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Generalized Schottky groups, oriented flag manifolds and proper actions

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Abstract

This work is dedicated to the study of proper actions of discrete subgroups of Lie groups on subsets of associated homogeneous spaces.

In the first part, we study actions of discrete subgroups Γ of semi–simple Lie groups G on associated oriented flag manifolds. These are quotients G/P, where the subgroup P lies between a parabolic subgroup and its identity component. Our main result is a description of (cocompact) domains of discontinuity for Anosov representations, in terms of combinatorial data. This generalizes results of Kapovich–Leeb–Porti to the oriented setting. We show that this generalization gives rise to new domains of discontinuity that are not lifts of known ones, e.g. in the case of Hitchin representations acting on oriented Grassmannians. We also apply the finer information inherent to the oriented setup to distinguish some connected components of Anosov representations. This part constitutes joint work with Florian Stecker.

The second part of this thesis, consisting of two chapters, focuses on a method of generalizing classical Schottky groups in $\mathrm{PSL}(2,\mathbb{R})$ using partial cyclic orders. We investigate two families of spaces carrying partial cyclic orders, namely Shilov boundaries of Hermitian symmetric spaces and complete oriented flags in $\mathbb{R}^n$, and prove that they both satisfy a number of topological properties. These spaces are then used to construct generalized Schottky representations into Hermitian Lie groups and $\mathrm{PSL}(n,\mathbb{R})$. We show that in the first case, generalized Schottky representations coincide with maximal representations (for surfaces with boundary) and that they yield examples of Anosov representations in both cases. Several of the results in this part are joint work with Jean–Philippe Burelle. The description of the partial cyclic order on Shilov boundaries, the relation of generalized Schottky representations in Hermitian Lie groups with maximal representations, and the analysis of generalized Schottky groups in $\mathrm{Sp}(2n,\mathbb{R})$ appeared in [BT17]. The definition of the partial cyclic order on complete oriented flags is based on discussions during a visit of Jean-Philippe to Heidelberg in September 2016.

The final part of this thesis is concerned with discrete subgroups of the group of invertible affine transformations of $\mathbb{R}^n$. Let $\rho\colon \Gamma \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ be a representation of a word hyperbolic group whose linear part is Anosov with respect to the stabilizer of a maximal isotropic subspace. We prove that properness of the induced affine action is equivalent to the nonvanishing of a generalized version of the Margulis invariant. This generalizes a theorem of Goldman–Labourie–Margulis, who proved this equivalence for representations of surface groups with Fuchsian linear parts. Our results on affine actions are joint work with Sourav Ghosh and are available on arXiv [GT17].

Zusammenfassung

Diese Arbeit ist dem Studium eigentlich diskontinuierlicher Wirkungen diskreter Untergruppen von Liegruppen auf Teilmengen zugehöriger homogener Räumen gewidmet.

Im ersten Teil untersuchen wir Wirkungen diskreter Untergruppen Γ von halbeinfachen Liegruppen G auf zugehörigen orientierten Fahnenmannigfaltigkeiten. Darunter verstehen wir Quotienten G/P, wobei P eine Untergruppe ist, die zwischen einer parabolischen Untergruppe und deren Identitätskomponente liegt. Unser Hauptresultat ist eine Beschreibung (kokompakter) Diskontinuitätsbereiche für Anosov–Darstellungen auf kombinatorische Weise, welche eine Verallgemeinerung eines Resultats von Kapovich–Leeb–Porti auf den orientierten Fall darstellt. Wir zeigen, dass wir dadurch neue Diskontinuitätsbereiche erhalten, die nicht lediglich Hochhebungen von Bereichen im unorientierten Fall sind. Beispiele hierfür beinhalten kokompakte Diskontinuitätsbereiche in orientierten Grassmannschen für Hitchin–Darstellungen. Darüber hinaus nutzen wir die durch die Orientierung zusätzlich gegebene Information, um Zusammenhangskomponenten von Anosov–Darstellungen zu unterscheiden. Dieser Teil basiert auf Zusammenarbeit mit Florian Stecker.

Das zentrale Thema des zweiten Teils (Kapitel 3 und 4) ist eine Verallgemeinerung klassischer Schottky–Gruppen in $\mathrm{PSL}(2,\mathbb{R})$ unter Benutzung partieller zyklischer Ordnungen. Wir untersuchen zwei Klassen von Räumen, die mit partiellen zyklischen Ordnungen ausgestattet sind, und beweisen diverse topologische Eigenschaften, die sie erfüllen: Shilov–Ränder von Hermiteschen symmetrischen Räumen und vollständige orientierte Fahnen in $\mathbb{R}^n$. Entsprechend konstruieren wir verallgemeinerte Schottky–Darstellungen in Hermiteschen Liegruppen und $\mathrm{PSL}(n,\mathbb{R})$ und beweisen, dass sie im ersten Fall mit maximalen Darstellungen (von Flächengruppen mit Rand) übereinstimmen. Darüber hinaus liefern sie in beiden Fällen Beispiele für Anosov–Darstellungen. Einige der Resultate sind in Zusammenarbeit mit Jean–Philippe Burelle entstanden. Die Beschreibung der partiellen zyklischen Ordnung auf Shilov–Rändern, die Relation zwischen verallgemeinerten Schottky–Darstellungen und maximalen Darstellungen sowie die Beschreibung verallgemeinerter Schottky–Gruppen in $\mathrm{Sp}(2n,\mathbb{R})$ sind in [BT17] erschienen. Die Definition der partiellen zyklischen Ordnung auf vollständigen orientierten Fahnen basiert auf Diskussionen während eines Besuchs Jean–Philippes in Heidelberg im September 2016.

Der letzte Teil dieser Arbeit behandelt diskrete Untergruppen der Gruppe invertierbarer affiner Abbildungen des $\mathbb{R}^n$. Sei $\rho\colon \Gamma \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ eine Darstellung einer Wort–hyperbolischen Gruppe, deren Linearteil Anosov bezüglich des Stabilisators eines maximalen isotropen Unterraums ist. Wir zeigen, dass die induzierte Wirkung auf $\mathbb{R}^{2n+1}$ genau dann eigentlich diskontinuierlich ist, wenn eine Verallgemeinerung der Margulis–Invariante keine Nullstelle hat. Dies verallgemeinert einen Satz von Goldman–Labourie–Margulis, die diese Äquivalenz im Fall von Darstellungen von Flächengruppen, deren Linearteil Fuchssch ist, bewiesen haben. Unsere Resultate

über affine Darstellungen sind in Zusammenarbeit mit Sourav Ghosh entstanden und auf arXiv verfügbar [GT17].

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Contents

1 Introduction

One topic that has followed me throughout my studies is that of *geometric structures*, in the sense of Ehresmann. Following Felix Klein's Erlangen program, a geometry is a pair (G, X) consisting of a manifold X and a Lie group G acting transitively on X. A (G, X)–*geometric structure* on a manifold M is an equivalence class of atlases whose charts map into X, with transition functions in G. This approach unifies many interesting types of geometries, including constant curvature Riemannian structures, flat affine structures and flat projective structures. Ehresmann initiated the study of these "locally homogeneous structures" in the general setting [Ehr36]. By piecing together charts, one associates a *developing map* from the universal cover of M to X to such a structure. There is a corresponding representation $\pi_1(M) \to G$ making the developing map equivariant, the *holonomy representation*.

Thurston showed that the task of understanding all the different ways a compact manifold M can be equipped with a (G, X)–structure is essentially equivalent to the task of understanding all holonomy representations of the fundamental group of M into the group of transformations G [Thu79]: Consider the space of *marked* (G, X)*–structures* on M, that is, of pairs (N, ϕ) consisting of a (G, X)–manifold N and a diffeomorphism $\phi\colon M \to N$. Two such structures (N_i, ϕ_i), $i = 1, 2$, are considered equivalent if there is a diffeomorphism $f\colon N_1 \to N_2$ respecting the (G, X)–structures such that ϕ_2 and $f \circ \phi_1$ are isotopic. Then the correspondence between a geometric structure and its holonomy representation induces "almost" a local homeomorphism between the space of marked (G, X)–structures on M and the space $\mathrm{Hom}(\pi_1(M), G)/G$, where G acts by conjugation (it is a local homemorphism in many cases, but a counterexample to the general statement can be found in [Bau14]).

This work is very much inspired by this correspondence. In various geometric settings, we study homomorphisms of word hyperbolic groups into Lie groups and identify domains in associated homogeneous spaces where the image group acts properly discontinuously. Interpreting such homomorphisms as holonomy representations of a geometric structure on the quotient gives them a geometric meaning.

1.1 Anosov representations and oriented flag manifolds

Anosov representations were originally introduced by Labourie in order to study the Hitchin component in $\mathrm{PSL}(n, \mathbb{R})$ [Lab06]. This is the component of the representation variety of a closed surface group containing *Fuchsian representations*, which are compositions of a Fuchsian representation into $\mathrm{PSL}(2,\mathbb{R})$ with the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(n,\mathbb{R})$. They have since been studied in their own right and have been established as a meaningful generalization of convex cocompact representations into Lie groups G of higher rank. While the original definition was restricted to representations of surface groups into $\mathrm{PSL}(n,\mathbb{R})$, it was extended to representations of any word hyperbolic group Γ into any semi–simple Lie group G in [GW12]. Furthermore, the original definition was dynamic in nature, involving the geodesic flow and contraction/expansion properties of associated bundles. By now, a range of different equivalent definitions exists. They were developed in [KLP14b], [KLP14a], [GGKW17] and [BPS16] (see [KLP17] for an overview). Throughout this work, several of these equivalent definitions of an Anosov representation are used, depending on which one is most convenient given the situation at hand (see Definition 2.3.2, Theorem 4.4.3, Definition 5.1.8 and Theorem 5.1.12).

Anosov representations have multiple desirable properties. They form an open subset of the representation variety, so any small deformation of an Anosov representation is still Anosov. Moreover, every Anosov representation is a quasiisometric embedding of Γ, equipped with the word metric, into G. They also admit continuous equivariant boundary maps (also called limit maps) $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ into a *flag manifold* $\mathcal{F}_\theta$ depending on the representation. A flag manifold is a G–homogeneous space $\mathcal{F}_\theta = G/P_\theta$, where $\theta \subset \Delta$ is a subset of the simple restricted roots and P_θ is the parabolic subgroup it determines. In the case of $\mathrm{SL}(n,\mathbb{R})$, these are exactly all spaces of (partial) flags in $\mathbb{R}^n$. The feature we want to emphasize most, however, is that Anosov representations give rise to properly discontinuous actions of Γ on some flag manifolds $\mathcal{F}_\eta$. Instances of such actions were described in [GW12] and investigated more systematically in [KLP18]. For many choices of such a flag manifold $\mathcal{F}_\eta$ they found cocompact domains of discontinuity $\Omega \subset \mathcal{F}_\eta$, open Γ–invariant subsets on which Γ acts properly discontinuously with compact quotient $\Gamma\backslash\Omega$. Interestingly, it is possible that no cocompact domain of discontinuity exists in the flag manifold $\mathcal{F}_\eta$, but in a special finite cover of $\mathcal{F}_\eta$ we call an *oriented flag manifold.* Before giving the definition, we consider the following example.

Let Σ be a compact surface with nonempty boundary and negative Euler characteristic, and $\rho\colon \pi_1(\Sigma) \xrightarrow{\rho_0} \mathrm{SO}_0(2,1) \to \mathrm{SL}(3,\mathbb{R})$ the holonomy representation of a complete hyperbolic structure such that all ends are funnels. It is Anosov with respect to the stabilizer of a complete flag, and the image of the limit map ξ is a Cantor set of flags (see Figure 1.1). The maximal domain of discontinuity in $\mathbb{RP}^2$ for this $\pi_1(\Sigma)$–action is obtained by removing the union of the projective lines determined by these flags. It consists of a countable number of open quadrilaterals and a central "big" connected component containing the Klein model of the hyperbolic plane, whose quotient is the noncompact surface $\mathbb{H}^2/\rho_0(\pi_1(\Sigma))$. This action has no cocompact domain of discontinuity in $\mathbb{RP}^2$ [Ste18].

However, a cocompact domain of discontinuity for this representation exists in the space of oriented lines, which is simply the double cover S^2 of $\mathbb{RP}^2$. This was first

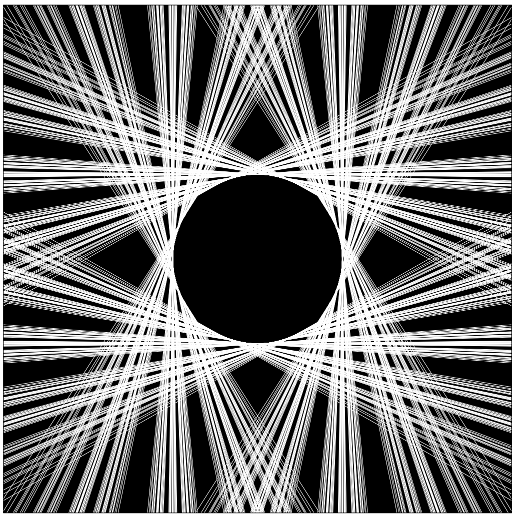


Figure 1.1: The domain of discontinuity (in black) in $\mathbb{RP}^2$ for a surface with boundary.

observed by Choi and can be found in [CG17]. The domain has the following structure: First, there are two copies of the "big" component from the unoriented case. For every point in the limit set of these two copies, we have to remove half of the tangent great circle, where the choices are made consistently so the half circles are all disjoint (see Figure 1.2). As a result, the two regions are joined by a countable number of "strips", one for every gap in the limit set. In the quotient, these strips become tubes connecting the two copies of the surface with open ends. It is therefore homeomorphic to the double of a surface with boundary and in particular compact. Note that the domain in S^2 is not simply connected and the fundamental group of the quotient is not the free group Γ.

With this example in mind, we define oriented flag manifolds as follows.

Definition 1.1.1 (Definition 2.2.1)**.** A *(standard) oriented parabolic subgroup* of G is a proper closed subgroup $P \subsetneq G$ containing B_0, the identity component of a Borel subgroup. The quotient G/P is an *oriented flag manifold.*

In the description of oriented parabolic subgroups, we work with the *extended Weyl group* $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$ instead of the regular Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Here, $K \subset G$ is a maximal compact subgroup and $\mathfrak{a}$ is a maximal abelian subalgebra in the Killing–orthogonal complement of its Lie algebra $\mathfrak{k}$.

Proposition 1.1.2 (Proposition 2.2.5)**.** *Every oriented parabolic subgroup P lies between a unique parabolic subgroup and its identity component. It is specified by the finite group $P \cap \widetilde{W}$ that we call its* oriented parabolic type*.*

The oriented parabolic subgroup of type $R \subset \widetilde{W}$ will be denoted by P_R and its associated oriented flag manifold by $\mathcal{F}_R$. Each oriented parabolic type R is of the

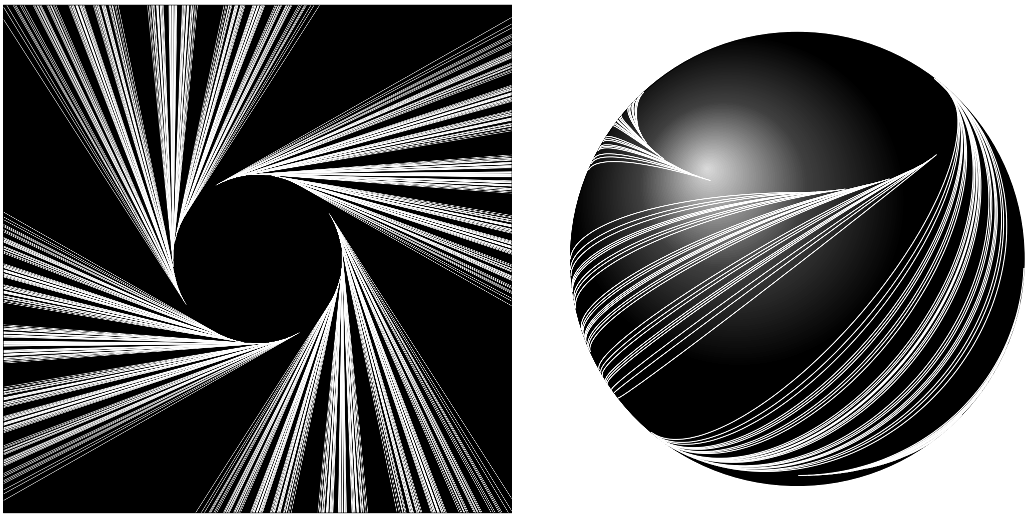


Figure 1.2: The domain of discontinuity (in black) in S^2.

form $R = \langle r(\theta), E\rangle$, where $r\colon \Delta \to \widetilde{W}$ realizes the simple restricted roots as reflections and E is a subgroup of $\overline{M} = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$.

We extend the notion of an Anosov representation to the oriented setting by requiring that the boundary map lifts to $\mathcal{F}_R$.

Definition 1.1.3 (Definition 2.3.3)**.** Let Γ be a non–elementary word hyperbolic group and $R = \langle r(\theta), E\rangle$ an oriented parabolic type such that θ is preserved by the opposition involution. The representation $\rho\colon \Gamma \to G$ is *P_R–Anosov* if it is P_θ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ and there is a continuous, ρ–equivariant lift $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of ξ.

Given such a boundary map $\widehat{\xi}$, its *transversality type* is the relative position between any two of its image points (the G–orbit in the space of pairs). It is represented by an element $w_0 \in \widetilde{W}$. Unlike the unoriented setting, there is in general more than one transverse position.

Our main result generalizes the description of domains of discontinuity in [KLP18] to the oriented setting. It uses a *(w_0–)balanced ideal* or *balanced thickening* (Definition 2.2.30) in $\widetilde{W}$ to determine a subset of the oriented flag manifold in question for every point in the limit set $\widehat{\xi}(\partial_\infty\Gamma)$. These subsets have to be removed, and what remains is the domain of discontinuity. The term "ideal" is defined using the *Bruhat order* (Definition 2.2.15) on $\widetilde{W}$: A subset $I \subset \widetilde{W}$ is an ideal if $w \in I$ and $w' \leq w$ implies $w' \in I$. "w_0–balanced" refers to the action of w_0 on $\widetilde{W}$. It is an involution, and I is balanced if and only if $\widetilde{W} = I \sqcup w_0 I$.

Theorem 1.1.4 (Theorem 2.4.1)**.** *Let Γ be a non–elementary word–hyperbolic group and G a connected, semi–simple, linear Lie group. Let $\rho\colon \Gamma \to G$ be a P_R–Anosov* *representation with limit map* $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$*. Furthermore, let* $w_0 \in \widetilde{W}$ *represent the transversality type of* $\widehat{\xi}$ *and* $I \subset \widetilde{W}$ *be a* w_0*–balanced,* R*–left invariant and* S*–right invariant ideal, where* $S \subset \widetilde{W}$ *is another oriented parabolic type. Define the subset* $\mathcal{K} \subset \mathcal{F}_S$ *by*

$$\mathcal{K} := \bigcup_{x\in\partial_\infty\Gamma} \bigcup_{w\in I} \{[gw] \in \mathcal{F}_S \mid g \in G, [g] = \widehat{\xi}(x)\}.$$

Then $\mathcal{K}$ *is* Γ*–invariant and closed, and* Γ *acts properly discontinuously and cocompactly on the domain* $\Omega = \mathcal{F}_S \setminus \mathcal{K}$*.*

A Hitchin representation $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ defines an action of Γ on the Grassmannian $\mathrm{Gr}(k,n)$ of k–dimensional subspaces of $\mathbb{R}^n$, and we can apply the previous theorem to find cocompact domains of discontinuity. If n is even and k is odd, there is a balanced ideal in W and therefore a cocompact domain of discontinuity $\Omega \subset \mathrm{Gr}(k,n)$ by [KLP18]. For odd $n \geq 5$, no cocompact domains exist in any $\mathrm{Gr}(k,n)$ [Ste18]. However, in several of these cases, we find cocompact domains of discontinuity in *oriented Grassmannians*, whose elements are k–dimensional subspaces that are equipped with an orientation.

Proposition 1.1.5 (Proposition 2.6.3)**.** *Let* $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ *be a Hitchin representation. Assume that either*

(i) n *is even and* k *is odd, or*

(ii) n *is odd and* $k(n+k+2)/2$ *is odd.*

Then there exists a nonempty, open, cocompact domain of discontinuity $\Omega \subset \mathrm{Gr}^+(k,n)$ *in the Grassmannian of oriented* k*–subspaces of* $\mathbb{R}^n$*.*

Another use of the theory we develop concerns connected components of Anosov representations, based on the following proposition.

Proposition 1.1.6 (Proposition 2.3.11)**.** *The set of* P_R*–Anosov representations is open and closed in the space of all* P_θ*–Anosov representations.*

Consequently, this notion can be used to distinguish connected components of Anosov representations: For a P_θ–Anosov representation ρ, there is a unique maximal choice of $\mathcal{F}_R$ such that the boundary map can be lifted (Proposition 2.3.6). If two P_θ–Anosov representations lie in the same connected component, then these maximal choices as well as the transversality types of their limit maps have to agree (Corollary 2.3.13). We apply this fact in Section 2.6.2 to B–Anosov representations of closed surface groups into $\mathrm{SL}(n,\mathbb{R})$ for odd n. Namely, we consider block embeddings constructed by composing a Fuchsian representation into $\mathrm{SL}(2,\mathbb{R})$ with irreducible representations into $\mathrm{SL}(k,\mathbb{R})$ and $\mathrm{SL}(n-k,\mathbb{R})$. For different choices of block sizes, we show that these representations lie in different connected components of B–Anosov representations. Together with an observation by Thierry Barbot and Jaejeong Lee, which is explained in [KK16, Section 4.1], this leads to the following lower bound for the number of connected components.

Proposition 1.1.7 (Corollary 2.6.6). *Let $\Gamma = \pi_1(\Sigma)$, where Σ is a closed surface of genus $g \geq 2$, and let n be odd. Then the space $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n, \mathbb{R}))$ has at least $2^{2g-1}(n-1)+1$ connected components.*

1.2 Generalizing Schottky groups through partial cyclic orders

The topic of the second part of this thesis (Chapter 3 and Chapter 4) is a generalization of classical Schottky groups in $\mathrm{PSL}(2, \mathbb{R})$. By this, we mean free subgroups of $\mathrm{PSL}(2, \mathbb{R})$ admitting a set of generators $\gamma_1 \ldots, \gamma_k$ satisfying the following property: Each generator has an associated pair of (open) intervals $I_i^{\pm} \subset \partial\mathbb{H}^2 \cong \mathbb{RP}^1$ and maps the complement of I_i^- to the closure of I_i^+. Furthermore, the intervals $I_i^{\pm},\ 1 \leq i \leq k$, have to be pairwise disjoint (often, their closures are required to be disjoint as well, but we want to allow this case).

Schottky groups have been generalized in various ways. One such way goes back to the application of the Ping–pong Lemma in the proof of Tits' alternative: If a symmetric set of generators of a linear group is in sufficiently generic position, and each of them has a unique largest eigenvalue, then some powers generate a free group. A similar, quantitative approach was used in [Ben96] to define ϵ–Schottky groups acting on projective spaces, and in [KLP14b] the machinery of Morse actions was used to construct Schottky subgroups of semi–simple Lie groups acting on symmetric spaces of noncompact type.

A slightly different method of generalizing Schottky groups is to start with a collection of disjoint subsets of a space X and to find a set of generators pairing them. Using this idea, Schottky subgroups were constructed in complex projective linear groups acting on $\mathbb{CP}^n$ [Nor86; SV03], in the conformal Lorentzian group $\mathrm{SO}(3, 2)$ acting on the Einstein universe [Fra03; CFL14] and in the affine group acting on $\mathbb{R}^3$ [Dru92].

The generalization we pursue here is in the spirit of the second approach. The structure it is based on is that of a *partial cyclic order* (Definition 3.1.1), which is a relation on triples in a set C generalizing the cyclic order on the circle. When a triple x, y, z is in this relation, we write $\overrightarrow{xyz}$ and call it an *increasing* triple. More generally, any map between sets carrying partial cyclic orders is called increasing if it respects these orders. The group of bijections of C preserving the partial cyclic order will be denoted $\mathrm{Aut}(C)$.

A partial cyclic order on C allows us to define the *interval* between x and z as

$$(\!(x, z)\!) := \{y \in C \mid \overrightarrow{xyz}\}.$$

The *opposite interval* of $(\!(x, z)\!)$ is $-(\!(x, z)\!) := (\!(z, x)\!)$. Now let $I_i^{\pm} \subset X,\ 1 \leq i \leq k$, be pairwise disjoint intervals such that some permutation of the set of their endpoints is in cyclic configuration. Each endpoint may be shared between two consecutive intervals. Let $\gamma_i \in \mathrm{Aut}(C)$, $1 \leq i \leq k$ map $-I_i^-$ to I_i^+. Then $\langle \gamma_1, \dots, \gamma_k \rangle$ is called a *generalized Schottky group*. By the Ping–pong Lemma, it is necessarily a nonabelian free group. We call a representation $\rho\colon \Gamma \to \mathrm{Aut}(C)$ of a free group Γ a *generalized Schottky representation* if it is constructed in this way.

To describe properties of generalized Schottky representations, it is useful to realize Γ as a Schottky subgroup of $\mathrm{PSL}(2, \mathbb{R})$ with the same combinatorial structure as the image in $\mathrm{Aut}(C)$ (see Section 4.1). Our focus lies on the following two extreme cases: A generalized Schottky representation is *purely hyperbolic* if no two intervals share an endpoint, and it is *exhaustive* if the model Schottky group in $\mathrm{PSL}(2, \mathbb{R})$ is a finite area hyperbolization of the interior of a compact surface with boundary.

Our first main results about generalized Schottky groups are the following constructions of boundary maps. Since we start with a very general setup, there is a number of topological properties we require of the partial cyclic order. The most significant of them is *increasing–completeness*, which asks that every increasing sequence has a unique limit. *Properness* is a consistency condition on a pair of nested intervals, while *regularity* describes the behavior of collapsing sequences of nested intervals (see Section 3.1).

Theorem 1.2.1 (Theorem 4.2.2, see also [BT17]). *Let $\rho\colon \Gamma \to G = \mathrm{Aut}(C)$ be an exhaustive generalized Schottky representation, and assume that C is first–countable, increasing–complete and proper.*
Then there is a left–continuous, equivariant, increasing boundary map $\xi : S^1 \to C$.

In the following theorem, we need an extra *contraction condition* for the partial cyclic order. Suppose that every interval is equipped with a canonical metric such that, if $\overline{\gamma(I)} \subset J$ for an element $\gamma \in \mathrm{Aut}(C)$ and intervals I, J, the induced map $\gamma\colon I \to J$ is a contraction.

Theorem 1.2.2 (Theorem 4.2.3). *Let $\rho\colon \Gamma \to G = \mathrm{Aut}(C)$ be a purely hyperbolic generalized Schottky representation. Assume that C is first–countable, increasing–complete and proper, and that it satisfies the contraction condition.*
Then there is a continuous, equivariant, increasing boundary map $\xi\colon \partial_\infty \Gamma \cong \Lambda_\Gamma \to C$.

We have two families of examples that we apply the above constructions to. The first family are Shilov boundaries of Hermitian symmetric spaces of tube type. Let $X = G/K$ be a Hermitian symmetric space of noncompact type of tube type with Shilov boundary $\mathcal{S}$. There is a G–invariant, skew–symmetric function M on triples of transverse points in $\mathcal{S}$, the generalized Maslov index introduced in [CØ01]. Its possible values are $-\mathrm{rk}(X), -\mathrm{rk}(X) + 2, \dots, \mathrm{rk}(X)$, and these values classify the G–orbits of transverse triples. The generalized Maslov index also satisfies a cocycle identity. As a consequence,

$$\overrightarrow{xyz} \quad \Leftrightarrow \quad \mathsf{M}(x, y, z) = \mathrm{rk}(X)$$

defines a G–invariant partial cyclic order on $\mathcal{S}$ (see Proposition 3.2.16). It satisfies all of the topological assumptions we require (the proof of increasing–completeness and properness can also be found in [BT17]):

Proposition 1.2.3 (Proposition 3.2.20)**.** *The partial cyclic order on $\mathcal{S}$ determined by the generalized Maslov index is increasing–complete, proper and regular.*

Our construction of an increasing boundary map for Schottky representations now connects them to *maximal representations* if the target is a Lie group of Hermitian type. Maximal representations were originally restricted to closed surfaces [Tol79; Tol89; DT87], but extended to surfaces with boundary using methods of bounded cohomology in [BIW10]. In particular, in the latter paper they are characterized in terms of increasing boundary maps, which leads to the following theorem.

Theorem 1.2.4 (Theorem 4.3.3, see also [BT17])**.** *Let Σ be a compact surface with nonempty boundary and negative Euler characteristic, $\Gamma = \pi_1(\Sigma)$ and G a Lie group of Hermitian type. Then a representation $\rho\colon \Gamma \to G$ is maximal if and only if it is an exhaustive Schottky representation.*

In the case of $\mathrm{Sp}(2n, \mathbb{R})$, the Shilov boundary identifies with $\mathrm{Lag}(\mathbb{R}^{2n})$, the space of Lagrangian subspaces of $\mathbb{R}^{2n}$. By showing that it satisfies the contraction condition mentioned earlier and with some extra work, we see that generalized Schottky representations provide examples of Anosov representations.

Theorem 1.2.5 (Theorem 4.5.17)**.** *Let Γ be a non–abelian free group and $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ a purely hyperbolic generalized Schottky representation. Then ρ is P–Anosov, where P is the stabilizer of a Lagrangian.*

The second family of examples are *complete oriented flags* in $\mathbb{R}^n$. A complete oriented flag is a sequence of nested subspaces

$$\{0\} \subset F^{(1)} \subset F^{(2)} \subset \ldots \subset F^{(n-1)} \subset \mathbb{R}^n,$$

where $\dim(F^{(i)}) = i$ and each subspace is equipped with an orientation. The space $\widehat{\mathcal{F}}_n$ of complete oriented flags is the oriented flag manifold $\mathrm{SL}(n, \mathbb{R})/B_0$, where B_0 is the identity component of a Borel subgroup of $\mathrm{SL}(n, \mathbb{R})$. It is more convenient to work with the oriented flag manifold $\mathcal{F}_n = \mathrm{PSL}(n, \mathbb{R})/B_0$, though. In even dimension, its elements can be interpreted as complete oriented flags up to simultaneously changing the orientation on each odd dimensional subspace.

The following relation on triples in $\mathcal{F}_n$ is an oriented version of Fock–Goncharov triple positivity [FG06], which in turn is based on Lusztig positivity for split real semisimple Lie groups.

Definition 1.2.6 (Definition 3.3.12)**.** Let (F_1, F_2, F_3) be a triple in $\mathcal{F}_n$.

- If n is odd, the triple is *increasing* if

$$F_1^{(i_1)} \oplus F_2^{(i_2)} \oplus F_3^{(i_3)} = \mathbb{R}^n$$

for all triples $0 \le i_1, i_2, i_3 \le n-1$ satisfying $i_1+i_2+i_3 = n$, and the orientation induced by the direct sum agrees with the orientation of $\mathbb{R}^n$.

- If n is even, the triple is *increasing* if there exist lifts $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3 \in \widehat{\mathcal{F}}_n$ such that

$$\widehat{F}_1^{(i_1)} \oplus \widehat{F}_2^{(i_2)} \oplus \widehat{F}_3^{(i_3)} = \mathbb{R}^n$$

for all triples $0 \le i_1, i_2, i_3 \le n-1$ satisfying $i_1+i_2+i_3 = n$, and the orientation induced by the direct sum agrees with the orientation of $\mathbb{R}^n$.

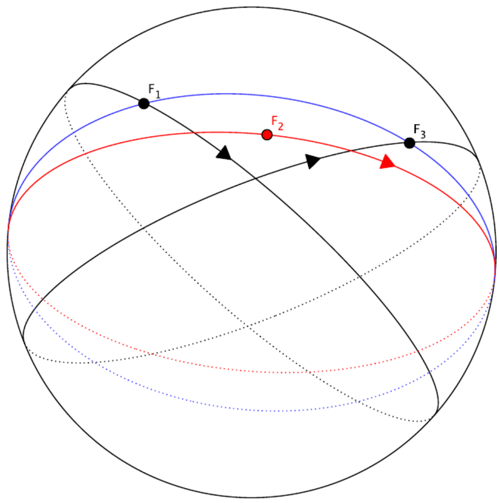


Figure 1.3: An increasing triple in $\mathcal{F}_3$, represented by points and oriented great circles on S^2. The point of F_2 must lie within the triangle defined by F_1 and F_3 as shown, and its oriented great circle must intersect the sides of this triangle in the right order.

We show that this relation satisfies the axioms of a partial cyclic order and that it satisfies all the topological properties required for our constructions.

