15 - Construction of the SL2 tree

Mittwoch, 14. Juni 2023 11:00

Construction of an affine building

Goal: Construct some 1-dim buildings from algebraic data.

We need a tiny bit of number theory to do so.

some prime or prime power

7.12 p-adic numbers

Take $K = \mathbb{Q}$ and consider the valuation

$v_p : \mathbb{Q}^* \to \mathbb{Z}$ given by $v_p(\frac{a}{b}) = n$ where $n \in \mathbb{Z}$ is s.th. $\frac{a}{b} = p^n \frac{a'}{b'}$ and a', b' not divisible by p.

Konvention: $v_p(0) = \infty$.

This satisfies $v_p(xy) = v_p(x) + v_p(y)$
and $v_p(x+y) \geq \min\{v_p(x), v_p(y)\}$ $\forall x, y \in K$.

i.e. is a discrete valuation.

Such a valuation defines a norm on $\mathbb{Q}$ via

$$\|x\|_p := p^{-v_p(x)}$$

numbers in $\mathbb{Q}$ are close if their difference is highly divisible by p

The p-adic numbers is the Cauchy-completion of $\mathbb{Q}$ wrt this norm. Denote them by $\mathbb{Q}_p$.

Put $\mathcal{O} = \mathbb{Z}_p = \{x \in K \mid \nu_p(x) \geq 0\}$ the valuation ring.

$\pi = p$ is a uniformizer ($\nu_p(\pi) = 1$).

$k := \mathcal{O}/\pi\mathcal{O}$ is the residue field ($\cong \mathbb{F}_p$)

7.13 Model for $\mathbb{Z}_p$

One may view $\mathbb{Z}_p$ as the set $\{\sum_{n\geq 0} a_n p^n \mid a_n \in \mathbb{F}_p\}$ of formal series with coefficients in the finite field $\mathbb{F}_p$.

Then $\mathbb{Q}_p$ is the set of fractions $\frac{s}{p^n}$, $n \in \mathbb{N}$, $s \in \mathbb{Z}_p$.

Or, in other words, $\mathbb{Q}_p$ corresponds to the set $\{\sum_{n\geq N} a_n p^n \mid N \in \mathbb{Z}, a_N \neq 0, a_n \in \mathbb{F}_p\}$.

Def. 7.14 (lattices and lattice classes)

A lattice in $\mathbb{Q}_p^2$ is a f.g. $\mathcal{O}$-module which generates $\mathbb{Q}_p^2$ (over $\mathbb{Q}_p$).

Two lattices L and L' are homothety-equivalent if $\exists\, k \in K^* = \mathbb{Q}_p^*$ s.th. $L' = k \cdot L$.

Denote such a class by $[L]$.

Rmk 7.15: Different bases may generate the same lattice. E.g. the $\mathcal{O}$-span of e_1, e_2 (=std basis) is the same as the $\mathcal{O}$-span of e_1, e_1+e_2.

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This is always the case in the following situation:

Let $\{b_1, b_2\}$ and $\{b_1', b_2'\}$ be two bases.

Then "B and B'" determine the same lattice iff there exists a matrix A in $SL_2(\mathbb{Z}_p)$ s.th.

$B' = AB$.

Warning $SL_2(\mathbb{Z}_p)$ are $\mathbb{Z}_p$-invertible mxes
in particular $SL_2(\mathbb{Z}_p) \neq \{A \in SL_2(\mathbb{Q}_p) \mid a_{ij} \in \mathbb{Z}_p\}$
but a proper subset!

Definition 7.16 (the SL_2-tree)

Fix p and define a graph X as follows:

vertices $\hat{=}$ Lattice classes $[L]$

edges: there is an edge between distinct classes $[L]$ and $[L']$ iff there exist representatives L of $[L]$ and L' of $[L']$ s.th.

$$\boxed{\pi L \subset L' \subset L}$$

Observation 7.17: $SL_2(\mathbb{Q}_p)$ acts on the graph X as follows: Let $\Lambda = [L]$ and b_1, b_2 be a basis of L. Let $A \in SL_2(\mathbb{Q}_p)$.

