14 - Buildings - the definition and first examples

Mittwoch, 22. Mai 2024 11:26

7 Buildings

Def 7.1 Buildings

A building is a simplicial complex X s.th. there exists a family $\mathcal{A}$ of subcomplexes A in X s.th. the following are satisfied

(B0) Every $A \in \mathcal{A}$ is isomorphic to some Coxeter complex $\Sigma = \Sigma(W, S)$.

(B1) $\forall$ pairs of simplices σ, τ in X there exists $A \in \mathcal{A}$ containing σ and τ.

(B2) If $\exists$ two $A \neq A' \in \mathcal{A}$ containing simplices σ, τ in X, then $\exists$ isom. $\varphi: A \to A'$ fixing $A \cap A'$ pointwise.

$\mathcal{A}$ is called atlas of X and an $A \in \mathcal{A}$ is called apartment of X.
Maximal simplices in X are called chambers.

Rem: One may equivalently replace (B2) by the slightly weaker axiom

(B2') If A, A' are apartments containing a simplex and a chamber, then there exists an iso $A \to A'$ fixing $A \cap A'$ pointwise.

or by (B2'') which only requires A and A' to contain a common chamber.

Example 7.2 "trivial" examples

1) A single apartment, e.g. a Coxeter complex, is a building. We call such buildings <u>thin</u>.

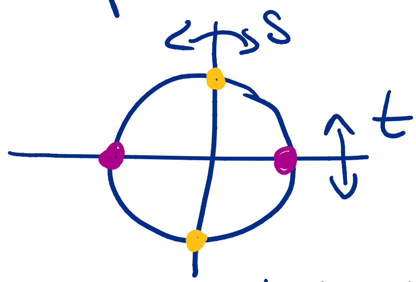
2) Any collection Δ of points is a building. Consider $W := \langle s \mid s^2 \rangle$ as a reflection of a 0-sphere swapping a pair of pts. Then Δ carries the structure of a type $(W, \{s\})$ building by putting $\mathcal{A}$ to be the collection of all 2-element subsets of Δ.

7.3 Complete bipartite graphs:

Let $W = \langle s, t \mid s^2, t^2, (st)^2 \rangle$.

Then $W \cong C_2 \times C_2$ with $C_2 \cong \langle s \mid s^2 \rangle$.

The Coxeter complex Σ of W is a 4-gon with vertices of alternating colors.



Let $K_{n,m}$ be a complete bipartite graph on $m+n$ vertices

e.g. $K_{3,4}$:

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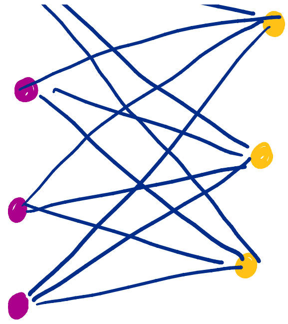


Take as $\mathcal{A}$ the collection of all embeddings of Σ into this $K_{3,4}$. It is not hard to check that the axioms are satisfied.

7.4 Def. generalized m-gons

think of this as a generalization of bipartite graphs

A generalized m-gon for some $m \geq 2$ is a connected bipartite graph of girth $2m$ and diameter m.

Recall: girth of a graph is the length of a shortest circuit inside the graph and the diameter is the maximal distance between a pair of vertices.

Rmk: generalized 2-gons are the same as bipartite graphs.

One can prove:

Prop. 7.5 (w/o proof)

Let (W,S) be a Coxeter system of type $I_2(m)$. Buildings of type (W,S) are the same as

Buildings of type (W,S) are the same as generalized m-gons.

Ex. 7.6 The Fano plane and its incidence graph

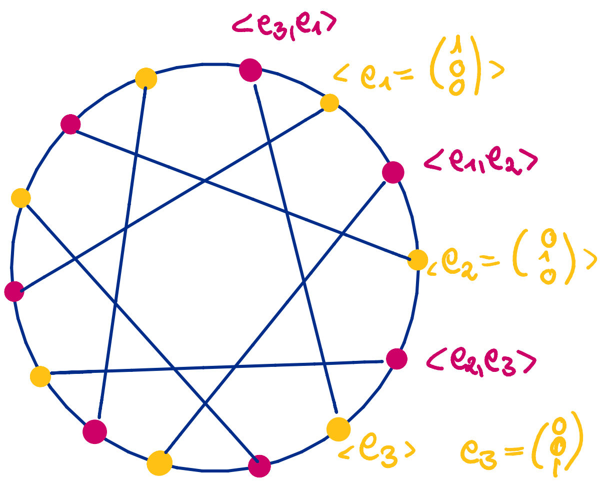
Recall from the very first week:

1.3 Example: $q=2$, $\mathbb{F}_2=\{0,1\}$

Δ is shown below. It contains $14=(2^2+2+1)\cdot 2$ vertices – 7 lines and 7 points.

Heawood graph Δ:

the spherical building ass. with $GL_3(\mathbb{F}_2)$.

