16 - Structure of buildings - part 1

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We will now have a closer look at some structure theory of buildings. First: local structure

<u>Recall</u>: the <u>link</u> of a simplex σ in a chamber complex Δ is the set of all simplices τ s.th. $\sigma \cup \tau$ is a chamber of Δ.

<u>7.30 Prop.</u> Links are buildings

Let Δ be a building of type (W,S) with labeling λ. Let b be a simplex in Δ. Then

$lk_\Delta(b)$ is a building of type (W',S')

where $S' := S \setminus \lambda(b)$, $W' := \langle S' \rangle \leq W$.

<u>Proof</u>: Fix a system $\mathcal{A}$ of apartments for Δ. And fix also a simplex b in Δ.

For every apartment $A \in \mathcal{A}$ containing b we obtain a subcomplex $lk_A(b)$ of $lk_\Delta(b)$. Collect those subcomplexes in a set $\mathcal{A}'$.

the analogous statement for apartments

By Prop 5.20 one has that all complexes in $\mathcal{A}'$ are Coxeter complexes of type (W',S').

It hence remains to prove the buildings axioms (B1) and (B2) for $lk_\Delta(b)$.

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Let b', b'' be two simplices in $lk_\Delta(b)$.
Join both with b to obtain simplices $b \cup b'$ and $b \cup b''$ in Δ.

Since Δ satisfies (B1) there exists an apartment $A \in \mathcal{A}$ containing $b \cup b'$ and $b \cup b''$.
This apartment by construction contains b and hence $lk_A(b) \in \mathcal{A}'$ and contains both b' and b''.

To see (B2) we argue as follows:
Let A and A' be two apartments containing $b \cup b'$ and $b \cup b''$ as above.
Then by (B2) of Δ $\exists$ iso $A \xrightarrow{\phi} A'$ fixing the intersection pointwise. Hence in particular b, b' and b'' are fixed.
We may restrict ϕ to $lk_A(b)$ which yields an isomorphism onto $lk_{A'}(b)$ which by construction still fixes b' and b''.

$\square$

We will now prove that every building retracts onto apartments. We need some notation.

<u>7.31 Def.</u> maps between simpl. cplxes

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1) A <u>simplicial map</u> $\phi : \Delta \to \Delta'$ from one simplicial cplx Δ to another Δ' is a function from the vertices of Δ to those of Δ' which sends simplices to simplices.

2) If the image of a simplex under a simplicial map always has the same dimension as its preimage we call the simplicial map <u>nondegenerate</u>.

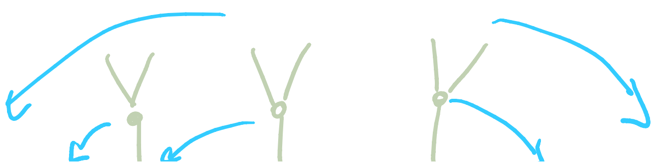
A nondegenerate simplicial map is the same as a poset map $\phi : \Delta \to \Delta'$ s.th. $\Delta_{\leq a}$ is send isomorphically to $\Delta'_{\leq \phi(a)}$ for every simplex a in Δ.

3) If Δ and Δ' are chamber complexes of the same dimension we call a nondegenerate simplicial map a <u>chamber map</u>.

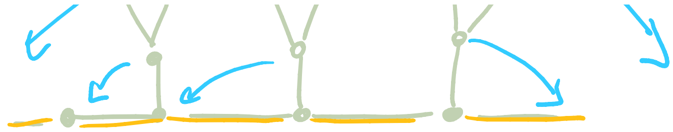
Note: chamber maps preserve adjacency! $\rightsquigarrow$ galleries are mapped to galleries.

4) If Δ' is a subcomplex of Δ and $\phi : \Delta \to \Delta'$ a chamber map s.th. $\phi|_{\Delta'} = id_{\Delta'}$, then we call ϕ a <u>retraction</u> and Δ' <u>retract</u> of Δ.

<u>Ex:</u>



is a retraction onto the



onto the yellow subcplx

7.32 Prop.

Let Δ be a building. Then every apartment A in Δ is a retract of Δ.