Proposition 1.2.7 (Proposition 3.3.22, Proposition 3.3.30, Corollary 3.3.42)**.** *The relation defined above is a partial cyclic order on $\mathcal{F}_n$. It is increasing–complete, proper and regular.*

Again, we can apply our constructions of boundary maps and conclude in particular:

Theorem 1.2.8 (Theorem 4.6.5)**.** *Let Γ be a non–abelian free group and $\rho\colon \Gamma \to$ $\mathrm{PSL}(n,\mathbb{R})$ a purely hyperbolic generalized Schottky representation. Then ρ is B_0–Anosov.*

In [GW16], the notion of Θ–positivity is introduced for real semi–simple Lie groups, where $\Theta \subset \Delta$ is a subset of the simple restricted roots. It includes as particular examples Lusztig positivity for split Lie groups and positivity for Lie groups of Hermitian type. Moreover, it gives rise to a new notion of positivity for an exceptional family and for $\mathrm{SO}(p,q)$, $3 \leq p < q$. The flag manifold carrying the positive structure in the latter case consists of isotropic flags containing subspaces of dimension 1 through $p-1$. It would be interesting to find out whether there is a partial cyclic order on either this flag manifold or one of its finite covers given by oriented flag manifolds.

1.3 Margulis spacetimes and higher–dimensional affine actions

The final part of this work deals with actions of discrete subgroups of the affine group $\mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$. The original motivation for the study of such groups comes from classical physics: The underlying symmetry group of a crystal is a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^3)$ with a compact fundamental domain and is called a *crystallographic* group. Fedorov, Schoenflies and Barlow independently classified all crystallographic groups in dimension 3 up to conjugation in $\mathrm{Aff}(\mathbb{R}^3)$ in the 19th century, showing that there are 219 different cases [Bar94; Fed91; Sch91]. Part of Hilbert's 18th Problem was the question whether a finite classification of crystallographic groups exists in all dimensions. It was answered positively by Bieberbach, who also showed that every crystallographic group G in dimension n contains a finite index subgroup generated by n independent translations. The quotient $\mathbb{R}^n/G$ is thus finitely covered by an n–torus.

One natural way of generalizing the above questions is to consider *affine crystallographic groups*, discrete subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ acting properly discontinuously and cocompactly on $\mathbb{R}^n$. Fairly elementary examples show that such groups are not virtually abelian in general (see [Abe01, Section 5], for example), but the question of whether they are always virtually solvable is an open conjecture due to Auslander [Aus64]. Since a finite index subgroup always acts freely and every compact complete flat affine manifold M is a quotient $\mathbb{R}^n/\pi_1(M)$ with $\pi_1(M)$ acting freely and properly discontinuously, the question of virtual solvability is equivalent to the question whether every compact complete flat affine manifold has a virtually solvable fundamental group. The case $n = 3$ was proved in [FG83], along with a classification of all virtually solvable discrete subgroups of $\mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously. More recently, the Auslander conjecture was shown to be true for $n \leq 6$ [AMS12]. A nice overview of further partial results (and the topic in general) can be found in [Abe01]. Even before these results were obtained, Milnor raised the question whether the conjecture might be true even without the assumption of cocompactness [Mil77]. This is indeed the case for $n = 1$ and $n = 2$, but Margulis found examples of discrete nonabelian free subgroups of $\mathrm{Aff}(\mathbb{R}^3)$ acting freely and properly discontinuously on $\mathbb{R}^3$ [Mar83; Mar84]. Quotients by such actions are called *Margulis spacetimes*.

By [FG83], if a discrete subgroup $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acting freely and properly discontinuously is not virtually solvable, the homomorphism $\Gamma \xrightarrow{L} \mathrm{GL}(3,\mathbb{R})$ taking linear parts embeds Γ as a discrete subgroup in a conjugate of $\mathrm{SO}(2,1)$. This discrete subgroup cannot be cocompact by [Mes07], so it is necessarily free. A different proof of this fact, including the case of subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ with Fuchsian linear part, is contained in [Lab06]. Thus Margulis spacetimes constitute the general not virtually solvable case, and their linear holonomy can be assumed to be in $\mathrm{SO}(2,1)$. Every non–cocompact discrete subgroup of $\mathrm{SO}(2,1)$ occurs as the linear part of a properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ [Dru93].

Margulis introduced an important invariant of an affine transformation to describe whether a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ acts properly. It is defined for any affine transformation γ with hyperbolic linear part, i.e. with linear part in $\mathrm{SO}(2,1)$ and eigenvalues $\lambda, 1, \lambda^{-1}$ for some $\lambda < 1$. Its value $\alpha(\gamma)$ is the signed translation length along the unique affine line fixed by γ. Using this invariant, Margulis proved the famous Opposite Sign Lemma: For a discrete group of affine transformations to act properly, the sign of the Margulis invariant of each element must be the same.

The definition of the Margulis invariant generalizes without modification to any affine transformation of $\mathbb{R}^{2n+1}$ whose linear part is in $\mathrm{SO}_0(n+1,n)$ and that admits a transverse pair of an attracting and a repelling maximal isotropic subspace (equivalently, that has 1 as a unique eigenvalue, n eigenvalues of modulus greater 1 and n eigenvalues of modulus smaller 1). The Opposite Sign Lemma holds in this setting as well.

Now let Σ be a compact hyperbolic surface with nonempty geodesic boundary, and $\varrho_0\colon \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ (a lift of) its holonomy representation. Consider a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ whose linear part $\varrho = L(\rho)$ is the composition $\iota \circ \varrho_0$, where $\iota\colon\ \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_0(n+1,n)$ is the irreducible representation. Labourie extended the Margulis invariant from a function defined on $\pi_1(\Sigma)$ to a continuous function Ψ_ρ defined on *geodesic currents* $\mu \in \mathscr{C}(\mathrm{T}^1_{\mathrm{rec}}\Sigma)$ [Lab01]. Here, the recurrent part $\mathrm{T}^1_{\mathrm{rec}}\Sigma \subset \mathrm{T}^1\Sigma$ consists of all tangent vectors such that the geodesic flow is defined on $\mathbb{R}$ (that is, the orbit does not hit the boundary transversely), and a geodesic current is a flow–invariant probability measure on the compact set $\mathrm{T}^1_{\mathrm{rec}}\Sigma$. The function Ψ_ρ extends the Margulis invariant in the following sense: Any element $\gamma \in \pi_1(\Sigma)$ has an associated closed orbit in $\mathrm{T}^1_{\mathrm{rec}}\Sigma$, and the value of Ψ_ρ on the flow–invariant measure supported on this orbit is $\frac{\alpha(\gamma)}{l(\gamma)}$, where $l(\gamma)$ is the length of the closed geodesic corresponding to γ.

As a beautiful application of the continuous version of the Margulis invariant, it is shown in [GLM09] that the converse of the Opposite Sign Lemma holds in the above setting: If $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ satisfies $\Psi_\rho(\mu) \neq 0$ for every geodesic current μ, then ρ acts properly discontinuously on $\mathbb{R}^{2n+1}$. We extend this result to representations of any word hyperbolic group whose linear part is Anosov with respect to the stabilizer of a maximal isotropic subspace.

Theorem 1.3.1 (Theorem 5.2.18, see also [GT17])**.** *Let Γ be a word hyperbolic group, $\mathsf{U}_0\Gamma$ the flow space of Γ, $\rho\colon \Gamma \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ a homomorphism and $\varrho = L(\rho)$*

its linear part. Let $P \subset \mathrm{SO}_0(n+1, n)$ be the stabilizer of a maximal isotropic subspace of $\mathbb{R}^{n+1,n}$. Then the following are equivalent.

(i) *ϱ is Anosov with respect to P and Γ acts properly on $\mathbb{R}^{2n+1}$ via ρ.*

(ii) *ϱ is Anosov with respect to P and $\Psi_\rho(\mu) \neq 0 \quad \forall \mu \in \mathscr{C}(\mathsf{U}_0\Gamma)$.*

Related to this equivalence, we give a definition of an *affine Anosov representation* using contraction/dilation properties of affine bundles associated to ρ and a positivity condition that is a priori stronger than $\Psi_\rho(\mu) \neq 0 \quad \forall \mu \in \mathscr{C}(\mathsf{U}_0\Gamma)$ (Definition 5.2.7). Related notions were explored before in [Gho17a; Gho17b]. We prove that this definition is in fact also equivalent to the two conditions in the previous theorem (see Corollary 5.2.15 and Theorem 5.2.17).

2 Domains of discontinuity in oriented flag manifolds

2.1 Preliminaries

2.1.1 Parabolic subgroups of Lie groups

Let G be a connected semi–simple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Choose a maximal compact subgroup $K \subset G$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K and $\mathfrak{p} = \mathfrak{k}^{\perp}$ its orthogonal complement with respect to the Killing form. Choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and let $\mathfrak{a}^*$ be its dual space.

For any $\alpha \in \mathfrak{a}^*$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a} \colon [H, X] = \alpha(H)X\}$$

and let $\Sigma \subset \mathfrak{a}^*$ be the set of restricted roots, that is the set of $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$. Σ is in general not reduced, i.e. there can be $\alpha \in \Sigma$ with $2\alpha \in \Sigma$ (but no other positive multiples except 2 or 1/2). Choose a simple system $\Delta \subset \Sigma$ (a basis of $\mathfrak{a}^*$ such that every element of Σ can be written as a linear combination in Δ with only non–negative or only non–positive integer coefficients). Let $\Sigma^{\pm}$ be the corresponding positive and negative roots, Σ_0 the indivisible roots (the roots $\alpha \in \Sigma$ with $\alpha/2 \notin \Sigma$) and let $\Sigma_0^{\pm} = \Sigma^{\pm} \cap \Sigma_0$. Let $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0\,\forall \alpha \in \Delta\}$ and $\overline{\mathfrak{a}^+} = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0\,\forall \alpha \in \Delta\}$.

The above choices are always possible and such a triple $(K, \mathfrak{a}, \Delta)$ is unique up to conjugation in G (see e.g. [Hel79, Theorem 2.1] and [Kna02, Theorems 2.63, 6.51, 6.57]).

For any proper subset $\theta \subsetneq \Delta$ define the Lie algebras

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha, \qquad \mathfrak{b} = \bigoplus_{\alpha \in \Sigma^+ \cup \{0\}} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_\theta = \bigoplus_{\alpha \in \Sigma^+ \cup \operatorname{span}(\theta)} \mathfrak{g}_\alpha.$$

The $\mathfrak{p}_\theta \subset \mathfrak{g}$ are the standard parabolic subalgebras and $\mathfrak{b} = \mathfrak{p}_\varnothing$ is the minimal standard parabolic subalgebra. A parabolic subalgebra is a subalgebra which is conjugate to a standard parabolic subalgebra.

Let $A, N, N^- \subset G$ be the connected subgroups with Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{n}^-$. The exponential map of G restricts to diffeomorphisms $\exp\colon \mathfrak{a} \to A$, $\exp\colon \mathfrak{n} \to N$ and

$\exp\colon \mathfrak{n}^- \to N^-$. Let $P_\theta = N_G(\mathfrak{p}_\theta) \subset G$ be the (standard) parabolic subgroups and $B = P_\varnothing$ the (standard) minimal parabolic subgroup. The Lie algebra of P_θ is $\mathfrak{p}_\theta$. A parabolic subgroup is typically defined to be a subgroup conjugate to some P_θ. However, when we write 'parabolic subgroup' here, we will just mean P_θ for some $\theta \subsetneq \Delta$. Using the Iwasawa decomposition $G = KAN$, the minimal parabolic can also be described as $B = Z_K(\mathfrak{a})AN$, with $Z_K(\mathfrak{a})$ being the centralizer of $\mathfrak{a}$ in K.

The quotients

$$\mathcal{F}_\theta := G/P_\theta$$

are compact homogeneous G-spaces and are called (partial) flag manifolds.

Define the groups

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}), \quad \widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0, \quad \overline{M} = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})_0,$$

where $Z_K(\mathfrak{a})_0$ is the identity component of the centralizer of $\mathfrak{a}$ in K. The group W is the Weyl group of G, and we sometimes call $\widetilde{W}$ the extended Weyl group. The set of simple roots Δ can be realized as a subset of W by identifying $\alpha \in \Delta$ with the Killing orthogonal reflection along $\ker\alpha$ in $\mathfrak{a}$. Then $\Delta \subset W$ is a generating set. We denote by $\ell\colon W \to \mathbb{N}$ the word length with respect to the generating set Δ. There is a unique longest element $w_0 \in W$. Conjugation by w_0 preserves Δ and defines a map $\iota\colon \Delta \to \Delta$ which is called the opposition involution. If we write $\ell(w)$ for $w \in \widetilde{W}$ we mean the length $\ell(\pi(w))$ of its projection to W.

Though the groups W, $\widetilde{W}$ and $\overline{M}$ are not really subgroups of G, we can often simplify notation by pretending they are. For example, if $H \subset G$ is a subgroup containing $Z_K(\mathfrak{a})_0$, we will write $H \cap \overline{M}$ as a shorthand for $(H \cap Z_K(\mathfrak{a}))/Z_K(\mathfrak{a})_0 \subset \overline{M}$, or wH for the coset nH where $n \in N_K(\mathfrak{a})$ is any choice of representative for $w \in \widetilde{W}$. Note that $Z_K(\mathfrak{a})$ is contained in B and therefore in every parabolic subgroup P_θ. As the identity component is a normal subgroup, conjugation by $z \in Z_K(\mathfrak{a})$ preserves $P_{\theta,0}$.

We write ℓ_g, r_g and c_g for left and right multiplication and conjugation by $g \in G$. For subsets $T_1, \ldots, T_n$ of a group we write $\langle T_1, \ldots, T_n\rangle$ for the smallest subgroup containing all of them. When speaking about quotients of groups, we usually denote equivalence classes by square brackets. To avoid confusion, we use double brackets $[\![\ ]\!]$ for all quotients of the group $\widetilde{W}$. It should always be clear from the context which quotient we are referring to.

Our attention in this paper will be restricted to semi–simple Lie groups G such that the group $\overline{M}$ is finite abelian and consists entirely of involutions. This holds for all Lie groups G which are linear, i.e. isomorphic to a closed subgroup of some $\mathrm{GL}(n,\mathbb{R})$ (see [Kna02, Theorem 7.53] and note that all connected linear Lie groups have a complexification). Also, every linear Lie group has a finite center [Kna02, Proposition 7.9]. All our arguments work equally well for Lie groups which are not linear, as long as their center is finite and $\overline{M}$ is finite abelian and consists of involutions. These assumptions on $\overline{M}$ do not appear to be essential for our theory, but they significantly simplify several arguments, e.g. the statement and proof of Lemma 2.1.8.

If $\overline{M}$ is trivial, the class of oriented flag manifolds (see Definition 2.2.1) reduces to ordinary flag manifolds. Our theory gives nothing new in this case. In particular, this happens whenever G is a complex Lie group, since the minimal parabolic subgroup is connected in this case.

2.1.2 Orbits of the $B_0 \times B_0$–action on G

Let $B \subset G$ be the minimal parabolic subgroup, and B_0 its identity component, i.e. the connected subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{b} = \bigoplus_{\alpha\in\Sigma^+\cup\{0\}} \mathfrak{g}_\alpha$. We consider the action of $B_0 \times B_0$ on G by left– and right–multiplication. In this section, we show that the double quotient $B_0\backslash G/B_0$ is described by the group $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$. This is a slight refinement of the Bruhat decomposition, which describes the double quotient $B\backslash G/B$ by the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. It will be an important ingredient for our description of relative positions of oriented flags.

As restricted root systems are not necessarily reduced, we will work with the set Σ_0 of indivisible roots, i.e. the roots $\alpha \in \Sigma$ such that $\alpha/2 \notin \Sigma$. For any $\alpha \in \Sigma_0^+ = \Sigma_0 \cap \Sigma^+$ let $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Then $\mathfrak{u}_\alpha$ is a subalgebra of $\mathfrak{g}$. Let $U_\alpha \subset G$ be the connected subgroup with Lie algebra $\mathfrak{u}_\alpha$.

For $\alpha, \beta \in \Sigma_0^+$ let $(\alpha, \beta) \subset \Sigma_0^+$ be the set of all indivisible roots which can be obtained as positive linear combinations of α and β. Then $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \bigoplus_{\gamma\in(\alpha,\beta)} \mathfrak{u}_\gamma$. For every $w \in W$ the set $\Psi_w = \Sigma_0^+ \cap w\Sigma_0^-$ has the property that $(\alpha, \beta) \subset \Psi_w$ for all $\alpha, \beta \in \Psi_w$. Let U_w be the connected subgroup of G with Lie algebra $\mathfrak{u}_w = \bigoplus_{\alpha\in\Psi_w} \mathfrak{u}_\alpha$.

Lemma 2.1.1. *Let $\Psi' \subset \Psi \subset \Sigma_0^+$ such that $(\alpha, \beta) \subset \Psi$ and $(\alpha, \gamma) \subset \Psi'$ for all $\alpha, \beta \in \Psi$ and $\gamma \in \Psi'$. Let $\mathfrak{u} = \bigoplus_{\alpha\in\Psi} \mathfrak{u}_\alpha$ and $\mathfrak{u}' = \bigoplus_{\alpha\in\Psi'} \mathfrak{u}_\alpha$ and let $U, U' \subset G$ be the corresponding connected subgroups. Let $\Psi \setminus \Psi' = \{\alpha_1, \dots, \alpha_n\}$, in arbitrary order. Then*

$$U = U'U_{\alpha_1} \cdots U_{\alpha_n}.$$

In particular, for $\Psi = \Psi_w$ and $\Psi' = \varnothing$, we have $U_w = \prod_{\alpha\in\Psi_w} U_\alpha$, where the product can be written in arbitrary order.

Proof. First note that the conditions ensure that $\mathfrak{u}, \mathfrak{u}' \subset \mathfrak{g}$ are subalgebras and that $\mathfrak{u}'$ is an ideal of $\mathfrak{u}$. We proceed by induction on $n = |\Psi \setminus \Psi'|$. The case $n = 0$ is trivial and for $n = 1$ the statement is shown in [Kna02, Lemma 7.97].

If $n \geq 1$ then choose a longest root α_k among $\alpha_1, \dots, \alpha_n$ and let $\Psi'' = \Psi' \cup \{\alpha_k\}$. Then $(\alpha, \beta) \subset \Psi' \subset \Psi''$ for all $\alpha \in \Psi$ and $\beta \in \Psi''$, since every element of (α, α_k) will be longer than α_k and therefore in Ψ'. So $\mathfrak{u}'' = \bigoplus_{\alpha\in\Psi''} \mathfrak{u}_\alpha$ is an ideal of $\mathfrak{u}$. Let $U'' \subset G$ be its connected subgroup. Since $\mathfrak{u}'$ and $\mathfrak{u}''$ are ideals of $\mathfrak{u}$, U' and U'' are normal subgroups of U. Now let $g \in U$. By the induction hypothesis there are $g'' \in U''$,

$g_- \in U_{\alpha_1} \cdots U_{\alpha_{k-1}}$ and $g_+ \in U_{\alpha_{k+1}} \cdots U_{\alpha_n}$ such that $g = g'' g_- g_+$. Since $U'' \subset U$ is normal, $g = g_- \overline{g}'' g_+$ for $\overline{g}'' = g_-^{-1} g'' g_- \in U''$. Since $\Psi' \subset \Psi''$ satisfy the assumptions of the Lemma for $n = 1$, we get $\overline{g}'' = g' g_0$ for some $g' \in U'$ and $g_0 \in U_{\alpha_k}$. Now set $\overline{g}' = g_- g' g_-^{-1} \in U'$, then $g = \overline{g}' g_- g_0 g_+ \in U' U_{\alpha_1} \cdots U_{\alpha_n}$, as required. □

Lemma 2.1.2. *Let $w \in N_K(\mathfrak{a})$. Then the map*

$$U_w \times B \to G, \quad (u, b) \mapsto uwb \tag{2.1.1}$$

is a smooth embedding with image BwB. The restriction of (2.1.1) *to $U_w \times B_0$ maps onto $U_w w B_0 = B_0 w B_0$.*

Proof. We get (2.1.1) as a composition

$$U_w \times B \xrightarrow{c_{w^{-1}} \times \mathrm{id}} w^{-1} U_w w \times B \hookrightarrow N^- \times B \to G \xrightarrow{\ell_w} G.$$

The first and last map are diffeomorphisms, the inclusion is a smooth embedding and the multiplication map $N^- \times B \to G$ is a diffeomorphism onto an open subset of G by [Kna02, Lemma 6.44, Proposition 7.83(e)]. So the composition is a smooth embedding. We only have to compute its image, i.e. that $U_w w B = BwB$.

To prove this, use the Iwasawa decomposition $B = NAZ_K(\mathfrak{a})$ to get $BwB = NwB$ and then write, using Lemma 2.1.1,

$$NwB = \Big(\prod_{\alpha \in \Psi_w} U_\alpha \Big) \Big(\prod_{\alpha \in \Sigma_0^+ \setminus \Psi_w} U_\alpha \Big) wB = U_w w \Big(\prod_{\alpha \in \Sigma_0^+ \setminus \Psi_w} w^{-1} U_\alpha w \Big) B.$$

For all $\alpha \in \Sigma_0^+ \setminus \Psi_w$ we have $w^{-1}\alpha \in \Sigma_0^+$, so $\mathrm{Ad}_{w^{-1}} \mathfrak{u}_\alpha = \mathfrak{u}_{w^{-1}\alpha} \subset \mathfrak{n}$ and thus $w^{-1} U_\alpha w \subset N \subset B$, so $BwB = U_w w B$. If we restrict (2.1.1) to the connected component $U_w \times B_0$, its image is $U_w w B_0$. The Iwasawa decomposition shows that $B_0 = NAZ_K(\mathfrak{a})_0$, so $B_0 w B_0 = N w B_0$ and this equals $U_w w B_0$ by the same argument as above. □

Proposition 2.1.3. *G decomposes into the disjoint union*

$$G = \bigsqcup_{w \in \widetilde{W}} B_0 w B_0.$$

Proof. Let $\pi \colon \widetilde{W} \to W$ be the projection to the Weyl group. By the Bruhat decomposition [Kna02, Theorem 7.40], G decomposes disjointly into BwB for $w \in W$, so we only have to show that

$$BwB = \bigsqcup_{w' \in \pi^{-1}(w)} B_0 w' B_0. \tag{2.1.2}$$

Lemma 2.1.2 identifies BwB with $U_w \times B$, the connected components of which are the sets $U_w \times m B_0$ for $m \in \overline{M}$. These correspond via the map from Lemma 2.1.2 to the subsets $U_w w m B_0 = B_0 w m B_0 \subset BwB$. Also $\pi^{-1}(w) = \{wm \mid m \in \overline{M}\}$, proving (2.1.2). □

2.1.3 Orbit closures of the $B_0 \times B_0$–action on G

Let $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$ as before and $\pi\colon \widetilde{W} \to W$ the projection to the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. As described in Section 2.1.1, Δ is realized as a generating set of W and we write ℓ for the word length with respect to Δ.

To describe the orbit structure of the $B_0 \times B_0$–action more closely, we will need a suitable lift of the generators $\Delta \subset W$ to the larger group $\widetilde{W}$. It will be given by Definition 2.1.4.

Let us first fix some notation. If B is the Killing form on $\mathfrak{g}$ and Θ the Cartan involution which is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$, then $\|X\|^2 = -B(X, \Theta X)$ defines a norm on $\mathfrak{g}$. Its restriction to $\mathfrak{a}$ is just $B(X, X)$. For $\alpha \in \mathfrak{a}^*$ let $H_\alpha \in \mathfrak{a}$ be its dual with respect to B, i.e. $B(H_\alpha, X) = \alpha(X)$ for all $X \in \mathfrak{a}$. We use the norm on $\mathfrak{a}^*$ defined by $\|\alpha\|^2 = B(H_\alpha, H_\alpha)$.

Definition 2.1.4. For every $\alpha \in \Delta$ choose a vector $E_\alpha \in \mathfrak{g}_\alpha$ such that $\|E_\alpha\|^2 = 2\|\alpha\|^{-2}$. Then we define

$$r(\alpha) = \exp\Big(\frac{\pi}{2}\,(E_\alpha + \Theta E_\alpha)\Big).$$

By [Kna02, Proposition 6.52(c)] this is in $N_K(\mathfrak{a})$ and acts on $\mathfrak{a}$ as a reflection along the kernel of the simple root α. We will regard $r(\alpha)$ as an element of $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$.

Remarks 2.1.5.

(i) $r(\alpha) \in \widetilde{W}$ is almost independent of the choice of E_α: If $\dim \mathfrak{g}_\alpha > 1$, then the set of admissible E_α is connected. Since $\widetilde{W}$ is discrete and $r(\alpha)$ depends continuously on E_α this means that $r(\alpha) \in \widetilde{W}$ is independent of E_α. In particular, we get the same $r(\alpha)$ when substituting E_α by $-E_\alpha$, so $r(\alpha) = r(\alpha)^{-1}$. On the other hand, if $\dim \mathfrak{g}_\alpha = 1$ then $r(\alpha)$ need not be of order 2, and there can be two different choices for $r(\alpha)$, which are inverses of each other. If they do not coincide, $r(\alpha)$ is of order 4 since $r(\alpha)^2$ acts trivially on $\mathfrak{a}$ and is therefore contained in $\overline{M}$. By our assumption of G being linear, $\overline{M}$ consists of involutions. In the group $\mathrm{SL}(n, \mathbb{R})$ for example, $r(\alpha)$ is of order 4 for all simple restricted roots α, while in $\mathrm{SO}_0(p, q),\ p < q$, the image of the "last" simple root α_p is of order 2.

(ii) For every $\alpha \in \Delta$, we have $\pi(r(\alpha)) = \alpha \in W$, so the projection of $r(\Delta)$ to W is just the usual generating set Δ. In fact, $r(\Delta)$ also generates the group $\widetilde{W}$ (this can be seen using Lemma 2.2.10 with $\theta = \Delta$).

Now we want to understand the closures of the $B_0 \times B_0$–orbits. This part is similar to [BT72, Section 3], where Borel and Tits describe the left and right action of the Borel subgroup for an algebraic group G. Some of their arguments also work in our setting. Recall that $\Psi_w = \Sigma_0^+ \cap w\Sigma_0^-$.

Lemma 2.1.6. *Let $w_1, w_2 \in W$. Then*

$$\Psi_{w_1} \cap w_1(\Psi_{w_2}) = \varnothing, \quad \Psi_{w_1 w_2} \subset \Psi_{w_1} \cup w_1(\Psi_{w_2}), \quad |\Psi_{w_1}| = \ell(w_1).$$

Furthermore, $\Psi_w \subset \operatorname{span}\theta$ for $\theta \subset \Delta$ if and only if $w \in \langle\theta\rangle \subset W$.

Proof. First observe that $\alpha\Sigma_0^+ = \Sigma_0^+ \setminus \{\alpha\} \cup \{-\alpha\}$ and therefore $\Psi_\alpha = \{\alpha\}$ for any $\alpha \in \Delta$. The first two identities follow easily from the definition of Ψ_w and the inequality $|\Psi_w| \leq \ell(w)$ is a direct consequence of $\Psi_{w\alpha} \subset \Psi_w \cup w\Psi_\alpha$ for every $\alpha \in \Delta$.

We want to show that $|\Psi_w| = r$ implies $\ell(w) = r$ by induction on $r \in \mathbb{N}$. For $r = 0$ this follows from the fact that W acts freely on positive systems of roots. If $|\Psi_w| = r > 0$, then $\Delta \not\subset w\Sigma_0^+$, as otherwise $\Sigma_0^+ \subset w\Sigma_0^+$ and thus $\Psi_w = \varnothing$. So choose $\alpha \in \Delta \cap w\Sigma_0^- \subset \Psi_w$. Then

$$\alpha\Psi_{\alpha w} = \alpha\Sigma_0^+ \cap w\Sigma_0^- = (\Sigma_0^+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap w\Sigma_0^- = \Psi_w \setminus \{\alpha\},$$

so $|\Psi_{\alpha w}| = r - 1$ and thus $\ell(\alpha w) = r - 1$ by the induction hypothesis. So $\ell(w) = r$.

To prove the remaining statement, note that the reflection along a root α maps every other root β into $\operatorname{span}(\alpha, \beta)$. So $\operatorname{span}(\theta)$ is invariant by every $w \in \langle\theta\rangle$. Assume the equivalence of $\Psi_w \subset \operatorname{span}\theta$ and $w \in \langle\theta\rangle$ was already proved for $w \in W$ and let $\ell(\alpha w) = \ell(w) + 1$. Then $\Psi_{\alpha w} = \Psi_\alpha \sqcup \alpha\Psi_w = \{\alpha\} \sqcup \alpha\Psi_w$ is contained in $\operatorname{span}\theta$ if and only if $\alpha \in \theta$ and $w \in \langle\theta\rangle$, proving what we wanted. □

Lemma 2.1.7. *Let $\alpha \in \Delta$ and E_α as in Definition 2.1.4. Then there is a Lie group homomorphism $\Phi\colon \mathrm{SL}(2,\mathbb{R}) \to G$, which is an immersion and satisfies*

(i) $d_1\Phi\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ *maps* $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ *to* E_α, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ *to* $-\Theta E_\alpha$, *and* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *to* $2\|\alpha\|^{-2} H_\alpha$,

(ii) $\Phi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = r(\alpha)$,

(iii) Φ *is an isomorphism if* $\operatorname{ord}(r(\alpha)) = 4$, *and* $\ker\Phi = \{\pm 1\}$ *if* $\operatorname{ord}(r(\alpha)) = 2$.

Proof. $d_1\Phi$ as defined in (i) is a monomorphism of Lie algebras [Kna02, Proposition 6.52], so it integrates to an immersive Lie group homomorphism $\widetilde{\Phi}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to G$. Since $\ker\widetilde{\Phi} \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$ is a normal subgroup and discrete and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is connected, conjugation actually fixes $\ker\widetilde{\Phi}$ pointwise, i.e.

$$\ker\widetilde{\Phi} \subset Z(\widetilde{\mathrm{SL}(2,\mathbb{R})}) = \exp(\mathbb{Z}X), \quad X = \left(\begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix}\right)$$

Now $\widetilde{\Phi}(\exp(kX)) = \exp(d_1\Phi(kX)) = \exp(k\pi(E_\alpha + \Theta E_\alpha)) = r(\alpha)^{2k}$. If $\operatorname{ord}(r(\alpha)) = 2$, then $\ker\widetilde{\Phi} = \exp(\mathbb{Z}X)$ and if $\operatorname{ord}(r(\alpha)) = 4$, then $\ker\widetilde{\Phi} = \exp(2\mathbb{Z}X)$. By Remark 2.1.5(i), these are the only possibilities. In any case, $\widetilde{\Phi}$ descends to a homomorphism Φ on $\mathrm{SL}(2,\mathbb{R}) = \widetilde{\mathrm{SL}(2,\mathbb{R})}/\exp(2\mathbb{Z}X)$, having the desired properties. □

Lemma 2.1.8. *Let $\alpha \in \Delta$ and let $s = r(\alpha) \in \widetilde{W}$. Then*

$$P_{\Delta\backslash\alpha,0} = B_0 \cup B_0 s B_0 \cup B_0 s^2 B_0 \cup B_0 s^3 B_0, \tag{2.1.3}$$
$$\overline{B_0 s B_0} = B_0 \cup B_0 s B_0 \cup B_0 s^2 B_0, \tag{2.1.4}$$
$$B_0 s B_0 s^{-1} B_0 = B_0 \cup B_0 s B_0 \cup B_0 s^{-1} B_0. \tag{2.1.5}$$

Note that these unions are disjoint unless s has order 2.

Proof. First note that $P_{\Delta\backslash\alpha} = B \cup BsB$. This follows from the Bruhat decomposition and the following argument: An element $w \in W$ is contained in $P_{\Delta\backslash\alpha} = N_G(\mathfrak{p}_{\Delta\backslash\alpha})$ if and only if $\mathrm{Ad}_w\, \mathfrak{p}_{\Delta\backslash\alpha} \subset \mathfrak{p}_{\Delta\backslash\alpha}$. This holds if and only if w preserves $\Sigma_0^+ \cup \mathrm{span}(\alpha)$, or equivalently $\Psi_w \subset \{\alpha\}$. By Lemma 2.1.6 this is true if and only if $w \in \{1, \alpha\}$.