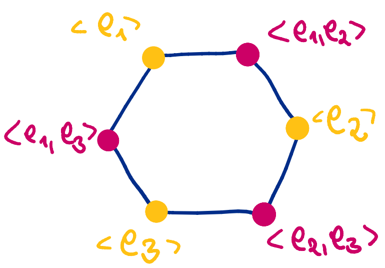


The 7 points are: $\mathcal{P}=\{\underbrace{\langle e_i\rangle}_{3}, \underbrace{\langle e_i+e_j\rangle}_{+\,3}, \underbrace{\langle e_1+e_2+e_3\rangle}_{+\,1}, i\neq j\in\{1,2,3\}\}$

The 7 lines are the spans of all pairs of distinct points.

Ex.1 live Exercise: 1) complete the picture.
2) what changes if we choose q to be a bigger prime? (power)

Observe that this graph is a union of hexagons:



where the vertices correspond to all linear subspaces that can be formed by a fixed basis, e.g. $\{e_1, e_2, e_3\}$

We will call these embedded hexagons apartments.

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Note that S_3 acts as a group of symmetries on this sphere permuting the basis' elements.

We may now verify that this graph is indeed a building of type (W,S).

(B0): true as any sub-VS in $\mathbb{F}_2^3$ has a basis which extends to one of $\mathbb{F}_2^3$.

(B1): follows from the fact that we may find a common basis for all pairs of 2-dim VS in $\mathbb{F}_2^3$.

(B2): is a consequence from the classical base change theorem.

It is not hard to see that the Heawood graph is a generalized 3-gon in the sense defined above.

Ex. 7.7.

Consider (W,S) of type D_∞, 

.
Then its Coxeter complex is a biinfinite line:

Σ

Check that every simplicial tree without leafs (i.e. no vertex contained in only one edge) is a building of type D_∞, where we may take W

of type D_∞, where we may take $\mathcal{A}$ to be the collection of all possible embeddings of Σ into the tree.

Note, this may not be the only choice for the apartment system $\mathcal{A}$!

ÜA: Come up with at least two sets $\mathcal{A}_1$ and $\mathcal{A}_2$ of apartments which turn the following tree into a building of type D_∞:

We collect some first properties:

Lemma 7.8

Let C be a fixed chamber in a building Δ. Let $\mathcal{A}_C$ be the set of apartments of Δ containing C. Then, as a set,

$$\Delta = \bigcup_{A \in \mathcal{A}_C} A$$

Beweis:

Since Δ is a (connected) chamber complex it suffices to prove that all chambers of Δ are in this union.

By (B1) there exists for any chamber C' an

By (B1) there exists for any chambers C' an apartment containing C and C' which is in $\mathcal{A}_C$. Hence the Lemma. $\square$

Prop. 7.9

Every building Δ is labellable.
Moreover, the isomorphism in (B2) may be taken to be lable preserving.

Cor 7.10 All apartments in a building are isomorphic to the same Coxeter Complex.

Proof of Prop 7.9

By definition a building is a chamber complex. Fix any chamber C of Δ. Label its vertices by a set I. Let A be an apartment containing C, then there exists a unique labeling λ_A of A s.th. λ_A agrees on C with the chosen one. (This holds since A is isomorphic to some Coxeter complex).

For any two apartments A, A' containing C the labelings λ_A and $\lambda_{A'}$ must agree on $A \cap A'$. This is due to the fact that

$\lambda_{A'} = \lambda_A \circ \phi$ where $\phi : A' \to A$ is the iso-

$A \cap A'$. This is ...

$\lambda_{A'}|_C = \lambda_A \circ \phi|_C$ where $\phi : A' \to A$ is the isomorphism from (B2) fixing $A \cap A'$.
Hence there is no other choice (by labelability and connectedness of A, A') as putting

$$\lambda_{A'} = \lambda_A \circ \phi \text{ on all of } A'.$$

Hence all labelings of Δ must fit together to give one labeling λ on the union of all apartments containing a fixed chamber C. By Lemma 7.8 the union of all these apartments is Δ.
Hence the first assertion.

To prove the second assertion consider an isomorphism as in (B2') or (B2'').
Such an iso is automatically label preserving as it fixes at least one chamber pointwise.

$\square$

<u>Proof of 7.10</u>

Choose a fixed labeling λ of Δ by a set I.
The labeling yields for a fixed apartment A a Coxeter matrix $M_A = (m_{ij})_{ij \in I}$ by putting

$$m_{ij} := \operatorname{diam}(\operatorname{lk}_A(\sigma)),$$

where σ is a simplex in A of class $I - \{i, j\}$

$$m_{ij} := \mathrm{diam}(\mathrm{lk}_A(\sigma)),$$
where σ is a simplex in A of type $I - \{i,j\}$.

Proposition 7.9 implies that any two apartments are isomorphic in a label preserving way. Hence these m_{ij} remain the same in all apartments and the Cor. follows. $\square$

<u>Def 7.11</u> Type of a building:

Given a building X with atlas $\mathcal{A}$, then all its apartments are isomorphic to the same Coxeter complex.
We refer to (W,S) as the <u>type</u> of X and say X is <u>spherical</u> if W is finite and <u>affine</u> (resp. <u>hyperbolic</u>) if W is infinite and acts cocompactly on some Euclidean (hyperbolic) space.