↙ compare with proof of labelability!

Proof

Fix a chamber C of a given apartment A.
Consider all apartments A' containing C.
For any such A' there exists a unique iso $\phi_{A'}: A' \longrightarrow A$ fixing C (in fact $A \cap A'$).
Existence follows from (B2), uniqueness from the top-dimensionality of C and the labelability of Δ.

Let A'' be a second such apartment. Its iso $\phi_{A''}: A'' \longrightarrow A$ may be constructed as follows: there is an iso $A'' \xrightarrow{\phi} A'$ (by (B2) again as both contain C).
Compose $\phi_{A'} \circ \phi$ to obtain $\phi_{A''}$ as ϕ fixes $A'' \cap A'$ (and hence in particular C).
Therefore:
For any pair A', A'' of such apartments the isomorphisms $\phi_{A'}$ and $\phi_{A''}$ agree on $A' \cap A''$

the isomorphisms $\varphi_{A'}$ and $\varphi_{A''}$ agree on $A' \cap A''$.

Those isos all fit together to form a map $\rho : \Delta \to A$ defined by

$$b' \longmapsto \phi_{A'}(b')$$

where A' is an apartment containing both b and b'. Such an apartment exists by (B1).

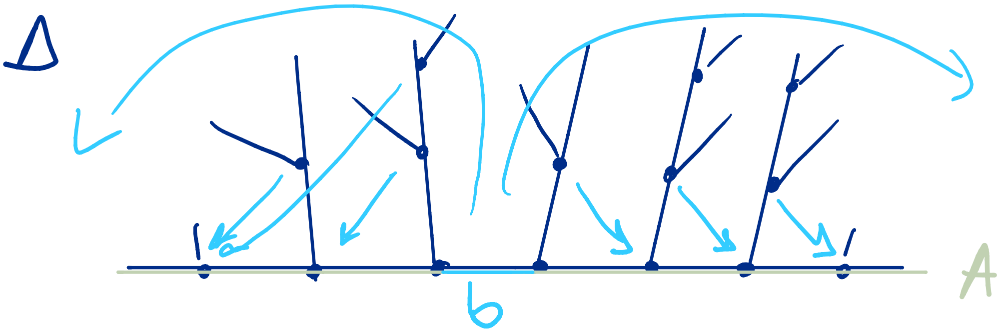
By construction $\rho|_A$ is the identity on A. Hence ρ is a retraction in the sense defined above. $\square$

7.33 Example

We have seen that trees w/o leafs are buildings of type D_∞.

Retractions wrt chambers are as follows:

Δ



Rmk 7.34 canonical retraction $\rho_{A,C}$

The proof of 7.32 yields for every pair of a chamber c contained in an apartment of a chamber c contained in an apartment A a canonical chamber retraction which we denote by

$$\begin{aligned} \rho_{A,c} : \Delta &\longrightarrow A \\ d &\longmapsto \phi_{A'}(d) \end{aligned}$$

where $\phi_{A'} : A' \to A$ iso for any A' containing both c and the simplex d.
(independence of the choice of A' was shown in the proof of 7.32)

7.35 Prop. distances on buildings

Let c, d be chambers in a building Δ.
Let A be an apartment containing c, d.
Then $d_\Delta(c,d) = d_A(c,d)$.
In particular $\mathrm{diam}(\Delta) = \mathrm{diam}(A)$ for any A.

#chambers in a minimal gallery from c to d

Proof:
Suppose γ is a minimal gallery from c to d in the apartment A. Then γ is also minimal in Δ.
If there was a shorter one, say γ', in Δ we could retract it onto A to obtain a shorter one in A.
Hence $d_\Delta(c,d) = d_A(c,d)$ and also

Hence $d_\Delta(c,d) = d_A(c,d)$ and also $\operatorname{diam}(A) \leq \operatorname{diam}(\Delta)$.

To prove the opposite inequality let c', d' be arbitrary chambers of Δ and let A' be an apartment containing them.

Then $d_\Delta(c', d') = d_{A'}(c', d') \leq \operatorname{diam}(A')$.

But $A' \cong A$ and $\operatorname{diam}(A) = \operatorname{diam}(A')$.

