10 - The Coxeter complex

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5.11 Thm

The poset $\Sigma = \Sigma(W,S)$ from Def. 5.6 is a simplicial complex.
In addition it is a thin, labellable chamber complex of rank $n = |S|$ and the W-action on Σ is type preserving.

Def. 5.12 Properties of simplicial complexes

Let Δ be a simplicial complex. Then

(1) Δ is a chamber complex if all maximal simplices have the same rank (or dimension).
In this case we call the maximal simplices chambers of Δ.
The codimension one faces of a chamber are its panels.

(2) a chamber complex is thin if every codimension one face of a chamber is contained in precisely two chambers.

(3) a gallery in a chamber complex is a sequence of chambers and

(3) a gallery in a chamber complex is a sequence of chambers such that any two consecutive ones have (at least) a panel in common.

(4) A labeling of a chamber complex Δ by a set I is a map from the vertex set V of Δ to I s.th. the restriction of this map to every chamber in Δ is a bijection.

A chamber has vertices colored by I.

We say a chamber complex is labellable if there exists a set I and a labeling of Δ by I.

(5) Given a labeling $l: V(\Delta) \to I$ the type of a vertex $v \in V(\Delta)$ is the image $l(v) \in I$.

An action $G \curvearrowright \Delta$ is type preserving if $\forall g \in G$ and all vertices v in Δ one has $l(v) = l(g \cdot v)$.

Group action preserves the color of vertices (and hence any simplex if the action is simplicial).

We will now prove Thm 5.10

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To see that Σ is a simplicial cplx
we need to verify:

(a) any two elements of Σ have a greatest lower bound

(b) for any $A \in \Sigma$ the poset $\Sigma_{\leq A}$ is isomorphic to the set of subsets of some finite set.

<u>Proof of a)</u>

We use the W-action on Σ to restrict to the case that one of the two elements, say A, is a face of the fundamental chamber $C = \{1\}$.
We must hence prove, that a special subgrp $A = \langle S' \rangle$ and a special coset $\mathcal{B} = w\langle S'' \rangle$ have a least upper bound in the set Σ of special cosets (wrt the ordering by inclusion).

Such a least upper bound is another special coset containg A and hence in particular the identity element of W.
It must hence be a special subgroup.

Moreover this subgroup contains $\mathcal{B}$ and

Moreover, this subgroup contains $\mathcal{B}$ and hence $w \cdot 1$ (as $1 \in \langle S'' \rangle$).
Therefore also w^{-1} and thus

$$w^{-1}\mathcal{B} = w^{-1} w \langle S'' \rangle = \langle S'' \rangle$$

must be contained.

We conclude that the upper bound of A and $\mathcal{B}$ must hence be a special subgrp containing w, $\langle S' \rangle$ and $\langle S'' \rangle$.
To identify this subgroup we consider the following Lemma:

<u>Lemma 5.13</u> (Brown II §3 Cor 3)

Suppose (W,S) satisfies the deletion condition.
(so as we know (W,S) is a Coxeter system)
Then for any $w \in W$ there is a subset $S(w) \subset S$ s.th. all reduced expressions of w involve precisely the letters in $S(w)$.
Moreover, $S(w)$ is the smallest subset S' of S s.th. $\langle S' \rangle$ contains w.

<u>Proof</u> The existence of the set $S(w)$ follows from Tits' solution to the word problem. Since all reduced decomp. can be transformed into one another using braid

transformed into one another using braid moves. But braid moves don't change the set of letters used.

To prove the second suppose S' is some subset of S with $w \in \langle S' \rangle$.
Now repeatedly apply the deletion condition to any (fixed) S'- word representing w.
This results in a reduced word for w which only uses letters in $S(w)$.
Thus $S(w) \subset S'$. $\square$

Back to the proof of a):

This Lemma 5.13 tells us, that the smallest possible special subgroup satisfying our needs is spanned by $S' \cup S'' \cup S(w)$ $\square$

Proof of b):

It suffices to prove this assertion for $A = C = \{1\}$.
In this case $\Sigma_{\leq C}$ is the poset of special subgroups of W ordered by reverse inclusion.
Consider the following Lemma:

Lemma 5.14

The map $S' \longmapsto \langle S' \rangle$ is a poset isomorphism

The map $S' \mapsto \langle S' \rangle$ is a poset isomorphism from the set of subsets of S to the set $\Sigma_{\leq C}$ of special subgrps of W.

