12 - Bruhat order

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6 Orders on Coxeter groups

Part I: Bruhat order

Throughout this section let (W,S) be a Coxeter system.

Def 6.1 Let $T := WSW^{-1}$ be the set of reflections in W. Let u, w be in W. We put

- $u \xrightarrow{t} w$ if $w = u \cdot t$ and $\ell(w) > \ell(u)$, for a fixed $t \in T$

  u is covered by w

- $u \to w$ if $u \xrightarrow{t} w$ for some $t \in T$
- $u \leq w$ if there exists a sequence $u_i \in W$ s.th. $u = u_0 \to u_1 \to u_2 \to \ldots \to u_k = w$.

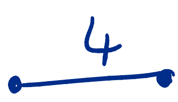
We call the so defined partial order on W the <u>Bruhat order</u>.

(ÜA): Prove that the defn. $u \overset{t}{\rightsquigarrow} w$ if $w = tu$ and $\ell(w) > \ell(u)$ yields the same order

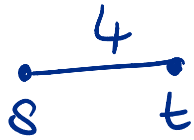
The <u>Bruhat graph</u> is the (directed) graph with vertices $w \in W$ and an edge from u to w if $u \to w$ as defined above.

Example 6.2

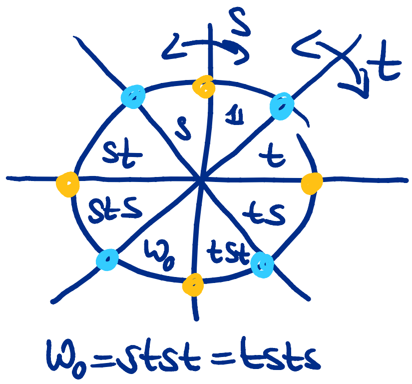
1) W of type $B_2 = I_2(4)$



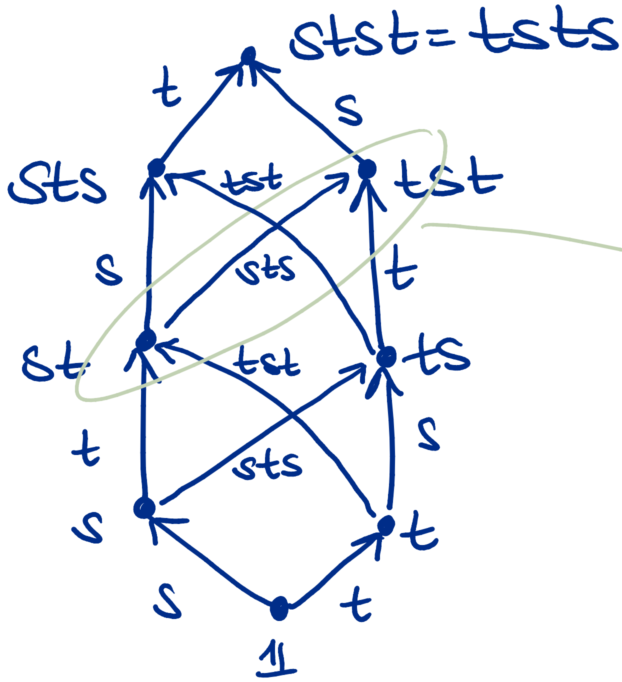
1) W of type $B_2 = I_2(4)$



Then: $T = \{s, t, sts, tst\}$

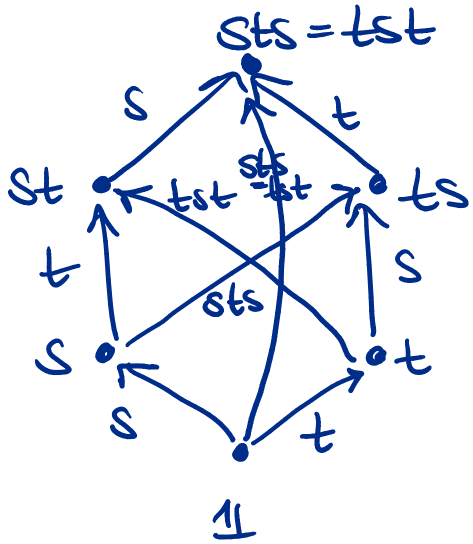


and the Bruhat graph looks as follows:



$tst = st \cdot (\underbrace{tstst})$
$= ststt$
$= sts \in T$

2) W of type A_2, i.e. $W \cong Sym(3)$



3) the poset of Bruhat order for the dihedral group of type $I_2(m)$

⇒ The poset of Bruhat order for the dihedral group of type $I_2(m)$ $\underset{s}{\circ}\overset{m}{-\!-\!-\!-}\underset{t}{\circ}$ is always a poset of length m (i.e. m levels/ranks) with two elements in each rank – except for top & bottom where there is always exactly one element.