$\Rightarrow \operatorname{diam}(A) = \operatorname{diam}(A') = \operatorname{diam}(\Delta)$. $\square$

7.36 Prop. properties of $\rho_{A,C}$

Let $\rho_{A,C}$ be the chamber retraction of Δ onto A wrt c. Then:

(1) For any face d of c: $\rho_{A,C}^{-1}(d) = \{d\}$.

(2) $\rho_{A,C}$ preserves distances to c, i.e.

$$d_\Delta(c, \rho_{A,C}(d)) = d_\Delta(c,d)$$

for all chambers d in Δ.

(3) $\rho_{A,C}$ is the unique chamber map which fixes c pointwise and preserves distance from c.

Beweis: Put $\rho := \rho_{A,C}$.

To see (1) suppose e is a simplex s.th. $\rho(e) = d$.

Details: For $\rho := \rho_{A,c}$.

To see (1) suppose e is a simplex sith. $\rho(e) = d$. Choose an apartment A' containing e and c. Then $\rho|_{A'}$ is an isomorphism and maps both d and e to d. Hence $d = e$.

For (2) let d be a chamber in Δ and let A' be an apartment containing d and c. Since $\rho|_{A'}$ is an isom. by constr. of ρ we have that $d_A(c, \rho(d)) = d_{A'}(c, d)$.

By 7.35 we conclude that this equals the distance in Δ.

To prove (3) suppose $\phi : \Delta \to A$ is another chamber map with these properties. Then both maps preserve adjacency in the simplicial complex and take a minimal gallery in Δ starting in C to a minimal gallery in A starting in C.
In particular, the image galleries do not repeat chambers (are non-stammering).
From this uniqueness directly follows as in A there is a unique gallery starting in C for any fixed reduced word obtained from a given min. gallery in Δ starting in C.
Hence $\phi = \rho$. $\square$

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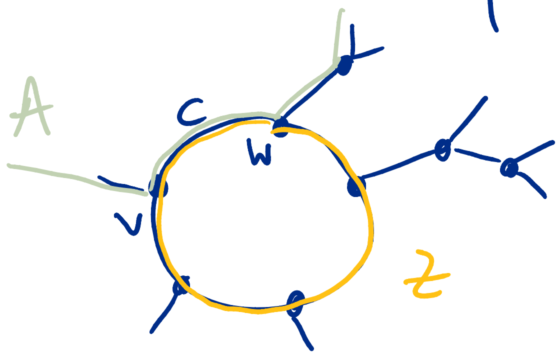
<u>7.37 Prop.</u> Every building of type D_∞ •—∞—• is a tree without leafs.

<u>Proof</u>: Suppose for a contradiction that such a building Δ contains a k-gon Z for some $3 \leq k \lneq \infty$. Let C be a chamber in the subcomplex Z of Δ.



Denote by v, w the vertices of C.
Let A be an apartment containing C and denote by ρ the chamber retraction of Δ onto A based at C.
We traverse Z starting at v, then w, etc. The successive images of these vertices are adjacent vertices in A. More precisely, the complex Z is mapped onto a closed loop (possibly degenerate) in A starting and ending in v where the second vertex is w.
Moreover, by Prop. 7.36 (1) the preimage of w under ρ only contains w. Hence no

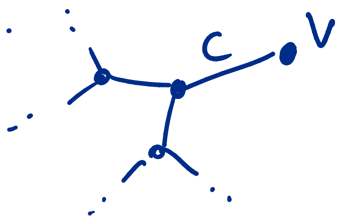
of w under γ only contains w. Hence no other vertex in Z is mapped onto w. The same holds true for v.

But A is a line, so every curve which starts  and returns to v either either passes w or v twice. ↯.

Hence Z can not exist in a graph Δ which is a building.

Thus Δ is a tree.

Suppose Δ has a leaf, i.e. a vertex v which is only contained in a single chamber c.



But then c is, by axiom (B1), contained in some apartment A which is an isometric copy of the D_∞ Coxeter complex as a subcplx of Δ $\leadsto$ $\exists$ a chamber c' on the other side of v. ↯ $\square$