17 - Structure of buildings - part 2

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The complete apartment system

Thm 7.38

Let Δ be a building. Then the union of any family of apartment systems for Δ is again an apartment system.
Hence Δ admits a largest system of apmts.

Proof:

It is clear that (B0) all apmts are Coxeter cplxes and (B1) pairs of chambers are in a common apmt hold for unions of apartment systems.
We will prove (B2''). A, A' apmts sharing a chamber then $\exists$ iso $A \to A'$ fixing $A \cap A'$ pointwise.

Obviously (B2'') holds if A, A' are in a same apartment system. So suppose $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$ for different apartment systems.
Let the chamber $C \in A \cap A'$.
We know that A and A' must have the same Coxeter matrix by Prop 7.9 saying that Δ is labelable and the proof of Cor 7.10 showing that the Coxeter matrix may be read off of the links of a simplex.

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Hence we may find a label-preserving isomorphism $\phi: A' \to A$ fixing c pointwise (as c is top-dimensional).

It remains to prove that ϕ fixes the entire intersection $A \cap A'$ pointwise.

There is a second way to construct an iso $\psi: A' \to A$ as the restriction of the chamber retraction $\rho_{A,C}: \Delta \to A$.
This retraction obviously fixes $A \cap A'$.
(since $\rho_{A,C}$ is a retraction).
It is not clear however, that ψ is an iso as A' may not be part of the apartment system of where A belongs.

There is a "standard uniqueness argument" showing that ψ and φ are in fact the same map.

(ÜA) read about the standard uniqueness argument in Brown, p.69, sec III 4 proof of Lemma 5.

The main idea is to track what both ψ and φ do along minimal galleries starting at c. (Use that minimality [illegible] starting at c. (Use that minimality of galleries in apartments is the same as in Δ and that f preserves distances to c). □

Def 7.39

The maximal system of apartments is called the <u>complete</u> apartment system.

Here are more useful descriptions:

Prop. 7.40

Let A be a chamber subcplx (chamber cplx, subcplx of Δ, chambers have same dim.) of a bldg Δ.
Let Σ be the Coxeter complex of (W,S) where (W,S) is the type of Δ.
Denote by $\mathcal{A}$ the complete system of apmts in Δ.
Then: $A \in \mathcal{A} \iff A \cong \Sigma$ as a labelled chamber complex

w/o proof. □

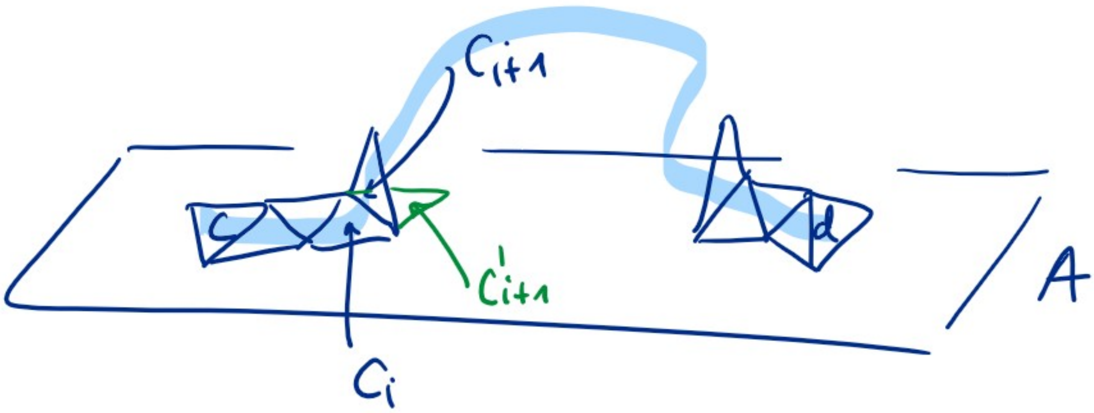
Prop 7.41

Every apartment is convex, that is for any pair of chambers c, d contained in an apartment A all minimal galleries in Δ from c to d are also entirely contained in A.

Proof:

Proof:

Let $c_0 = c, c_1, c_2, \dots, c_k = d$ be a minimal gallery γ from c to d in Δ. If this gallery is not contained in an apartment A containing c and d there exists an index i s.th. $c_i \in A$ and $c_{i+1} \notin A$.



Let c'_{i+1} be the chamber in A adjacent to c_i along the panel $c_i \cap c_{i+1}$.

Let ρ be the retraction onto A centered at the chamber c'_{i+1}.

Then $\rho(c_{i+1}) = \rho(c_i)$. Hence ρ maps the gallery γ to a stammering gallery in A.

But this contradicts minimality of γ in Δ.

$\square$

One can show:

7.42 Prop

Every subcomplex B in Δ which is isometric in a label-preserving way to a ..

metric in a label-preserving way to a subset of an apartment in Δ is actually contained in some (possibly other) apmt.

Spherical buildings

Recall that a building is spherical if its apmts are. Let from now on in this section all Δ be spherical. unless otherwise stated!