6.3 Def. Let $s_1 s_2 \ldots s_q$ be a word in S.
A *subword* of this word is a word of the form $s_{i_1}, \ldots s_{i_j}$ with $1 \leq i_1 \leq \ldots \leq i_j \leq q$.

Lemma 6.4 [BB Le 2.2.1]
For $u, w \in W$, $u \neq w$, let $w = s_1 s_2 \ldots s_q$ be a reduced expression and suppose there exists a subword of this expression which is a word for u.
Then there exists $v \in W$ s.th. all of the following:
(i) $v > u$
(ii) $l(v) = l(u) + 1$
(iii) Some reduced expr. of v is a subword of $s_1, s_2, \ldots s_q$.

Proof: Of all reduced subwords for u in the word for w pick the one where the last deleted index is minimal, i.e.

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deleted index is minimal, i.e.

$$u = s_1 \ldots \hat{s}_{i_1} \ldots \hat{s}_{i_k} \ldots s_q \qquad 1 \leq i_1 < \ldots < i_k \leq i_q$$

with i_k minimal. Let

$$t := s_q s_{q-1} \ldots s_{i_k} \ldots s_{q-1} s_q$$

Then $u \cdot t = s_1 \ldots \hat{s}_{i_1} \ldots \hat{s}_{i_{k-1}} \ldots s_{i_k} \ldots s_q$.

Hence $l(ut) \leq l(u) + 1$.

We claim that in fact $ut > u$.

If so the element $ut =: v$ satisfies (i)–(iii) and we are done.

Suppose for a contradiction that $ut < u$.
Then, using the (strong) exchange property we either have

$$(*_1) \qquad t = s_q s_{q-1} \ldots s_p \ldots s_{q-1} s_q$$

for some $p > i_k$, or

$$(*_2) \qquad t = s_q \ldots \hat{s}_{i_k} \ldots \hat{s}_{i_j} \ldots s_r \ldots \hat{s}_{i_j} \ldots \hat{s}_{i_k} \ldots s_q$$

for some $r < i_k$, $r \neq j$.

In the first case $(*_1)$

$$\begin{aligned} w &= wt^2 \\ &= (s_1 s_2 \ldots s_q)(s_q \ldots s_{i_k} \ldots s_q)(s_q \ldots s_p \ldots s_q) \end{aligned}$$

$$= (s_1 s_2 \dots s_q)(s_q \dots s_{i_k} \dots s_q)(s_q \dots s_p \dots s_q)$$
$$= s_1 \dots \hat{s}_{i_k} \dots \hat{s}_p \dots s_q$$

which contradicts the fact that the word for w was reduced and $\ell(w) = q$.

In the second case $(*_2)$

$$u = ut^2$$
$$= (s_1 \dots \hat{s}_{i_1} \dots \hat{s}_{i_k} \dots s_q)(s_q \dots \hat{s}_{i_k} \dots s_r \dots \hat{s}_{i_k} \dots s_q)$$
$$\cdot (s_q \dots s_{i_k} \dots s_q).$$
$$= s_1 \dots \hat{s}_{i_1} \dots \hat{s}_r \dots s_{i_k} \dots s_q.$$

But this contradicts minimality of the index i_k.

Note that it is possible that $r < i_1$ in this situation.

$\square$

<u>Thm 6.5</u> <u>The subword property</u>

Let $w = s_1 \dots s_q$ be a reduced expression. Then $u \leq w$ if and only if there exists a reduced expression of u which is a subword of the given one of w.

one of w.

<u>Proof:</u> „$\Leftarrow$" follows directly from Le 6.4 with a quick induction on $\ell(w)-\ell(u)$.

„$\Rightarrow$" Suppose $u \leq w$. Then there exists a sequence of x_i in W and t_i in T s.th.

$$u = x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} x_3 \longrightarrow \dots \xrightarrow{t_m} x_m = w.$$

By construction $x_{m-1} = w t_m$

$$= s_1 \dots \hat{s}_i \dots s_q$$

for some i, by the strong exchange property for W.

