_i$
$\Leftrightarrow g^{-1}h \in P_i$.

Each i-equivalence class then contains
$[P_i : B]$ elements.

This is also a chamber system.

Note: Example 2 is a special case of 1
by putting $G := W$, $B := W_\emptyset = \{1\}$
and $P_i := \langle s_i \rangle = W_{\{s_i\}}$.

9.4 Def. A *gallery* in a chamber system $\mathcal{C}$ over I
is a sequence of chambers c_j $j = 0, \dots, n$ s.th.
c_j is adjacent to c_{j-1} $\forall j \geq 1$.
We say the gallery is *non-stammering* if

We say the gallery is <u>non-stammering</u> if
$c_j \neq c_{j-1}$ $\forall j = 1 \dots n$.
Given a gallery $(c_0, c_1, \dots, c_n)$ and indices
$(i_1, \dots, i_n)$ we call the latter <u>a type of the gallery</u> if $c_j \sim_{i_j} c_{j-1}$ $\forall j = 1, \dots, n$.

<u>Note</u>: in general a gallery may have more than one type.
A non-stammering gallery in Example 1 has a unique type.

9.5 <u>Def</u>. Let $\mathcal{C}$ be a chamber system over I and J a subset of I.
A <u>J-residue</u> is a J-connected component of the chamber system $\mathcal{C}$, i.e. an inclusion-maximal subset of $\mathcal{C}$ s.th. any pair of chambers in the subset are J-connected.
We refer to $\{i\}$-residues as <u>panels</u> or, more precisely <u>i-panels</u>.

Fits with the notion of a panel in a Cox. cplx.

<u>Back to Example 1</u>

Each gallery in W corresponds to a word in S and may be identified with a path in

Each gallery in W corresponds to a word in S and may be identified with a path in $\mathrm{Cay}(W,S)$ which is the same as the dual graph of the Coxeter complex $\Sigma = \Sigma(W,S)$.

If $(w_0, w_1, \dots w_k)$ is a gallery of type $(i_1, \dots, i_k)$ then the corresponding word is $(s_{i_1}, \dots s_{i_k})$ with $w_j = w_{j-1} s_{i_j}$ $\forall j = 1, \dots, k$.
The vertices in the Cayley graph are $w_0, w_1, \dots w_k$.
Since S generates W this chamber system is connected.
The i-panels all consist of two i-adj. vertices, corresponding to the edges in $\mathrm{Cay}(W,S)$ labeled by s_i.

For some $J \subseteq I$ the J-residues are the left-cosets of $\langle s_j \mid j \in J \rangle$ in G.

<u>Back to Example 2:</u>
The chamber system is connected iff the P_i generate G.
J-residues are left-cosets of $\langle P_j \mid j \in J \rangle$.

9.6 <u>Def.</u> Let $\mathcal{C}$ be a chamber system over I and let (W,S) be a Coxeter system and let (W,S) be a Coxeter system with $S=\{s_i \mid i\in I\}$. Then:

A <u>W-valued distance function</u> on $\mathcal{C}$ is a map $\delta: \mathcal{C}\times\mathcal{C} \longrightarrow W$ s.th.

for all <u>reduced</u> words $s_{i_1}\cdots s_{i_k}$ and all $x,y\in\mathcal{C}$ the following holds:

$$\delta(x,y) = s_{i_1}\cdots s_{i_k} \iff \exists \text{ a gallery from } x \text{ to } y \text{ in } \mathcal{C} \text{ of type } (i_1,\ldots,i_k).$$

Not every chamber system admits a W-distance. We want to view buildings as chamber systems. They will have a W-distance.

<u>Back to Example 1</u>

For $x,y\in W$ put $\delta(x,y):=x^{-1}y$.

This defines a W-distance on $\mathcal{C}=W$.

The word-metric d_S and word length ℓ_S on W satisfy:

$$d_S(x,y) = \ell_S(x^{-1}y) = \ell_S(\underbrace{\delta(x,y)}_{\text{refined version of } d_S!}).$$

9.7 <u>Rmk</u> reduced words is a necessary assumption in the def of W-distance!