We now distinguish two cases, depending on the dimension of $\mathfrak{u}_\alpha$. First assume that $\dim \mathfrak{u}_\alpha > 1$. In this case, $P_{\Delta\backslash\alpha,0} \cap B \subset P_{\Delta\backslash\alpha,0}$ is a closed subgroup of codimension at least 2. Therefore, its complement $P_{\Delta\backslash\alpha,0} \cap BsB$ in $P_{\Delta\backslash\alpha,0}$ is connected, thus equal to $B_0 s B_0$, which is a connected component of BsB by Lemma 2.1.2. This implies $P_{\Delta\backslash\alpha,0} \cap Z_K(\mathfrak{a}) = Z_K(\mathfrak{a})_0$, as otherwise there would be $m \in \overline{M} \setminus \{1\}$ with $B_0 s B_0 m = B_0 s m B_0 \subset P_{\Delta\backslash\alpha,0}$, but this is disjoint from $B_0 s B_0$ by Proposition 2.1.3. So $P_{\Delta\backslash\alpha,0} = B_0 \cup B_0 s B_0$. Since $s \in \widetilde{W}$ must have order 2 in this case by Proposition 2.1.3, this is (2.1.3) as we wanted.

To see (2.1.4) and (2.1.5) in this case, we only have to prove that the inclusions $B_0 s B_0 \subset \overline{B_0 s B_0}$ and $B_0 \subset B_0 s B_0 s^{-1} B_0$ are strict: Using B_0–invariance from both sides, this will imply $\overline{B_0 s B_0} = B_0 s B_0 s^{-1} B_0 = P_{\Delta\backslash\alpha,0} = B_0 \cup B_0 s B_0$. And indeed, $B_0 s B_0 = \overline{B_0 s B_0}$ would imply that $B_0 s B_0$ and B_0 are closed, so $B_0 s B_0 \cup B_0$ would not be connected. If $B_0 = B_0 s B_0 s^{-1} B_0$, then $s B_0 s^{-1} \subset B_0$, so $\mathfrak{g}_{-\alpha} = \mathrm{Ad}_s\, \mathfrak{g}_\alpha \subset \mathrm{Ad}_s\, \mathfrak{b} \subset \mathfrak{b}$, a contradiction.

Now we consider the case $\dim \mathfrak{u}_\alpha = 1$. Then $P_{\Delta\backslash\alpha}/B_0$ is a compact 1–dimensional manifold, i.e. a disjoint union of circles. Denote by π the projection from $P_{\Delta\backslash\alpha}$ to the quotient. Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and let Φ be the map from Lemma 2.1.7. The map $\gamma\colon \mathbb{R} \to P_{\Delta\backslash\alpha}/B_0$ defined by $\gamma(t) = \pi(\Phi(\exp(te))s)$ is an injective smooth curve in this 1–manifold. This is because the map $\mathbb{R} \to U_\alpha,\ t \mapsto \Phi(\exp(te)) = \exp(tE_\alpha)$ is injective, and the map $U_\alpha \to P_{\Delta\backslash\alpha}/B_0,\ u \mapsto \pi(us)$ is injective as a consequence of Lemma 2.1.2. Therefore, its limits for $t \to \pm\infty$ exist and $\overline{\gamma(\mathbb{R})} = \gamma(\mathbb{R}) \cup \{\gamma(\pm\infty)\}$. Now by Lemma 2.1.2

$$B_0 s B_0 = U_\alpha s B_0 = \exp(\mathfrak{u}_\alpha) s B_0 = \pi^{-1}(\pi(\exp(\mathfrak{u}_\alpha)s)) = \pi^{-1}(\gamma(\mathbb{R})),$$

so

$$\overline{B_0 s B_0} = \pi^{-1}(\overline{\gamma(\mathbb{R})}) = B_0 s B_0 \cup \pi^{-1}(\gamma(\infty)) \cup \pi^{-1}(\gamma(-\infty)).$$

To compute the limits, note that

$$\Phi \begin{pmatrix} |t|^{-1} & \mathrm{sgn}(t) \\ 0 & |t| \end{pmatrix} = \exp\left[\mathrm{d}_1\Phi \begin{pmatrix} -\log|t| & 0 \\ 0 & \log|t| \end{pmatrix}\right] \exp\left[\mathrm{d}_1\Phi \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right] \in B_0,$$

so

$$\begin{aligned}\lim_{t\to\pm\infty}\gamma(t) &= \lim_{t\to\pm\infty}\pi(\Phi(\exp(te)s)) = \lim_{t\to\pm\infty}\pi\left[\Phi\left[\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}|t|^{-1} & \operatorname{sgn}(t)\\ 0 & |t|\end{pmatrix}\right]\right]\\ &= \pi\left[\Phi\left[\lim_{t\to\pm\infty}\begin{pmatrix}-\operatorname{sgn}(t) & 0\\ -|t|^{-1} & -\operatorname{sgn}(t)\end{pmatrix}\right]\right] = \pi(\Phi(\mp 1)) = \pi(s^{1\pm 1}).\end{aligned}$$

So $\overline{B_0 s B_0} = B_0 s B_0 \cup B_0 \cup B_0 s^2 B_0$, which is (2.1.4).

Since $B_0 \cup B_0 s B_0 \cup B_0 s^2 B_0 \cup B_0 s^3 B_0 = \overline{B_0 s B_0} \cup \overline{B_0 s^3 B_0} \subset P_{\Delta\backslash\alpha}$, $P_{\Delta\backslash\alpha}$ decomposes into the disjoint union

$$P_{\Delta\backslash\alpha} = \bigsqcup_{m\in\overline{M}}(B_0 \cup B_0 s B_0)\, m = \bigsqcup_{m\in\langle s^2\rangle\backslash\overline{M}}(\overline{B_0 s B_0} \cup \overline{B_0 s^3 B_0})\, m.$$

Therefore, $\overline{B_0 s B_0} \cup \overline{B_0 s^3 B_0}$ is closed and open in $P_{\Delta\backslash\alpha}$, hence equal to $P_{\Delta\backslash\alpha,0}$.

Finally, to prove (2.1.5), we claim that, for $t, \tau \in \mathbb{R}$,

$$\pi(\exp(tE_\alpha)s\exp(\tau E_\alpha)s^{-1}) = \begin{cases}\pi(1) & \text{if } \tau = 0,\\ \pi(\exp((t-\tau^{-1})E_\alpha)s^{\operatorname{sgn}(\tau)}) & \text{if } \tau \neq 0.\end{cases} \tag{2.1.6}$$

This then shows $\pi(U_\alpha s U_\alpha s^{-1}) = \pi(1) \cup \pi(U_\alpha s) \cup \pi(U_\alpha s^{-1})$ and therefore

$$B_0 s B_0 s^{-1} B_0 = U_\alpha s U_\alpha s^{-1} B_0 = B_0 \cup U_\alpha s B_0 \cup U_\alpha s^{-1} B_0 = B_0 \cup B_0 s B_0 \cup B_0 s^{-1} B_0.$$

The claim (2.1.6) is clear if $\tau = 0$, since $\exp(tE_\alpha) \in B_0$. So let $\tau \neq 0$. Then

$$\begin{aligned}\pi(e^{tE_\alpha}se^{\tau E_\alpha}s^{-1}) &= \pi\left[\Phi\left[\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]\right]\\ &= \pi\left[\Phi\left[\begin{pmatrix}1-t\tau & t\\ -\tau & 1\end{pmatrix}\begin{pmatrix}|\tau|^{-1} & \operatorname{sgn}(\tau)\\ 0 & |\tau|\end{pmatrix}\right]\right]\\ &= \pi\left[\Phi\left[\begin{pmatrix}|\tau|^{-1} - t\operatorname{sgn}(\tau) & \operatorname{sgn}(\tau)\\ -\operatorname{sgn}(\tau) & 0\end{pmatrix}\begin{pmatrix}0 & -\operatorname{sgn}(\tau)\\ \operatorname{sgn}(\tau) & 0\end{pmatrix}\right]s^{\operatorname{sgn}(\tau)}\right]\\ &= \pi\left[\Phi\left[\begin{pmatrix}1 & t-\tau^{-1}\\ 0 & 1\end{pmatrix}\right]s^{\operatorname{sgn}(\tau)}\right] = \pi(\exp((\tau^{-1}-t)E_\alpha)s^{\operatorname{sgn}(\tau)}). \qquad \square\end{aligned}$$

Lemma 2.1.9. *For any $w_1, w_2 \in \widetilde{W}$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ we have*

$$B_0 w_1 w_2 B_0 = B_0 w_1 B_0 w_2 B_0.$$

Proof. We get $B_0 w_1 B_0 w_2 B_0 = U_{w_1} w_1 U_{w_2} w_2 B_0$ and $B_0 w_1 w_2 B_0 = U_{w_1 w_2} w_1 w_2 B_0$ by Lemma 2.1.2. We want to show that $U_{w_1w_2} = U_{w_1} w_1 U_{w_2} w_1^{-1}$. By Lemma 2.1.1 both sides can be written as products of U_α for some set of α. For the left hand side, the product is taken over all $\alpha \in \Psi_{w_1w_2}$ while for the right hand side we need all $\alpha \in \Psi_{w_1} \cup w_1\Psi_{w_2}$. But it follows from Lemma 2.1.6 that $\Psi_{w_1w_2} = \Psi_{w_1} \cup w_1\Psi_{w_2}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. $\square$

Lemma 2.1.10. *Let $w \in \widetilde{W}$ and $s = r(\alpha) \in \widetilde{W}$ for some $\alpha \in \Delta$. Then $\ell(sw) = \ell(w) \pm 1$ and*

$$B_0 s B_0 w B_0 = \begin{cases} B_0 sw B_0 & \text{if } \ell(sw) = \ell(w) + 1, \\ B_0 w B_0 \cup B_0 sw B_0 \cup B_0 s^2 w B_0 & \text{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

Proof. Clearly $|\ell(sw) - \ell(w)| \leq 1$ since $\pi(s) \in W$ is in the generating system, but also $\ell(sw) \neq \ell(w)$ by the property of Coxeter groups that only words with an even number of letters can represent the identity. If $\ell(sw) = \ell(w) + 1$, then the statement follows from Lemma 2.1.9. Assume $\ell(sw) = \ell(w) - 1$. Then $\ell(s^{-1}w) = \ell(s^{-2}sw) = \ell(w) - 1$ since $\ell(s^{-2}) = 0$. So $B_0 s B_0 s^{-1} w B_0 = B_0 w B_0$ by the first part and also $B_0 s B_0 s B_0 = B_0 s B_0 \cup B_0 s^2 B_0 \cup B_0 s^3 B_0$ by Lemma 2.1.8. We thus get

$$\begin{aligned} B_0 s B_0 w B_0 &= B_0 s B_0 s B_0 s^{-1} w B_0 = B_0 s B_0 s^{-1} w B_0 \cup B_0 s^2 B_0 s^{-1} w B_0 \cup B_0 s^3 B_0 s^{-1} w B_0 \\ &= B_0 w B_0 \cup B_0 sw B_0 \cup B_0 s^2 w B_0, \end{aligned}$$

again using the first part of the lemma for the last equality. □

Lemma 2.1.11. *Let $w \in \widetilde{W}$ and $s = r(\alpha) \in \widetilde{W}$ for some $\alpha \in \Delta$. Then $\overline{B_0 s B_0 w B_0} = \overline{B_0 s B_0}\ \overline{B_0 w B_0}$ and $\overline{B_0 s B_0} B_0 w B_0 = B_0 w B_0 \cup B_0 sw B_0 \cup B_0 s^2 w B_0$.*

Proof. For the first part, note that $\overline{B_0 s B_0}\ \overline{B_0 w B_0} \subset \overline{B_0 s B_0 w B_0}$. We want to prove that $\overline{B_0 s B_0}\ \overline{B_0 w B_0}$ is closed. Consider the map

$$f \colon G/B_0 \to \mathcal{C}(G/B_0), \quad gB_0 \mapsto g\overline{B_0 w B_0}$$

where $\mathcal{C}(G/B_0)$ is the set of closed subsets of G/B_0. Since G/B_0 is compact and f is G–equivariant, the space $\mathcal{C}(G/B_0)$ is compact with the Hausdorff metric, f is continuous and for any closed subset $A \subset G/B_0$ the union $\bigcup_{x \in A} f(x)$ is closed (see e.g. Proposition 2.4.10, Lemma 2.4.16(i), and Lemma 2.4.16(ii)).

In particular, the union of all elements of $f(\overline{B_0 s B_0}/B_0)$ is a closed subset of G/B_0, and so is its preimage in G. But this is just $\overline{B_0 s B_0}\ \overline{B_0 w B_0}$, which is therefore a closed set containing $B_0 s B_0 w B_0$, hence equal to $\overline{B_0 s B_0 w B_0}$.

For the second part, Lemma 2.1.8 implies that

$$\overline{B_0 s B_0} B_0 w B_0 = B_0 w B_0 \cup B_0 s B_0 w B_0 \cup B_0 s^2 B_0 w B_0$$

and in both cases of Lemma 2.1.10 this equals what we want. □

Proposition 2.1.12. *Let $w \in \widetilde{W}$ and $\pi(w) = \alpha_1 \dots \alpha_k$ a reduced expression by simple root reflections for the projection $\pi(w) \in W$ of w to the Weyl group. Then $w = r(\alpha_1) \dots r(\alpha_k) m$ for some $m \in \overline{M}$. Let*

$$A_w = \{r(\alpha_1)^{i_1} \dots r(\alpha_k)^{i_k}\, m \mid i_1, \dots, i_k \in \{0, 1, 2\}\} \subset \widetilde{W}.$$

be the set of words that can be obtained by deleting or squaring some of the letters. Then

$$\overline{B_0 w B_0} = \bigcup_{w' \in A_w} B_0 w' B_0.$$

In particular, A_w does not depend on the choice of reduced word for $\pi(w)$.

Proof. First of all, since $\pi(r(\alpha_1)\dots r(\alpha_k)) = \alpha_1 \dots \alpha_k = \pi(w)$, there exists $m \in \overline{M}$ such that $w = r(\alpha_1)\dots r(\alpha_k)m$.
We now prove the second statement by induction on $\ell(w)$. If $\ell(w) = 0$, then $w \in \overline{M}$, so $B_0 w B_0 = wB_0$ is already closed, and $A_w = \{w\}$. Now let $\ell(w) > 0$ and assume the statement is already proven for all $\widetilde{w} \in \widetilde{W}$ with $\ell(\widetilde{w}) < \ell(w)$. Assume that $w = r(\alpha_1)\dots r(\alpha_k)m$ as above. Then we can write $w = s\widetilde{w}$ with $s = r(\alpha_1)$ and $\ell(\widetilde{w}) = \ell(w) - 1$. Using Lemma 2.1.10 and Lemma 2.1.11 we get

$$\begin{aligned}\overline{B_0 w B_0} = \overline{B_0 s \widetilde{w} B_0} = \overline{B_0 s B_0 \widetilde{w} B_0} = \overline{B_0 s B_0}\,\overline{B_0 \widetilde{w} B_0} &= \bigcup_{w' \in A_{\widetilde{w}}} \overline{B_0 s B_0}\, B_0 w' B_0 \\ &= \bigcup_{w' \in A_{\widetilde{w}}} B_0 w' B_0 \cup B_0 s w' B_0 \cup B_0 s^2 w' B_0 = \bigcup_{w' \in A_w} B_0 w' B_0.\end{aligned}$$

□

2.2 Oriented relative positions

2.2.1 Oriented flag manifolds

Let B be the minimal parabolic subgroup as defined in Section 2.1.1 and B_0 its identity component. Note that a proper closed subgroup $B_0 \subset P \subsetneq G$ containing B_0 has a parabolic Lie algebra and is thus a union of connected components of a parabolic subgroup.

Definition 2.2.1. Let $B_0 \subset P \subsetneq G$ be a proper closed subgroup containing B_0. We call such a group *(standard) oriented parabolic subgroup* and the quotient G/P an *oriented flag manifold.*

Example 2.2.2. Let $G = \mathrm{SL}(n, \mathbb{R})$ be the special linear group. Then B_0 is the set of upper triangular matrices with positive diagonal entries $\lambda_1, \dots, \lambda_n$. The space G/B_0 can be identified with the space of complete oriented flags, i.e. complete flags with a choice of orientation in every dimension. An example of a closed subgroup $B_0 \subsetneq P \subsetneq G$ is the group of upper triangular matrices where λ_1 and λ_2 are allowed to be negative, while the remaining entries are positive. The space G/P identifies with the space of complete flags with a choice of orientation on every component except the 1-dimensional one. In this way, all partial flag manifolds with a choice of orientation on a subset of the components of the flags can be obtained. However, we can also consider e.g. the group $P' = \langle B_0, -I \rangle$. Its corresponding oriented flag manifold G/P' is the space of complete oriented flags up to simultaneously changing the orientation on every odd-dimensional component.

The parabolic subgroups of G are parametrized by proper subsets θ of Δ. As the oriented parabolics are unions of connected components of these, we will need some more information to specify them. Recall that we defined lifts $r\colon \Delta \to \widetilde{W}$ of the standard generators Δ of W in Section 2.1.3.

Definition 2.2.3. Let $\theta \subsetneq \Delta$ and $\overline{M}_\theta = \langle r(\theta)\rangle \cap \overline{M}$. Let $\overline{M}_\theta \subset E \subset \overline{M}$ be a subgroup. Then we call the group $R = \langle r(\theta), E\rangle \subset \widetilde{W}$ an *oriented parabolic type*.

Remarks 2.2.4.

(i) This definition does not depend on the choices involved in r (see Remark 2.1.5(i)).

(ii) For every oriented parabolic type R, there is a unique pair (θ, E) with $\theta \subsetneq \Delta$, $\overline{M}_\theta \subset E \subset \overline{M}$, and $R = \langle r(\theta), E\rangle$. In fact, using Lemma 2.2.7 below, we can recover θ and E from R by

$$R \cap \overline{M} = \langle r(\theta)\rangle E \cap \overline{M} = \overline{M}_\theta E = E$$

and

$$\pi(R) \cap \Delta = \pi(\langle r(\theta)\rangle) \cap \Delta = \langle \theta\rangle \cap \Delta = \theta.$$

Proposition 2.2.5. *The map*

$$\{\textit{oriented parabolic types}\} \to \{\textit{oriented parabolic subgroups}\}$$

mapping R to $P_R = B_0 R B_0$ is a bijection. Its inverse maps P to $P \cap \widetilde{W}$. We will call $P \cap \widetilde{W}$ the type of P.

Definition 2.2.6. Let P_R be the oriented parabolic of type $R = P \cap \widetilde{W} = \langle r(\theta), E\rangle$. Then we write

$$\mathcal{F}_R = G/P_R, \qquad \mathcal{F}_\theta = G/P_\theta$$

for the associated oriented and unoriented flag manifolds.

To prove Proposition 2.2.5, we first need a few lemmas.

Lemma 2.2.7. *Let $\alpha \in \Delta$ and $w \in \widetilde{W}$ such that $\pi(w)$ and α are commuting elements of W. Then $w\, r(\alpha)\, w^{-1} \in \{r(\alpha), r(\alpha)^{-1}\} \subset \widetilde{W}$. In particular, this holds for any $w \in \overline{M}$. As a consequence, for any $\theta \subset \Delta$ and any subgroup $E \subset \overline{M}$*

$$\langle r(\theta), E\rangle = \langle r(\theta)\rangle\, E = E\, \langle r(\theta)\rangle.$$

Proof. We compute, using that Ad_w commutes with the Cartan involution,

$$w\, r(\alpha)\, w^{-1} = \exp\left(\tfrac{\pi}{2}\left(\mathrm{Ad}_w E_\alpha + \Theta\, \mathrm{Ad}_w E_\alpha\right)\right).$$

Since Ad_w preserves $\|\cdot\|$ and the root α is preserved by w this just corresponds to a different choice of $E_\alpha \in \mathfrak{g}_\alpha$ in the definition of $r(\alpha)$, so $w\, r(\alpha)\, w^{-1}$ must be either $r(\alpha)$ or $r(\alpha)^{-1}$ by Remark 2.1.5(i). So in particular $m\, \langle r(\theta)\rangle\, m^{-1} \subset \langle r(\theta)\rangle$ for any $m \in \overline{M}$ and $\theta \subset \Delta$, which shows the second statement. □

Lemma 2.2.8. *Let R, S be oriented parabolic types and $w \in \widetilde{W}$. Then*

$$B_0RB_0wB_0SB_0 = B_0RwSB_0.$$

Proof. Let $R = \langle r(\theta), E\rangle$. We first prove $B_0w'B_0wB_0 \subset B_0RwB_0$ for all $w \in \widetilde{W}$ and $w' \in R$ by induction on $\ell(w')$. If $\ell(w') = 0$, then $w' \in \overline{M}$, so $B_0w'B_0wB_0 = B_0w'wB_0 \subset B_0RwB_0$. If $\ell(w') > 0$ then we can find $\alpha \in \theta$ and $s = r(\alpha)$ with $w' = w''s$ and $\ell(w') = \ell(w'') + 1$. So by Lemma 2.1.10

$$\begin{aligned} B_0w'B_0wB_0 &= B_0w''sB_0wB_0 = B_0w''B_0sB_0wB_0 \\ &\subset B_0w''B_0wB_0 \cup B_0w''B_0swB_0 \cup B_0w''B_0s^2wB_0, \end{aligned}$$

which is in B_0RwB_0 by the induction hypothesis, since $s, w'' \in R$. So $B_0RB_0wB_0 \subset B_0RwB_0$. By the same argument, we get $B_0SB_0w^{-1}B_0 \subset B_0Sw^{-1}B_0$, and therefore $B_0wB_0SB_0 \subset B_0wSB_0$. Together, this shows the lemma. □

Lemma 2.2.9. *If R is an oriented parabolic type, then B_0RB_0 is a closed subgroup of G.*

Proof. Closedness follows from Proposition 2.1.12 as $A_w \subset R$ for every $w \in R$. This is because we can write $w = w'm$ with $w' \in \langle r(\theta)\rangle$ and $m \in E$, and then $A_w = A_{w'}m \subset R$. To see that B_0RB_0 is a subgroup we take $w, w' \in R$ and need to prove that $B_0wB_0w'B_0 \subset B_0RB_0$. But this follows from Lemma 2.2.8 (with $S = 1$) □

Lemma 2.2.10. *Let $\theta \subsetneq \Delta$. Then $P_\theta \cap \widetilde{W} = \langle r(\theta), \overline{M}\rangle$ and $P_{\theta,0} \cap \widetilde{W} = \langle r(\theta)\rangle$.*

Proof. Since P_θ is B–invariant from both sides, it is a union of Bruhat cells, so $P_\theta \cap \widetilde{W} = \pi^{-1}(P_\theta \cap W)$. Recall that $P_\theta = N_G(\mathfrak{p}_\theta)$, so $w \in W$ is in P_θ if and only if $\mathrm{Ad}_w\, \mathfrak{p}_\theta \subset \mathfrak{p}_\theta$. This holds if and only if w preserves $\Sigma_0^+ \cup \mathrm{span}(\theta)$. A simple computation shows that this is equivalent to $\Psi_w \subset \mathrm{span}(\theta)$, which in turn is equivalent to $w \in \langle\theta\rangle \subset W$ by Lemma 2.1.6. This proves the first equality.

For the second one, note that $r(\alpha) \in P_{\{\alpha\},0} \subset P_{\theta,0}$ by Lemma 2.1.8 for every $\alpha \in \theta$, so $\langle r(\theta)\rangle \subset P_{\theta,0} \cap \widetilde{W}$. By Lemma 2.2.9, $P_{\langle r(\theta)\rangle}$ is a closed subgroup of G and $P_{\langle r(\theta)\rangle} \subset P_{\theta,0}$. But by the preceding paragraph, $P_\theta = P_{\langle r(\theta), \overline{M}\rangle} = P_{\langle r(\theta)\rangle}\overline{M}$ is a union of finitely many copies of $P_{\langle r(\theta)\rangle}$. This is only possible if $P_{\langle r(\theta)\rangle} = P_{\theta,0}$. □

Proof of Proposition 2.2.5. Lemma 2.2.9 shows that $P_R = B_0RB_0$ is a closed subgroup containing B_0 for every oriented parabolic type R. On the other hand, the Lie algebra of such a subgroup P contains $\mathfrak{b}$ and is therefore of the form $\mathfrak{p}_\theta$ for some $\theta \subset \Delta$ [Kna02, Proposition 7.76]. So $P_{\theta,0} \subset P \subset P_\theta$ and, by Lemma 2.2.10, $\langle r(\theta)\rangle \subset P \cap \widetilde{W} \subset \langle r(\theta)\rangle\overline{M}$. Let $E = P \cap \overline{M}$. Then $\overline{M}_\theta \subset E \subset \overline{M}$ and $P \cap \widetilde{W} = \langle r(\theta)\rangle E$ is an oriented parabolic type.

So the maps in both directions are well–defined. It is clear by Proposition 2.1.3 that they are inverses of each other. □

2.2.2 Relative positions

Let P_R and P_S be the oriented parabolic subgroups of types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$ and let $\mathcal{F}_R$, $\mathcal{F}_S$ be the oriented flag manifolds.

Definition 2.2.11. The *set of relative positions* is the quotient

$$\widetilde{W}_{R,S} = G\backslash(\mathcal{F}_R \times \mathcal{F}_S),$$

where G acts diagonally on $\mathcal{F}_R \times \mathcal{F}_S$. The projection

$$\mathrm{pos}_{R,S} \colon \mathcal{F}_R \times \mathcal{F}_S \to \widetilde{W}_{R,S}$$

is called the *relative position map.*

Example 2.2.12. Consider the group $G = \mathrm{SL}(2,\mathbb{R})$ and $R = S = \{1\}$, so that both $\mathcal{F}_R$ and $\mathcal{F}_S$ are identified with S^1, the space of oriented lines in $\mathbb{R}^2$. Then there are two 1-dimensional and two 0-dimensional G-orbits in the space $S^1 \times S^1$. The 1-dimensional orbits consist of all transverse pairs (v, w) defining a positive or a negative basis of $\mathbb{R}^2$. The 0-dimensional orbits consist of pairs $(v, \pm v)$.

The relative positions admit a combinatorial description in the framework of the preceding sections. This is the main reason why we consider the types of parabolics as subgroups of $\widetilde{W}$, and the reason for the notation $\widetilde{W}_{R,S}$. When we write $\widetilde{W}_{R,S}$ in the following, we will usually regard it as $R\backslash\widetilde{W}/S$ and we will write double brackets $[\![\,\cdot\,]\!]$ for equivalence classes in these quotients.

Proposition 2.2.13. *There are bijections*

$$R\backslash\widetilde{W}/S \to P_R\backslash G/P_S \to G\backslash(\mathcal{F}_R \times \mathcal{F}_S).$$

The first map is induced by the inclusion of $N_K(\mathfrak{a})$ into G and the second by mapping $g \in G$ to $([1],[g]) \in \mathcal{F}_R \times \mathcal{F}_S$. In particular, $\widetilde{W}_{R,S}$ is a finite set.

Proof. It is clear from the definitions that the maps are well–defined and the second map is injective. It is surjective since G acts transitively on $\mathcal{F}_R$ so every pair in $\mathcal{F}_R \times \mathcal{F}_S$ can thus be brought into the form $([1],[g])$ by the diagonal action. To see that the first map is surjective, let $P_R g P_S \in P_R\backslash G/P_S$. By Proposition 2.1.3 $g \in B_0 w B_0$ for some $w \in \widetilde{W}$. Then $[\![w]\!] \in R\backslash\widetilde{W}/S$ maps to $[g]$.

To prove injectivity of the first map, let $w, w' \in \widetilde{W}$ with $P_R w P_S = P_R w' P_S$. Since $P_R = B_0 R B_0$ and $P_S = B_0 S B_0$ by Lemma 2.2.9, we can write $w' \in P_R w P_S = B_0 R B_0 w B_0 S B_0$. By Lemma 2.2.8, $B_0 R B_0 w B_0 S B_0 = B_0 R w S B_0$, and by Proposition 2.1.3 this implies $w' \in RwS$, proving injectivity. $\square$

Definition 2.2.14. Let $f \in \mathcal{F}_R$ and $[\![w]\!] \in \widetilde{W}_{R,S}$. Then we write

$$C^{R,S}_{[\![w]\!]}(f) := \{f' \in \mathcal{F}_S \mid \mathrm{pos}_{R,S}(f,f') = [\![w]\!]\}$$

for the set of oriented flags in $\mathcal{F}_S$ at position $[\![w]\!]$ with respect to f. We sometimes omit the superscript R, S if it is clear from the context.

2.2.3 The Bruhat order

Again, let P_R and P_S be the oriented parabolic subgroups of types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$ and let $\mathcal{F}_R$, $\mathcal{F}_S$ be the oriented flag manifolds.

Definition 2.2.15. The *Bruhat order* on $\widetilde{W}_{R,S} = G\backslash(\mathcal{F}_R \times \mathcal{F}_S)$ is the inclusion order on closures, i.e.

$$G(f_1, f_2) \leq G(f_1', f_2') \quad \Leftrightarrow \quad \overline{G(f_1, f_2)} \subset \overline{G(f_1', f_2')}.$$

In other words, if we have sequences of flags $f_n \to f \in \mathcal{F}_R$ and $f_n' \to f' \in \mathcal{F}_S$ with $\mathrm{pos}_{R,S}(f_n, f_n')$ constant, then

$$\mathrm{pos}_{R,S}(f, f') \leq \mathrm{pos}_{R,S}(f_n, f_n').$$

If we view the set of relative positions as the double quotient $R\backslash\widetilde{W}/S$ via the correspondence in Proposition 2.2.13, the Bruhat order is given as follows.

Lemma 2.2.16. *Let $w, w' \in \widetilde{W}$ and denote by $[\![w]\!], [\![w']\!] \in \widetilde{W}_{R,S}$ the equivalence classes they represent. Then we have*

$$[\![w]\!] \leq [\![w']\!] \quad \Leftrightarrow \quad \overline{P_R w P_S} \subset \overline{P_R w' P_S}.$$

Proof. Recall that w represents the orbit $G([1],[w])$. The inequality $[\![w]\!] \leq [\![w']\!]$ is equivalent to the existence of a sequence $g_n \in G$ such that $[g_n] \to [1]$ in $\mathcal{F}_R$ and $[g_n w'] \to [w]$ in $\mathcal{F}_S$. This means that there exist sequences $p_n \in P_R$ and $q_n \in P_S$ such that $g_n p_n \to 1$ and $g_n w' q_n \to w$. Writing

$$g_n p_n p_n^{-1} w' q_n \to w$$

shows that $p_n^{-1} w' q_n \to w$.
Conversely, if such sequences p_n and q_n are given, we can simply take $g_n = p_n$ in the description above. □

In the case $R = S = \{1\}$, the Bruhat order on $\widetilde{W}$ is defined by orbit closures of the $B_0 \times B_0$–action. Proposition 2.1.12 describes this in combinatorial terms. The following lemma shows how the Bruhat order on $\widetilde{W}$ relates to that on the quotients $\widetilde{W}_{R,S} = R\backslash\widetilde{W}/S$.

Lemma 2.2.17. *Let $R \subset R'$ and $S \subset S'$ be oriented parabolic types. In this lemma, we write $[\![w]\!]$ for the equivalence class of $w \in \widetilde{W}$ in $\widetilde{W}_{R,S}$ and $[\![w]\!]'$ for its equivalence class in $\widetilde{W}_{R',S'}$. Then for every $w_1, w_2 \in \widetilde{W}$*

(i) If $[\![w_1]\!] \leq [\![w_2]\!]$, then $[\![w_1]\!]' \leq [\![w_2]\!]'$.

(ii) If $[\![w_1]\!]' \leq [\![w_2]\!]'$, then there exists $w_3 \in \widetilde{W}$ with $[\![w_3]\!]' = [\![w_2]\!]'$ and $[\![w_1]\!] \leq [\![w_3]\!]$.

(iii) If $[\![w_1]\!]' \leq [\![w_2]\!]'$, then there exists $w_3 \in \widetilde{W}$ with $[\![w_3]\!]' = [\![w_1]\!]'$ and $[\![w_3]\!] \leq [\![w_2]\!]$.

Proof. If $[\![w_1]\!] \leq [\![w_2]\!]$ then $w_1 \in \overline{P_R w_2 P_S} \subset \overline{P_{R'} w_2 P_{S'}}$. As the last term is $P_{R'}$–left and $P_{S'}$–right invariant and closed, this implies $\overline{P_{R'} w_1 P_{S'}} \subset \overline{P_{R'} w_2 P_{S'}}$, hence (i).

The assumption in (ii) is equivalent to $w_1 \in \overline{P_{R'} w_2 P_{S'}}$. By Lemma 2.2.8 and Lemma 2.2.9, $\overline{P_{R'} w_2 P_{S'}} = \overline{B_0 R' w_2 S' B_0}$, so there exist $r \in R'$ and $s \in S'$ such that $w_1 \in \overline{B_0 r w_2 s B_0}$. So $w_3 = r w_2 s$ satisfies the properties we want.