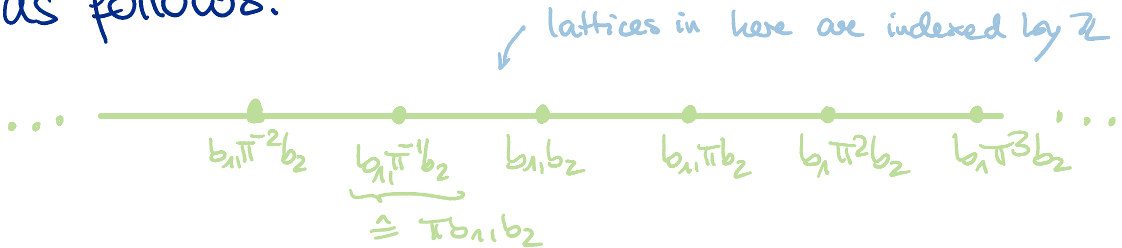
Then $A.\Lambda = [\langle Ab_1, Ab_2 \rangle_{\mathcal{O}}]$.

In fact $GL_2(\mathbb{Q}_p)$ also acts with same action.

Then $A \cdot \mathcal{L} = [\langle A e_1, \ldots A e_2 \rangle_{\mathcal{O}}]$.

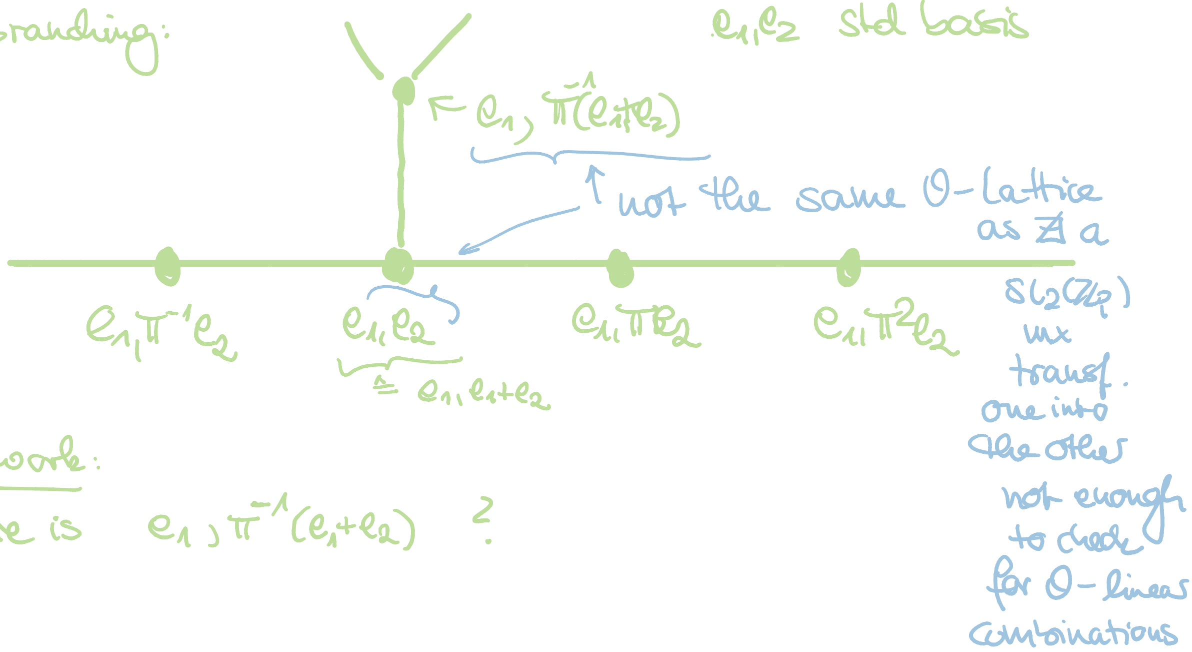
In fact $GL_2(\mathbb{Q}_p)$ also acts with same action.

<u>Observation 7.18:</u> Given a fixed basis b_1, b_2 we obtain a copy of the Coxeter complex of D_∞ as follows.



Here label b_1, b_2 means the lattice class of the $\mathcal{O}$-lattice spanned by b_1 and b_2, i.e. $[\langle b_1, b_2 \rangle_{\mathcal{O}}]$.

Ex: of branching:



<u>Homework:</u>

where is $e_1, \pi^{-1}(e_1+e_2)$?

<u>Lemma 7.19:</u>

Fix the std basis e_1, e_2 of $\mathbb{Q}_p^2$ and let L be the $\mathcal{O}$-lattice spanned by it. Let $A \in SL_2(\mathbb{Q}_p)$. Then Ae_1, Ae_2 spans a lattice L' with $[L] = [L']$ if and only if $A \in SL_2(\mathcal{O})$.