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if $x \sim_i y \sim_i z$ $\quad x \neq y$ and $y \neq z$

then (x, y, z) is a gallery of type (i, i).

The word $s_i s_i$ is not reduced but if the i-panel has more than 2 elements we could have $x = z$ or $x \neq z$.

If $x \neq z$ and non-reduced galleries are allowed we will have

$$\delta(x,z) = 1 \quad \text{using } (x,y,z) \text{ of type } (i,i)$$

and

$$\delta(x,z) = s_i \quad \text{since } x \sim_i z.$$

↯ we want this to hold only if $x = z$

↯ we want δ to be a function.

9.8 Thm 1 (Buildings are chamber systems)

Every building X of type (W, S) with $S = \{s_i \mid i \in I\}$ carries the structure of a chamber system $\mathcal{C}$ over I equipped with a W-valued distance function δ.
Every panel in $\mathcal{C}$ has at least two chambers.

Here $\mathcal{C} = \{$ max simplices in $X \}$

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The converse also holds:

9.10 <u>Thm 2</u> (Certain chamber systems are buildings)

Let (W,S) be a Coxeter system with $S = \{s_i \mid i \in I\}$. Then every chamber system $\mathcal{C}$ over I equipped with a W-distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$ in which every panel has at least two chambers is a building.

In the sense that $\mathcal{C}$ = Chambers of a simpl. complex X and we find an atlas $\mathcal{A}$ s.th. $(X, \mathcal{A})$ satisfy the simplicial definition.

<u>Proof of Thm 1:</u>

For details see Ronan: Chapter 3, Section 3

X is a building of type (W,S). So apartments are copies of Coxeter complexes of (W,S).

Put $\mathcal{C} := \{c \mid c \text{ chamber in } X\}$ max. simplex

Recall that panels in a Coxeter complex are codim 1 faces of pairs of chambers and each panel is colored by some $s_i \in S$.

Notion of i-equivalence:

For chambers $c, c' \in \mathcal{C}$ define

$$c \sim_i c' \iff \begin{cases} c = c' \\ c \neq c' \text{ and } c \cap c' \text{ is a panel colored } s_i \end{cases}$$

This yields an equivalence relation on $\mathcal{C}$.
By construction each i-equivalence class has at least 2 chambers.

Next we need to define a <u>W-valued distance function $\delta: \mathcal{C} \times \mathcal{C} \to W$</u>:

Let A be an apartment and $\overset{\subset\mathcal{C}}{\mathcal{C}_A}$ be the set of chambers in A.
Then each $c \in \mathcal{C}_A$ is of the form $wW_\emptyset$ for some $w \in W$.
Define $\delta_A : \mathcal{C}_A \times \mathcal{C}_A \longrightarrow W$ by putting

$$\delta_A(c, c') = x^{-1}y$$

where $c \mathrel{\hat{=}} xW_\emptyset$ and $c' \mathrel{\hat{=}} yW_\emptyset$.
We have checked above in the example that this yields a W-distance on $\mathcal{C}_A$.

Let now $c, c' \in \mathcal{C}$. By axiom (B1) there exist an apartment A containing c and c'.

exist an apartment A containing C and C'.
Put $\delta(C,C') := \delta_A(C,C')$.

We need to prove that this is independent of the chosen apmt A.

Let A_1 and A_2 be two apartments containing C and C'. Let $\varphi: A_1 \to A_2$ be an isomorphism fixing $A_1 \cap A_2$ pointwise. Such a φ exists by (B2).
In particular $\varphi(C) = C$ and $\varphi(C') = C'$.

Since δ_{A_1} is a W-distance there exists a gallery γ from C to C' in A_1. The type of γ is $(i_1, \dots, i_k)$ where $s_{i_1} \cdots s_{i_k}$ is a word for $\delta_{A_1}(C,C')$.

The isomorphism φ has to preserve types of panels.
Hence $\varphi(\gamma)$ is a gallery from $\varphi(C) = C$ to $\varphi(C') = C'$ of the same type as γ.
Moreover $\varphi(\gamma)$ is contained in A_2. So

$$\begin{aligned}\delta_{A_2}(\varphi(C), \varphi(C')) = \delta_{A_2}(C,C') &= s_{i_1} \cdots s_{i_k} \\ &= \delta_{A_1}(C,C')\end{aligned}$$

and the map δ is well defined.