In part (iii), as $[\![w_1]\!]' \leq [\![w_2]\!]'$ there is a sequence $g_n \in G$ such that $[g_n] \to [1]$ in $\mathcal{F}_{R'}$ and $[g_n w_2] \to [w_1]$ in $\mathcal{F}_{S'}$. Passing to a subsequence, we can also assume that $[g_n] \to f_1 \in \mathcal{F}_R$ and $[g_n w_2] \to f_2 \in \mathcal{F}_S$. Let $[\![w_3]\!] = \mathrm{pos}_{R,S}(f_1, f_2)$. Then $[\![w_3]\!] \leq [\![w_2]\!]$ and $[\![w_3]\!]' = \mathrm{pos}_{R',S'}(\pi_{R'}(f_1), \pi_{S'}(f_2)) = \mathrm{pos}_{R',S'}([1],[w_1]) = [\![w_1]\!]'$. □

This allows us to describe the Bruhat order on $\widetilde{W}_{R,S}$ solely in terms of $\widetilde{W}$, the types R and S, the projection to W and the generating system $r(\Delta)$. Recall from Proposition 2.1.12 the definition of $A_w \subset \widetilde{W}$ for $w \in \widetilde{W}$: If $w = r(\alpha_1) \cdots r(\alpha_k) m$ with $k = \ell(w)$, $\alpha_1, \dots, \alpha_k \in \Delta$ and $m \in \widetilde{M}$, then

$$A_w = \{ r(\alpha_1)^{i_1} \cdots r(\alpha_k)^{i_k} m \mid i_1, \dots, i_k \in \{0,1,2\} \} \subset \widetilde{W},$$

i.e. the set of words obtained from w by deleting or squaring some of the letters. By Proposition 2.1.12 this is independent of the chosen representative of w.

Proposition 2.2.18. *The Bruhat order on $\widetilde{W}_{R,S}$ is a partial order. For $w, w' \in \widetilde{W}$ with projections $[\![w]\!], [\![w']\!] \in \widetilde{W}_{R,S}$ it is given by*

$$[\![w']\!] \leq [\![w]\!] \quad \Leftrightarrow \quad w' \in R A_w S = \bigcup_{r \in R,\ s \in S} A_{rws}. \tag{2.2.1}$$

Proof. In the case $R = S = \{1\}$ the description (2.2.1) holds by Proposition 2.1.12. In the general case, if $w' \in A_{rws}$ or $w' \in r^{-1} A_w s^{-1}$ for some $r \in R$, $s \in S$, then $w' \leq rws$ or $rw's \leq w$ in $\widetilde{W}$, respectively. By Lemma 2.2.17(i) both inequalities imply $[\![w']\!] \leq [\![w]\!]$. Conversely, if $[\![w']\!] \leq [\![w]\!]$, then $w' \leq rws$ and $r'w's' \leq w$ for some $r, r' \in R$ and $s, s' \in S$ by Lemma 2.2.17(ii) and Lemma 2.2.17(iii), so $w' \in A_{rws}$ and $w' \in r^{-1} A_w s^{-1}$.

The Bruhat order is transitive and reflexive by its definition. To show antisymmetry, suppose $[\![w]\!] \leq [\![w']\!] \leq [\![w]\!]$ in $\widetilde{W}_{R,S}$. Then $R A_w S = R A_{w'} S$. Let $\mathcal{L}_w \subset R A_w S$ be the subset of elements which are maximal in $R A_w S = \bigcup_{r,s} A_{rws}$ with respect to ℓ. Then every element of $\mathcal{L}_w$ is also maximal in A_{rws} for some $r \in R$, $s \in S$. But the unique longest element of A_{rws} is rws, since squaring and deleting letters only reduces ℓ. So $\mathcal{L}_w \subset RwS$ and, since $\mathcal{L}_w = \mathcal{L}_{w'} \neq \varnothing$, $RwS = Rw'S$. □

The following characterization of the Bruhat order will also be useful later:

Lemma 2.2.19. *Let $w, w' \in \widetilde{W}$ with $\ell(w') = \ell(w) + 1$. Let*

$$Q = \{ w\, r(\alpha)^{\pm 1}\, w^{-1} \mid w \in \widetilde{W}, \alpha \in \Delta \} \subset \widetilde{W}$$

be the set of conjugates of the standard generators or their inverses. Then

$$w \leq w' \quad \Longleftrightarrow \quad \exists q \in Q \colon w = qw'.$$

Proof. The implication '⇒' follows from Proposition 2.1.12 by choosing q of the form

$$q = r(\alpha_1) \dots r(\alpha_{i-1}) r(\alpha_i)^{\pm 1} r(\alpha_{i-1})^{-1} \dots r(\alpha_1)^{-1}$$

for some i. For the other direction, assume that $w = qw'$ and write

$$w' = r(\alpha_1) \dots r(\alpha_k)\, m$$

for some $\alpha_1, \dots, \alpha_k \in \Delta$ with $k = \ell(w')$ and $m \in \overline{M}$. Then $\pi(w') = \alpha_1 \dots \alpha_k$ and by the strong exchange property of Coxeter groups [BB06, Theorem 1.4.3]

$$\pi(w) = \pi(q)\pi(w') = \alpha_1 \dots \widehat{\alpha_i} \dots \alpha_k$$

for some i, so $\pi(q) = (\alpha_1 \dots \alpha_{i-1})\alpha_i(\alpha_1 \dots \alpha_{i-1})^{-1}$. Set $c = r(\alpha_1) \dots r(\alpha_{i-1}) \in \widetilde{W}$. Then $c^{-1}qc \in \pi^{-1}(\alpha_i) \cap Q = \{r(\alpha_i)^{\pm 1}\}$ by Lemma 2.2.7. So

$$w = qw' = c\, r(\alpha_i)^{\pm 1} c^{-1} w' = r(\alpha_1) \dots r(\alpha_{i-1}) r(\alpha_i)^{1 \pm 1} r(\alpha_{i+1}) \dots r(\alpha_k)\, m \leq w',$$

where the inequality at the end follows by Proposition 2.1.12. □

In our later discussion of group actions, we will need the notion of an ideal for a partial order.

Definition 2.2.20. Let $(X, \leq)$ be a set equipped with a partial order. Then a subset $I \subset X$ is called an *ideal* if for every $x \in I$ and $y \in X$ with $y \leq x$, we have $y \in I$.

In the case of the Bruhat order on the extended Weyl group, an ideal corresponds to a G-invariant closed subset of $\mathcal{F}_R \times \mathcal{F}_S$. Intuitively, if a specific relative position is contained in the ideal, then all less generic relative positions are contained in it as well. Note that "less generic" is somewhat more subtle in the oriented case (a discussion of examples is found in Section 2.5).

The following lemmas will be useful when calculating with relative positions.

Lemma 2.2.21.

(i) *$Z_K(\mathfrak{a})$ normalizes the subgroups P_R, P_S. Consequently, the (finite abelian) group $\overline{M}/E$ acts on $\mathcal{F}_R$ by right multiplication and on $\widetilde{W}_{R,S}$ by left multiplication. $\overline{M}/F$ acts on $\mathcal{F}_S$ and $\widetilde{W}_{R,S}$ by right multiplication. The actions on $\widetilde{W}_{R,S}$ preserve the Bruhat order.*

(ii) *For any $f_1 \in \mathcal{F}_R$, $f_2 \in \mathcal{F}_S$, $m_1 \in \overline{M}/E$, $m_2 \in \overline{M}/F$, the following holds:*

$$\mathrm{pos}_{R,S}(r_{m_1}(f_1), r_{m_2}(f_2)) = m_1^{-1}\, \mathrm{pos}_{R,S}(f_1, f_2) m_2$$

Proof.

(i) It follows e.g. from Lemma 2.2.7 that $\overline{M}$ normalizes $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$. Furthermore, $Z_K(\mathfrak{a})$ normalizes B_0 and thus also $P_R = B_0RB_0$ and $P_S = B_0SB_0$. This implies that the actions of $\overline{M}$ on $\mathcal{F}_R$ and $\mathcal{F}_S$ by right multiplication and on $\widetilde{W}_{R,S}$ by left and right multiplication are well-defined. Since E resp. F acts trivially, we obtain the induced actions of $\overline{M}/E$ resp. $\overline{M}/F$. They preserve the Bruhat order since, for $m, m' \in \overline{M}$,

$$\begin{aligned}\llbracket w\rrbracket \leq \llbracket w'\rrbracket &\Leftrightarrow \overline{P_RwP_S} \subset \overline{P_Rw'P_S} \Leftrightarrow m\overline{P_RwP_S}m' \subset m\overline{P_Rw'P_S}m' \\ &\Leftrightarrow \overline{P_Rmwm'P_S} \subset \overline{P_Rmw'm'P_S} \Leftrightarrow \llbracket mwm'\rrbracket \leq \llbracket mw'm'\rrbracket.\end{aligned}$$

(ii) Let $\mathrm{pos}_{R,S}(f_1, f_2) = \llbracket w\rrbracket \in \widetilde{W}_{R,S}$. This means that there exists some $g \in G$ such that $g(f_1, f_2) = ([1], [w])$. It follows that

$$m_1^{-1}g(r_{m_1}(f_1), r_{m_2}(f_2)) = m_1^{-1}([m_1], [wm_2]) = ([1], [m_1^{-1}wm_2]).$$

So we obtain $\mathrm{pos}_{R,S}(r_{m_1}(f_1), r_{m_2}(f_2)) = \llbracket m_1^{-1}wm_2\rrbracket$. □

Corollary 2.2.22. *Let $f \in \mathcal{F}_R$, $w \in \widetilde{W}$ and $m \in \overline{M}$. Then we have*

$$r_m(C^{R,S}_{\llbracket w\rrbracket}(f)) = C^{R,S}_{\llbracket wm\rrbracket}(f) = C^{R,S}_{\llbracket w\rrbracket}(r_{wm^{-1}w^{-1}}(f)).$$

Proof. From the previous lemma we obtain

$$\begin{aligned}\mathrm{pos}_{R,S}(f, r_{m^{-1}}(f')) = \llbracket w\rrbracket &\Leftrightarrow \mathrm{pos}_{R,S}(f, f') = \llbracket wm\rrbracket \\ &\Leftrightarrow \mathrm{pos}_{R,S}(r_{wm^{-1}w^{-1}}(f), f') = \llbracket w\rrbracket.\end{aligned}$$ □

Lemma 2.2.23. *Let $w \in \widetilde{W}$. Then*

$$C^{R,S}_{\llbracket w\rrbracket}([1]) = NR[w] \subset \mathcal{F}_S.$$

Proof. Let $g \in G$ such that $[g] \in C_{\llbracket w\rrbracket}([1])$. Then by Lemma 2.2.8 we have

$$g \in P_RwP_S = B_0RwP_S.$$

Using the Iwasawa decomposition $B_0 = NAZ_K(\mathfrak{a})_0$ and the fact that both A and $Z_K(\mathfrak{a})$ are normalized by $\widetilde{W} = N_K(\mathfrak{a})$, this implies

$$B_0RwP_S = NAZ_K(\mathfrak{a})_0RwP_S = NRwAZ_K(\mathfrak{a})_0P_S = NRwP_S.$$ □

Lemma 2.2.24. *Let $w_0, w_1, w_2 \in \widetilde{W}$ with $w_0Rw_0^{-1} = R$. Assume there are $f_1, f_2 \in \mathcal{F}_R$ and $f_3 \in \mathcal{F}_S$ such that*

$$\begin{aligned}\mathrm{pos}_{R,R}(f_1, f_2) &= \llbracket w_0\rrbracket \in \widetilde{W}_{R,R},\\ \mathrm{pos}_{R,S}(f_1, f_3) &= \llbracket w_1\rrbracket \in \widetilde{W}_{R,S},\\ \mathrm{pos}_{R,S}(f_2, f_3) &= \llbracket w_2\rrbracket \in \widetilde{W}_{R,S}.\end{aligned}$$

Then

$$\llbracket w_1\rrbracket \geq \llbracket w_0w_2\rrbracket$$

in $\widetilde{W}_{R,S}$.

Proof. Using the G-action on pairs, we can assume that $(f_1, f_2) = ([1], [w_0])$. Then, since $\mathrm{pos}_{R,S}(f_2, f_3) = \mathrm{pos}_{R,S}([w_0], f_3) = [\![w_2]\!]$, Lemma 2.2.23 implies that f_3 has a representative in G of the form $w_0 u r w_2$ for some $u \in N$ and $r \in R$. We want to find elements $g_n \in G$ such that

$$g_n(f_1, f_3) = g_n([1], [w_0 u r w_2]) \xrightarrow{n\to\infty} ([1], [w_0 w_2]).$$

Let $A_n \in \overline{\mathfrak{a}^+}$ be a sequence with $\alpha(A_n) \to \infty$ for every $\alpha \in \Delta$ and $g_n = w_0 r^{-1} e^{-A_n} w_0^{-1}$. Then $g_n \in P_R$ since A and R are normalized by w_0. Observe that $w_2^{-1} r^{-1} e^{A_n} r w_2 \in A \subset P_S$, since A is normalized by all of $\widetilde{W}$. Then g_n stabilizes $[1] \in \mathcal{F}_R$, and we calculate

$$\begin{aligned} g_n[w_0 u r w_2] &= [(w_0 r^{-1} e^{-A_n} w_0^{-1}) w_0 u r w_2 (w_2^{-1} r^{-1} e^{A_n} r w_2)] \\ &= [w_0 r^{-1} e^{-A_n} u e^{A_n} r w_2] \xrightarrow{n\to\infty} [w_0 w_2], \end{aligned}$$

where we used that $e^{-A_n} u e^{A_n} \xrightarrow{n\to\infty} 1$. $\square$

2.2.4 Transverse positions

In this section, we consider the action of elements $w_0 \in \widetilde{W}$ which are lifts of the longest element of the Weyl group on the set of relative positions. The aim is to find a “good” lift, which acts in particular as an involution. There does not seem to be a unique candidate among the lifts, and the validity of any choice depends on the oriented flag manifold under consideration.

Again, let P_R and P_S be oriented parabolic subgroups of types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$ where $\theta, \eta \subsetneq \Delta$ and $\overline{M}_\theta \subset E \subset \overline{M}$ and $\overline{M}_\eta \subset F \subset \overline{M}$ are subgroups. Now we also assume that $\iota(\theta) = \theta$.

Definition 2.2.25. A relative position $[\![w_0]\!] \in \widetilde{W}_{R,S}$ is called *transverse* if it is a projection of an element $w_0 \in \widetilde{W}$ which is a lift of the longest element of W. The set of transverse positions in $\widetilde{W}_{R,S}$ will be denoted by $T_{R,S}$ (if $R = S = \{1\}$, the index will be omitted).

In the oriented setting, there can be multiple transverse positions which are maximal in the Bruhat order and therefore incomparable.

Lemma 2.2.26. *Let $[\![w]\!] \in \widetilde{W}_{R,S}$ be a transverse position. Then $[\![w]\!]$ is maximal in the Bruhat order.*

Proof. Since $[\![w]\!]$ is transverse, we may assume that $\ell(w)$ is maximal. Let $[\![w']\!] \in \widetilde{W}_{R,S}$ be such that $[\![w]\!] \leq [\![w']\!]$. By Proposition 2.2.18, this implies the following. There exist $\tilde{r} \in R, s \in S$, and we can write $\tilde{r} w' s = r(\alpha_1) \dots r(\alpha_k) m$, where $r(\alpha_i)$, $\alpha_i \in \Delta$ are our preferred choice of generators for $\widetilde{W}$, $k = \ell(\tilde{r} w' s)$, and $m \in \overline{M}$. Furthermore, a word representing w is obtained from $\tilde{r} w' s = r(\alpha_1) \dots r(\alpha_k) m$ by squaring or deleting some of the $r(\alpha_i)$. However, if any letters were indeed squared or deleted, $\ell(w)$ would by strictly smaller and thus not maximal, so the two words must be equal. $\square$

Lemma 2.2.27. *Let $w_0 \in T \subset \widetilde{W}$ be a transverse position such that $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$. Then w_0 acts as an involution on $\widetilde{W}_{R,S}$.*

Proof. We want w_0 to act by left–multiplication on $\widetilde{W}_{R,S} = R\backslash\widetilde{W}/S$. For this action to be well–defined, we need to show that w_0 normalizes R. The induced action of w_0 on reduced roots is given by ι and θ is ι–invariant. Moreover, Remark 2.1.5(i) implies that for every $\alpha \in \theta$, we have $w_0 r(\alpha) w_0^{-1} = r(\iota(\alpha))$ or $w_0 r(\alpha) w_0^{-1} = r(\iota(\alpha))^{-1}$. Therefore, $\langle r(\theta)\rangle$ is normalized by w_0. Since $w_0 E w_0^{-1} = E$ by assumption and $R = \langle r(\theta), E\rangle$, all of R is normalized by w_0. Clearly, $w_0^2 \in E$ implies that the induced action is an involution. $\square$

Lemma 2.2.28. *Let $w_0 \in T \subset \widetilde{W}$ be a transverse position. Then $w \leq w'$ implies $w_0 w' \leq w_0 w$ for any $w, w' \in \widetilde{W}$. If $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$, then w_0 acts as an order–reversing involution on $\widetilde{W}_{R,S}$.*

Proof. Assume $w \leq w'$. Then by Proposition 2.1.12 there exists a sequence $w = w_1 \leq \cdots \leq w_k = w'$ with $\ell(w_{i+1}) = \ell(w_i) + 1$. So by Lemma 2.2.19 $w_i = q_i w_{i+1}$ for some $q_i \in Q$. Therefore, $w_0 w_{i+1} = q_i' w_0 w_i$ for $q_i' = w_0 t_i^{-1} w_0^{-1} \in Q$ and $\ell(w_0 w_i) = \ell(w_0) - \ell(w_i) = \ell(w_0) - \ell(w_{i+1}) + 1 = \ell(w_0 w_{i+1}) + 1$, so $w_0 w_k \leq \cdots \leq w_0 w_1$ by the same lemma. Now Lemma 2.2.27 shows that w_0 defines an involution of the quotient $\widetilde{W}_{R,S}$ and it is an easy consequence of Lemma 2.2.17 that the action on this quotient still reverses the order. $\square$

Remark 2.2.29. When $w_0 \in T \subset \widetilde{W}$ is a transverse position, the conditions $\iota(\theta) = \theta$ and $w_0 E w_0^{-1} = E$ are equivalent to $w_0 R w_0^{-1} = R$. Moreover, $w_0^2 \in E$ is equivalent to $w_0^2 \in R$.

The existence of an involution w_0 on $\widetilde{W}_{R,S}$ gives rise to the following properties an ideal $I \subset \widetilde{W}_{R,S}$ can have. They play a crucial role in the description of properly discontinuous and cocompact group actions of oriented flag manifolds in Section 2.4.1 and Section 2.4.2.

Definition 2.2.30. Let $I \subset \widetilde{W}_{R,S}$ be an ideal and $w_0 \in T \subset \widetilde{W}$ a transverse position satisfying $w_0 E w_0^{-1} = E$. Then I is called w_0–*fat* if $x \notin I$ implies $w_0 x \in I$. I is called w_0–*slim* if $x \in I$ implies $w_0 x \notin I$. I is called w_0–*balanced* if it is fat and slim.

Observe that there can be no w_0–balanced ideal if w_0 has a fixed point. Conversely, if w_0 has no fixed points, there will be w_0–balanced ideals by the following lemma. For the case of the Weyl group, this is proved in [KLP18, Proposition 3.29].

Lemma 2.2.31. *Let X be a partially ordered set and $\sigma\colon X \to X$ an order–reversing involution without fixed points. Then every minimal σ–fat ideal and every maximal σ–slim ideal is σ–balanced.*

Proof. The two statements are equivalent by replacing an ideal I by $X \setminus \sigma(I)$. So assume that $I \subset X$ is a σ–fat ideal which is not σ–balanced. Choose a maximal element $x \in I \cap \sigma(I) \neq \varnothing$ and let $I' = I \setminus \{x\}$. Assume I' is not an ideal. Then there exist $x_1 \leq x_2$ with $x_2 \in I'$ but $x_1 \notin I'$. So $x_1 = x$ since I is an ideal. Furthermore $\sigma(x_2) \leq \sigma(x_1) = \sigma(x)$ and $\sigma(x) \in I$, so $\sigma(x_2) \in I$ and therefore $x_2 \in I \cap \sigma(I)$. Since x is maximal in $I \cap \sigma(I)$ and $x \leq x_2$, this implies $x_2 = x \notin I'$, a contradiction. So I' is an ideal, and it is clearly σ–fat. So I is not a minimal σ–fat ideal. □

Example 2.2.32. Consider $G = \mathrm{SL}(3,\mathbb{R})$ with its maximal compact $K = \mathrm{SO}(3,\mathbb{R})$ and $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -\lambda_1 - \lambda_2 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{R} \right\}$. The extended Weyl group $\widetilde{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})_0$ consists of all permutation matrices A with determinant 1 – i.e. all matrices with exactly one ± 1 entry per line and row and all other entries 0, such that $\det(A) = 1$. The transverse positions are

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

The first two of these are actually involutions in $\widetilde{W}$, so the condition $w_0^2 \in E$ is empty. The condition $w_0 E w_0^{-1} = E$ is a symmetry condition on E, similar to the condition $\iota(\theta) = \theta$. It does not depend on the choice of lift: Any other w_0' is of the form $w_0' = w_0 m$ for some $m \in \overline{M}$, and $\overline{M}$ is abelian. In Proposition 2.3.7, we will show that we can always assume it to hold in our setting.
The last two are not involutions in $\widetilde{W}$. The smallest possible choice of E containing their square is

$$E = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}.$$

A discussion of balanced ideals for $\mathrm{SL}(3,\mathbb{R})$ is found in Section 2.5.3.

2.3 P_R–Anosov representations

Let P_R be the oriented parabolic of type $R = \langle r(\theta), E \rangle$, with $\iota(\theta) = \theta$. Moreover, let $w_0 \in T \subset \widetilde{W}$ be a transverse position. Let Γ be a finitely generated group and $\rho \colon \Gamma \to G$ a representation. We denote the Cartan projection by

$$\mu \colon G \to \overline{\mathfrak{a}^+}.$$

It maps $g \in G$ to the unique element $\mu(g) \in \overline{\mathfrak{a}^+}$ with $g \in K \exp(\mu(g)) K$. We need the following notions in order to define Anosov representations.

Definition 2.3.1. Let Γ be a word hyperbolic group, $\partial_\infty\Gamma$ its Gromov boundary and $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ a map.

(i) A sequence $g_n \in G$ is called *P_θ–divergent* if

$$\alpha(\mu(g_n)) \to \infty \qquad \forall \alpha \in \Delta\backslash\theta.$$

The representation ρ is *P_θ–divergent* if for every divergent sequence $\gamma_n \to \infty$ in Γ its image $\rho(\gamma_n)$ is P_θ–divergent.

(ii) Two elements $x, y \in \mathcal{F}_\theta$ are called *transverse* if the pair $(x, y) \in \mathcal{F}_\theta \times \mathcal{F}_\theta$ lies in the unique open G–orbit. Equivalently, their relative position is represented by the longest element of W.

The map ξ is called *transverse* if for every pair $x \neq y \in \partial_\infty\Gamma$, the images $\xi(x), \xi(y) \in \mathcal{F}_\theta$ are transverse.

(iii) ξ is called *dynamics–preserving* if, for every element $\gamma \in \Gamma$ of infinite order, its unique attracting fixed point $\gamma^+ \in \partial_\infty\Gamma$ is mapped to an attracting fixed point of $\rho(\gamma)$.

Definition 2.3.2. The representation $\rho\colon \Gamma \to G$ is *P_θ–Anosov* if Γ is word hyperbolic, ρ is P_θ–divergent and there is a continuous, transverse, dynamics–preserving, ρ–equivariant map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ called *limit map* or *boundary map.*

Definition 2.3.3. Assume that Γ is non–elementary. The representation $\rho\colon \Gamma \to G$ is *P_R–Anosov* if it is P_θ–Anosov with limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ and there is a continuous, ρ–equivariant lift $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of ξ. Such a map $\widehat{\xi}$ will be called a limit map or boundary map of ρ as an P_R–Anosov representation. The relative position $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ for $x \neq y \in \partial_\infty\Gamma$ is its *transversality type.*

We should verify that the transversality type is in fact well–defined.

Lemma 2.3.4. *The relative position* $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ *in the above definition does not depend on the choice of x and y.*

Proof. By [Gro87, 8.2.I], there exists a dense orbit in $\partial_\infty\Gamma \times \partial_\infty\Gamma$. By equivariance of $\widehat{\xi}$, the relative position $[\![w_0]\!]$ of pairs in this orbit is constant. It is a transverse position because this orbit contains (only) pairs of distinct points. An arbitrary pair (x, y) of distinct points in $\partial_\infty\Gamma$ can be approximated by pairs in the dense orbit, so by continuity of $\widehat{\xi}$, we have

$$\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y)) \leq [\![w_0]\!].$$

But $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$ is a transverse position, thus equality holds by Lemma 2.2.26. $\square$

Remarks 2.3.5.

(i) The definition (apart from the transversality type) makes sense for elementary hyperbolic groups, but it is not a very interesting notion in this case: The boundary has at most two points. Consequently, after restricting to a finite index subgroup, the boundary map always lifts to the maximally oriented setting. Moreover, after restricting to the subgroup preserving the boundary pointwise, the lifted boundary map holds no additional information.

(ii) In the oriented setting, the boundary map $\widehat{\xi} : \partial_\infty\Gamma \to \mathcal{F}_R$ is not unique: For any element $[m] \in \overline{M}/E$, the map $r_m \circ \widehat{\xi}$ is also continuous and equivariant. This gives all possible boundary maps in $\mathcal{F}_R$:
Since the unoriented boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_\theta$ is unique [GW12, Lemma 3.3], an oriented boundary map $\widehat{\xi}'$ must be a lift of it. But if $\widehat{\xi}'$ agrees with $r_m \circ \widehat{\xi}$ at a single point, it must agree everywhere by equivariance and continuity since any orbit is dense in $\partial_\infty\Gamma$ [KB02, Proposition 4.2]. If the transversality type of $\widehat{\xi}$ was $[\![w_0]\!]$, then that of $r_m \circ \widehat{\xi}$ is $[\![m^{-1}w_0m]\!]$ by Lemma 2.2.21.

The oriented flag manifold $\mathcal{F}_R$ in Definition 2.3.3 which is the target of the lift $\widehat{\xi}$ is not unique. However, there is a unique maximal choice of such a $\mathcal{F}_R$ (or equivalently, minimal choice of R), similar to the fact that an Anosov representation admits a unique minimal choice of θ such that it is P_θ–Anosov.

Proposition 2.3.6. *Let $\rho\colon \Gamma \to G$ be P_θ–Anosov. Then there is a unique minimal choice of E such that $\overline{M}_\theta \subset E \subset \overline{M}$ and ρ is P_R–Anosov, where $R = \langle r(\theta), E\rangle$.*

Proof. Assume that there are two different choices E_1 and E_2 such that ρ is both P_{R_1}– and P_{R_2}–Anosov, where $R_i = \langle r(\theta), E_i\rangle$. Let $E_3 = E_1 \cap E_2$. We will show that ρ is also P_{R_3}–Anosov. To do so, we have to construct a boundary map into $\mathcal{F}_{R_3}$ from the boundary maps into $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$.

Let $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$ be the boundary map of ρ as a P_θ–Anosov representation, and let $\xi_1\colon \partial_\infty\Gamma \to \mathcal{F}_{R_1}$, $\xi_2\colon \partial_\infty\Gamma \to \mathcal{F}_{R_2}$ be the two lifts we are given. Fix a point $x \in \partial_\infty\Gamma$, and let $F_x \in \mathcal{F}_{R_3}$ be a lift of $\xi(x) \in \mathcal{F}_\theta$. Denote by $\pi_1\colon \mathcal{F}_3 \to \mathcal{F}_1$ and $\pi_2\colon \mathcal{F}_3 \to \mathcal{F}_2$ the two projections. After right–multiplying ξ_1 with an element of $\overline{M}/E_1$ and ξ_2 with an element of $\overline{M}/E_2$, we may assume that $\pi_i(F) = \xi_i(x)$ for $i = 1, 2$. Set $\xi_3(x) := F_x$ and observe the following general property:

For every point $y \in \partial_\infty\Gamma$, there is at most one flag $F_y \in \mathcal{F}_{R_3}$ satisfying $\pi_i(F_y) = \xi_i(y)$ for $i = 1, 2$. Indeed, if gP_{R_3} and hP_{R_3} satisfy $gP_{R_i} = hP_{R_i}$ for $i = 1, 2$, there are elements $p_i \in P_{R_i}$ such that $g = hp_1 = hp_2$. This implies that $h^{-1}g \in P_{R_1} \cap P_{R_2} = P_{R_3}$.

By equivariance of ξ_1, ξ_2 and uniqueness of lifts to $\mathcal{F}_{R_3}$, we can extend ξ_3 equivariantly to a map $\xi_3\colon \Gamma x \to \mathcal{F}_{R_3}$. It is a lift of both $\xi_1|_{\Gamma x}$ and $\xi_2|_{\Gamma x}$. Recall that the orbit Γx is dense in $\partial_\infty\Gamma$ for any choice of x ([KB02, Proposition 4.2]). Using the corresponding properties of ξ_1 and ξ_2, we now show that this map is continuous and extends continuously to all of $\partial_\infty\Gamma$. Let $x_n \in \Gamma x$ and assume that $x_n \to x_\infty \in \partial_\infty\Gamma$.

Then $\xi_i(x_n) \to \xi_i(x_\infty)$ for $i = 1, 2$. Therefore, there exist $m_n \in \overline{M}/E_1 \cap E_2$ such that $\xi_3(x_n)m_n$ converges in $\mathcal{F}_{R_3}$. By injectivity of the map

$$\overline{M}/E_1 \cap E_2 \to \overline{M}/E_1 \times \overline{M}/E_2$$

and convergence of $\xi_i(x_n)$, $i = 1, 2$, m_n must eventually be constant. Thus the limit $\xi_3(x_\infty) := \lim_{n\to\infty} \xi_3(x_n)$ exists and is the unique lift of $\xi_1(x_\infty), \xi_2(x_\infty)$ to $\mathcal{F}_{R_3}$. $\square$

The following proposition shows that given a P_R–Anosov representation ρ of transversality type $[\![w_0]\!]$, we may always assume that R is stable under conjugation by w_0. This appeared as an assumption in Section 2.2.4 and plays a role later on when showing that balanced ideals give rise to cocompact domains of discontinuity.

Proposition 2.3.7. *Let $w_0 \in \widetilde{W}$ and $E' = E \cap w_0 E w_0^{-1}$, and let $R = \langle r(\theta), E\rangle$ and $R' = \langle r(\theta), E'\rangle$. Assume that $\rho\colon \Gamma \to G$ is P_R–Anosov with a limit map $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Then ρ is $P_{R'}$–Anosov.*

Proof. Let $x \neq z \in \partial_\infty\Gamma$, and consider the images $\widehat{\xi}(x), \widehat{\xi}(z) \in \mathcal{F}_R$. We claim that there is a unique lift $\eta_x(z) \in \mathcal{F}_{R'}$ satisfying

- $\eta_x(z)$ projects to $\widehat{\xi}(z)$.
- $\mathrm{pos}_{R,R'}(\widehat{\xi}(x), \eta_x(z)) = [\![w_0]\!]$.

