18 BN pairs

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8 Buildings from groups (with valuation)

Goal: Introduce an algebraic notion of a BN-pair & a way to construct from it a building (spherical or affine)

Def. 8.1 (BN-pair)

Let G be a group. A BN-pair (or Tits system) for G is a pair of subgroups (B,N) s.th. the following holds:

(BN0) G is generated by B and N.

(BN1) $T = B \cap N$ is normal in N and $W = N/T$ is a Coxeter group with distinguished Coxeter generating set $S = \{ s_i \mid i \in I \}$.

(BN2) $\forall\, w \in W$ and all $s_i \in S$ we have

$$BwB \cdot Bs_iB = BwBs_iB \subseteq BwB \cup Bws_iB$$

(BN3) $\forall\, i \in I$

$$s_i B s_i^{-1} = s_i B s_i \neq B$$

(cont.) $\forall i \in I$

$$s_i B s_i^{-1} = s_i B s_i \neq B.$$

Rmk: The set S will be uniquely determined by these axioms.

Lemma 8.2

(BN2) and (BN3) are well defined.

Proof

This follows from the fact that each $w \in W$ is of the form $n \cdot T$ and T is a subgroup of B. $\square$

Rmk 1) Alternative formulation for (BN2):

$$B s_i B w B \subseteq BwB \cup B s_i w B.$$

3) (BN3) is not always needed and will correspond to the fact that the building constructed from the BN-pair is thick.

Example 8.3 Recall: Fano plane

$G = GL_3(\mathbb{F}_q) \quad q = 2$

$B := \mathrm{Stab}(C_0) = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \in G \right\}$

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$$C_0 = \{ \langle e_1 \rangle, \langle e_1, e_2 \rangle \}$$

$N := \mathrm{Stab}(A_0) =$ monomial mxes

apmt corr. to std basis

This B and N forms a BN-pair.
(Requires some checking)

(BN0) clear by construction

(BN1) : $T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \in G \right\}$ torus

$= B \cap N$

is normal in N

$N/T = W$ = monomial mxes with nonzero entries $= 1$

$\cong \mathrm{Sym}(3)$

$$s_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$\rightsquigarrow S = \{ s_1, s_2 \}$ generates W

(BN2) Homework

(BN3) $s_1 B s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \neq \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

similar for s_2.

$$P_2 = \mathrm{stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\}$$

$$P_1 = \mathrm{stab}_G(\langle e_1, e_2 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}.$$

Similarly one can take upper triangular and monomial matrices in the groups $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ for any field $\mathbb{K}$.

Proposition 8.4 (Bruhat decomposition)

If G has a BN-pair, then

$$G = \bigsqcup_{w \in W} BwB.$$

Proof Let $g \in G$.

Then $g = b_1 n_1 b_2 n_2 \dots b_k n_k b_{k+1}$

where $b_i \in B$, $n_i \in N$.

where $\check{b}_i \in B$, $n_i \in N$.

Hence $g \in Bn_1Bn_2B \dots Bn_kB$

$$= Bw_1Bw_2B \dots w_kB$$

where $w_i := n_iT$.

One can deduce that

$$g \in \bigcup_{w \in W} BwB.$$

use (BN2)

To see that it is a disjoint union write l for the word-length in S and suppose $BwB = Bw'B$.

Induction on $d = \min\{l(w), l(w')\}$

wolg $d = l(w')$.

Suppose $d=0$, then $w' = 1$ and $Bw'B = B$.

And hence $BwB = B$ as well.

But then $w = 1$ in $W = N/T = N/(B \cap N)$.

$\Rightarrow w = 1 = w'$.

$d>0$ Write $w' = sw''$, $s \in S$, $l(w'') = d-1$.

$d = l(w')$.

Then $sw''B \subseteq Bsw''B = BwB$.