To complete the proof that δ is a W-distance function we need to

W-distance function we need to prove the following:

Let γ be a gallery from c to c' of type $(i_1, \dots, i_k)$ with $s_i \dots s_k$ reduced.
Then $\delta(c, c') = s_1 \dots s_k$.

We prove this by induction:

$k=1$: Consider an apartment A containing c and c'.
Since $k=1$ we have $c \sim_i c'$
and hence $\delta(c, c') = s_i = s_{i_1}$.

$k \geq 2$: Let c'' be the second last chamber in the gallery γ.
Then $c'' \sim_i c'$ and $c'' \cap c'$ is an i-panel in some apartment A containing c and c'.
By induction $\delta(c, c'') = s_{i_1} \cdots s_{i_{k-1}}$
and there is a gallery γ' of type $(i_1 \dots i_{k-1})$ connecting c and c'' in some Apartment A'.
Take $\varphi : A' \to A$ be an isomorphism fixing $A \cap A'$. Then $\varphi(\gamma')$ is a gallery from $\varphi(c)$ to $\varphi(c'')$ in $\mathcal{C}_A$.
The map φ fixes $c'' \cap c'$.

the map φ fixes $c'' \cap c'$.
We may hence concatenate to γ' the additional chamber c' which is i-adjacent to c'' to obtain a gallery of type $(i_1 \dots i_{k-1}, i_k)$ with $i_k := i$, where $c' \sim_i c''$.
Hence the claim.

With this we conclude the statement of Thm 1. $\square$

To prove Thm 2 we need to work a lot harder since we need to construct an apartment system first.

Proof of Thm 2

We need to introduce apartments first in order to show that such a chamber complex is a building.
We use the following:

9.11 **Def.** Let Δ be a chamber system with W-distance, let X be a subset of W and $\alpha : X \to \Delta$ a map. Then α is a W-isometric embedding if $\forall x, y \in X$

and $\alpha: X' \to \Delta$ a map. Then α is a
<u>W-isometric embedding</u> if $\forall x, y \in X'$
$$\delta(\alpha(x), \alpha(y)) = x^{-1}y.$$
An <u>apartment</u> in Δ is the image under any W-isometric embedding of W.

Write $\mathcal{A}$ for the collection of all apartments in $\mathcal{C}$.
We first prove:

9.12 <u>Prop.</u> Let $\mathcal{C}$, (W,S), I and δ be as in Thm 2.
Let $X \subseteq W$ be a subset and $\alpha: X' \to \mathcal{C}$ a W-isometric embedding of X'.
Then α extends to a W-isometric embedding from $W \to \mathcal{C}$.

<u>Proof</u>

every partially ordered set in which every chain has an upper bound has at least one maximal element

By Zorn's lemma it is enough to prove that α extends to a strictly larger superset of X' inside W.
If $X' = \emptyset$ we are done (extension to a 1-elem. set is possible).
There is nothing to prove if $X' = W$.
So suppose $\emptyset \subsetneq X' \subsetneq W$.
Then there exists an $x_0 \in X'$ and an $s_i \in S$

Then there exists an $x_0 \in X$ and an $s_i \in S$ s.th. $x_0 s_i \notin X$.

We may pre-compose α by the left-multiplication by x_0^{-1} and so obtain another W-isometric embedding of X.

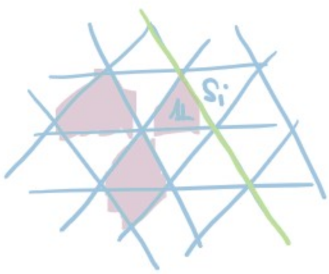
We may hence assume w.l.o.g. that $x_0 = 1\!\!1$ and $x_0 s_i = s_i \notin X$.

We now extend α to a map

$$X \cup \{s_i\} \longrightarrow \mathcal{C}.$$

Case 1: $\ell(s_i x) > \ell(x)$ for all $x \in X$.