To show this, we have to determine which lifts of $\widehat{\xi}(z)$ to $\mathcal{F}_{R'}$ satisfy the second condition. Let us first fix a good representative in G for $\widehat{\xi}(z)$: Since $\mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(z)) = [\![w_0]\!]$, there exists $h \in G$ such that

$$h(\widehat{\xi}(x), \widehat{\xi}(z)) = ([1], [w_0]).$$

Then $h^{-1}w_0 =: g \in G$ represents the flag $\widehat{\xi}(z)$ and also satisfies

$$\mathrm{pos}_{R,R'}(\widehat{\xi}(x), [g]) = [\![w_0]\!].$$

Any lift of $\widehat{\xi}(z)$ into $\mathcal{F}_{R'}$ can be written as $[gm] \in \mathcal{F}_{R'}$ for some $m \in E$. By Lemma 2.2.21, we have $\mathrm{pos}_{R,R'}(\widehat{\xi}(x), [gm]) = [\![w_0 m]\!]$. We claim that $[\![w_0 m]\!] = [\![w_0]\!] \in \widetilde{W}_{R,R'}$ implies that $m \in E'$ and therefore $[gm] = [g] \in \mathcal{F}_{R'}$, proving uniqueness of $\eta_x(z)$. Indeed, if $w_0 m = r w_0 r'$ for some $r \in R, r' \in R'$, we obtain

$$m = w_0^{-1} r w_0 r' \in w_0 R w_0^{-1} \cdot R' \subset w_0 R w_0^{-1}.$$

Since

$$E \cap w_0 R w_0^{-1} = E \cap w_0 E w_0^{-1} = E',$$

it follows that $m \in E'$ and the lift $[g] \in \mathcal{F}_{R'}$ is unique.
This defines a map

$$\eta_x\colon \partial_\infty\Gamma \setminus \{x\} \to \mathcal{F}_{R'}$$

which is continuous since $\widehat{\xi}$ is continuous. We will show that it is independent of the choice of x, i.e. if $y \neq z$ is another point, we have $\eta_x(z) = \eta_y(z)$. Let $\gamma \in \Gamma$ be an element of infinite order with fixed points $\gamma^{\pm} \in \partial_\infty\Gamma$ such that $x \neq \gamma^-$ and $y \neq \gamma^-$. Then we have

$$[\![w_0]\!] = \mathrm{pos}_{R,R'}(\widehat{\xi}(x), \eta_x(\gamma^-)) = \mathrm{pos}_{R,R'}(\rho(\gamma)^n\widehat{\xi}(x), \rho(\gamma)^n\eta_x(\gamma^-))$$

for every $n \in \mathbb{N}$. Moreover, $\rho(\gamma)^n\widehat{\xi}(x) \to \widehat{\xi}(\gamma^+)$ and $\rho(\gamma)^n\eta_x(\gamma^-)$ is a lift of $\widehat{\xi}(\gamma^-)$. For every subsequence n_k such that $\rho(\gamma)^{n_k}\eta_x(\gamma^-)$ is constant, it follows that

$$\mathrm{pos}_{R,R'}(\widehat{\xi}(\gamma^+), \rho(\gamma)^{n_k}\eta_x(\gamma^-)) \leq [\![w_0]\!]. \tag{2.3.1}$$

But as $\mathrm{pos}_{R,R}(\widehat{\xi}(\gamma^+), \widehat{\xi}(\gamma^-)) = [\![w_0]\!]$, the position in (2.3.1) must be a transverse one, thus equality holds by Lemma 2.2.26. As seen before, this uniquely determines $\rho(\gamma)^{n_k}\eta_x(\gamma^-)$ among the lifts of $\widehat{\xi}(\gamma^-)$. Since the same holds for any subsequence n_k such that $\rho(\gamma)^{n_k}\eta_x(\gamma^-)$ is constant, $\rho(\gamma)$ fixes $\eta_x(\gamma^-)$ and we obtain

$$\mathrm{pos}_{R,R'}(\widehat{\xi}(\gamma^+), \eta_x(\gamma^-)) = [\![w_0]\!].$$

Applying the same argument to $y \neq \gamma^-$ shows that $\eta_x(\gamma^-) = \eta_y(\gamma^-)$.
Therefore, η_x and η_y are continuous functions on $\partial_\infty\Gamma \setminus \{x, y\}$ which agree on the dense subset of poles, hence they agree everywhere. We denote by

$$\eta \colon \partial_\infty\Gamma \to \mathcal{F}_{R'}$$

the continuous function defined by $\eta(y) = \eta_x(y)$ for any choice of $x \neq y$. It is ρ–equivariant because $\eta(\gamma y) = \eta_{\gamma x}(\gamma y) \in \mathcal{F}_{R'}$ is the lift of $\widehat{\xi}(\gamma y)$ defined by

$$\mathrm{pos}_{R,R'}(\widehat{\xi}(\gamma x), \eta_{\gamma x}(\gamma y)) = \mathrm{pos}_{R,R'}(\rho(\gamma)\widehat{\xi}(x), \eta_{\gamma x}(\gamma y)) = [\![w_0]\!],$$

which is $\rho(\gamma)\eta_x(y) = \rho(\gamma)\eta(y)$. □

Remark 2.3.8. It is worth noting that the independence of $\eta_x(z)$ of the point x simplifies greatly if $\partial_\infty\Gamma$ is connected: If x and y can be connected by a path x_t in $\partial_\infty\Gamma$, we consider the lifts $\eta_{x_t}(z)$ along the path. They need to be constant by continuity of $\widehat{\xi}$, so $\eta_x(z)$ and $\eta_y(z)$ agree.

Example 2.3.9. Let us illustrate Proposition 2.3.7 with an example. Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\rho : \Gamma \to G$ a representation which is P_θ–Anosov with $\theta = \{\alpha_2, \ldots, \alpha_{n-2}\}$, so that we have a boundary map $\xi : \partial_\infty\Gamma \to \mathcal{F}_{1,n-1}$ into the space of partial flags comprising a line and a hyperplane. Assume that ρ is P_R–Anosov, where $R = \langle r(\theta), r(\alpha_{n-1})^2\rangle$. Then there is a boundary map $\widehat{\xi}$ into the space $\mathcal{F}_R$ of flags comprising an oriented line and an unoriented hyperplane. Let $x, z \in \partial_\infty\Gamma$ be two points as in the proof of the proposition. We can fix an orientation on $\widehat{\xi}(z)^{(n-1)}$ by requiring that $(\widehat{\xi}(x)^{(1)}, \widehat{\xi}(z)^{(n-1)})$, written in this order, induces the standard orientation on $\mathbb{R}^n$ (or the opposite orientation, depending on which element $w_0 \in \widetilde{W}$ we chose to represent the transversality type $[\![w_0]\!]$ of $\widehat{\xi}$). Doing so for all points $z \in \partial_\infty\Gamma$ extends the boundary map to a map into the space $\mathcal{F}_{R'}$ of flags comprising an oriented line and an oriented hyperplane.

As a consequence of the previous two propositions, the minimal oriented parabolic type associated to a P_θ–Anosov representation automatically has certain properties.

Proposition 2.3.10. *Let $\theta \subsetneq \Delta$ be stable under ι, $R = \langle r(\theta), E\rangle$ an oriented parabolic type, $w_0 \in T$, and $\rho\colon \Gamma \to G$ be P_R–Anosov with transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Then $w_0^2 \in E$.*

Proof. Let $x \neq y \in \partial_\infty\Gamma$ be two points in the boundary. Then we have $[\![w_0]\!] = \mathrm{pos}_{R,R}(\widehat{\xi}(x), \widehat{\xi}(y))$, where $\widehat{\xi}\colon \partial_\infty\Gamma \to F_R$ is the limit map. Then $\mathrm{pos}_{R,R}(\widehat{\xi}(y), \widehat{\xi}(x)) = [\![w_0^{-1}]\!]$. As observed in the proof of Lemma 2.3.4, there is a dense orbit in $\partial_\infty\Gamma \times \partial_\infty\Gamma$; let (a, b) be an element of this orbit. Since we can approximate both (x, y) and (y, x) by this orbit, continuity of $\widehat{\xi}$ implies that $[\![w_0]\!] \leq \mathrm{pos}_{R,R}(\widehat{\xi}(a), \widehat{\xi}(b))$ and $[\![w_0^{-1}]\!] \leq \mathrm{pos}_{R,R}(\widehat{\xi}(a), \widehat{\xi}(b))$. All of these are transverse positions, thus equality must hold in both cases by Lemma 2.2.26 and we conclude $[\![w_0^2]\!] = [\![1]\!] \in \widetilde{W}_{R,R}$, so $w_0^2 \in R$. □

The final part of this chapter is aimed at distinguishing connected components of Anosov representations by comparing the possible lifts of the limit map.

Proposition 2.3.11. *The set of P_R–Anosov representations is open and closed in the space of P_θ–Anosov representations $\mathrm{Hom}_{P_\theta\text{–}Anosov}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$.*

To prove this proposition, we will make use of the following technical lemma. Choose an auxiliary Riemannian metric on $\mathcal{F}_\theta$ and equip $\mathcal{F}_R$ with the metric which makes the finite covering $\pi_R\colon \mathcal{F}_R \to \mathcal{F}_\theta$ a local isometry.

Lemma 2.3.12. *Let $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ be a limit map of an P_R–Anosov representation and $\xi = \pi_R \circ \widehat{\xi}$. Then there exists $\delta > 0$ such that for every $x \in \partial_\infty\Gamma$*

(i) $\pi_R^{-1}(B_\delta(\xi(x))) = \displaystyle\bigsqcup_{[m]\in \overline{M}/E} B_\delta(r_m(\widehat{\xi}(x)))$, *and π_R maps any of these components isometrically to $B_\delta(\xi(x))$,*

(ii) and the set $r_m(B_\delta(\widehat{\xi}(x))) \subset \mathcal{F}_R$ intersects $\widehat{\xi}(\partial_\infty\Gamma)$ if and only if $[m] = 1 \in \overline{M}/E$.

Proof. By compactness of $\mathcal{F}_\theta$ there is an $\varepsilon > 0$ such that, for every $f \in \mathcal{F}_\theta$, the preimage of $B_\varepsilon(f)$ under π_R is the disjoint union of ε–balls around the preimages of x. Together with the choice of metric on $\mathcal{F}_R$, this shows (i) for any $\delta \leq \varepsilon$.

Now for every $x \in \partial_\infty\Gamma$ define

$$\mathcal{R}_x = \xi(\widehat{\xi}^{-1}(\mathcal{F}_R \setminus B_\varepsilon(\widehat{\xi}(x)))), \qquad \delta_x = \min\{\varepsilon,\ \tfrac{1}{2}d(\xi(x), \mathcal{R}_x)\}.$$

This is positive since $\mathcal{R}_x \subset \mathcal{F}_\theta$ is closed. By compactness there is a finite collection $x_1, \dots, x_m \in \partial_\infty\Gamma$ such that the sets $B_{\delta_{x_i}}(\xi(x_i)) \subset \mathcal{F}_\theta$ cover $\xi(\partial_\infty\Gamma)$. Let $\delta = \min_i \delta_{x_i}$. Then $U = B_\delta(\xi(x)) \subset B_\varepsilon(\xi(x))$ for every $x \in \partial_\infty\Gamma$, so $\pi_R^{-1}(U)$ decomposes into disjoint δ–balls as in (i). One of these is $V = B_\delta(\widehat{\xi}(x))$, and it is indeed the only one intersecting $\widehat{\xi}(\partial_\infty\Gamma)$:

If $y \in \partial_\infty\Gamma$ with $\widehat{\xi}(y) \in \pi_R^{-1}(U)$, then $\xi(y) \in U = B_\delta(\xi(x)) \subset B_{2\delta_{x_i}}(\xi(x_i))$ for some i. So

$$d(\xi(x_i), \xi(y)) < 2\delta_{x_i} \leq d(\xi(x_i), \mathcal{R}_{x_i}),$$

thus $\xi(y) \notin \mathcal{R}_{x_i}$ or equivalently $\widehat{\xi}(y) \in B_\varepsilon(\widehat{\xi}(x_i))$. So $\widehat{\xi}(y) \in \pi_R^{-1}(U) \cap B_\varepsilon(\widehat{\xi}(x_i))$, which is exactly V. □

Proof of Proposition 2.3.11. To show openness, let ρ_0 be P_R–Anosov with limit map $\widehat{\xi}_0 \colon \partial_\infty\Gamma \to \mathcal{F}_R$ and $\xi_0 = \pi_R \circ \widehat{\xi}_0$. Let δ be the constant from Lemma 2.3.12 for $\widehat{\xi}_0$. Choose $x_1, \ldots, x_k \in \partial_\infty\Gamma$ such that $B_{\delta/4}(\xi_0(x_i))$ cover $\xi_0(\partial_\infty\Gamma)$ and let $U_i = B_{\delta/2}(\xi_0(x_i))$ and $V_i = B_{\delta/2}(\widehat{\xi}_0(x_i))$. For every i we get a local section $s_i \colon U_i \to V_i$. If U_i and U_j intersect, then s_i and s_j coincide on the intersection, since $U_i \cup U_j$ is contained in a δ–ball, of which only a single lift can intersect $\widehat{\xi}_0(\partial_\infty\Gamma)$, so V_i and V_j both have to be contained in this lift. Therefore, the s_i combine to a smooth section $s \colon \bigcup_i U_i \to \bigcup_i V_i$.

For every $\rho_1 \in \mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ which is close enough to ρ_0, there is a path $\rho_t \in \mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ connecting ρ_0 and ρ_1 such that $d_{C^0}(\xi_t, \xi_0) < \delta/4$ for every $t \in [0,1]$. This is because $\mathrm{Hom}_{P_\theta\text{–Anosov}}(\Gamma, G)$ is open and the limit map depends continuously on the representation [GW12, Theorem 5.13]. Then for every $x \in \partial_\infty\Gamma$ there is an i such that

$$d(\xi_1(x), \xi_0(x_i)) \leq d_{C^0}(\xi_1, \xi_0) + d(\xi_0(x), \xi_0(x_i)) < \delta/2,$$

hence $\xi_1(x) \in U_i$. So $\xi_1(\partial_\infty\Gamma) \subset \bigcup_i U_i$ and we can define $\widehat{\xi}_1 = s \circ \xi_1$. This is a continuous lift of ξ_1. Note that also $\widehat{\xi}_0 = s \circ \xi_0$ and that we can equally define $\widehat{\xi}_t = s \circ \xi_t$ for every $t \in [0,1]$.

To show ρ_1–equivariance of $\widehat{\xi}_1$, let $\gamma \in \Gamma$, $x \in \partial_\infty\Gamma$ and consider the curves

$$\alpha(t) = \rho_t(\gamma)^{-1}\widehat{\xi}_t(\gamma x), \quad \beta(t) = \widehat{\xi}_t(x).$$

They are continuous and $\pi_R(\alpha(t)) = \rho_t(\gamma)^{-1}\pi_R(\widehat{\xi}_t(\gamma x)) = \xi_t(x) = \pi_R(\beta(t))$. Also $\alpha(0) = \rho_0(\gamma)^{-1}\widehat{\xi}_0(\gamma x) = \widehat{\xi}_0(x) = \beta(0)$ by ρ_0–equivariance of ξ_0. Therefore, the curves α and β coincide, so in particular $\widehat{\xi}_1$ is ρ_1–equivariant.

For closedness, let ρ_n be a sequence of P_R–Anosov representations with limit maps $\widehat{\xi}_n$ converging to the P_θ–Anosov representation ρ. Then the unoriented limit maps $\xi_n = \pi_R \circ \widehat{\xi}_n$ converge uniformly to ξ, the limit map of ρ. Let $\gamma \in \Gamma$ be an element of infinite order and $\gamma^-, \gamma^+ \in \partial_\infty\Gamma$ its poles. Since π_R is a finite covering, up to taking a subsequence, we can assume that $\widehat{\xi}_n(\gamma^+)$ converges to a point we call $\widehat{\xi}(\gamma^+)$. First, we are going to show that there is a neighborhood of γ^+ in $\partial_\infty\Gamma$ on which the maps $\widehat{\xi}_n$ converge uniformly to some limit.

As ρ is Anosov, the points $\xi(\gamma^-), \xi(\gamma^+) \in \mathcal{F}_\theta$ are transverse fixed points of $\rho(\gamma)$. Since $\xi_n(\gamma^\pm) \to \xi(\gamma^\pm)$, we can find an $\epsilon > 0$ such that all elements of $B_\epsilon(\xi(\gamma^+))$ are transverse to $\xi_n(\gamma^-)$ for sufficiently large n. In particular, $\rho_n(\gamma^k)$ restricted to this ball converges locally uniformly to $\xi_n(\gamma^+)$ as $k \to \infty$. After shrinking ϵ, the preimage $\pi_R^{-1}(B_\epsilon(\xi(\gamma^+)))$ is a union of finitely many disjoint copies of $B_\epsilon(\xi(\gamma^+))$:

$$\pi_R^{-1}(B_\epsilon(\xi(\gamma^+))) = \bigsqcup_{[m]\in\overline{M}/E} B_\epsilon(\widehat{\xi}(\gamma^+))m$$

For n large, $\widehat{\xi}_n(\gamma^+) \in B_\epsilon(\widehat{\xi}(\gamma^+))$. Since $\rho_n(\gamma^k)|_{B_\epsilon(\xi(\gamma^+))}$ converges locally uniformly to $\xi(\gamma^+)$, when seen as maps on $\mathcal{F}_R$, $\rho_n(\gamma^k)|_{B_\epsilon(\widehat{\xi}(\gamma^+))}$ converges locally uniformly to a lift of $\xi_n(\gamma^+)$. This lift must be $\widehat{\xi}_n(\gamma^+)$ as this point is fixed by $\rho_n(\gamma)$. We claim that this implies the existence of a $\delta > 0$ such that $\widehat{\xi}_n|_{B_\delta(\gamma^+)}$ converges uniformly to a lift of $\xi|_{B_\delta(\gamma^+)}$.

To see this claim, choose δ such that $\xi(B_\delta(\gamma^+)) \subset B_{\epsilon/2}(\xi(\gamma^+))$ and $\gamma^- \notin B_\delta(\gamma^+)$, and let n be large enough so that $d_{C^0}(\xi_n, \xi) < \epsilon/2$. Then $\xi_n(B_\delta(\gamma^+)) \subset B_\epsilon(\xi(\gamma^+))$. Let $y \in \widehat{\xi}_n(B_\delta(\gamma^+))$ be any point, and let $m \in \overline{M}$ be chosen such that $y \in B_\epsilon(\widehat{\xi}(\gamma^+))m$. It follows that $\rho_n(\gamma^k)(y) \xrightarrow{k\to\infty} \widehat{\xi}_n(\gamma^+)m$. So by $\rho(\Gamma)$–invariance and closedness of $\widehat{\xi}_n(\partial_\infty\Gamma)$, $\widehat{\xi}_n(\gamma^+)m \in \widehat{\xi}_n(\partial_\infty\Gamma)$, so $[m] = 1 \in \overline{M}/E$ by transversality. Thus for all sufficiently large n, the image of $\widehat{\xi}_n|_{B_\delta(\gamma^+)}$ is entirely contained in $B_\epsilon(\widehat{\xi}(\gamma^+))$, so we can use the local section $s : B_\epsilon(\xi(\gamma^+)) \to B_\epsilon(\widehat{\xi}(\gamma^+))$ to write $\widehat{\xi}_n|_{B_\delta(\gamma^+)} = s \circ \xi_n|_{B_\delta(\gamma^+)}$. This proves the stated uniform convergence on $B_\delta(\gamma^+)$.

Now we use local uniform convergence at γ^+ to obtain uniform convergence everywhere. For any point $y \in \partial_\infty\Gamma\backslash\{\gamma^-\}$ and any neighborhood $U \ni y$ whose closure does not contain γ^-, there is an integer $k(U)$ such that $\gamma^k(U) \subset B_\delta(\gamma^+)$ for all $k \geq k(U)$ [KB02, Theorem 4.3]. Then, for $z \in U$,

$$\widehat{\xi}_n(z) = \rho_n(\gamma)^{-k(U)}\widehat{\xi}_n(\gamma^{k(U)}z) \xrightarrow{n\to\infty} \rho(\gamma)^{-k(U)}\widehat{\xi}(\gamma^{k(U)}z),$$

so we get local uniform convergence on $\partial_\infty\Gamma\backslash\{\gamma^-\}$. Similarly, since $\widehat{\xi}_n = \rho_n(\zeta)^{-1}\circ\widehat{\xi}\circ\zeta$ for some $\zeta \in \Gamma$ with $\zeta\gamma^- \neq \gamma^-$, $\widehat{\xi}_n$ also converges uniformly in a neighborhood γ^-. So the maps $\widehat{\xi}_n$ converge uniformly to a limit $\widehat{\xi}$, which is continuous and equivariant. □

From the previous proposition, we obtain the following two criteria to distinguish connected components of Anosov representations.

Corollary 2.3.13. *Let $\rho, \rho' \colon \Gamma \to G$ be P_θ–Anosov. Furthermore, let $R, R' \subset \widetilde{W}$ be the minimal oriented parabolic types such that ρ is P_R–Anosov and ρ' is $P_{R'}$–Anosov (see Proposition 2.3.6). Assume that ρ and ρ' lie in the same connected component of $\mathrm{Hom}_{P_\theta\text{–}Anosov}(\Gamma, G)$. Then the types R and R' agree. Furthermore, if $\widehat{\xi}, \widehat{\xi}' \colon \partial_\infty\Gamma \to \mathcal{F}_R$ are limit maps of ρ, ρ' of transversality types $[\![w_0]\!], [\![w_0']\!] \in \widetilde{W}_{R,R}$, then $[\![w_0]\!], [\![w_0']\!]$ are conjugate by an element of $\overline{M}$.*

Proof. By Proposition 2.3.11, ρ is also $P_{R'}$–Anosov and ρ' is P_R–Anosov. If R and R' were not equal, either R' would not be minimal for ρ' or R would not be minimal for ρ.
By Remark 2.3.5(ii), the transversality type of any limit map $\xi_\rho \colon \partial_\infty\Gamma \to \mathcal{F}_R$ of ρ

is conjugate to $[\![w_0]\!]$ by an element of $\overline{M}$. By (the proof of) Proposition 2.3.11, ρ also admits a limit map of transversality type $[\![w_0']\!]$, so they must be conjugate by an element of $\overline{M}$. $\square$

2.4 Domains of discontinuity

In this section, we extend the machinery developed in [KLP18] to the setting of oriented flag manifolds (Definition 2.2.1). More specifically, we show that their description of cocompact domains of discontinuity for the action of Anosov representations on flag manifolds can be applied with some adjustments to oriented flag manifolds. Our main result is the following theorem, which is analogous to [KLP18, Theorem 7.14]:

Theorem 2.4.1. *Let Γ be a non-elementary word hyperbolic group and G a connected, semi-simple, linear Lie group. Furthermore, let $R, S \subset \widetilde{W}$ be oriented parabolic types and $w_0 \in T \subset \widetilde{W}$ a transverse position such that $w_0 R w_0^{-1} = R$ and $w_0^2 \in R$.*
Let $\rho\colon \Gamma \to G$ be an P_R–Anosov representation and $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ a limit map of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–balanced ideal, and define $\mathcal{K} \subset \mathcal{F}_S$ as

$$\mathcal{K} := \bigcup_{x \in \partial_\infty \Gamma} \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ-invariant and closed, and Γ acts properly discontinuously and cocompactly on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

The formulation here is slightly different than in Theorem 1.1.4, so let us show first how that theorem follows: By Proposition 2.3.7, when starting with a P_R–Anosov representation such that $w_0 R w_0^{-1} \neq R$, then it is actually $P_{R'}$–Anosov, with $R' = R \cap w_0 R w_0^{-1}$. Then the conditions $w_0 R' w_0^{-1} = R'$ and also $w_0^2 \in R$ are automatically satisfied by Proposition 2.3.10. Also, if a balanced ideal is invariant by R from the left and by S from the right, then it is also invariant by R' and gives a balanced ideal in $\widetilde{W}_{R',S}$.

A large part of the work required to prove this version, namely extending the Bruhat order to the extended Weyl group $\widetilde{W}$, was already done in Sections 2.1 and 2.2. We prove proper discontinuity and cocompactness of the action of Γ on Ω separately in the following two subsections (Theorems 2.4.9 and 2.4.21). The part about cocompactness follows [KLP18] in all key arguments. Since oriented flag manifolds are not as established and well–studied as their unoriented counterparts, we reprove all the required technical lemmas in the setting of compact G-homogeneous spaces X, Y and G–equivariant maps between X and $\mathcal{C}(Y)$, the space of closed subsets of Y.

2.4.1 Proper discontinuity

Let P_R and P_S be oriented parabolic subgroups of types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$. Furthermore, let $w_0 \in T \subset \widetilde{W}$ be a transverse position. We assume that $\iota(\theta) = \theta$, $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$, so that w_0 acts involutively on $\widetilde{W}_{R,S}$ (see Section 2.2.4).

The following definition of w_0–related limits is an oriented version of the one used for contracting sequences in [KLP18, Definition 6.1]. The idea goes back to the study of discrete quasiconformal groups in [GM87]. Apart from the dependence on the choice of w_0, we will see later that pairs of such limits are not unique in this setting (Lemma 2.4.7).

Definition 2.4.2. Let $g_n \in G$ be a diverging sequence. A pair $F^-, F^+ \in \mathcal{F}_R$ is called a pair of w_0*–related limits* of the sequence g_n if

$$g_n|_{C_{[\![w_0]\!]}(F^-)} \xrightarrow{n\to\infty} F^+$$

locally uniformly.

We will also make use of the notion of dynamically related points, which was given in [Fra05, Definition 1].

Definition 2.4.3. Let X be a topological space. Two points $x, y \in X$ are called *dynamically related via the sequence* $g_n \in \mathrm{Homeo}(X)$ if g_n is a divergent sequence and there exists a sequence $x_n \to x$ such that

$$g_n(x_n) \to y.$$

Using similar arguments as in the unoriented case, we can prove the following useful relative position inequality.

Lemma 2.4.4 ([KLP18, Proposition 6.5])**.** *Let $g_n \in G$ be a sequence admitting a pair $F^\pm \in \mathcal{F}_R$ of w_0–related limits. Assume that $F, F' \in \mathcal{F}_S$ are dynamically related via (g_n). Then*

$$\mathrm{pos}_{R,S}(F^+, F') \le w_0 \, \mathrm{pos}_{R,S}(F^-, F).$$

Proof. Let $F_n \in \mathcal{F}_S$ be a sequence such that $F_n \to F$ and $g_n F_n \to F'$. We pick elements $h_n \in G$ satisfying $F_n = h_n F$ and $h_n \to 1$. Writing $[\![w]\!] = \mathrm{pos}_{R,S}(F^-, F)$, it follows that there exists some $g \in G$ such that $g(F^-, F) = ([1], [w])$. Define $f \in \mathcal{F}_R$ as $f = [g^{-1} w_0]$, so that we obtain the following relative positions:

- $\mathrm{pos}_{R,R}(F^-, f) = [\![w_0]\!]$
- $\mathrm{pos}_{R,S}(F^-, F) = [\![w]\!]$
- $\mathrm{pos}_{R,S}(f, F) = \mathrm{pos}_{R,S}([w_0], [w]) = [\![w_0 w]\!]$

In other words, f is chosen such that $\mathrm{pos}_{R,R}(F^-, f) = [\![w_0]\!]$ and $\mathrm{pos}_{R,S}(f, F)$ is as small as possible. Then, since $h_n f \to f$, f lies in the open set $C_{[\![w_0]\!]}(F^-)$ and $F^\pm$ are w_0–related limits, it follows that $g_n h_n f \to F^+$. Finally, observe that

$$\mathrm{pos}_{R,S}(g_n h_n f, g_n h_n F) = \mathrm{pos}_{R,S}(f, F)$$

is constant. We thus obtain the following inequalities:

$$\mathrm{pos}_{R,S}(F^+, F') \leq \mathrm{pos}_{R,S}(g_n h_n f, g_n h_n F) = \mathrm{pos}_{R,S}(f, F) = [\![w_0 w]\!] \qquad \square$$

One consequence of this inequality is that being w_0–related limits is a symmetric condition.

Lemma 2.4.5 ([KLP18, (6.7)])**.** *If (F^-, F^+) is a pair of w_0–related limits in $\mathcal{F}_R$ of a sequence (g_n) then (F^+, F^-) is a pair of w_0–related limits of (g_n^{-1}).*

Proof. Let $F_n \to F$ be a convergent sequence in $C_{[\![w_0]\!]}(F^+) \subset \mathcal{F}_R$ and $g_{n_k}^{-1} F_{n_k} \to F' \in \mathcal{F}_R$ a convergent subsequence of $g_n^{-1} F_n$. This means F is dynamically related to F' via $(g_{n_k}^{-1})$ or equivalently F' is dynamically related to F via (g_{n_k}). So by Lemma 2.4.4 (with $S = R$)

$$[\![w_0]\!] = \mathrm{pos}_{R,R}(F^+, F) \leq w_0 \, \mathrm{pos}_{R,R}(F^-, F').$$

Since $[\![w_0]\!]$ is maximal in the Bruhat order, this implies that $w_0 \, \mathrm{pos}_{R,R}(F^-, F') = [\![w_0]\!]$ by Lemma 2.2.17. As w_0 induces an involution on $\widetilde{W}_{R,S}$, we obtain $\mathrm{pos}_{R,R}(F^-, F') = [\![1]\!]$, i.e. $g_{n_k} F_{n_k} \to F' = F^-$. By the same argument every subsequence of $g_n^{-1} F_n$ accumulates at F^- and thus $g_n^{-1} F_n \to F^-$, which shows that (F^+, F^-) are w_0–related limits of (g_n^{-1}). $\square$

Lemma 2.4.6. *Let*

$$\mathfrak{n}_\theta^- = \bigoplus_{\alpha \in \Sigma^- \setminus \mathrm{span}(\theta)} \mathfrak{g}_\alpha$$

and consider the set $C_{[\![w_0]\!]}([w_0^{-1}]) \subset \mathcal{F}_R$ of flags at relative position $[\![w_0]\!]$ to $[w_0^{-1}] \in \mathcal{F}_R$. Then the map

$$\varphi\colon \mathfrak{n}_\theta^- \to C_{[\![w_0]\!]}([w_0^{-1}]), \qquad X \mapsto [e^X]$$

is a diffeomorphism.

Proof. Let $N_\theta^- \subset G$ be the connected subgroup with Lie algebra $\mathfrak{n}_\theta^-$. As a subgroup of N^- its exponential map $\mathfrak{n}_\theta^- \to N_\theta^-$ is a diffeomorphism. So it suffices to show that the projection map $\widetilde{\varphi}\colon N_\theta^- \to \mathcal{F}_R$ is a diffeomorphism onto $C_{[\![w_0]\!]}([w_0^{-1}])$.

First we verify that $C_{[\![w_0]\!]}([w_0^{-1}]) = \{[n] \mid n \in N_\theta^-\}$. We have

$$\begin{aligned} C_{[\![w_0]\!]}([w_0^{-1}]) &= \{f \in \mathcal{F}_R \mid \mathrm{pos}_{R,R}([1], w_0 f) = [\![w_0]\!]\} \\ &= \{f \in \mathcal{F}_R \mid \exists p \in G\colon [1] = [p],\ w_0 f = [p w_0]\} = \{[w_0^{-1} p w_0] \mid p \in P_R\} \end{aligned}$$

and $N_\theta^- \subset w_0^{-1}P_R w_0$, so it remains to show that $w_0^{-1}P_R w_0 \subset N_\theta^- P_R$. As a consequence of the Langlands decomposition [Kna02, Propositions 7.82(a) and 7.83(d)] we can write $P_\theta = N_\theta^+ Z_G(\mathfrak{a}_\theta)$ where $N_\theta^+ = w_0 N_\theta^- w_0^{-1}$ and $\mathfrak{a}_\theta = \bigcap_{\beta\in\theta}\ker\beta$. Now Ad_{w_0} preserves $\mathfrak{a}_\theta$ and thus $w_0^{-1}Z_G(\mathfrak{a}_\theta)w_0 = Z_G(\mathfrak{a}_\theta)$, hence $w_0^{-1}P_\theta w_0 = N_\theta^- Z_G(\mathfrak{a}_\theta)$. As N_θ^- is connected, we even get $w_0^{-1}P_{\theta,0}w_0 = N_\theta^- Z_G(\mathfrak{a}_\theta)_0$ and therefore

$$w_0^{-1}P_R w_0 = w_0^{-1}P_{\theta,0}Ew_0 = w_0^{-1}P_{\theta,0}w_0E = N_\theta^- Z_G(\mathfrak{a}_\theta)_0E \subset N_\theta^- P_R.$$

To prove injectivity of $\widetilde{\varphi}$, let $n, n' \in N_\theta^-$ with $[n] = [n']$. Then $n^{-1}n' \in N_\theta^- \cap P_R = \{1\}$ by [Kna02, Proposition 7.83(e)], so $\widetilde{\varphi}$ is injective. To see that $\widetilde{\varphi}$ is regular, we observe that $\mathfrak{n}_\theta^-$ is composed of the root spaces of roots in $\Sigma^- \setminus \mathrm{span}(\theta)$ while $\mathfrak{p}_\theta$ has the root spaces $\Sigma^+ \cup \mathrm{span}(\theta)$. So $\mathfrak{g} = \mathfrak{n}_\theta^- \oplus \mathfrak{p}_\theta$ and $D_1\widetilde{\varphi}\colon \mathfrak{n}_\theta^- \to \mathfrak{g}/\mathfrak{p}_\theta$ is an isomorphism. By equivariance we then see that $\widetilde{\varphi}$ is a diffeomorphism onto its image. □

In the unoriented case, a P_θ–divergent sequence admits subsequences with unique attracting limits in $\mathcal{F}_\theta$. In the oriented case, however, this uniqueness is lost and all lifts of such a limit will be attracting on an open set.

Lemma 2.4.7. *Let $g_n \in G$ be a P_θ–divergent sequence. Then there is a subsequence g_{n_k} admitting $|\overline{M}/E|$ pairs of w_0-related limits in $\mathcal{F}_R$. More precisely, the action*

$$\overline{M}/E \times \mathcal{F}_R^2 \to \mathcal{F}_R^2, \quad ([m],(F^-,F^+)) \mapsto (r_{w_0 m w_0^{-1}}(F^-), r_m(F^+)) \tag{2.4.1}$$

is simply transitive on the pairs of w_0–related limits of g_{n_k}.

