13 - The weak order

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Part II The weak order

Again (W,S) is always a Coxeter system.

6.14 Def. left/right weak order

(i) $u \leq_R w :\Leftrightarrow w = u s_1 s_2 \dots s_k$ for some $s_i \in S$, $i = 1,\dots,k$, where $\ell(u s_1 \dots s_i) = \ell(u) + i$ for all i.

(ii) $u \leq_L w :\Leftrightarrow w = s_k s_{k-1} \dots s_1 \cdot u$ for some $s_i \in S$, $i = 1,\dots,k$, where $\ell(s_i \dots s_1 u) = \ell(u) + i$ for all i.

Rmks 6.15

1) These two orders are in general distinct. The order posets are isomorphic via

$$w \longmapsto w^{-1}.$$

2) Weak order is strictly weaker than Bruhat order in the sense that they have fewer relations. More precisely:

$$u \leq_R w \text{ or } u \leq_L w \quad \underset{\not\Leftarrow}{\Longrightarrow} \quad u \underset{\text{Bruhat order}}{\leq} w.$$

$u \leq_R w$ or $u \leq_L w$ $\Longrightarrow$ $u \leq w$.
$\not\Longleftarrow$ $\qquad$ Bruhat order

3) ⚠ All partial order notation will refer to Bruhat order unless it has a "R" or "L" subscript.
E.g. $[u,v]$ is an interval in Bruhat order while $[u,v]_R$ is an interval in right weak order.

<u>Def 6.16</u>: $D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$
$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$
the <u>left- / right descent set</u> of w.

vergl BB Prop 3.1.2

<u>6.17 Proposition</u> Let $u, w \in W$.

(i) There is a bijection between

$$\left\{\begin{matrix}\text{reduced}\\ \text{decompositions}\\ \text{of } w\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{maximal}\\ \text{chains in}\\ [1,w]_R\end{matrix}\right\}$$

(ii) $u \leq_R w \Longleftrightarrow \ell(u) + \ell(u^{-1}w) = \ell(w)$

(iii) if W is finite, then:

$w \leq_R w_0$ for all $w \in W$

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where w_0 is length-maximal in W.

(iv) weak order satisfies the <u>prefix property</u>:

$u \leq_R w \iff$ there exist reduced expressions
$u = s_1 \dots s_k$ and
$w = s_1 \dots s_k s_1' \dots s_q'$.

(v) weak order satisfies the chain property analogous to Proposition 6.11 above for Bruhat order.

(vi) Suppose $s \in D_L(w) \cap D_L(u)$. Then:
$u \leq_R w \iff su \leq_R sw$.

<u>Rmk 6.18</u>
By 6.17(v) the weak order poset is also a graded poset ranked by the length function on W. And so is every interval $[u,w]_R$.

<u>Proof of 6.17</u>
(i) The map from left to right is given by:

given by:

$w = s_1 \dots s_q$ reduced, then associate to it the chain

$$\underset{=w_0}{\mathbb{1}},\ \underset{=w_1}{s_1},\ \underset{=w_2}{s_1 s_2},\ \dots,\ \overbrace{s_1 \dots s_{q-1}}^{w_{q-1}},\ w = w_q$$

it is clear that $w_i \leq_R w_{i+1}$ in this chain.

This map clearly is injective.

To see it is surjective observe that in a maximal chain

$$u_0 \leq_R u_1 \leq_R \dots \leq_R u_q \qquad (*)$$

one has to have that $u_0 = \mathbb{1}$ and $u_q = w$.

Moreover if $u_{i+1} \neq u_i \cdot s$ for some $s \in S$, then $u_{i+1} = u_i \cdot s_{i_1} \dots s_{i_k}$ for $s_{i_j} \in S$, $k \geq 2$.

But this contradicts maximality of $(*)$.

$\leadsto$ insert $u_i s_{i_1} \dots s_{i_{k-1}}$ between u_i and u_{i+1}.

(ii) $u \leq_R w$ then $w = u \cdot s_1 \dots s_k$

Hence $l(w) = l(u) + k$ and $s_1 \dots s_k = u^{-1} w$.

Hence the right-hand side.

The converse uses similar observations.

(iii) This uses the fact that in every

(iii) This uses the fact that in every finite Coxeter group there exists a unique longest element w_0.
But then put, for $w \in W$ fix,
$v := w^{-1} w_0$. Then, using (ii), we may observe that $l(w_0) = l(w^{-1} w_0) + l(w)$ implies $w \leq_R w_0$.

(iv) follows directly from the definitions.

(v) similar to the proof of the statement for Bruhat order.