Def 7.43

We call two chambers in a spherical building Δ <u>opposite</u> if $d(c,d) = \mathrm{diam}(\Delta)$.

Recall also:

on a sphere the geodesics between opposite points cover the entire sphere

combin. version of this

Lemma 7.43

Let c,d be opposite in Δ. Let A be an apmt containing c and d. Then for any chamber e in A there exists a minimal gallery γ from c to d containing e.

<u>proof</u> Identify A with the Coxeter Cplx Σ of the same type as Δ. We may do so such that $c \hat{=} 1$ and $d \hat{=} \omega$ for some

such that $c \hat{=} \mathbb{1}$ and $d \hat{=} w$ for some w in the associated Coxeter group W.
Then a minimal gallery γ from c to d in A corresponds uniquely to a minimal gallery γ_Σ from $\mathbb{1}$ to w in Σ. This latter gallery γ_Σ corresponds in turn to a reduced word $s_{i_1} \dots s_{i_k}$ in the generators S of W.
We may associate to this word a set of reflections R_w of w. $R_w = \{ s_{i_1} \dots s_{i_j} s_{i_{j-1}} \dots s_{i_1} \mid j = 1 \dots k \}$

Each of the walls H_r for $r \in R_w$ is crossed exactly once by the path γ_Σ as this path is minimal.

By results in Section 3 (Le 3.20 and 3.22) each such wall separates Σ into two components.
So any other chamber e in A corresponds to an element $u \in W$ and hence chamber u in Σ.
Then for H_r, $r \in R_w$, the chamber u is either in the half-space of H_r containing $\mathbb{1}$ or w. Hence H_r separates u from $\mathbb{1}$ or from w but not both.
Since the combinatorial distance between chambers is the same as the number

chambers is the same as the number of separating walls we have

$$d_S(1, \omega) = d(1, u) + d(u, w).$$

This uses that the dist. betw. 1 and ω is maximal

We may hence construct a minimal gallery from 1 to ω by concatenating a minimal gallery from 1 to u with one from u to ω. □

Thm 7.44

A spherical bldg contains a unique system of apartments. The apartments are exactly the convex hulls of opposite chambers in Δ.

Proof: Let $\mathcal{A}$ be an arbitrary system of apartments. Every apmt in $\mathcal{A}$ contains a pair of opposite chambers and is hence, by 7.43, their convex hull.

Conversely let c, c' be two opposite chambers. Then there must be an apmt in $\mathcal{A}$ containing them by (B1). Hence this apmt is the convex hull of c and c', which is in $\mathcal{A}$. □

Thm 7.45

Let Δ be a spherical bldg of rank n.

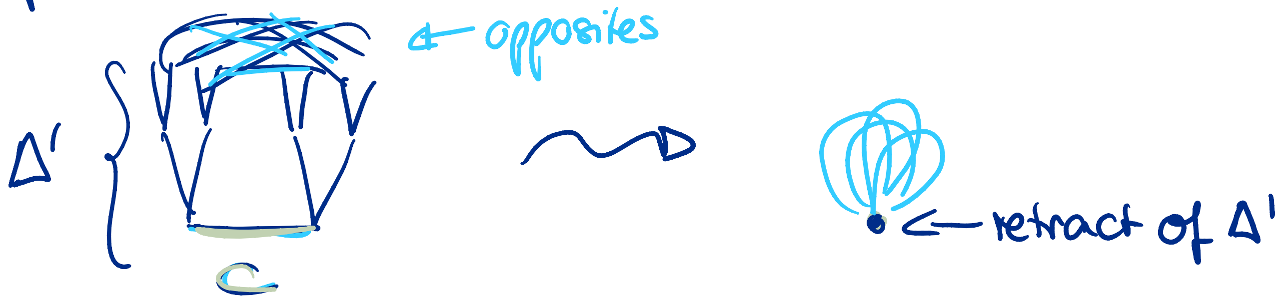
Let Δ be a spherical bldg of rank n.
Then $|\Delta|$, its geometric realization, has the homotopy type of a bouquet of spheres.
There is one sphere for every apartment containing a fixed chamber C in Δ.

Sketch of proof:

Fix a chamber C of Δ. Let Δ' be the subcplx of Δ where we remove all chambers opposite C.

Claim: Δ' is contractible.

If the claim is true, then we may contract Δ' to a point without affecting the homotopy type of Δ and we are done.



For the claim:

Every apartment containing C admits a canonical label-preserving iso to Σ, the Coxeter complex underlying the type π of Δ.
Now $|\Sigma|$ is a unit sphere of dimension $n-1$ and carries the structure of a topol. sphere. Hence $|\Sigma \cap \Delta'|$ admits a canonical

sphere. Hence $|\Sigma \cap \Delta'|$ admits a canonical retraction onto the barycenter of C along geodesics in the sphere.
= arcs of great circles

One shows that the various homotopies, one for each apartment, are compatible by the gluing of apmts in Δ.
This yields the desired homotopy of Δ to a bouquet of spheres of the desired size.

„$\square$"

<u>Rmk</u>: In contrast buildings of type (W,S) where (W,S) is irreducible and infinite are all contractible.