<u>Proof:</u> Homework. $\square$

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Lemma 1 implies that $Sl_2(\mathbb{Z}_p)$ is the point-stabilizer of $[L]$ in the graph X.

<u>Thm 7.20</u>: The graph X is a tree.

More precisely an affine building of type D_∞.

In order to prove this we introduce a distance function on lattice classes:

<u>Fact 7.21</u>: For any two lattices L and L' there is an $\mathcal{O}$-basis b_1, b_2 of L such that L' has $\mathcal{O}$-basis $b_1\pi^a, b_2\pi^b$. The pair a, b does not depend on choices of bases.

<u>Homework</u>: Find out why this is true!
(Read: Casselmann-tree.pdf Sec3)

- One can see that $L' \subset L$ iff $a, b \geq 0$.
  In this case $L/L' \cong \mathcal{O}/\pi^a\mathcal{O} \oplus \mathcal{O}/\pi^b\mathcal{O}$.

- Scaling the lattices L and L' by some $x, y \in K^*$ we obtain Lx and $L'y$ and (frome above) the integers a and b will be replaced by $a+c$ and $b+c$ where $c = v_\pi(\frac{x}{y})$.

the integers a and b will be replaced by $a+c$ and $b+c$ where $c = v_p(\frac{x}{y})$.
Hence $|a-b|$ does not depend on the homothety class of L and L'.

This observation implies that the following is a well defined function:

Def. 7.22 (distance of lattices)
Let $\Lambda = [L]$ and $\Lambda' = [L']$ be two lattice classes. Put $d(\Lambda, \Lambda') := |a-b|$ where a and b are as above, and call it the <u>distance of Λ and Λ'</u>.

Lemma 7.23
The distance $d(\Lambda, \Lambda') = 1$ for two lattice classes Λ and Λ' if and only if Λ and Λ' are adjacent in X'.

Proof
Suppose $d(\Lambda, \Lambda') = 1$. Then $\exists$ basis b_1, b_2 of L with $[L] = \Lambda$ s.th. $b_1\pi^a, b_2\pi^b$ is an $\mathcal{O}$-basis for L' with $[L'] = \Lambda$.
Here $|a-b| = 1$. But then $b_1, b_2\pi^{b-a}$ spans a lattice also representing Λ.
This lattice is of the form $b_1, b_2\pi$ or $b_1, b_2\pi^{-1}$ and hence its class is adjacent to Λ in X'.

b_1, b_2" and hence its class is adjacent to Λ in X.

Conversely two adjacent classes have representatives L and L' satisfying

$$\pi L \subset L' \subset L$$

and hence the condition in the Lemma holds. $\square$

We obtain:

Lemma 7.24:

Let $L = \langle b_1, b_2 \rangle_{\mathcal{O}}$ be a lattice.
The neighbours of $[L]$ in X are the classes of lattices L' of the form

$$\langle b_1, \pi b_2 \rangle_{\mathcal{O}} \text{ and } \langle \pi b_1, x b_1 + b_2 \rangle_{\mathcal{O}}$$

where x ranges over $\mathcal{O}/\pi\mathcal{O} = \mathbb{F}_p$.

i.e. has $p+1$ neighbours!

In order to prove that the graph at hand is a tree we more closely investigate the action of $Sl_2(\mathbb{Q}_p)$.

Denote by Λ_0 the vertex given by the class ass. to the lattice for the std basis e_1, e_2. where $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$.

and by Λ_m the vertex corresponding to the class

$[\langle e_1, \pi^{-m} e_2 \rangle]$

to the class
$[\langle e_1, \pi^{-m} e_2\rangle_{\mathcal{O}}]$.

in fact $GL_2(\mathbb{Q}_p)$

7.25 Observe: The action of $SL_2(\mathbb{Q}_p)$ on X has the following properties:

a) every $A \in SL_2(\mathbb{Q}_p)$ maps lattices to lattices

b) $\forall A \in SL_2(\mathbb{Q}_p)$ the images $A.\Lambda, A.\Lambda'$ of two adjacent lattices are adjacent.
Hence A is an automorphism of X.

c) $\forall A \in SL_2(\mathbb{Q}_p)$ and lattices Λ, Λ' one has:
$$d(\Lambda,\Lambda') \bmod 2 = d(A\Lambda, A.\Lambda') \bmod 2$$
i.e. parity is preserved.

d) There are two orbits of vertices in X. One contains Λ_0 (and all vertices at even graph-distance to it) and one contains $[\langle e_1, \pi e_2\rangle_{\mathcal{O}}]$ or, equivalently, all vertices at odd distance to Λ_0.