Proof. Observe that since $w_0 E w_0^{-1} = E$, conjugation by w_0 defines an action on $\overline{M}/E$. In other words, the choice of the representative $m \in \overline{M}$ in (2.4.1) does not matter.
Let us first prove that $\overline{M}/E$ acts simply transitively on the w_0–related limits of g_{n_k}, assuming such limits exist. We know from Corollary 2.2.22 that

$$C_{[\![w_0]\!]}(r_{w_0 m w_0^{-1}}(F^-)) = C_{[\![w_0 m]\!]}(F^-) = r_m(C_{[\![w_0]\!]}(F^-)).$$

Because of this and since left and right multiplication commute, (2.4.1) restricts to an action on w_0–related limits of g_{n_k}. It is free by the definition of E and $\mathcal{F}_R$. For transitivity, let $F^\pm$ and $F'^\pm$ be two w_0–related limit pairs for g_{n_k}. Then

$$\bigcup_{[m]\in\overline{M}/E} r_m(C_{[\![w_0]\!]}(F^-)) = \bigcup_{[m]\in\overline{M}/E} C_{[\![w_0 m]\!]}(F^-)$$

is dense in $\mathcal{F}_R$ since its closure is all of $\mathcal{F}_R$ by Proposition 2.2.18. So for some $[m] \in \overline{M}/E$, $r_m(C_{[\![w_0]\!]}(F^-))$ must intersect the open set $C_{[\![w_0]\!]}(F'^-)$. On this intersection, g_{n_k} converges locally uniformly to $r_m(F^+)$ and F'^+, so $F'^+ = r_m(F^+)$. By Lemma 2.4.5, (F'^+, F'^-) and $(r_m(F^+), r_{w_0 m w_0^{-1}}(F^-))$ are w_0–related limits for $g_{n_k}^{-1}$, so on $C_{[\![w_0]\!]}(F'^+) = C_{[\![w_0]\!]}(r_m(F^+))$ the sequence $g_{n_k}^{-1}$ converges to both F'^- and $r_{w_0 m w_0^{-1}}(F^-)$, so $F'^- = r_{w_0 m w_0^{-1}}(F^-)$.

What is left to show is the existence of w_0–related limits. This is done by an argument similar to [GKW15, Lemma 4.7] Decompose the sequence g_n as $g_n = k_n e^{A_n} \ell_n$ with $k_n, \ell_n \in K$ and $A_n \in \overline{\mathfrak{a}^+}$. After taking a subsequence, we can assume that $k_n \to k$ and $\ell_n \to \ell$. We want to show that $F^- = [\ell^{-1} w_0^{-1}] \in \mathcal{F}_R$ and $F^+ = [k] \in \mathcal{F}_R$ are w_0–related limits of (g_n). We use the following characterization of locally uniform convergence: For every sequence $F_n \to F$ converging inside $C_{[\![w_0]\!]}(F^-)$ we want to show that $g_n F_n$ converges to F^+. For sufficiently large n the sequence $\ell_n F_n$ will be inside $C_{[\![w_0]\!]}(\ell F^-) = C_{[\![w_0]\!]}([w_0^{-1}])$, so by Lemma 2.4.6 we can write $\ell_n F_n = [e^{X_n}]$ with

$$X_n \in \bigoplus_{\alpha \in \Sigma^- \setminus \operatorname{span}(\theta)} \mathfrak{g}_\alpha$$

converging to some X from the same space. So

$$g_n F_n = [k_n e^{A_n} e^{X_n}] = [k_n e^{A_n} e^{X_n} e^{-A_n}] = [k_n \exp(\operatorname{Ad}_{e^{A_n}} X_n)] = [k_n \exp(e^{\operatorname{ad} A_n} X_n)].$$

If we decompose $X_n = \sum_\alpha X_n^\alpha$ into root spaces then

$$e^{\operatorname{ad} A_n} X_n = \sum_{\alpha \in \Sigma^- \setminus \operatorname{span}(\theta)} e^{\alpha(A_n)} X_n^\alpha.$$

Now every $\alpha \in \Sigma^- \setminus \operatorname{span}(\theta)$ can be written as a linear combination of simple roots with non–positive coefficients and with the coefficient of at least one simple root $\beta \in \Delta \setminus \theta$ being strictly negative. As $\beta(A_n) \to \infty$ by P_θ–divergence, $\alpha(A_n)$ must converge to $-\infty$ and therefore $e^{\operatorname{ad} A_n} X_n$ goes to 0. This implies $g_n F_n \to [k] = F^+$, so $g_n|_{C_{[\![w_0]\!]}(F^-)} \to F^+$ locally uniformly. □

Let Γ be a non–elementary word hyperbolic group and G a connected, semi–simple, linear Lie group (see Section 2.1.1 for some remarks on these assumptions).

Lemma 2.4.8. *Let $\rho : \Gamma \to G$ be a P_R–Anosov representation and let $\xi : \partial_\infty \Gamma \to \mathcal{F}_R$ be a continuous, equivariant limit map of transversality type $[\![w_0]\!]$. Then every P_θ–divergent sequence $(\rho(\gamma_n))$ has a subsequence admitting a pair $F^\pm \in \mathcal{F}_R$ of w_0–related limits such that $F^\pm \in \xi(\partial_\infty \Gamma)$.*

Proof. By Lemma 2.4.7, there exists a subsequence $\rho(\gamma_{n_k})$ with w_0-related limits $F^\pm \in \mathcal{F}_R$. Then F^- is a lift of the unique repelling limit $\pi(F^-) \in \mathcal{F}_\theta$, and we have $\pi(F^-) \in \pi(\xi(\partial_\infty \Gamma))$ (see the description of the boundary map in [GGKW17, Theorem 5.3]). By right–multiplying with an element $m \in \overline{M}/E$ if necessary, we may assume that $F^- = \xi(x)$ for some $x \in \partial_\infty \Gamma$. For any $x \neq y \in \partial_\infty \Gamma$, we have $\operatorname{pos}_{R,R}(F^-, \xi(y)) = \operatorname{pos}_{R,R}(\xi(x), \xi(y)) = [\![w_0]\!]$. Since the corresponding w_0–related attracting limit $F^+ \in \mathcal{F}_R$ is characterized by

$$\rho(\gamma_n)|_{C_{[\![w_0]\!]}(F^-)} \xrightarrow{n \to \infty} F^+,$$

it follows that $\rho(\gamma_n)(\xi(y)) \xrightarrow{n\to\infty} F^+$. But $\xi(\partial_\infty \Gamma)$ is a closed, Γ–invariant set, so $F^+ \in \xi(\partial_\infty \Gamma)$. □

Recall from Section 2.2.3 and Section 2.2.4 that a subset $I \subset \widetilde{W}_{R,S}$ is an ideal if $[\![w]\!] \in I$ and $[\![w']\!] \leq [\![w]\!]$ implies $[\![w']\!] \in I$, and it is w_0–fat if $[\![w]\!] \notin I$ implies $[\![w_0 w]\!] \in I$.

Theorem 2.4.9. *Let $\rho : \Gamma \to G$ be an P_R–Anosov representation and let $\xi : \partial_\infty \Gamma \to \mathcal{F}_R$ be a limit map of transversality type $[\![w_0]\!]$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–fat ideal, and define $\mathcal{K} \subset \mathcal{F}_S$ as*

$$\mathcal{K} := \bigcup_{x \in \partial_\infty \Gamma} \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ–invariant and closed, and Γ acts properly discontinuously on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

Proof. Γ–invariance and closedness of $\mathcal{K}$ follows from Lemma 2.4.16(ii) and Example 2.4.17.

Assume that the action of Γ on Ω is not proper. Then there exist $F, F' \in \Omega$ which are dynamically related by some sequence $\rho(\gamma_n)$. This sequence is P_θ–divergent and by Lemma 2.4.8, a subsequence admits a pair of w_0–related limits of the form $\xi(x^\pm)$, where $x^\pm \in \partial_\infty \Gamma$. So Lemma 2.4.4 shows that

$$\mathrm{pos}_{R,S}(\xi(x^+), F') \leq w_0 \, \mathrm{pos}_{R,S}(\xi(x^-), F). \tag{2.4.2}$$

Since $F, F' \notin \mathcal{K}$, neither $\mathrm{pos}_{R,S}(\xi(x^+), F')$ nor $\mathrm{pos}_{R,S}(\xi(x^-), F)$ can be in I. As I is w_0–fat, this implies in particular $w_0 \, \mathrm{pos}_{R,S}(\xi(x^-), F) \in I$. But I is an ideal, so (2.4.2) implies $\mathrm{pos}_{R,S}(\xi(x^+), F') \in I$, a contradiction. $\square$

2.4.2 Cocompactness

We now come to the cocompactness part of Theorem 2.4.1. Owing to the fact that we want to apply everything to oriented flag manifolds, our setup here is slightly more general than in [KLP18]. Nevertheless, we will show that all the key arguments from that paper still work. This includes in particular the idea of using expansion to prove cocompactness, which is stated in a basic version in Lemma 2.4.19. The connection between (convex) cocompactness and expansion at the limit set was originally observed for Kleinian groups in [Sul85].

For a compact metric space Z let $\mathcal{C}(Z)$ be the space of closed subsets, equipped with the Hausdorff metric. The following fact will be useful to us later on (see for example [BH99, Lemma 5.31] for a proof).

Proposition 2.4.10. *The space $\mathcal{C}(Z)$, equipped with the Hausdorff metric, is a compact metric space.*

The following notion of an expanding action was introduced by Sullivan in [Sul85, §9]. In [KLP18], it was used together with a new and more general definition of transverse expansion (Definition 5.28) to prove cocompactness. The version of transverse expansion we use here is almost identical to that definition.

Definition 2.4.11. Let Z be a metric space, g a homeomorphism of Z, Γ a group acting on Z by homeomorphisms and $K\colon \Lambda \to \mathcal{C}(Z)$ a map from any set Λ.

(i) g is *expanding at* $z \in Z$ if there exists an open neighbourhood $z \in U \subset Z$ and a constant $c > 1$ such that

$$d(gx, gy) \geq c\, d(x, y)$$

for all $x, y \in U$.

(ii) Let $A \subset Z$ be a subset. The action of Γ on Z is *expanding at* A if for every $z \in A$ there is a $\gamma \in \Gamma$ which is expanding at z.

(iii) g is *expanding at* $z \in Z$ *transversely to* K if there is an open neighbourhood $z \in U$ and a constant $c > 1$ such that

$$d(gx, gK(\lambda)) \geq c\, d(x, K(\lambda))$$

for all $x \in U$ and all $\lambda \in \Lambda$ with $K(\lambda) \cap U \neq \varnothing$.

Lemma 2.4.12 ([KLP18, Remark 5.22])**.** *If the action of* Γ *on* Z *is expanding at a closed* Γ*–invariant subset* $A \subset Z$ *then it is arbitrarily strongly expanding, i.e. for every* $z \in A$ *and* $c > 1$ *there is a* $\gamma \in \Gamma$ *which is expanding at* z *with expansion factor* c*.*

Proof. If the action is expanding at $z \in A$ with some expansion factor, then it is expanding by the same factor in a neighbourhood of z. By covering A with finitely many such neighborhoods, we can assume that the action is expanding with a uniform expansion factor $C > 1$. Now let $z \in A$ and let $\gamma_1 \in \Gamma$ be expanding at z by the factor C. Let $\gamma_2 \in \Gamma$ be expanding at $\gamma_1 z \in A$ by C. Then $\gamma_2\gamma_1$ expands at z by C^2. Iterating this, we get an element $\gamma_n \cdots \gamma_1 \in \Gamma$ which is expanding with expansion factor $C^n \geq c$ at z. □

Let G be a Lie group and X, Y be compact G–homogeneous spaces. Fix Riemannian metrics on X, Y and a left–invariant Riemannian metric on G. Recall that smooth maps between manifolds are locally Lipschitz with respect to any Riemannian distances.

Lemma 2.4.13. *There exists a compact subset* $S \subset G$ *such that for every pair* $(x, y) \in X$*, there exists* $s_{xy} \in S$ *satisfying* $s_{xy}x = y$*.*

Proof. Fix a basepoint $x_0 \in X$ and let V be a precompact open neighborhood of the identity in G. Since $G \to X$, $g \mapsto gx_0$ is a submersion, Vx_0 is a neighborhood of x_0. Then by compactness there are finitely many $g_1, \ldots, g_n \in G$ such that the sets g_iVx_0 cover X. So $S = \overline{g_1V} \cup \cdots \cup \overline{g_nV}$ is a compact subset of G which maps x_0 to any point in X. The set SS^{-1} is compact and satisfies the desired transitivity property. □

Lemma 2.4.14. *There exists a constant $C > 0$ such that the following holds: For any two points $x, y \in X$, there exists $g \in G$ satisfying $gx = y$ and $d(1, g) \leq Cd(x, y)$.*

Proof. Assume by contradiction that there are sequences $x_n, y_n \in X$ such that every g_n sending x_n to y_n must satisfy $d(1, g_n) > nd(x_n, y_n)$. After taking subsequences, we have $x_n \to x,\ y_n \to y$. If $x \neq y$, we obtain in particular that $d(1, g_n) \to \infty$ for every choice of g_n sending x_n to y_n. But by Lemma 2.4.13, a compact subset of G already acts transitively on X, so g_n can be chosen such that $d(1, g_n)$ remains bounded. We are thus left with the case $x = y$. Since the map $G \to X,\ g \mapsto gx$ is a smooth submersion, there exists a local section at x: There is a neighborhood $x \in U$ and a smooth map $s : U \to G$ satisfying $s(x) = 1$ and $s(z)x = z$ for every $z \in U$. After shrinking U if necessary, s is C'–Lipschitz for some $C' > 0$. For large n, x_n and y_n are inside U, and we have $s(y_n)s(x_n)^{-1}x_n = y_n$. Since inversion in G is a smooth map and therefore C''–Lipschitz close to the identity, it follows that (after possibly shrinking U some more)

$$d\big(1, s(y_n)s(x_n)^{-1}\big) = d\big(s(y_n)^{-1}, s(x_n)^{-1}\big) \leq C'' d(s(y_n), s(x_n)) \leq C'C'' d(y_n, x_n),$$

a contradiction. □

Lemma 2.4.15. *Let $A \subset G$ be a compact set. Then there exists a constant $C > 0$ such that:*

- *The map $A \to X,\ g \mapsto gx$ is C–Lipschitz.*
- *For every $g \in A$, the diffeomorphism $X \to X,\ x \mapsto gx$ is C–Lipschitz.*

Proof. The map $G \times X \to X,\ (g, x) \mapsto gx$ is smooth and thus locally Lipschitz. Its restriction to the compact set $A \times X$ is therefore Lipschitz. This implies both parts of the claim. □

The following auxiliary lemma is a combination of the corresponding statements in Lemmas 7.1, 7.3 and 7.4 in [KLP18], transferred to our setting.

Lemma 2.4.16. *Let*

$$K \colon X \to \mathcal{C}(Y)$$

be a G–equivariant map. Then there are constants $L, D > 0$ such that

(i) K is L–Lipschitz.

(ii) If $A \subset X$ is compact, then $\bigcup_{x \in A} K(x)$ is compact.

(iii) For all $x \in X$ and $y \in Y$ there exists $x' \in X$ such that $y \in K(x')$ and

$$d(x', x) \leq D\, d(y, K(x)).$$

(iv) If $\Lambda \subset X$ is compact with $K(\lambda) \cap K(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$, then the map

$$\pi \colon \bigcup_{\lambda \in \Lambda} K(\lambda) \to \Lambda$$

mapping every point of $K(\lambda)$ to λ is a uniformly continuous fiber bundle (in the subspace topologies).

Proof.

(i) Let $x, y \in X$ be arbitrary. Using Lemma 2.4.14, we choose $g \in G$ such that $gx = y$ and $d(1, g) \leq Cd(x, y)$. By equivariance, $gK(x) = K(y)$ holds. As $\operatorname{diam}(X)$ is finite, Lemma 2.4.15 implies that $d(gz, hz) \leq C'd(g, h)$ for any $g, h \in \overline{B_{C \cdot \operatorname{diam}(X)}(1)}$ and $z \in Y$. Both constants C, C' do not depend on the choice of x and y. Therefore,

$$\begin{aligned} d_H(K(x), K(y)) &= \max \left\{ \max_{a \in K(x)} d(a, gK(x)), \max_{gb \in gK(x)} d(K(x), gb) \right\} \\ &\leq \max \left\{ \max_{a \in K(x)} d(a, ga), \max_{gb \in gK(x)} d(b, gb) \right\} \leq C'd(1, g) \leq CC'd(x, y). \end{aligned}$$

(ii) Let $y_n \in \bigcup_{x \in A} K(x)$ be a sequence and $x_n \in A$ such that $y_n \in K(x_n)$. Passing to a subsequence we can assume that $y_n \to y \in Y$ and $x_n \to x \in A$. But

$$d(y_n, K(x)) \leq d_H(K(x_n), K(x)) \to 0$$

by (i), so $d(y, K(x)) = 0$, which means $y \in K(x)$ since $K(x)$ is closed.

(iii) Let $a \in K(x)$ be such that $d(y, K(x)) = d(y, a)$. By Lemma 2.4.14, there is an element g with $ga = y$ and $d(1, g) \leq Cd(y, a)$. Moreover, since $\operatorname{diam}(Y)$ is finite, Lemma 2.4.15 implies that $d(x, gx) \leq C'd(1, g)$. Therefore, gx is the point x' we were looking for.

(iv) We start by showing continuity of π. Assume that $y_n \in K(x_n)$, $y_n \to y \in \bigcup_{\lambda \in \Lambda} K(\lambda)$ and $\pi(y_n) =: x_n \to x \in \Lambda$. We need to show that $\pi(y) = x$. Since K is continuous, we have $K(x_n) \to K(x)$ in $\mathcal{C}(Y)$. Therefore, $d(y, K(x)) = 0$, so $y \in K(x)$ and $\pi(y) = x$ as $K(x)$ is closed. By compactness of $\bigcup_{\lambda \in \Lambda} K(\lambda)$ (according to (ii)), π is uniformly continuous.

Now we construct a local trivialization. Let $x \in \Lambda$ be a point, U a neighborhood of x in X, and $s : U \to G$ a smooth local section of the submersion $G \to X$, $g \mapsto gx$. Then the map

$$\begin{aligned} (\Lambda \cap U) \times K(x) &\to \bigcup_{\lambda \in \Lambda} K(\lambda) \\ (\lambda, y) &\mapsto s(\lambda)y \end{aligned}$$

is a homeomorphism onto its image, since its inverse is given by

$$y \mapsto (\pi(y), s(\pi(y))^{-1}y). \qquad \square$$

Example 2.4.17. Let us describe the main example of such a map K that we are interested in. Let $\mathcal{F}_R$ and $\mathcal{F}_S$ be two oriented flag manifolds, where $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$. For any $f \in \mathcal{F}_R$ and $[\![w]\!] \in \widetilde{W}_{R,S}$, we defined the subset

$$C_{[\![w]\!]}(f) = \{f' \in \mathcal{F}_S \mid \mathrm{pos}_{R,S}(f, f') = [\![w]\!]\}.$$

Observe that for any element $g \in G$ satisfying $[g] = f \in \mathcal{F}_R$, we have $C_{[\![w]\!]}(f) = gC_{[\![w]\!]}([1])$; in other words, the map $f \mapsto C_{[\![w]\!]}(f)$ from $\mathcal{F}_R$ to subsets of $\mathcal{F}_S$ is equivariant. By definition of the Bruhat order on $\widetilde{W}_{R,S}$, the closure of $C_w(f)$ is given by

$$\overline{C_{[\![w]\!]}(f)} = \bigcup_{[\![w']\!] \leq [\![w]\!]} C_{[\![w']\!]}(f).$$

In particular, if $I \subset \widetilde{W}_{R,S}$ is an ideal, then $\bigcup_{[\![w']\!] \in I} C_{[\![w']\!]}(f)$ is closed for any $f \in \mathcal{F}_R$. Therefore, we obtain the equivariant map

$$\begin{aligned} K \colon \mathcal{F}_R &\to \mathcal{C}(\mathcal{F}_S) \\ f &\mapsto \bigcup_{[\![w']\!] \in I} C_{[\![w']\!]}(f). \end{aligned}$$

The following key lemma shows how expansion in X leads to expansion transverse to the map $K \colon X \to \mathcal{C}(Y)$ in Y (compare [KLP18, Lemma 7.5]).

Lemma 2.4.18. *Let $K \colon X \to \mathcal{C}(Y)$ be G–equivariant and $\Lambda \subset X$ compact with $K(\lambda) \cap K(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$. Let $g \in G$ be expanding at $\lambda \in \Lambda$ with expansion factor $c > LD$, where L and D are the constants from Lemma 2.4.16. Then g is expanding at every $y \in K(\lambda)$ transversely to $K|_\Lambda$.*

Proof. We give a short outline of the proof before delving into the details. Let V be a neighborhood of the point $y \in K(\lambda)$, $y' \in V$ and $\lambda' \in \Lambda$ such that $K(\lambda') \cap V \neq \varnothing$. We want to choose $x \in X$ with $y' \in K(x)$ such that the following string of inequalities holds:

$$\begin{aligned} d(y', K(\lambda')) \leq d_H(K(x), K(\lambda')) &\leq L\, d(x, \lambda') \\ &\overset{(*)}{\leq} c^{-1} L\, d(gx, g\lambda') \overset{(**)}{\leq} c^{-1} LD\, d(gy', K(g\lambda')) \end{aligned}$$

The first two inequalities are true for any choice of x. For $(*)$, x and λ' need to be close to λ so g is expanding. For $(**)$, we need gx to be a "good" choice in the sense of Lemma 2.4.16(iii). Our task is to make sure that the choices of V and x can be made accordingly.

Since g is expanding, there is an open neighbourhood $U \subset X$ of λ with $d(gz, gz') \geq c\, d(z, z')$ for all $z, z' \in U$. We can assume that $U = B_\varepsilon(\lambda)$ for some $\varepsilon > 0$. Let π

be the bundle map from Lemma 2.4.16(iv). Since π is uniformly continuous there is $\delta > 0$ with

$$d(\pi(y), \pi(y')) < \frac{\varepsilon}{2} \quad \text{whenever} \quad d(y, y') < \delta. \tag{2.4.3}$$

We can assume that $\delta \leq \frac{\varepsilon}{2}\alpha\beta D$ where α and β are Lipschitz constants for the action of g^{-1} on X and the action of g on Y.

Let $V = B_\delta(y)$ and let $\lambda' \in \Lambda$ with $K(\lambda') \cap V \neq \varnothing$ and $y' \in V$. Then by Lemma 2.4.16(iii) (applied to $g\lambda'$ and gy') there is gx such that $gy' \in K(gx)$ and thus $y' \in K(x)$ and

$$d(gx, g\lambda') \overset{(**)}{\leq} D\, d(gy', K(g\lambda')).$$

Next we want to show that $x, \lambda' \in U = B_\varepsilon(\lambda)$ by bounding $d(\lambda, \lambda')$ and $d(x, \lambda')$. First, since $K(\lambda')$ intersects V, there is a point $p \in K(\lambda')$ with $d(p, y) < \delta$. So $d(\lambda, \lambda') = d(\pi(y), \pi(p)) < \varepsilon/2$ by (2.4.3). Second,

$$d(x, \lambda') \leq \alpha\, d(gx, g\lambda') \leq \alpha D\, d(gy', gK(\lambda')) \leq \alpha\beta D\, d(y', K(\lambda')) \leq \alpha\beta D\delta \leq \varepsilon/2,$$

so $x \in U$ and also $\lambda' \in U$. This implies $(*)$. Therefore,

$$d(y', K(\lambda')) \leq c^{-1} LD\, d(gy', K(g\lambda')),$$

and $c^{-1}LD < 1$, so g is transversely expanding. □

After these preparations, we now turn to our goal for this section: A criterion for group actions to be cocompact. Let $\rho\colon \Gamma \to G$ be a representation of a discrete group. This defines an action of Γ on X and Y. The following lemma illustrates the basic idea of using expansion to prove cocompactness. The notion of expansion used here is slightly different from the ones in Definition 2.4.11.

Lemma 2.4.19. *Let $\Xi \subset Y$ be a compact, Γ-invariant set. Assume that for every $y \in \Xi$, there exists a neighborhood U, an element $\gamma \in \Gamma$ and a constant $c > 1$ such that*

$$d(\gamma y', \Xi) \geq c d(y', \Xi) \quad \forall y' \in U.$$

Then Γ acts cocompactly on $Y \setminus \Xi$.

Proof. By compactness of Ξ, we can find finitely many points $y_1, \ldots, y_n \in Y$ such that their associated neighborhoods U_{y_i} cover Ξ. Moreover, there exists a $\delta > 0$ such that their union $\bigcup_i U_{y_i}$ contains the δ–neighborhood $N_\delta(\Xi)$. Let $c > 1$ be the minimal expansion factor of the corresponding elements γ_i. We will show that every orbit Γy, $y \in Y \setminus \Xi$ has a representative in $Y \setminus N_\delta(\Xi)$. This will prove the lemma since $Y \setminus N_\delta(\Xi)$ is compact.
Indeed, if $y \in N_\delta(\Xi) \setminus \Xi$, there exists $i_0 \in \{1, \ldots, n\}$ such that $y \in U_{y_{i_0}}$. Therefore, $d(\gamma_{i_0} y, \Xi) \geq c d(y, \Xi)$. If $d(\gamma_{i_0} y, \Xi) \geq \delta$, we are done. Else, we repeat the procedure until we obtain a point in the orbit which does not lie in $N_\delta(\Xi)$. □

Connecting expansion, transverse expansion and the previous lemma yields the following useful result (compare [KLP18, Proposition 5.30]):

Proposition 2.4.20. *Let $K\colon X \to \mathcal{C}(Y)$ be G–equivariant, $\Lambda \subset X$ compact and Γ–invariant with $K(\lambda) \cap K(\lambda') = \varnothing$ for all distinct $\lambda, \lambda' \in \Lambda$. Also assume that the action of Γ on X is expanding at Λ. Then Γ acts cocompactly on $\Omega = Y \setminus \bigcup_{\lambda\in\Lambda} K(\lambda)$.*

Proof. Since Λ is closed and Γ–invariant, we know by Lemma 2.4.12 that the action of Γ is expanding arbitrarily strongly at every point $\lambda \in \Lambda$. Lemma 2.4.18 therefore shows that the action of Γ is expanding at every point of $\bigcup_{\lambda\in\Lambda} K(\lambda)$ transversely to K. We will show that this implies the prerequisites of Lemma 2.4.19 and thus the action on Ω is cocompact.
First of all, we observe that for any point $z \in Y$, we have

$$d\Big(z, \bigcup_{\lambda\in\Lambda} K(\lambda)\Big) = d(z, K(\lambda_z)) \tag{2.4.4}$$

for some $\lambda_z \in \Lambda$ (this follows from compactness of $\bigcup_{\lambda\in\Lambda} K(\lambda)$, Lemma 2.4.16(ii)). Now let $y \in \bigcup_{\lambda\in\Lambda} K(\lambda)$ and let a neighborhood $U \ni y$ and $\gamma \in \Gamma$ be chosen such that γ is c–expanding on U transversely to K. There exists $\epsilon > 0$ satisfying

$$B_\epsilon(\gamma y) \subset \gamma U.$$

Let $\delta > 0$ be sufficiently small such that

$$\gamma B_\delta(y) \subset B_{\epsilon/2}(\gamma y).$$

For any point $y' \in B_\delta(y)$, let $\lambda_{\gamma y'}$ be chosen as in (2.4.4). Since $d(\gamma y', \gamma y) < \epsilon/2$ and $\gamma y \in \bigcup_{\lambda\in\Lambda} K(\lambda)$, we necessarily have $K(\lambda_{\gamma y'}) \cap B_\epsilon(\gamma y) \neq \varnothing$. Therefore,

$$\gamma^{-1} K(\lambda_{\gamma y'}) \cap U \neq \varnothing.$$

Since $y' \in B_\delta(y) \subset U$, transverse expansion now implies

$$d\Big(\gamma y', \bigcup_{\lambda\in\Lambda} K(\lambda)\Big) = d(\gamma y', K(\lambda_{\gamma y'})) \geq c d(y', \gamma^{-1} K(\lambda_{\gamma y'})) \geq c d\Big(y', \bigcup_{\lambda\in\Lambda} K(\lambda)\Big). \qquad \square$$

We can now apply the previous proposition to the setting of Anosov representations and oriented flag manifolds, which is our main result of this section. Once more, we recall the notation we use:

Let Γ be a non-elementary word hyperbolic group and G a connected, semi-simple, linear Lie group. Let P_R and P_S be oriented parabolic subgroups of types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$. Furthermore, let $w_0 \in T \subset \widetilde{W}$ be a transverse position. We assume that $\iota(\theta) = \theta$, $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$, so that w_0 acts involutively on $\widetilde{W}_{R,S}$ (see Section 2.2.4). Recall from Section 2.2.3 and Section 2.2.4 that a subset $I \subset \widetilde{W}_{R,S}$ is an ideal if $[\![w]\!] \in I$ and $[\![w']\!] \leq [\![w]\!]$ implies $[\![w']\!] \in I$, and it is w_0–slim if $[\![w]\!] \in I$ implies $[\![w_0 w]\!] \notin I$.

Theorem 2.4.21. *Let $\rho : \Gamma \to G$ be an P_R–Anosov representation and $\xi : \partial_\infty\Gamma \to \mathcal{F}_R$ a limit map of transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–slim ideal, and define the set $\mathcal{K} \subset \mathcal{F}_S$ as*

$$\mathcal{K} := \bigcup_{x\in\partial_\infty\Gamma} \bigcup_{[\![w]\!]\in I} C_{[\![w]\!]}(\xi(x)).$$

Then $\mathcal{K}$ is Γ–invariant and closed, and Γ acts cocompactly on the domain $\Omega = \mathcal{F}_S \setminus \mathcal{K}$.

Proof. Γ–invariance and closedness of $\mathcal{K}$ follows from Lemma 2.4.16(ii) and Example 2.4.17. As discussed in Example 2.4.17, the map

$$\begin{aligned} K\colon \mathcal{F}_R &\to \mathcal{C}(\mathcal{F}_S) \\ f &\mapsto \bigcup_{[\![w]\!]\in I} C_{[\![w]\!]}(f) \end{aligned}$$

is G–equivariant. Moreover, the image $\xi(\partial_\infty\Gamma)$ is compact and the action of Γ on $\mathcal{F}_R$ is expanding at $\xi(\partial_\infty\Gamma)$ since ρ is P_R–Anosov. If we can show that $K(\xi(x))\cap K(\xi(x')) = \varnothing$ for $x \neq x'$, cocompactness will follow from Proposition 2.4.20.
Let $y' \in K(\xi(x'))$ be any point. Since ξ has transversality type $[\![w_0]\!]$ and by definition of K, we have the relative positions

$$\begin{aligned} \mathrm{pos}_{R,R}(\xi(x), \xi(x')) &= [\![w_0]\!], \\ \mathrm{pos}_{R,S}(\xi(x'), y') &=: [\![w]\!] \in I. \end{aligned}$$

By Lemma 2.2.24, this implies that $\mathrm{pos}_{R,S}(\xi(x), y') \geq [\![w_0 w]\!]$. Since $[\![w_0 w]\!] \notin I$ by w_0–slimness and I is an ideal, $y' \notin K(\xi(x))$. □

2.5 Examples of balanced ideals

In this section, we will describe explicitly the Bruhat order on $\widetilde{W}$ and the possible balanced ideals for the group $G = \mathrm{SL}(3,\mathbb{R})$. These examples already show how passing from W to $\widetilde{W}$ vastly increases the number of balanced ideals and therefore the possibilities to build cocompact domains of discontinuity. We have no classification of all balanced ideals, so explicit examples of balanced ideals will be restricted to low dimensions and some special cases in higher dimension.

2.5.1 Reduction to $R = \{1, w_0^2\}$ and $S = \{1\}$

In applications, we are usually given a fixed representation $\rho\colon \Gamma \to G$. If it is Anosov, then by Proposition 2.3.6, there is a unique minimal oriented parabolic type $R = \langle r(\theta), E\rangle$ for ρ. If we also fix one of the possible lifts of the boundary map to $\mathcal{F}_R$, then we get a transversality type $[\![w_0]\!] \in \widetilde{W}_{R,R}$. To apply Theorem 2.4.1 and find cocompact domains of discontinuity in a given flag manifold $\mathcal{F}_S$, we need to look for w_0–balanced ideals in $\widetilde{W}_{R,S}$. Note that this notion of w_0–balanced only depends on the equivalence class $[\![w_0]\!]$.