<u>proof:</u>
Define a map in the other direction by putting
$$W' \mapsto W' \cap S.$$
It is clear that $W' = \langle W' \cap S \rangle$ if W' is a special subgroup.
Moreover, for any $S' \subseteq S$ we have $S' \subseteq \langle S' \rangle \cap S$.
To prove the opposite inclusion suppose that $s \in \langle S' \rangle \cap S$.
Then s is a product of elements in S'.
Repeated application of the deletion condition yields a reduced expression for s with all letters in S'. This expression must be of length 1 as all reduced expressions have the same length. Hence $s \in S'$ and
$$\langle S' \rangle \cap S \subset S'.$$
But then $S' = \langle S' \rangle \cap S$ and the map we have constructed is the inverse to the map sending S' to $\langle S' \rangle$. Hence the assertion. $\square$

Using the lemma we obtain

using the lemma we obtain

$$\Sigma_{\leq C} \cong (\text{subsets of } S)^{op} \quad \phi$$
$$\cong \text{subsets of } S$$

The iso ϕ is given by: $S' \longmapsto S \setminus S'$.
Hence b) and therefore Σ is indeed a simplicial complex. ∎

To see the rest of the claim:

The group W acts transitively on the simplices of maximal rank $|S| = n$.
(All of them are of the form $w \cdot W_\emptyset = \{w\}$).
Hence Σ is a chamber complex.

To see that Σ is thin argue as follows:

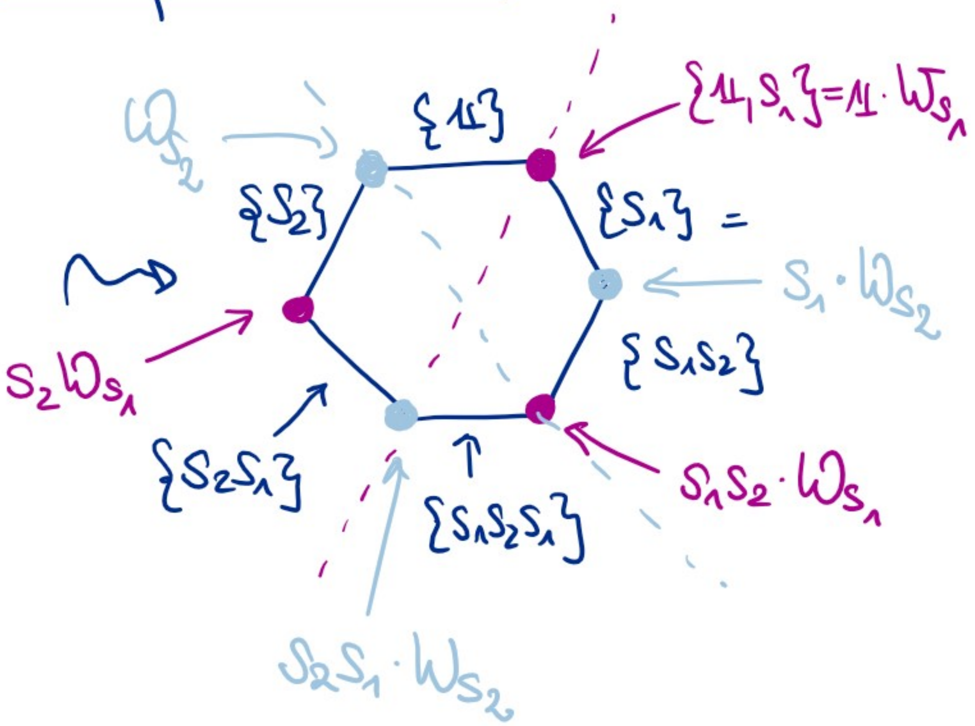
Recall that the chambers are all of the form $w \cdot W_\emptyset = \{w\}$ for some $w \in W$.
The simplices one rank lower / of codim 1 are the sets $u \cdot \langle s \rangle$ for $s \in S$, $u \in W$.
Such a panel is a face of exactly two chambers, namely precisely $\{us\} = us \cdot W_\emptyset$ and $\{u \cdot 1\!\!1\} = u W_\emptyset$. Hence Σ is thin.

Finally, a W-invariant labeling $\lambda : \Sigma \to I$ can be defined as follows:
Put $I := S$ and let $\lambda(w \cdot \langle S' \rangle) = S - S'$.

can be defined as follows:
Put $I := S$ and let $\lambda(w \cdot \langle S' \rangle) = S - S'$.
It is clear that this is W-invariant.
And $\lambda(\text{chamber}) = S$, i.e. the desired bijection.
$\square$

<u>Def</u>. 5.15
We call the labeling from the proof above the <u>canonical labeling</u> of Σ.
We say two distinct chambers c, d in Σ <u>s-adjacent</u> if $c \cap d$ is a panel and $\lambda(c \cap d) = S - \{s\}$.