This is the case when in $\Sigma(W,S)$ all $x \in X$ lie on one side of the hyperplane spanned by the panel between the $1\!\!1 = x_0$ and s_i chamber.



Since every panel of $\mathcal{C}$ contains at least two chambers we define $\alpha(s_i)$ to be a chamber of $\mathcal{C}$ which is i-adjacent to $\alpha(1\!\!1)$ but not equal to $\alpha(1\!\!1)$.

[illegible]

Case 2: $\exists\, x_1 \in X$ s.th. $\ell(s_i x_1) < \ell(x_1)$

We use exchange condition:

> if $s_1 \ldots s_k$ is a reduced word for w and $s \in S$, then either $\ell(sw) = k+1$ or there exists j s.th. $w = s s_1 \ldots \hat{s}_j \ldots s_k$

to see that there exists a reduced word for x_1 starting with s_i, e.g. the word $s_i s_{i_2} \cdots s_{i_k}$.

Since $\mathbb{1}^{-1} x_1 = x_1$ there exists a gallery of type $(i, i_2, \ldots, i_k)$ in $\mathcal{C}$ from $\alpha(\mathbb{1})$ to $\alpha(x_1)$.

Define $\alpha(s_i)$ to be the second chamber in this gallery. This is well defined since there is at most one gallery of a given type between a fixed pair of chambers. See Ronan (3.1)

Remains to check:

In both cases: $\delta(\alpha(s_i), \alpha(x)) = s_i^{-1} x$
$= s_i x$

for all $x \in X$.

$\square$ Prop.

Using this Proposition we may prove, that $\mathcal{C}$ is a building.

First construct a simplicial complex $\hat{X}$ by taking $\hat{X} = \bigsqcup_{A \in \mathcal{A}} \Sigma_A \Big/ \substack{\text{gluing} \\ \text{inherited} \\ \text{from } \mathcal{C}}$

i.e. for every apartment in $\mathcal{C}$ take a copy Σ_A of $\Sigma(W, S)$ together with a fixed W-isometric identification of elements of the apartment with max simplices in Σ_A.

Identify two maximal simplices in the quotient in a color-preserving way whenever they agree as elements of $\mathcal{C}$.

The resulting complex X has the chamber graph of $\mathcal{C}$ as its dual graph.

Colors of edges match the labels of codim 1 faces of chambers in X.

We freely walk between these two viewpoints and prove the axioms using the chamber complex language.

<u>To prove (B0):</u>

By construction and the Prop. the set $\mathcal{C}$

By construction and the Prop. the set $\mathcal{C}$ is the union of W-isom. embeddings of W.

To prove (B1):

Let c, d be two chambers at distance $w = \delta(c,d)$. Then take $X' := \{1\!\!1, w\}$ and $\alpha : \begin{array}{l} 1\!\!1 \longmapsto c \\ w \longmapsto d \end{array}$. This is W-isometric

as $\delta(1\!\!1, w) = 1\!\!1^{-1} w = w = \delta(c,d)$.

By the proposition α extends to all of W yielding an apartment containing c, d.

To prove (B2):

Suppose $\alpha : W \to \mathcal{C}$ and $\beta : W \to \mathcal{C}$ are two W-isometric embeddings such that the apartments $A = \mathrm{im}(\alpha)$ and $B = \mathrm{im}(\beta)$ both contain chambers c, d in $\mathcal{C}$.

We may pre-compose both α and β with elements in W to obtain W-isom. embeddings $\tilde{\alpha}, \tilde{\beta}$ that satisfy

$$\tilde{\alpha}(1\!\!1) = \tilde{\beta}(1\!\!1) = c \quad \text{and} \quad \tilde{\alpha}(w) = \tilde{\beta}(w) = d$$

where $w = \delta(c,d)$.

Then $\tilde{\beta} \circ \tilde{\alpha}^{-1} : A \to B$ induces the

Then $\tilde{\beta} \circ \tilde{\alpha}^{-1} : A \to B$ induces the desired isomorphism. $\square$