To enumerate all balanced ideals as we want to do in this section, a different approach is more convenient: We first determine the set $T \subset \widetilde{W}$ of transverse positions. Then, we want to list all w_0–balanced ideals in $\widetilde{W}_{R,S}$ for $w_0 \in T$ and all possible oriented parabolic types $R = \langle r(\theta), E\rangle$ and $S = \langle r(\eta), F\rangle$. For this to be well–defined, w_0 must act as an involution on $\widetilde{W}_{R,S}$, which by Section 2.2.4 happens if $\iota(\theta) = \theta$, $w_0 E w_0^{-1} = E$ and $w_0^2 \in E$.

Note that the unique smallest E satisfying these conditions is $E = \{1, w_0^2\}$. The following lemma implies that when listing all possible w_0–balanced ideals, one can restrict to the minimal choice $R = \{1, w_0^2\}$ and $S = \{1\}$.

Lemma 2.5.1. *Let R and S be oriented parabolic types as above, and consider the projection $\pi\colon \widetilde{W}_{\{1,w_0^2\},\{1\}} \to \widetilde{W}_{R,S}$. Assume that $I \subset \widetilde{W}_{R,S}$ is a w_0–balanced ideal. Then $\pi^{-1}(I) \subset \widetilde{W}_{\{1,w_0^2\},\{1\}}$ is a w_0–balanced ideal as well.*

Proof. By Lemma 2.2.17(i), $\pi^{-1}(I)$ is again an ideal. Let $[\![w]\!] \in \widetilde{W}_{\{1,w_0^2\},\{1\}}$, and recall that w_0 acts by left multiplication on both $\widetilde{W}_{\{1,w_0^2\},\{1\}}$ and $\widetilde{W}_{R,S}$, satisfying

$$\pi([\![w_0 w]\!]) = w_0 \pi([\![w]\!]).$$

Therefore, we obtain the following equivalences:

$$[\![w]\!] \in \pi^{-1}(I) \Leftrightarrow \pi([\![w]\!]) \in I \Leftrightarrow w_0\pi([\![w]\!]) \notin I \Leftrightarrow [\![w_0 w]\!] \notin \pi^{-1}(I) \qquad \square$$

By this lemma, every w_0–balanced ideal in $\widetilde{W}_{R,S}$ is obtained by projecting a R–left invariant and S–right invariant w_0–balanced ideal of $\widetilde{W}_{\{1,w_0^2\},\{1\}}$. We can further reduce the number of w_0 we have to consider by observing that choices of w_0 conjugate by an element in $\overline{M}$ lead to essentially the same balanced ideals:

Lemma 2.5.2. *Let $I \subset \widetilde{W}_{\{1,w_0^2\},\{1\}}$ be a w_0–balanced ideal and $m \in \overline{M}$. Then mI is a mw_0m^{-1}–balanced ideal.*

Proof. mI is again an ideal by Lemma 2.2.21(i). It is mw_0m^{-1}–balanced because

$$[\![w]\!] \in mI \Leftrightarrow [\![m^{-1}w]\!] \in I \Leftrightarrow w_0[\![m^{-1}w]\!] \notin I \Leftrightarrow (mw_0m^{-1})[\![w]\!] \notin mI. \qquad \square$$

Given a w_0–balanced ideal I and an element $m \in \overline{M}$ such that $mw_0m^{-1} = w_0$, mI is again w_0–balanced. The cocompact domains obtained via Theorem 2.4.1 for I and mI are in general different. In contrast to this, the action of $\overline{M}$ by right–multiplication is easy to describe: An ideal I is w_0–balanced if and only if Im is. Moreover, by Lemma 2.2.21, the domain will simply change by global right–multiplication with m, i.e. by changing some orientations.

2.5.2 The extended Weyl group of $\mathrm{SL}(n,\mathbb{R})$

Let $G = \mathrm{SL}(n,\mathbb{R})$ with maximal compact $K = \mathrm{SO}(n,\mathbb{R})$ and $\mathfrak{a} \subset \mathfrak{sl}(n,\mathbb{R})$ the set of diagonal matrices with trace 0. Then $\Sigma = \{\lambda_i - \lambda_j \mid i \neq j\} \subset \mathfrak{a}^*$, where $\lambda_i \colon \mathfrak{a} \to \mathbb{R}$ is the i–th diagonal entry. Choose the simple system Δ consisting of all roots $\alpha_i :=$ $\lambda_i - \lambda_{i+1}$ with $i \leq i \leq n-1$. Then B_0 is the subgroup of upper triangular matrices with positive diagonal. The group $Z_K(\mathfrak{a})$ is the group of diagonal matrices with ± 1 entries and det $= 1$. Its identity component is trivial, so $\overline{M} = Z_K(\mathfrak{a})$. The extended Weyl group $\widetilde{W} = N_K(\mathfrak{a})$ consists of all permutation matrices with determinant 1 – i.e. all matrices with exactly one ± 1 entry per line and row and all other entries 0, such that det $= 1$.

A generating set $r(\Delta)$ in the sense of Definition 2.1.4 is given by

$$r(\alpha_i) = \begin{pmatrix} I_{i-1} & & & \\ & & -1 & \\ & 1 & & \\ & & & I_{n-i-1} \end{pmatrix}.$$

The transverse positions $T \subset \widetilde{W}$ are antidiagonal matrices with ± 1 entries. The number of -1 entries has to be even if n is equal to 0 or 1 mod 4, and odd otherwise. In one formula, it has the same parity as $(n-1)n/2$.

The group $\overline{M}$ is generated by diagonal matrices with exactly two -1 entries and the remaining entries $+1$. Conjugating $w_0 \in T$ by such an element negates the two lines and the two columns corresponding to the minus signs. This yields the following standard representatives for equivalence classes in T under conjugation by $\overline{M}$:

(i) If n is odd, the $(n-1)/2$–block in the upper right corner can be normalized to have $+1$–entries.

(ii) If n is even, the $(n-2)/2$–block in the upper right corner can be normalized to have $+1$–entries.

If $w_0, w_0' \in T$ of this form are different, they are not conjugate by an element of $\overline{M}$.

2.5.3 Balanced ideals for $\mathrm{SL}(3,\mathbb{R})$

Let $G = \mathrm{SL}(3,\mathbb{R})$ and $\Delta = \{\alpha_1, \alpha_2\}$ the set of simple roots, viewed as their associated reflections. These generate the Weyl group

$$W = \langle \alpha_1, \alpha_2 \mid \alpha_1^2 = \alpha_2^2 = (\alpha_1\alpha_2)^3 = 1 \rangle = \{1, \alpha_1, \alpha_2, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_1\alpha_2\alpha_1\}.$$

The Bruhat order on W is just the order by word length and there is a unique longest element $w_0 = \alpha_1\alpha_2\alpha_1$ which acts on W from the left, reversing the order (see Figure 2.1).

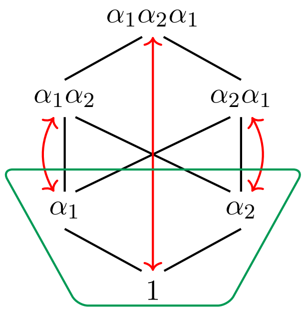


Figure 2.1: The Weyl group of $\mathrm{SL}(3,\mathbb{R})$. The black lines indicate the Bruhat order, in the sense that a line going downward from x to y means that x covers y in the Bruhat order. The red arrows show the involution induced by w_0. The subset surrounded by the green box is the only balanced ideal.

There is only one w_0–balanced ideal in this case, which is indicated by the green box in Figure 2.1.

Since $|\overline{M}| = 4$, each of the 6 elements of W has 4 preimages in $\widetilde{W}$, corresponding to different signs in the permutation matrix. The Bruhat order on $\widetilde{W}$ can be determined using Proposition 2.2.18 and is shown in Figure 2.2. See Section 2.5.4 for a geometric interpretation in terms of oriented flags.

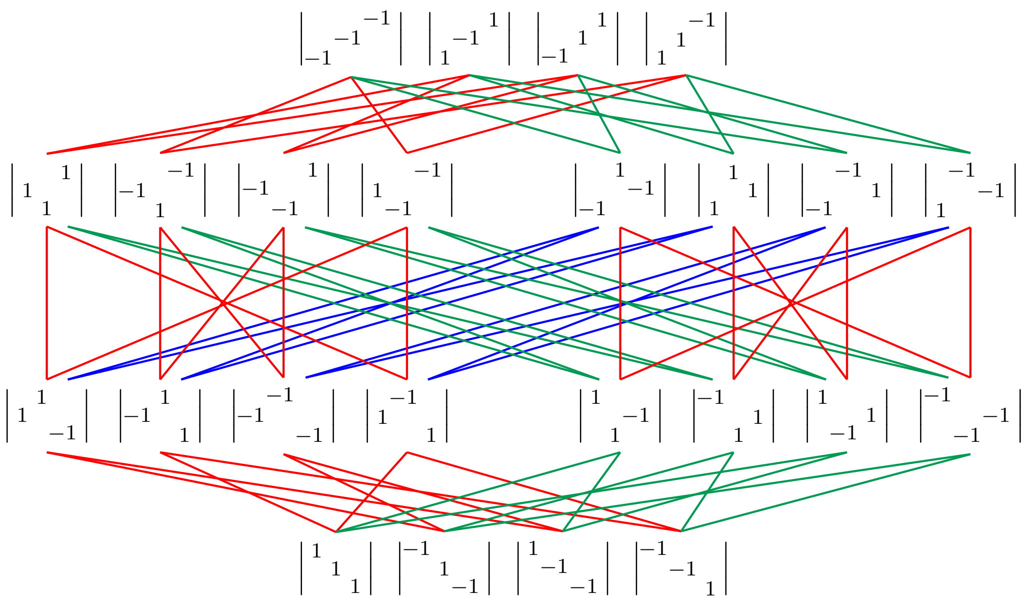


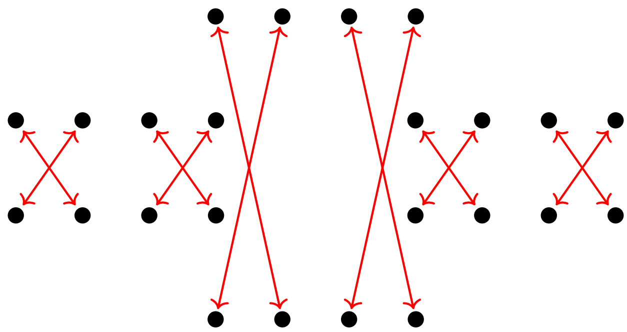
Figure 2.2: The Bruhat order on the extended Weyl group of $\mathrm{SL}(3,\mathbb{R})$. Different colors are used purely for better visibility.

To find balanced ideals, we first list the possible transverse positions $w_0 \in T$. These are

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

The first two of these are conjugate by $\overline{M}$, as are the last two. So we have to distinguish two cases.

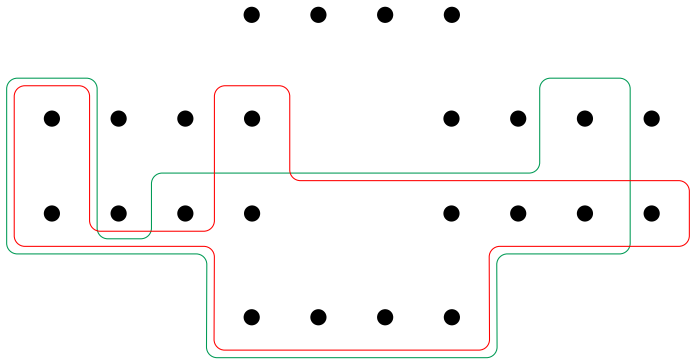
(i) $w_0 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$. Since $w_0^2 = 1$, the minimal choice of E (or R) is the trivial group, so $\widetilde{W}_{R,S} = \widetilde{W}_{\{1\},\{1\}} = \widetilde{W}$. The involution induced by w_0 acts on $\widetilde{W}$ in the following way (each dot represents the corresponding matrix from the above picture):



Combining this with Figure 2.2, we obtain the following balanced ideals:

a) The lift of the unoriented balanced ideal contains all relative positions in the bottom half of the picture.

b) Ideals containing two positions from the third level and everything below these two positions in the Bruhat order. The possible pairs of positions from the third level that can be chosen are (ideals equivalent by right-multiplication by $\overline{M}$ are in curly brackets): $\{(1,4),(2,3)\}, \{(5,8),(6,7)\}$ as well as $\{(1,7),(2,8),(3,6),(4,5)\}, \{(1,8),(2,7),(3,5),(4,6)\}$. In the following picture, we drew the examples $(1,4)$ in red and $(1,7)$ in green.



c) Ideals containing one relative position from the third level and everything except its w_0-image from the second level and below. There are 8 balanced ideals of this type, determined by the element on the third level. Right multiplication by $\overline{M}$ identfies the first 4 and the last 4 ideals.

In total, we find 21 w_0–balanced ideals, which form 7 equivalence classes with respect to right–multiplication with $\overline{M}$. Let us emphasize again that the balanced ideals in (b) and (c) are not lifts of balanced ideals from the unoriented setting. If a representation satisfies the prerequisites of Theorem 2.4.1, we therefore obtain new cocompact domains of discontinuity in oriented flag manifolds.

(ii) $w_0 = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Since $w_0^2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ is nontrivial, the minimal choice of R is $\{1, w_0^2\}$ and we consider $R\backslash\widetilde{W}$. By Lemma 2.2.17 we get the Bruhat order on $R\backslash\widetilde{W}$ as the projection of Figure 2.2. It is shown in Figure 2.3 alongside the action of w_0 on $R\backslash\widetilde{W}$.

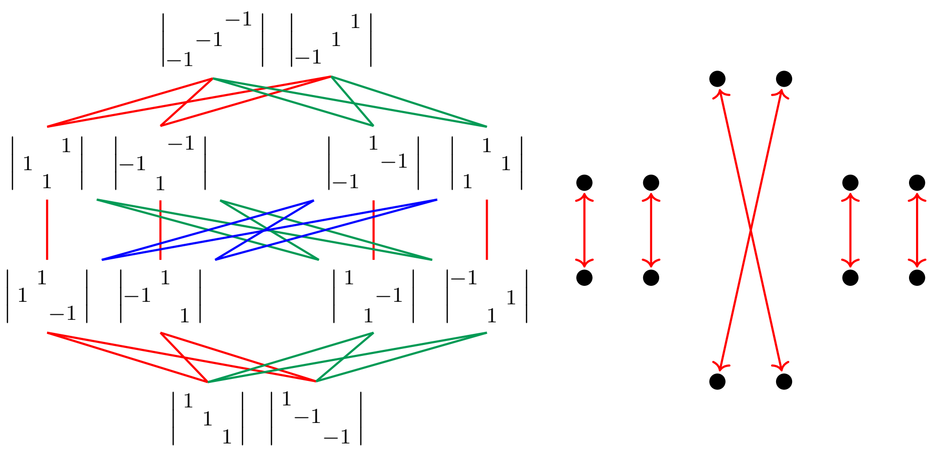


Figure 2.3: The Bruhat order on $R\backslash\widetilde{W}$ and the involution given by w_0. Different colors are used purely for better visibility.

In this case, the only balanced ideal is the lift of the unoriented one, and we do not obtain any new cocompact domains of discontinuity in oriented flag manifolds.

2.5.4 Geometric interpretation of relative positions for $\mathrm{SL}(3,\mathbb{R})$

In order to give a hands–on description of the various relative positions we saw in the previous two subsections, we need notions of direct sums and intersections that take orientations into account. These notions appeared already in [Gui05], where they are used to describe curves of flags. First of all, let us fix some notation for oriented subspaces.

Definition 2.5.3. Let $A, B \subset \mathbb{R}^n$ be oriented subspaces. Then we denote by $-A$ the subspace A with the opposite orientation. If A and B agree as oriented subspaces, we write $A \overset{\pm}{=} B$.

Now let $A, B \subset \mathbb{R}^n$ be oriented subspaces. If they are (unoriented) transverse, taking a positive basis of A and extending it by a positive basis of B yields a basis of $A \oplus B$. Declaring this basis to be positive defines an orientation on $A \oplus B$. The orientation on the direct sum depends on the order we write the two subspaces in,

$$A \oplus B \overset{\pm}{=} (-1)^{\dim(A)\dim(B)} B \oplus A.$$

The case of intersections is slightly more difficult. Assume that $A, B \subset \mathbb{R}^n$ are oriented subspaces such that $A + B = \mathbb{R}^n$, and fix a standard orientation on $\mathbb{R}^n$. Let $A' \subset A$ be a subspace complementary to $A \cap B$ and analogously $B' \subset B$ a subspace complementary to $A \cap B$. We fix orientations on these two subspaces by requiring that

$$A' \oplus B \overset{\pm}{=} \mathbb{R}^n$$

and

$$A \oplus B' \overset{\pm}{=} \mathbb{R}^n.$$

Then there is a unique orientation on $A \cap B$ satisfying

$$A' \oplus (A \cap B) \oplus B' \overset{\pm}{=} \mathbb{R}^n.$$

This is the induced orientation on the intersection. Since the set of subspaces of A complementary to $A \cap B$ can be identified with $\mathrm{Hom}(A', A \cap B)$ and is therefore (simply) connected, the result does not depend on the choice of A', and analogously does not depend on the choice of B'. Like the oriented sum, it depends on the order we write the two subspaces in,

$$B \cap A \overset{\pm}{=} (-1)^{\operatorname{codim}(A)\operatorname{codim}(B)} A \cap B.$$

With this terminology at hand, consider the oriented relative positions shown in Figure 2.2. Let $f \in G/B_0$ be a reference complete oriented flag. We denote by $f^{(k)}$ the k–dimensional part of the flag f. Let $w = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} \in \widetilde{W}$. Then we have

$$C_w(f) = \{F \in G/B_0 \mid f^{(2)} \oplus F^{(1)} \stackrel{\pm}{=} -\mathbb{R}^3,\ f^{(1)} \oplus F^{(2)} \stackrel{\pm}{=} -\mathbb{R}^3\}.$$

The other three positions of the highest level are characterized by the other choices of the two signs. Similarly, for the position $w' = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \in \widetilde{W}$, we obtain

$$C_{w'}(f) = \{F \in G/B_0 \mid f^{(1)} \oplus F^{(1)} \stackrel{\pm}{=} f^{(2)},\ f^{(2)} \cap F^{(2)} \stackrel{\pm}{=} F^{(1)}\},$$

and the other three positions are characterized by the other choices of the two signs. Similar descriptions hold for the remaining oriented relative positions.

2.5.5 A simple example in odd dimension: Halfspaces in spheres

As the previous subsection demonstrated, calculating the most general relative positions and the Bruhat order gets out of hand very quickly as one increases the dimension. For example, in $\mathrm{SL}(5,\mathbb{R})$, there are 120 unoriented relative positions between complete flags and 1920 oriented relative positions between complete oriented flags. For practical reasons, it thus makes sense to restrict to more special cases, i.e. to consider relative positions $\widetilde{W}_{R,S}$ for bigger R, S than strictly necessary.

The example we describe now is a balanced ideal corresponding to a domain in an even–dimensional sphere, obtained by removing half–dimensional half spaces. It is not a lift of an unoriented balanced ideal. In fact, it is not hard to see that relative positions between an unoriented line and a complete flag simply describe the smallest subspace of the flag containing the line. The Bruhat order is a total order on this set. From the simple fact that the number of relative positions is odd, it follows that every involution has a fixed point and there can be no balanced ideal.
In contrast, when considering $\mathrm{SL}(2n,\mathbb{R})$, by the same argument, there is a balanced ideal corresponding to a domain in $\mathbb{RP}^{2n-1}$: The Bruhat order on relative positions between unoriented lines and complete flags is a total order on a set with $2n$ elements, so the lower half forms a balanced ideal.

Let $G = \mathrm{SL}(2n+1,\mathbb{R})$ and $\theta, \eta \subset \Delta$ be the complements of $\Delta \setminus \theta = \{\alpha_n, \alpha_{n+1}\}$, $\Delta \backslash \eta = \{\alpha_1\}$, so that $\mathcal{F}_\theta$ is the space of partial flags consisting of the dimension n and $n+1$ parts, and $\mathcal{F}_\eta$ is $\mathbb{RP}^{2n}$. Furthermore, let

$$E = \langle \overline{M}_\theta, r(\alpha_n)^2 r(\alpha_{n+1})^2 \rangle = \{m \in \overline{M} \mid m_{n+1,n+1} = +1\},$$

$F = \overline{M}_\eta$, $R = \langle r(\theta), E\rangle$, and $S = \langle r(\eta), F\rangle$. Then $\mathcal{F}_R$ is the space of oriented partial flags consisting of oriented n– and $(n+1)$–dimensional subspaces up to changing both orientations simultaneously, and $\mathcal{F}_S$ is S^{2n}, the space of oriented lines on $\mathbb{R}^{2n+1}$.

Choose w_0 to be antidiagonal with -1 as the middle entry. The remaining entries are irrelevant for this example; if $2n+1 \in 4\mathbb{Z}+3$, there should be an odd number of minus signs, if $2n+1 \in 4\mathbb{Z}+1$, it has to be even.

The Bruhat order on the space $\widetilde{W}_{R,S}$ of relative positions as well as the involution w_0 are shown in Figure 2.4. We only need to keep track of the first column of the matrix representative since we right quotient by S. The left quotient by R then reduces the possible relative positions further. There are thus two balanced ideals, determined by

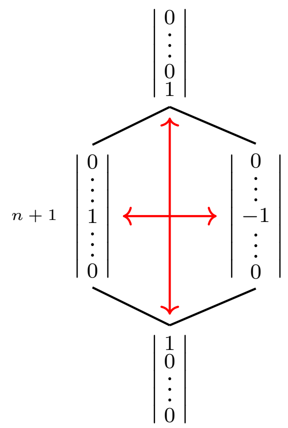


Figure 2.4: Oriented relative positions between $\mathcal{F}_R$ and $\mathcal{F}_S$

choosing one of the two middle positions. This is in contrast to the unoriented case, where the two middle positions coincide and are a fixed point of the involution.

The geometric description of the relative positions is as follows. Let $f \in \mathcal{F}_R$ be a reference flag. Then $[\![w]\!] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and $[\![w']\!] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ correspond to

$$C_{[\![w]\!]}(f) = \{F \in S^{2n} \mid F \notin f^{(n+1)}\}, \qquad C_{[\![w']\!]}(f) = \{F \in S^{2n} \mid F \in f^{(n)}\},$$

and $[\![w'']\!] = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ corresponds to

$$C_{[\![w'']\!]}(f) = \{F \in S^{2n} \mid f^{(n)} \oplus F \overset{\pm}{=} f^{(n+1)}\}.$$

This can be rephrased slightly: The codimension 1 subspace $f^{(n)} \subset f^{(n+1)}$ decomposes $f^{(n+1)}$ into two half–spaces, and we say that an oriented line $l \subset f^{(n+1)}$ is in the positive half–space if $f^{(n)} \oplus l \overset{\pm}{=} f^{(n+1)}$. This is invariant under simultaneously changing the orientations of both $f^{(n)}$ and $f^{(n+1)}$ and therefore well–defined. Then $C_{[\![w']\!]}(f)$ is the spherical projectivization of the positive half of $f^{(n+1)}$.

The half great circles in Figure 1.2 in the introduction are an example of a "bad set" determined by this balanced ideal, so their complement is a cocompact domain of discontinuity in S^2 for a convex cocompact representation $\rho\colon F_k \to \mathrm{SO}_0(2,1)$. In Chapter 4, we consider a generalization of Schottky groups in $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(2,1)$. This yields examples of representations (of free groups) admitting a suitable boundary map of the right transversality type into $\mathrm{PSL}(4n+3,\mathbb{R})$. The balanced ideals also apply to Hitchin representations (of closed surface groups) into $\mathrm{PSL}(4n+3,\mathbb{R})$ (see Section 2.6.1).

2.6 Applications

We now give first applications our theory of P_R–Anosov representations and domains of discontinuity in oriented flag manifolds.

Our first result is that Hitchin representations into $\mathrm{PSL}(n,\mathbb{R})$ are B_0–Anosov, with transversality type

$$w_0 = \begin{pmatrix} & & & & \iddots \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} \in \widetilde{W}^{\mathrm{PSL}(n,\mathbb{R})}$$

(Proposition 2.6.1). Consequently, every w_0–balanced ideal in $\widetilde{W}^{\mathrm{PSL}(n,\mathbb{R})}$ describes a cocompact domain of discontinuity for Hitchin representations. We provide a list of all the oriented Grassmannians appearing in this way (Proposition 2.6.3). This includes cases where the domain is a lift of a domain of discontinuity in the unoriented Grassmannian, but also new ones.

The second application presented here is a lower bound on the number of connected components of B–Anosov representations of a closed surface group into $\mathrm{SL}(n,\mathbb{R})$ (Corollary 2.6.6). We consider a special family of B–Anosov representations and use Corollary 2.3.13 to show that they must lie in different components. These representations are obtained by composing a discrete and faithful representation into $\mathrm{SL}(2,\mathbb{R})$ with irreducible representations into $\mathrm{SL}(k,\mathbb{R})$ and $\mathrm{SL}(n-k,\mathbb{R})$ and block embedding into $\mathrm{SL}(n,\mathbb{R})$. An additional variation based on a remark by Thierry Barbot and Jaejeong Lee (see [KK16, Section 4.1]) increases the lower bound further.

Finally, while not discussed in this section, let us also mention another class of representations admitting cocompact domains of discontinuity in oriented flag manifolds: In Chapter 4, we introduce and study a generalized version of Schottky groups in $\mathrm{PSL}(2,\mathbb{R})$. In particular, Theorem 4.6.5 shows that purely hyperbolic generalized Schottky groups in $\mathrm{PSL}(n,\mathbb{R})$ are B_0–Anosov, with transversality type w_0.

2.6.1 Domains of discontinuity for Hitchin representations

Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed surface S of genus at least 2. Particularly simple examples of representations of Γ into $\mathrm{PSL}(n,\mathbb{R})$ are the Fuchsian ones: These are of the form $\iota \circ \rho_0$, where $\rho_0 \colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is injective with discrete image and $\iota\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ is the irreducible representation. Connected components of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ which contain Fuchsian representations are called Hitchin components and their elements Hitchin representations.

If n is odd, then the space $\mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ has 3 connected components, one of which is Hitchin. If n is even, there are 6 components in total, and 2 of them are Hitchin components [Hit87]. Every Hitchin representation is B–Anosov [Lab06].

To find out if the limit map of a Hitchin representation lifts to an oriented flag manifold, let us first take a closer look at the irreducible representation. The standard Euclidean scalar product on $\mathbb{R}^2$ induces a scalar product on the symmetric product $\mathrm{Sym}^{n-1}\mathbb{R}^2$ by restricting the induced scalar product on the tensor power to symmetric tensors. Let $X = \binom{1}{0}$ and $Y = \binom{0}{1}$ be the standard orthonormal basis of $\mathbb{R}^2$. Then $e_i = \sqrt{\binom{n-1}{i-1}} X^{n-i} Y^{i-1}$ for $1 \leq i \leq n$ is an orthonormal basis of $\mathrm{Sym}^{n-1}\mathbb{R}^2$ and provides an identification $\mathbb{R}^n \cong \mathrm{Sym}^{n-1}\mathbb{R}^2$. For $A \in \mathrm{SL}(2,\mathbb{R})$ let $\iota(A) \in \mathrm{SL}(n,\mathbb{R})$ be the induced action on $\mathrm{Sym}^{n-1}\mathbb{R}^2$ in this basis. The homomorphism

$$\iota\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R}),$$

defined this way is the (up to conjugation) unique irreducible representation. It maps -1 to $(-1)^{n-1}$ and is therefore also well–defined as a map $\iota\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$. The induced action on $\mathrm{Sym}^{n-1}\mathbb{R}^2$ preserves the scalar product described above, so $\iota(\mathrm{PSO}(2)) \subset \mathrm{PSO}(n)$.

It is easy to see that ι maps diagonal matrices to diagonal matrices. It also maps upper triangular matrices into $B_0 \subset \mathrm{PSL}(n,\mathbb{R})$ (that is, upper triangular matrices with the diagonal entries either all positive or all negative). Therefore, ι induces a smooth equivariant map

$$\varphi\colon \mathbb{RP}^1 \to \mathcal{F}_{\{1\}}$$

between the complete oriented flag manifolds of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(n,\mathbb{R})$.

Proposition 2.6.1. *Let $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Then its limit map $\xi\colon \partial_\infty\Gamma \to \mathcal{F}_{\overline{M}} = G/B$ lifts to the fully oriented flag manifold $\mathcal{F}_{\{1\}} = G/B_0$ with transversality type*

$$w_0 = \begin{pmatrix} & & & \iddots \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

So all Hitchin representations are B_0–Anosov.

Proof. Since the B_0–Anosov representations are a union of connected components of B–Anosov representations by Proposition 2.3.11, we can assume that $\rho = \iota \circ \rho_0$ is Fuchsian.

Let $\xi_0 \colon \partial_\infty \Gamma \to \mathbb{RP}^1$ be the limit map of ρ_0 and $\pi \colon \mathcal{F}_{\{1\}} \to \mathcal{F}_{\overline{M}}$ the projection forgetting all orientations. Then the limit map $\xi \colon \partial_\infty \Gamma \to \mathcal{F}_{\overline{M}}$ of ρ is just $\pi \circ \varphi \circ \xi_0$ (this is the unique continuous and dynamics–preserving map, see [GGKW17, Remark 2.32b]). So $\widehat{\xi} = \varphi \circ \xi_0$ is a continuous and ρ–equivariant lift to $\mathcal{F}_{\{1\}}$. To calculate the transversality type, let $x, y \in \partial_\infty \Gamma$ with $\xi_0(x) = [1]$ and $\xi_0(y) = [w] \in \mathbb{RP}^1$, where $w \in \mathrm{PSL}(2,\mathbb{R})$ is the anti–diagonal matrix with ± 1 entries. Then, since $\iota(w) = w_0$,

$$\mathrm{pos}(\widehat{\xi}(x), \widehat{\xi}(y)) = \mathrm{pos}(\varphi([1]), \varphi([w])) = \mathrm{pos}([1], [\iota(w)]) = w_0 \in \widetilde{W}. \qquad \square$$

Remark 2.6.2. Note that Hitchin representations map into $\mathrm{PSL}(n,\mathbb{R})$ instead of $\mathrm{SL}(n,\mathbb{R})$. If n is even, the fully oriented flag manifold $\mathcal{F}_{\{1\}}$ in $\mathrm{PSL}(n,\mathbb{R})$ is the space of flags $f^{(1)} \subset \cdots \subset f^{(n-1)}$ with a choice of orientation on every part, but up to simultaneously reversing the orientation in every odd dimension (the action of -1). While we could lift ρ to $\mathrm{SL}(n,\mathbb{R})$, its limit map would still only lift to $\mathcal{F}^{\mathrm{SL}(n,\mathbb{R})}_{\{\pm 1\}} = \mathcal{F}^{\mathrm{PSL}(n,\mathbb{R})}_{\{1\}}$ and not give us any extra information.

Now that we know that Hitchin representations are B_0–Anosov, we can apply Theorem 2.4.9 and Theorem 2.4.21. For every w_0–balanced ideal in $\widetilde{W}$ we get a cocompact domain of discontinuity in the oriented flag manifold $\mathcal{F}_{\{1\}}$ of $\mathrm{PSL}(n,\mathbb{R})$. These include lifts of the domains in unoriented flag manifolds constructed in [KLP18], but also some new examples.

There are 21 different such w_0–balanced ideals if $n = 3$ (see Section 2.5.3) and already 4732 of them if $n = 4$, which makes it infeasible to list all of them here. For oriented Grassmannians, however, it is not difficult.

Proposition 2.6.3. *Let $\rho \colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. Assume that either*

(i) n is even and k is odd, or

(ii) n is odd and $k(n+k+2)/2$ is odd.

Then there exists a nonempty, open Γ–invariant subset $\Omega \subset \mathrm{Gr}^+(k,n)$ of the Grassmannian of oriented k–subspaces of $\mathbb{R}^n$, such that the action of Γ on Ω is properly discontinuous and cocompact.

Remarks 2.6.4.

(i) The domain Ω is not unique, unless n is even and $k \in \{1, n-1\}$.

(ii) In case (i) of Proposition 2.6.3, there is a cocompact domain of discontinuity also in the unoriented Grassmannian, and Ω is just the lift of one of these. The domains in case (ii) are new (see [Ste18]).

Proof. In the light of Theorem 2.4.1 it suffices to show that there is a w_0–balanced ideal in the set $\widetilde{W}/S$ where $S = \langle r(\Delta \setminus \{\alpha_k\})\rangle$. A w_0–balanced ideal exists if and only if the action of w_0 on $\widetilde{W}/S$ has no fixed points (see Lemma 2.2.28 and Lemma 2.2.31).