<u>Example 5.16 (5.8 continued)</u>

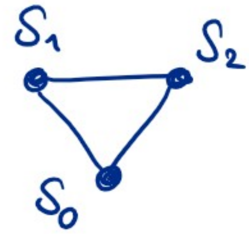


This complex is a thin chamber complex
with chambers being edges in six lines

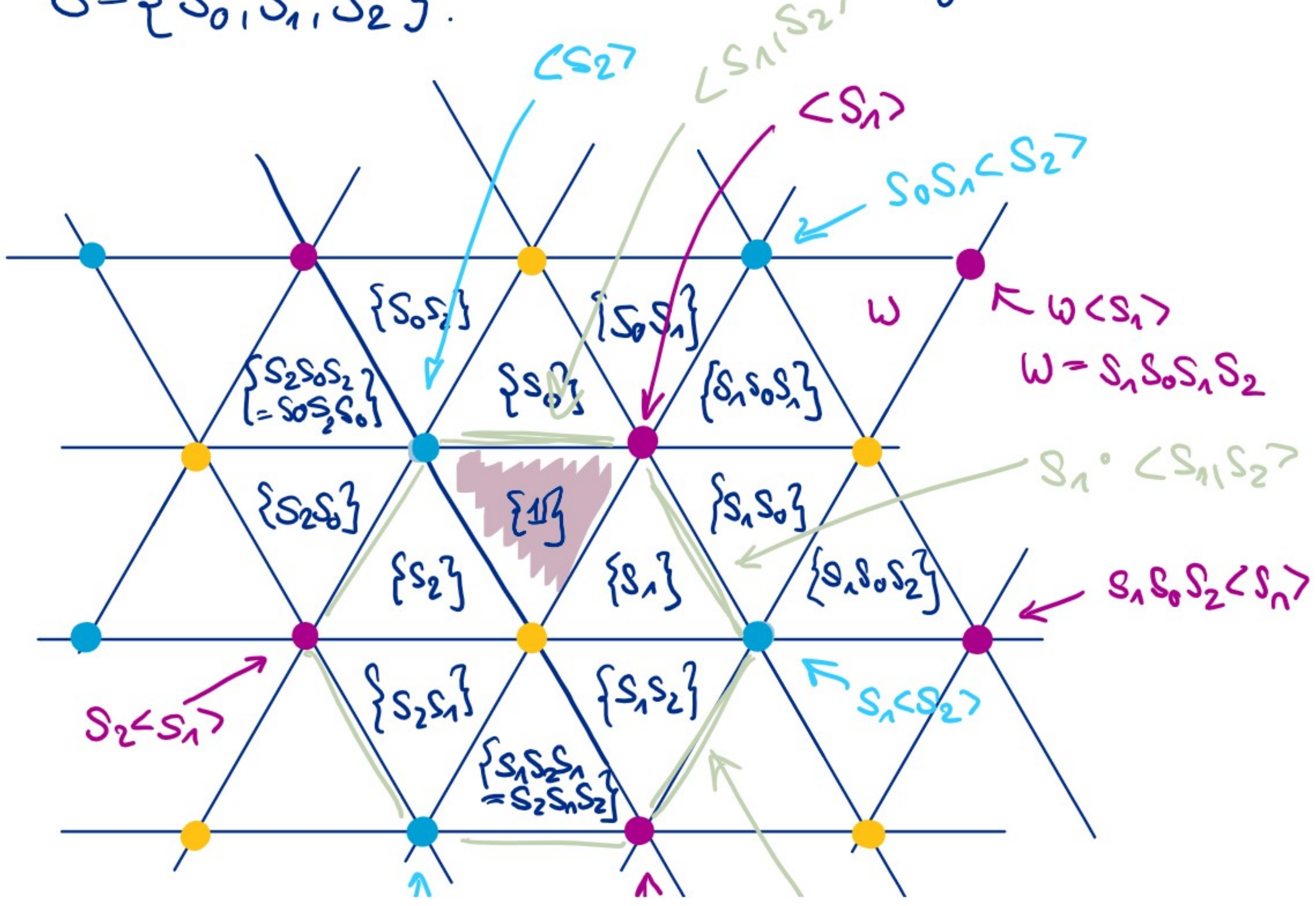
This complex is a thin chamber complex with chambers being edges, i.e. simplices of rank 2 and dimension one.
Each panel is either
of the form $w\cdot\langle\{s_1\}\rangle$ and label $S-\{s_1\}$ $=\{s_2\}$ (shown in pink)

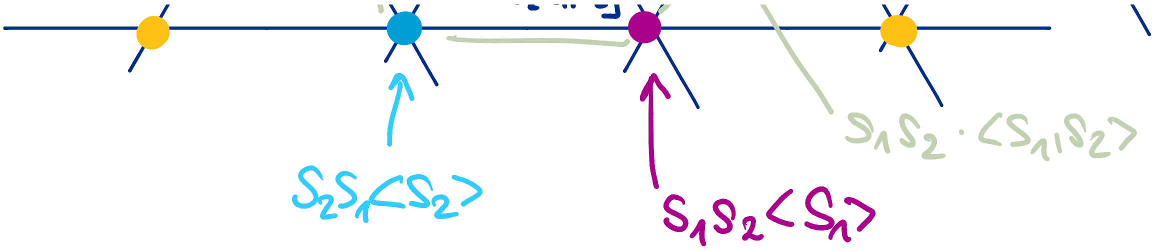
or of the form $w\cdot\langle\{s_2\}\rangle$ and with label $S-\{s_2\}=\{s_1\}$ (shown in blue).

Example 5.16 Type $\tilde{A}_2$:



$S=\{s_0,s_1,s_2\}$.




$s_2s_1\langle s_2\rangle$
$s_1s_2\langle s_1\rangle$
$s_1s_2\cdot\langle s_1 s_2\rangle$