Multiply the equation on the left

Multiply the equation on the left by s^{-1}:

$$w'' B \subseteq s^{-1} B s w'' B = s B w B$$
$$\subseteq \; BwB \cup BswB.$$

alternative (BN2)

$s^{-1} = s$

But then (by induction hypothesis)

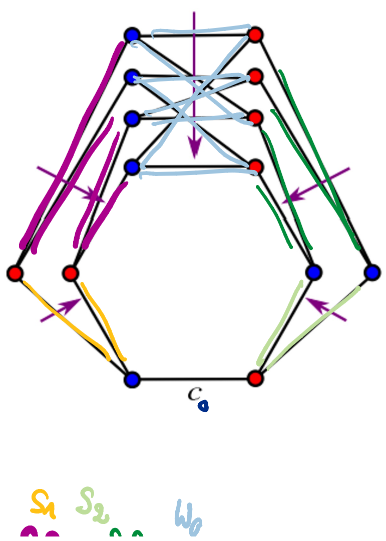
$$Bw''B = BwB \text{ or } BswB.$$

Hence $w'' = w$ or sw.

If $w'' = w$, then (since $\ell(w'') < d \leq \ell(w)$) we arrive at a contradiction.

So $w'' = sw$ and we get $w = sw'' = w'$ as was required. $\square$

8.5 Geometric interpretation of Bruhat decomposition:



Recall in case $G = GL_3(\mathbb{F}_2)$

$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ stabilizes C_0 and permutes apartments (i.e. hexagons) containing C_0.

G acts transitively on all chambers in the

s_1 s_2 w_0
s_1s_2 s_2s_1

all chambers in the building.

Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB$$

Every $g \cdot C_0 = C$ (i.e. every chamber in fact) is obtained from C_0 by

- not acting on C_0 (i.e. elements in B)
- then taking the image under a unique $w \in W$
- then swapping apartments containing C_0, i.e. keeping the relative W-position to C_0

This allows us to define a <u>retraction</u> from Δ to $\Sigma(W,S)$ by mapping

$$Ch(\Delta) \ni C = gC_0 \longmapsto w \cdot W_\emptyset \in \Sigma(W,S)$$

unique w s.th. $g \in BwB$.

If we identify $\Sigma(W,S)$ with the std

If we identify $\Sigma(W,S)$ with the std apartment A_0 in Δ then this collapses the whole building Δ to A_0 and keeps relative dist. to c_0 infact. $\rightsquigarrow$ must be the one we saw in Prop. 7.32 as by Prop 7.36 (3) there is a unique map with these properties.

There is a general method to obtain examples of BN-pairs:

Def 8.6 (strongly transitive actions)
Let Δ be a building of type (W,S).
Write $\mathrm{Aut}_c(\Delta)$ for the color-preserving automorphisms of Δ.
Then a subgroup $G \leq \mathrm{Aut}_c(\Delta)$ acts <u>strongly transitively</u> on Δ if it is transitive on the set of pairs

$$\{(c, A) \mid c \text{ is a chamber in the apartment } A\}$$

8.7 <u>Equivalent formulations:</u>
G acts strongly transitively on the bldg Δ
$\Leftrightarrow$ G is transitive on the chambers

$\Leftrightarrow$ G is transitive on the chambers of Δ and $\forall$ chambers c in Δ $\mathrm{Stab}_G(c)$ acts transitively on all apartments containing c

automatically pointwise

$\Leftrightarrow$ G is transitive on apartments of Δ and $\mathrm{Stab}_G(A)$ is transitive on the chambers in A for all apartments A.

setwise

Ex. The group $GL_3(\mathbb{F}_2)$ acts transitively on the Heawood graph.

Thm 8.8 (Tits) See e.g. Ronan, Thm 5.2

Let X be a building of type (W,S) with atlas $\mathcal{A}$ and let $G < \mathrm{Aut}_C(X)$ act strongly transitively on X.
Let $B := \mathrm{Stab}_G(C_0)$, C_0 a fixed chamber and $N := \mathrm{Stab}_G(A_0)$, A apmt, $C_0 \in A$.

setwise

Then (B,N) is a (weak) BN-pair and satisfies axioms (BN0) – (BN2).
The axiom (BN3) is satisfied if X is thick

The axiom (BN3) is satisfied if X is thick.

<u>Rmk</u> For $SL_2(\mathbb{Q}_p)$ we obtain two BN-pairs. One via the thm as stabilizers of an edge inside an apartment of the tree T_{p+1} and the other via $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ and monomial mxes $N = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix}, \begin{pmatrix} & * \\ * & \end{pmatrix} \right\}$.

We will see next week how they relate.

Exercise: compute the stabilizer of an edge in the SL_2-tree