The action of $GL_2(\mathbb{Q}_p)$ is transitive on vertices.

Def. 7.26 (Chains of lattices)

A chain of lattices is a finite or half-infinite sequence of lattices

$$L_0 \supset L_1 \supset L_2 \supset \dots \supset L_n \supset L_{n+1} \supset \dots$$

with $\quad L_n \supset L_{n+1} \supset \pi L_n \quad \forall n.$

Call such a chain simple if it does not backtrack, i.e. none of the L_i are equivalent (define the same class).

The copies of $\mathcal{D}_\infty$ associated with bases are good sources of lattice chains!

Def 7.27 (Chain of nodes)

A chain of nodes is the sequence of vertices in X corresponding to a chain of lattices.

Rmk: A chain of nodes is without back-tracking if the chain of lattices it comes from is simple.

Def 7.28 (Std chain)

A standard chain (finite or infinite) is a lattice chain of the form lattice chain of the form

$$\langle e_1,e_2\rangle_{\mathcal{O}} \supset \langle \pi e_1,e_2\rangle_{\mathcal{O}} \supset \ldots \supset \langle \pi^k e_1,e_2\rangle_{\mathcal{O}} \ldots$$

$$\Lambda_0 \qquad \Lambda_1 \qquad \Lambda_k$$

associated chain of nodes

Proposition 7.29

Every finite simple chain of lattices may be transformed to a standard chain by an element of $GL_2(\mathbb{Q}_p)$.

We only sketch the proof.

„Proof" is by induction on the length of the chain.

Let $L_0 \supset L_1 \supset \ldots \supset L_n$ be the given chain.

Here $L_k \supset L_{k+1} \supset \pi L_k \ \forall k$.

Since $GL_2(\mathbb{Q}_p)$ is transitive on all nodes we find some $g \in GL_2(\mathbb{Q}_p)$ s.th

$gL_0 = \langle e_1,e_2\rangle_{\mathcal{O}}$.

We may hence wlog assume $L_0 = \langle e_1,e_2\rangle_{\mathcal{O}}$.

For $\underline{n=1}$: one can explicitly find an element $g \in GL_2(\mathbb{Q}_p)$ that maps $[L_1]$ to $[\langle \pi e_1,e_2\rangle_{\mathcal{O}}]$ and stabilizes $[L_0]$.

and stabilizes $[L_0]$.

Such a matrix will be of the form $\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}$ with $x \in \pi\mathcal{O}$.

For a longer chain proceed inductively
$\leadsto$ transform a length $n+1$ chain to a standard chain by some $g_1 g_2 \cdots g_n \in GL_2(\mathbb{Q}_p)$
then the last edge by an additional g_{n+1}
observation that

neighbours of Λ_m are nodes corresponding to lattices of the form $\langle \pi^{m-1} \cdot e_1, e_2 \rangle$ and $\langle \pi^{m+1} e_1, x e_1 + e_2 \rangle$ with $x \in \pi^m \mathcal{O} / \pi^{m+1} \mathcal{O}$.

Matrices of the form $\begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix}$ with $x \in (\pi\mathcal{O})^n$ will do the trick.

To see the main Prop. 7.29 we need to check that the product $g_1 \cdot g_2 \cdot \ldots \cdot g_{n+1}$ converges

This is a product of matrices $\begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$ with $x_i \in (\pi\mathcal{O})^i$. „$\square$"

Proof that X is a tree

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Let L be any lattice. Let b_1, b_2 be a basis of L_0 s.th. $\pi^m b_1, \pi^n b_2$ is a basis of L. We may take a multiple L' of L representing the same class $\Lambda = [L]$ s.th. L' has $b_1, \pi^n b_2$ as basis with $n \geq 0$.

$\Rightarrow$ We find a chain of lattices from Λ_0 to Λ.

This implies connectedness.

Since any chain can be transformed into a standard chain it has to be a tree, since standard chains have no loops.

But then we have:

X is a connected graph without loops in which every vertex has $q+1$ neighbors.

And the claim follows. $\square$