Similarly, $x_{m-2} = x_{m-1} t_{m-1}$

$$= s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_q .$$

Argue inductively to obtain a word for u which is obtained from the word for w by deleting m letters, i.e.

$$u = s_1 \dots \hat{s}_{i_1} \dots \hat{s}_{i_m} \dots s_q .$$

By the deletion property this word for u must contain a reduced expression for u — which is the desired one. $\square$

Cor 6.6

For u, w the following are equivalent:

- $u \leq w$
- every reduced expression of w has a subexpression which is a word for u.
- some reduced expression of w has a subexpression which is a word for u.

This is immediate from Thm 6.5 & Def.

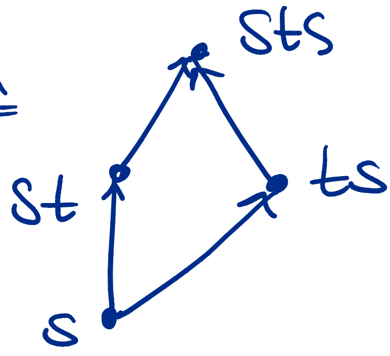
Def 6.7

Suppose u, w in W are comparable wrt „$\leq$". The Bruhat interval between u, w in W is defined as follows:

$$[u,w] := \{ v \in W \mid u \leq v \leq w \}.$$

Bsp. 6.8

1) A_2: $[s, sts] \;\hat{=}$



Lemma 6.9

Bruhat intervals are finite. (even if W is infinite)

Proof. Every reduced expression for w has

Proof. Every reduced expression for w has $2^{l(w)}$ subwords. And there is an injective map from $[u,w]$ into this set. $\square$

Rmk. The proof of the lemma shows that $\operatorname{card}([u,w]) \leq 2^{l(w)-l(u)}$

Lemma 6.10

The map $w \longmapsto w^{-1}$ on W is an order preserving automorphism of the poset on W wrt Bruhat order.

We often say: "autom. of Bruhat order".

Proof. Being a subword is not affected by reversing both the word for w and the word for u.

Prop. 6.11 The chain property

Suppose $u < w$. Then there exists a chain $u = x_0 < x_1 \dots < x_k = w$ sth. $l(x_i) = l(u) + i$ for all $1 \leq i \leq k$.

Proof This is a direct consequence [illegible] subword property.

Proof This is a direct consequence of Lemma 6.4 and the subword property. $\square$

Rmk 6.12

The chain property means that the Bruhat order poset is graded where the rank is given by the length-function on W.

The same holds true for every Bruhat interval.

very useful technical statement

Prop. 6.13 The lifting property

Suppose $u < w$ and s is such that $\underbrace{l(sw) < l(w)}_{s \in D_L(w)}$ and $\underbrace{l(su) \not< l(u)}_{s \notin D_L(u)}$.

the left-descend set of w.

Then $u \leq sw$ and $su \leq w$.

Proof: Choose a reduced expression for $sw = s_1 s_2 \dots s_q$.

Then $w = s \cdot s_1 s_2 \cdots s_q$ is also reduced and there exists a subword of $s, s_1, \dots, s_q$ s.th. $u = s_{i_1} s_{i_2} \cdots s_{i_k}$.

s.t.h. $u = s_{i_1} s_{i_2} \cdots s_{i_k}$.

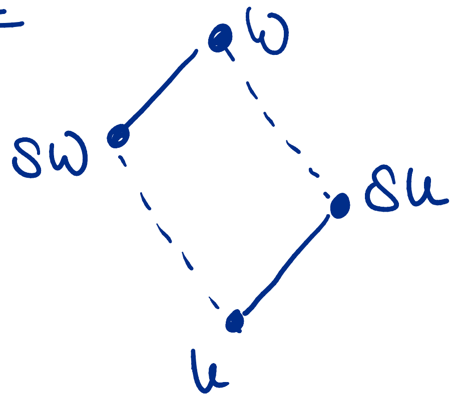
Now s_{i_1} can not be s as $s \notin D_L(u)$.

Hence $s_{i_1} \cdots s_{i_k}$ is in fact a subword of the word $s_1 \cdots s_q$ for sw.

This implies $u \leq sw$.

Moreover $\overbrace{s \cdot s_{i_1} \cdots s_{i_k}}^{= s \cdot u}$ is a subword of $s \cdot s_1 s_2 \cdots s_q$ and hence $su \leq w$. $\square$

Rmk



lift of an edge in the Bruhat graph

We will revisit Bruhat order and its more refined properties when talking about buildings later in the course.