(vi) if $s \in D_L(w) \cap D_L(u)$, then
$l(su) < l(u)$ and $l(sw) < l(w)$.
Therefore (by the exchange condition) there exist reduced expressions for both u and w which start with s.
Using this it directly follows from the Definition of $\leq_R$ that $u \leq_R w$ implies $su \leq_R sw$. $\square$

We will now characterize weak order in terms of associated reflections.

all reflections, $T = W^{-1} S W$

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<u>Def 6.19</u> Put $T_L(w) := \{ t \in T \mid tw < w \}$,
and $T_R(w) := \{ t \in T \mid wt < w \}$,
the <u>left-/right-associated reflections</u>.

<u>Prop. 6.20</u> $u \leq_R w \iff T_L(u) \subseteq T_L(w)$.

<u>Proof</u>: If $u = s_1 \dots s_k$ and $w = s_1 \dots s_k s_{k+1} \dots s_q$ are reduced expressions. Then

$$T_L(u) = \{ s_1 s_2 \dots s_i \dots s_2 s_1 \mid 1 \leq i \leq k \}$$
$$\subseteq \{ s_1 s_2 \dots s_i \dots s_2 s_1 \mid 1 \leq i \leq q \} = T_L(w).$$

Conversely, if $T_L(u) \subseteq T_L(w)$, then suppose $u = s_1 \dots s_k$ is reduced.

Put $t_i := s_1 s_2 \dots s_i \dots s_2 s_1$ for $1 \leq i \leq k$.
Then $T_L(u) = \{ t_i, 1 \leq i \leq k \} \subseteq T_L(w)$.

We need to prove (by induction!) the following claim for $1 \leq i \leq k$:

$C(i)$: there exists a reduced expression
$$w = s_1 s_2 \dots s_i s_1' \dots s_{q-i}'.$$

Clearly $C(0)$ is true.

Now suppose $C(i)$ is true for some i ...

Clearly $C(0)$ is true.

Now suppose $C(i)$ is true for some i. Then the fact that $t_{i+1} \in T_L(w)$ and that $t_{i+1} \neq t_j \ \forall j \leq i$ shows that

$$t_{i+1} = s_1 \dots s_i s_1' \cdots s_m' \dots s_1' s_i \dots s_1$$

for some $1 \leq m \leq q-i$.

Hence $w = t_{i+1}^2 w$

$$= (s_1 \dots s_{i+1} \dots s_1)(s_1 \dots s_i s_1' \dots \hat{s}_m' \dots s_{q-i}')$$

$$= s_1 \dots s_{i+1} s_1' \dots \hat{s}_m' \dots s_{q-i}' \ .$$

But this proves $C(i+1)$.

Hence inductively $C(i)$ holds true for all $1 \leq i \leq k$. $\square$

Cor 6.21

The map $w \mapsto T_L(w)$ is an order-preserving and rank-preserving embedding of the weak order poset on W into the poset of all finite subsets of T.

this is in fact a lattice!

This implies:

Cor 6.22 Let $u \leq_R w$. Put $m := \ell(w) - \ell(u)$.
Then for $0 \leq k \leq m$:

$$\#\{v \in [u,w]_R : \ell(v) = \ell(u) + k\} \leq \binom{m}{k}.$$

Prop. 6.23 Symmetry of weak order

If $u \leq_R w$, then $[\mathbb{1}, u^{-1}w]_R \cong [u,w]_R$.

Proof: We prove that the map: $x \mapsto ux$ is a poset isomorphism from $[u,w]_R$ to $[\mathbb{1}, u^{-1}w]_R$. Basic properties of the length function imply:

$$\ell(w) = \ell(u) + \ell(u^{-1}w)$$
$$\overset{a)}{\leq} \ell(u) + \underbrace{\ell(x) + \ell(x^{-1}u^{-1}w)}$$
$$\overset{b)}{\geq} \ell(ux) + \ell(x^{-1}u^{-1}w) \overset{c)}{\geq} \ell(w).$$

by the triangle inequality

We can conclude:

$x \leq_R u^{-1}w \iff$ there is an equality in a)
$\iff$ there is an equality in b) and c)

and c)

$\Leftrightarrow u \leq_R ux \leq_R w.$

Hence $x \in [\mathbb{1}, u^{-1}w]_R \Leftrightarrow ux \in [u, w]_R$.

And if this is true, then $\ell(ux) = \ell(u) + \ell(x)$.

$\square$

If time permits comment on

→ word problem (sec 3.3 BB)

→ lattice property (sec 3.2 BB)

→ order complex (thm 3.2.7 BB)