To see that w_0 has no fixed points on $\widetilde{W}/S$, observe that every equivalence class in $\widetilde{W}/S$ has a representative whose first k columns are either the standard basis vectors $e_{i_1}, \dots e_{i_k}$ or $-e_{i_1}, e_{i_2}, \dots, e_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$. So we can identify $\widetilde{W}/S$ with the set

$$\{\pm 1\} \times \{k\text{–element subsets of } \{1, \dots, n\}\}.$$

The action of w_0 on this is given by

$$(\varepsilon, \{i_1, \dots, i_k\}) \quad \mapsto \quad ((-1)^{k(k-1)/2 + \sum_j (i_j+1)}\varepsilon, \{n+1-i_k, \dots, n+1-i_1\}).$$

Only looking at the second factor, this can have no fixed points if n is even and k is odd, showing case (i). Otherwise, to get a fixed point it is necessary that $i_j + i_{k+1-j} = n+1$ for all $j \leq k$. But then

$$\frac{k(k-1)}{2} + \sum_{j=1}^{k} (i_j + 1) = \frac{k(k-1)}{2} + \frac{k}{2}(n+3) = \frac{k(n+k+2)}{2},$$

so w_0 fixes these elements if and only if $k(n+k+2)/2$ is even. Note that this number is always even if n and k are both even, which is why assuming n odd in case (ii) does not weaken the statement.

It remains to show that every $\Omega \in \mathrm{Gr}^+(k,n)$ constructed from a balanced ideal $I \subset \widetilde{W}/S$ is nonempty. Consider the lifts $\Omega' \subset \mathcal{F}_{\{1\}}$ of Ω and $I' \subset \widetilde{W}$ of I. Then Ω' is the domain in $\mathcal{F}_{\{1\}}$ given by I'. We will show that $\mathcal{F}_{\{1\}} \setminus \Omega'$ has covering codimension[1] at least 1, so Ω' must be nonempty. See [Nag83] for more background on dimension theory. In this case, we could also use the dimension of $\mathcal{F}_{\{1\}} \setminus \Omega'$ as a CW-complex, but the present approach has the benefit of generalizing to word hyperbolic groups with more complicated boundaries.

By Lemma 2.4.16 and the proof of Theorem 2.4.21, $\mathcal{F}_{\{1\}} \setminus \Omega'$ is homeomorphic to a fiber bundle over $\partial_\infty \Gamma \cong S^1$ with fiber $\bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}([1])$. The covering dimension is invariant under homeomorphisms and has the following locality property: If a metric space is decomposed into open sets of dimension (at most) k, then the whole space has dimension (at most) k [2]. Therefore, the dimension of this fiber bundle equals the dimension of a local trivialization, that is, the dimension of the product

[1] The (Lebesgue) covering dimension of a topological space X is the smallest number n such that every open cover of X admits a refinement with the property that each point of X is contained in at most $n+1$ of its elements.

[2] This follows from the equivalence of covering dimension and strong inductive dimension ([Nag83, Theorem II.7]) and locality of the strong inductive dimension.

$\mathbb{R} \times \bigcup_{[\![w]\!] \in I} C_{[\![w]\!]}([1])$. By [Mor77, Theorem 2], the dimension of a product is the sum of the dimensions of the factors whenever one of the factors is a CW complex [3]. Thus

$$\dim(\mathcal{F}_{\{1\}} \setminus \Omega') = 1 + \max_{w \in I'} \dim C_w([1]) = 1 + \max_{w \in I'} \ell(w).$$

If we know that $\ell(w) \leq \ell(w_0) - 2$ for every $w \in I'$, then, since $\dim \mathcal{F}_{\{1\}} = \ell(w_0)$, the codimension of $\mathcal{F}_{\{1\}} \setminus \Omega'$ is at least 1, so $\Omega \neq \varnothing$.

For $k < n$, if we write $w_k = r(\alpha_1) r(\alpha_2) \cdots r(\alpha_k)$, and $\widetilde{w_k} = r(\alpha_2) \cdots r(\alpha_k)$, then by direct calculation, one verifies that

$$w_0 = w_{n-1} \cdots w_1$$

and

$$w_0 r(\alpha_k) = w_{n-1} \cdots w_{k+1} \widetilde{w_k} w_{k-1} \cdots w_1$$

(see Section 2.5.2 for an explicit description of $\widetilde{W}$). These are reduced expressions in the $r(\alpha_i)$. So if $n \geq 3$, then $r(\alpha_k) \leq w_0 r(\alpha_k)$ by Proposition 2.2.18. Therefore, the balanced ideal $I' \subset \widetilde{W}$ cannot contain $w_0 r(\alpha_k)$ and thus no element of length $\ell(w_0) - 1$. $\square$

A special case of such cocompact domains of discontinuity for Hitchin representations $\rho\colon \Gamma \to \mathrm{PSL}(4n+3, \mathbb{R})$ is described by the balanced ideals in Section 2.5.5: Let $\widehat{\xi}\colon \partial_\infty \Gamma \to \mathcal{F}_{\{1\}}$ be the boundary map of ρ, with image in complete oriented flags in $\mathbb{R}^{4n+3}$. Then the domain in S^{4n+2} is obtained by removing the spherical projectivizations of the positive halves of $\widehat{\xi}(x)^{(2n+2)}$, $x \in \partial_\infty \Gamma$. Note that in the case of $\mathrm{PSL}(3, \mathbb{R})$, the result is not very interesting: Consider the base case $\partial_\infty \Gamma \xrightarrow{\rho_0} \mathrm{PSL}(2, \mathbb{R}) \xrightarrow{\iota} \mathrm{PSL}(3, \mathbb{R})$, where ρ_0 is Fuchsian and ι is the irreducible representation. Since the limit set of ρ_0 is the full circle, the domain simply consists of two disjoint disks, and the quotient is two disjoint copies of the surface S (compare Figure 1.2). In higher dimension however, the domain is always connected and dense in S^{4n+2}.

2.6.2 Connected components of B–Anosov representations in $\mathrm{SL}(n, \mathbb{R})$

Let n be odd and $k \leq n$. Let $\iota_k\colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(k, \mathbb{R})$ be the irreducible representation (see Section 2.6.1). Then we define

$$b_k\colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R}), \quad A \mapsto \begin{pmatrix} \iota_k(A) & 0 \\ 0 & \iota_{n-k}(A) \end{pmatrix}.$$

Let Γ be the fundamental group of a closed surface of genus at least 2 and $\rho\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ a Fuchsian (i.e. discrete and faithful) representation. Let $\overline{\rho}\colon \Gamma \to \mathrm{SL}(2, \mathbb{R})$

[3] In that paper, Katětov–Smirnov covering dimension is used, which coincides with (Lebesgue) covering dimension for normal spaces.

be a lift of ρ. We get every other lift of ρ as $\overline{\rho}^{\varepsilon}$, where $\varepsilon\colon \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism and $\overline{\rho}^{\varepsilon}(\gamma) = (-1)^{\varepsilon(\gamma)}\overline{\rho}(\gamma)$.

In this section, we will consider representations $\rho_k^{\varepsilon} = b_k \circ \overline{\rho}^{\varepsilon}$ obtained by composing a Fuchsian representation with b_k. Our main result is the following proposition and its corollary: For different choices of k and ε, the representations ρ_k^{ε} land in different connected components of Anosov representations.

Proposition 2.6.5. *The representation ρ_k^{ε} is B–Anosov and there exists $w_k \in T$ such that its limit map lifts to a continuous, equivariant map into $\mathcal{F}_{\{1,w_k^2\}}$ of transversality type $[\![w_k]\!]$. Thus ρ_k^{ε} is $P_{\{1,w_k^2\}}$–Anosov. The choice $R = \{1, w_k^2\}$ is minimal in the sense of Proposition 2.3.6.*

Futhermore, w_k is up to conjugation by elements of $\overline{M}$ given by

$$w_k = \begin{pmatrix} & & & J \\ & & \delta & \\ & K & & \\ L & & & \end{pmatrix}$$

with

$$\delta = \begin{cases} (-1)^{(k-1)/2}, & k \text{ odd} \\ (-1)^{(n-k-1)/2}, & k \text{ even} \end{cases}$$

and $J \in \mathrm{GL}(\frac{n-1}{2}, \mathbb{R})$, $K \in \mathrm{GL}(q-1, \mathbb{R})$, and $L \in \mathrm{GL}(\frac{Q-q+1}{2}, \mathbb{R})$ denoting blocks of the form

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad K = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ \iddots & & & \end{pmatrix} \quad L = (-1)^{Q-1}\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

where $q = \min(k, n-k)$ and $Q = \max(k, n-k)$.

Proof. To simplify the description of the limit map, we will first modify the block embedding. For any $\lambda > 1$ the map b_k maps $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ to

$$g_\lambda = \begin{pmatrix} \lambda^{k-1} & & & & & & \\ & \lambda^{k-3} & & & & & \\ & & \ddots & & & & \\ & & & \lambda^{1-k} & & & \\ & & & & \lambda^{n-k-1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda^{k+1-n} \end{pmatrix}.$$

Let $z \in \mathrm{SO}(n)$ be the permutation matrix (or its negative) such that the entries of $zg_\lambda z^{-1}$ are in decreasing order, and consider

$$\rho' = z\rho_k^\varepsilon z^{-1} = \iota \circ \overline{\rho}^\varepsilon,$$

where ι is the composition of b_k and conjugation by z. The representation ρ' is B–Anosov if and only ρ_k^ε is, and, since $\mathrm{SO}(n) \subset \mathrm{SL}(n,\mathbb{R})$ is connected, ρ' then lies in the same component of $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ as ρ_k^ε. So we can consider ρ' instead of ρ_k^ε.

We first show that ρ' is B–Anosov. By [BPS16, Theorem 8.4], it suffices to show that there exist positive constants c, d such that for every $\alpha \in \Delta$ and every element $\gamma \in \Gamma$, we have

$$\alpha(\mu_n(\rho'(\gamma))) \geq c|\gamma| - d, \tag{2.6.1}$$

where μ_n is the Cartan projection in $\mathrm{SL}(n,\mathbb{R})$ and $|\cdot|$ denotes the word length in Γ. It follows from the description in Section 2.6.1 that ι maps $\mathrm{SO}(2)$ into $\mathrm{SO}(n)$, and it maps $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ to $zg_\lambda z^{-1}$. Let α_0 denote the (unique) simple root for $\mathrm{SL}(2,\mathbb{R})$ and α_i the i–th simple root for $\mathrm{SL}(n,\mathbb{R})$. Then by the above, for $h \in \mathrm{SL}(2,\mathbb{R})$,

$$\alpha_i(\mu_n(\iota(h))) = \begin{cases} \frac{1}{2}\alpha_0(\mu_2(h)) & \text{if } \frac{n+1}{2} - q \leq i \leq \frac{n-1}{2} + q, \\ \alpha_0(\mu_2(h)) & \text{otherwise} \end{cases}$$

Since $\overline{\rho}^\varepsilon$ is Fuchsian and therefore Anosov, there are positive constants c_0, d_0 such that

$$\alpha_0(\mu_2(\overline{\rho}^\varepsilon(\gamma))) \geq c_0|\gamma| - d_0 \qquad \forall \gamma \in \Gamma.$$

This implies (2.6.1) with $c = c_0/2$ and $d = d_0/2$, so ρ' is B–Anosov.

The map ι maps $B_0^{\mathrm{SL}(2,\mathbb{R})}$ into $B_0^{\mathrm{SL}(n,\mathbb{R})}$ and -1 to some diagonal matrix $m = \iota(-1) \in \overline{M}$ with ± 1 entries. So $\iota(B) \subset B_0 \cup mB_0 = P_{\{1,m\}}$ and ι therefore induces smooth maps

$$\varphi\colon \mathbb{RP}^1 \to \mathcal{F}_{\{1,m\}} = G/P_{\{1,m\}}, \qquad \psi\colon S^1 \to \mathcal{F}_{\{1\}} = G/B_0$$

which are ι–equivariant. Let $\pi\colon \mathcal{F}_{\{1,m\}} \to \mathcal{F}_{\overline{M}}$ be the projection which forgets orientations, and let $\overline{\xi}\colon \partial_\infty\Gamma \to \mathbb{RP}^1$ be the limit map of $\overline{\rho}^\varepsilon$, a homeomorphism (which does not depend on ε). Then the curve $\xi = \pi \circ \varphi \circ \overline{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_{\overline{M}}$ is ρ'–equivariant and continuous. The definition of z ensures that it is also dynamics–preserving. So by [GGKW17, Remark 2.32b] ξ is the limit map of ρ', and $\widehat{\xi} = \varphi \circ \overline{\xi}$ is a continuous and equivariant lift to $\mathcal{F}_{\{1,m\}}$.

We now show that ξ does not lift to $\mathcal{F}_{\{1\}}^{\mathrm{SL}(n,\mathbb{R})}$. Write $\pi'\colon \mathcal{F}_{\{1\}} \to \mathcal{F}_{\overline{M}}$ and $p\colon S^1 \to \mathbb{RP}^1$ for the projections. Then $\pi \circ \varphi \circ p = \pi' \circ \psi$. Now assume that ξ lifts to $\mathcal{F}_{\{1\}}$. Then the curve $\xi \circ \overline{\xi}^{-1} = \pi \circ \varphi\colon \mathbb{RP}^1 \to \mathcal{F}_{\overline{M}}$ also lifts to some curve $\widehat{\varphi}\colon \mathbb{RP}^1 \to \mathcal{F}_{\{1\}}$, i.e. $\pi' \circ \widehat{\varphi} = \pi \circ \varphi$. So

$$\pi' \circ \widehat{\varphi} \circ p = \pi \circ \varphi \circ p = \pi' \circ \psi.$$

By right–multiplication with an element of $\overline{M}$ we can assume that $\widehat{\varphi}([1]) = [1]$. So $\widehat{\varphi}(p([1])) = [1] = \psi([1])$, and uniqueness of lifts implies that $\widehat{\varphi} \circ p = \psi$. But $p([1]) = p([-1])$, so then $[1] = \psi([1]) = \psi([-1]) = [m] \in \mathcal{F}_{\{1\}}$, which is false since either k or $n-k$ has to be even and therefore $m \in \overline{M} \setminus \{1\}$.

To calculate the transversality type, let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. Then $\iota(w) \in \widetilde{W}$ and we can easily compute the relative position of $\widehat{\xi}$ at the points $x = \overline{\xi}^{-1}([1])$ and $y = \overline{\xi}^{-1}([w])$. It is

$$\mathrm{pos}(\widehat{\xi}(x), \widehat{\xi}(y)) = \mathrm{pos}(\varphi([1]), \varphi([w])) = \mathrm{pos}([1], [\iota(w)]) = [\![\iota(w)]\!].$$

Now $\iota_k(w)$ and $\iota_{n-k}(w)$ are antidiagonal, with alternating ± 1 entries and starting with $+1$ in the upper right corner. Conjugation by z interlaces the two blocks in the following way: The resulting matrix is antidiagonal, the middle entry is assigned to the odd–sized block and going towards the corners from there, entries are assigned alternatingly to the two blocks for as long as possible. Combined with the remarks on conjugation by $\overline{M}$ at the beginning of Section 2.5.2 and careful bookkeeping, this proves the claim about the transversality type $[\![w_k]\!]$. Since $\iota(w)^2 = \iota(-1) = m$, we have $w_k^2 = m$ (recall from Remark 2.3.5 (ii) that w_k is well–defined up to conjugation with $\overline{M}$, which does not change the square since $\overline{M}$ is abelian). □

Corollary 2.6.6. *Let n be odd, $0 \leq k_1 \leq k_2 \leq \frac{n-1}{2}$ and $\rho_{k_1}^{\varepsilon_1}, \rho_{k_2}^{\varepsilon_2}$ be as in the previous proposition. If $\rho_{k_1}^{\varepsilon_1}$ and $\rho_{k_2}^{\varepsilon_2}$ are contained in the same connected component of* $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$*, then $k_1 = k_2$ and either $k_1 = k_2 = 0$ or $\varepsilon_1 = \varepsilon_2$.*

As a consequence, $\mathrm{Hom}_{B\text{–Anosov}}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ *has at least* $2^{2g-1}(n-1)+1$ *components.*

Proof. We saw before that ρ_k^{ε} is $P_{\{1,w_k^2\}}$–Anosov, with a limit map of transversality type $[\![w_k]\!]$, and that this is the minimal oriented parabolic for which ρ_k^{ε} is Anosov. By Corollary 2.3.13, if $\rho_{k_1}^{\varepsilon_1}$ and $\rho_{k_2}^{\varepsilon_2}$ were in the same connected component, then $[\![w_{k_1}]\!]$ and $[\![w_{k_2}]\!]$ would be conjugate by $\overline{M}$, which only occurs when if $k_1 = k_2$ by the discussion at the beginning of Section 2.5.2.

Now assume that $k_1 = k_2 = k \neq 0$ but $\varepsilon_1(\gamma) \neq \varepsilon_2(\gamma)$ for some $\gamma \in \Gamma$. Then $\overline{\rho}^{\varepsilon_1}(\gamma) = -\overline{\rho}^{\varepsilon_2}(\gamma)$, so one of them, say $\overline{\rho}^{\varepsilon_1}(\gamma)$, has two negative eigenvalues while the eigenvalues of $\overline{\rho}^{\varepsilon_2}(\gamma)$ are both positive. Then $\rho_k^{\varepsilon_1}(\gamma)$ has k (if k is even) or $n-k$ (if k is odd) negative eigenvalues, while $\rho_k^{\varepsilon_2}(\gamma)$ has only positive eigenvalues. But since $\rho(\gamma)$ has only real non–zero eigenvalues for every B–Anosov representation ρ, there can be no continuous path from $\rho_k^{\varepsilon_1}(\gamma)$ to $\rho_k^{\varepsilon_2}(\gamma)$ in this case.

In summary, we have $\frac{n-1}{2}$ different possible non–zero values for k and 2^{2g} different choices for ε (its values on the generators of Γ), giving $2^{2g-1}(n-1)$ connected components, plus the Hitchin component, $k = 0$. □

3 Partial cyclic orders

A partial cyclic order is a relation on triples which is analogous to a partial order, but generalizing a cyclic order instead of a linear order. The definition we use was introduced in 1982 by Novák [Nov82]. After stating this definition, we list some topological properties of partial cyclic orders which will be useful later for constructing and analyzing generalized Schottky groups. Then we describe two classes of examples. The first consists of Shilov boundaries of Hermitian symmetric spaces of tube type, while the second consists of complete oriented flags in $\mathbb{R}^n$.

3.1 Definition and properties

Definition 3.1.1. A *partial cyclic order* (PCO) on a set C is a relation $\to$ on triples in C satisfying, for any $a, b, c, d \in C$:

- if $\overrightarrow{abc}$, then $\overrightarrow{bca}$ (*cyclicity*).
- if $\overrightarrow{abc}$, then not $\overrightarrow{cba}$ (*asymmetry*).
- if $\overrightarrow{abc}$ and $\overrightarrow{acd}$, then $\overrightarrow{abd}$ (*transitivity*).

If in addition the relation satisfies:

- If a, b, c are distinct, then either $\overrightarrow{abc}$ or $\overrightarrow{cba}$ (*totality*),

then it is called a *total cyclic order*.

Let C, D be partially cyclically ordered sets.

Definition 3.1.2. A map $f : C \to D$ is called *increasing* if $\overrightarrow{abc}$ implies $\overrightarrow{f(a)f(b)f(c)}$. An automorphism of a partial cyclic order is an increasing map $f : C \to C$ with an increasing inverse. We will denote by G the group of all automorphisms of C.

Any subset $X \subset C$ such that the restriction of the partial cyclic order is a total cyclic order on X will be called a *cycle*. We will also use the term cycle for (ordered) tuples $(x_1, \ldots, x_n) \in C^n$ if the cyclic order relations between the points in C agree with the cyclic order given by the ordering of the tuple.

Definition 3.1.3. Let $a, b \in C$. The *interval* between a and b is the set $((a,b)) := \{x \in C \mid \overrightarrow{axb}\}$. The set of all intervals generates a natural topology on C under which automorphisms of the partial cyclic order are homeomorphisms. We call this topology the *interval topology* on C. We call C *first–countable* when its interval topology is first–countable. First countability is satisfied by all the examples discussed in this paper and will be useful when we want to use the sequential definition of continuity in our proofs.

The *opposite* of an interval $I = ((a,b))$ is the interval $((b,a))$, also denoted by $-I$.

Example 3.1.4. The circle S^1 admits a total cyclic order. The relation on triples is : $\overrightarrow{abc}$ whenever (a,b,c) are in counterclockwise order around the circle. The automorphism group of this cyclic order is the group of orientation preserving homeomorphisms of the circle.

Example 3.1.5. We can define a product cyclic order on the torus $S^1 \times S^1$. Define the relation to be $\overrightarrow{xyz}$ whenever $\overrightarrow{x_1y_1z_1}$ and $\overrightarrow{x_2y_2z_2}$. This is not a total cyclic order. Some intervals in this cyclically ordered space are shown in Figure 4.2.

Example 3.1.6. Every strict partial order $<$ on a set X induces a partial cyclic order in the following way: define $\overrightarrow{abc}$ if and only if either $a < b < c$, $b < c < a$, or $c < a < b$. The cyclic permutation axiom is automatic and the two other axioms follow from the antisymmetry and transitivity axioms of a partial order.

Example 3.1.7. Assume that a set Y is equipped with a partial cyclic order and we have a map $f : X \to Y$. Then it induces a pullback partial cyclic order on X, defined by $\overrightarrow{xyz}$ whenever $\overrightarrow{f(x)f(y)f(z)}$.

The key topological property that we will need in Section 4.2 to construct increasing limit curves for representations is a notion of completeness that we can associate to a space carrying a PCO.

Definition 3.1.8. A sequence $a_1, a_2, \ldots \in C$ is *increasing* if and only if $\overrightarrow{a_ia_ja_k}$ whenever $i < j < k$.
Equivalently, the map $a : \mathbb{N} \to C$ defined by $a(i) = a_i$ is increasing, where the cyclic order on $\mathbb{N}$ is given by $\overrightarrow{ijk}$ whenever $i < j < k$, $j < k < i$ or $k < i < j$ (as in Example 3.1.6).
A sequence $a_1, a_2, \ldots$ is *decreasing* if and only if $\overrightarrow{a_ka_ja_i}$ whenever $i < j < k$.
If an increasing (resp. decreasing) sequence a_n converges to a point x, we denote this by $a_n \nearrow x$ (resp. $a_n \searrow x$).

Definition 3.1.9. A partially cyclically ordered set C is *increasing–complete* if every increasing sequence converges to a unique limit in the interval topology.

Example 3.1.10. Consider the PCO on $S^1 \times S^1$ obtained by pulling back the total cyclic order on S^1 by the projection to the first factor. This PCO is not increasing–complete.

The following is a natural equivalence relation for increasing sequences.

Definition 3.1.11. Two increasing sequences a_n and b_m are called *compatible* if they admit subsequences a_{n_k} and b_{m_l} making the combined sequence $a_{n_1}, b_{m_1}, a_{n_2}, b_{m_2}, \dots$ increasing.

Lemma 3.1.12. *Let C be an increasing–complete partially cyclically ordered set, and let a_n and b_m be compatible increasing sequences. Then their limits agree.*

Proof. Any increasing sequence has a unique limit, and any subsequence of an increasing sequence therefore has the same unique limit.
The combined sequence (see the previous definition) is increasing, hence its unique limit must agree with the unique limits of both subsequences a_{n_k} and b_{m_l}. $\square$

We finish our list of definitions related to PCOs with two further topological properties which will be useful in Chapter 4.

Definition 3.1.13. A partially cyclically ordered set C is *proper* if for any increasing quadruple $(a, b, c, d) \in C^4$, we have $\overline{(\!(b, c)\!)} \subset (\!(a, d)\!)$. Here, "bar" denotes the closure in the interval topology.

Definition 3.1.14. Two points $a, b \in C$ in a partially cyclically ordered set C are called *comparable* if there exists a point $c \in C$ with either $\overrightarrow{acb}$ or $\overrightarrow{bca}$. The set of all points comparable to a is its *comparable set*, denoted $\mathcal{C}(a)$.

Definition 3.1.15. Let C be a partially cyclically ordered set. We call C *regular* if, for any $x \in C$, any increasing sequence $a_n \nearrow x$ and any decreasing sequence $b_n \searrow x$ satisfying $x \in (\!(a_1, b_1)\!)$, we have

$$\bigcap (\!(a_n, b_n)\!) = \{x\}$$

and

$$\bigcup (\!(b_n, a_n)\!) = \mathcal{C}(x).$$

Lemma 3.1.16. *Let C be a proper partially cyclically ordered space, $x \in C$ and $a_n \nearrow x$ an increasing sequence converging to x. Then $\overrightarrow{a_k a_l x}$ for $k \geq 2, l > k$.*

Proof. Since a_n is increasing, $a_{l+m} \in (\!(a_{l+1}, a_1)\!)$ for any $m \geq 2$. As a_n converges to x, properness implies that $x \in \overline{(\!(a_{l+1}, a_1)\!)} \subset (\!(a_l, a_2)\!)$. Thus the claim holds by transitivity. $\square$

3.2 Shilov boundaries of Hermitian symmetric spaces of tube type

In this section, we show that the Shilov boundary of a Hermitian symmetric space of tube type X admits a partial cyclic order invariant under the biholomorphism group of

X. Moreover, we prove that this partial cyclic order on the Shilov boundary satisfies the topological properties introduced in the previous section. Our main sources for background material are [FK94] and [CØ01]; we will not provide exact references for every statement.

Let V be a real Euclidean vector space. That is, V is equipped with a scalar product $\langle \cdot, \cdot \rangle$.

Definition 3.2.1. A *symmetric cone* $\Omega \subset V$ is an open convex cone which is self–dual and homogeneous. More precisely, the dual cone

$$\Omega^* := \{v \in V \mid \langle u, v \rangle > 0, \forall u \in \overline{\Omega} \backslash \{0\}\}$$

equals Ω itself, and the subgroup of $\mathrm{GL}(V)$ preserving Ω acts transitively on Ω.

A *tube type domain* is a domain of the form $X = V + i\Omega \subset V_{\mathbb{C}}$ in the complexification of V, where Ω is a symmetric cone. Let G be the group of biholomorphisms of X.

The vector space V admits a *Euclidean Jordan algebra* structure associated to the symmetric cone Ω. The two structures (symmetric cone and Euclidean Jordan algebra) determine each other [FK94, chapter III].

Definition 3.2.2. A Jordan algebra is a vector space V over $\mathbb{R}$ together with a bilinear product $(u, v) \mapsto uv \in V$ satifsying:

$$uv = vu$$

and

$$u(u^2v) = u^2(uv)$$

for all $u, v \in V$.

Definition 3.2.3. A Jordan algebra V is *Euclidean* if it admits an identity element e, and there exists a positive definite inner product $\langle \cdot, \cdot \rangle$ on V such that

$$\langle uv, w \rangle = \langle v, uw \rangle$$

for all $u, v, w \in V$. The *cone of squares* of V is

$$C = \{v^2 \mid v \in V\}.$$

The interior C° of C is a symmetric cone, and coincides with Ω for the Jordan algebra structure induced by Ω.

Example 3.2.4. Consider $V = \mathbb{R}^{2,1}$, a 3-dimensional real vector space with Lorentzian inner product $u \cdot v = u_1v_1 + u_2v_2 - u_3v_3$. The set $\Omega = \{v \in V \mid v \cdot v < 0, v_3 > 0\}$ of future-pointing timelike vectors is a symmetric cone. The Jordan algebra structure associated to this cone is given by the product:

$$(u_1, u_2, u_3)(v_1, v_2, v_3) = (u_1v_3 - u_3v_1, u_2v_3 - u_3v_2, u_1v_1 + u_2v_2 + u_3v_3).$$

Example 3.2.5. The set of $n \times n$ real symmetric matrices is a Jordan algebra with product $A \star B = (AB + BA)/2$. The corresponding symmetric cone is the cone of positive definite matrices.

There is a spectral theorem for Euclidean Jordan algebras :

Proposition 3.2.6 ([FK94] Theorem III.1.2)**.** *Let $v \in V$ with* $\dim(V) = k$. *Then, there exist unique real numbers $\lambda_1, \ldots, \lambda_k$, and a* Jordan frame *of primitive orthogonal idempotents $c_1, \ldots, c_k$ (that is, $c_i^2 = c_i$, $c_i c_j = 0$ for $i \neq j$, $\sum c_i = e$, and no c_i is the sum of two non-zero orthogonal idempotents) such that*

$$v = \lambda_1 c_1 + \cdots + \lambda_k c_k.$$

The λ_i are called the eigenvalues *of v.*

The decomposition in this theorem is obtained by considering the symmetric endomorphism

$$\begin{aligned} L(v) \colon \mathbb{R}[v] &\to \mathbb{R}[v] \\ x &\mapsto vx \end{aligned}$$

of the associative algebra $\mathbb{R}[v]$ generated by e and v. It gives rise to the *determinant*

$$\det(v) = \prod_i \lambda_i.$$

If v is regular, that is, $\min\{m > 0 \mid e, x, \ldots, x^m$ are linearly dependent$\}$ takes on its maximal value, this determinant coincides with the determinant of the linear map $L(v)$. The element v is invertible if and only if $\det(v) \neq 0$.

Definition 3.2.7. Let V be a vector space over $\mathbb{R}$ and $\Omega \subset V$ an open convex cone. Assume that whenever $0 \neq v \in \Omega$, Ω does not contain $-v$. Then the partial order $<_\Omega$ on V is defined by $x <_\Omega y$ if and only if $y - x \in \Omega$.

Remark 3.2.8. By self–duality, a symmetric cone cannot contain any inverse pair $\{v, -v\}$. Therefore, a Euclidean Jordan algebra carries a partial order induced by its associated symmetric cone.

The Cayley transform is the classical biholomorphic map which sends the upper half plane to the unit disk in $\mathbb{C}$. We will use the following generalization to Jordan algebras in order to define a bounded realization of tube type domains.

Definition 3.2.9. Let V be a real vector space equipped with a Euclidean Jordan algebra structure. Let $\mathsf{D} = \{z \in V_\mathbb{C} \mid z + ie$ is invertible$\}$, where e is the identity of the Jordan algebra and we extend the multiplication linearly to the complexification of V.
The *Cayley transform* is the map $p : \mathsf{D} \to V_\mathbb{C}$ defined by

$$p(v) = (v - ie)(v + ie)^{-1}.$$

Proposition 3.2.10 ([FK94], Theorem X.4.3)**.** *The Cayley transform p maps the tube type domain $X = V \oplus i\Omega$ biholomorphically onto a bounded domain $B \subset V_{\mathbb{C}}$, which we call the* bounded domain realization *of X (also known as the* Harish–Chandra realization*).*

Definition 3.2.11. If B is a bounded domain in $\mathbb{C}^n$, denote by $C(B)$ the set of continuous complex-valued functions on $\overline{B}$ which are holomorphic on B. The *Shilov boundary* $\mathcal{S}$ of B is the smallest closed subset of ∂B such that, for all $f \in C(B)$ we have

$$\max_{z \in \overline{B}} |f(z)| = \max_{z \in \mathcal{S}} |f(z)|.$$

By extension, the Shilov boundary of a tube type domain X is the Shilov boundary of its bounded domain realization. The action of the group G of biholomorphisms of B extends smoothly to its Shilov boundary.

For a tube type domain $X = V \oplus i\Omega$, not all of the Shilov boundary of its bounded domain realization is visible in $V_{\mathbb{C}}$.

Definition 3.2.12. Two points $x, y \in \mathcal{S}$ are called *transverse* if the pair $(x, y) \in \mathcal{S} \times \mathcal{S}$ belongs to the unique open G–orbit for the diagonal action. Equivalently, $\det(x - y) \neq 0$.

Proposition 3.2.13 ([FK94], Proposition X.2.3)**.** *The Cayley transform p maps the vector space V into the Shilov boundary $\mathcal{S}$ and $\overline{p(V)} = \mathcal{S}$. The image $p(V)$ consists of all points $x \in \mathcal{S}$ which are transverse to a fixed point that we denote by ∞.*

The next object we need to define is the generalized Maslov index. This index is a function on ordered triples of points in $\mathcal{S}$, invariant under G. It will be used in order to define a partial cyclic order on $\mathcal{S}$, extending the partial cyclic order induced by $<_\Omega$ on $p(V) \subset \mathcal{S}$. The generalized Maslov index is defined in [CØ01, Section 4].

Definition 3.2.14. Let $x, y, z \in \mathcal{S}$. Applying an element of G, we may assume $x, y, z \in p(V)$. Let $v_x, v_y, v_z \in V$ be the vectors which map respectively to x, y, z under the Cayley transform p. Then the *generalized Maslov index* of x, y, z is the integer

$$\mathsf{M}(x, y, z) := \mathsf{k}(v_y - v_x) + \mathsf{k}(v_z - v_y) + \mathsf{k}(v_x - v_z),$$

where $\mathsf{k}(v)$ is the difference between the number of positive eigenvalues of v and the number of negative eigenvalues of v in its spectral decomposition.

When x and y are transverse to z, we can equivalently map z to ∞ using an element of G and define

$$\mathsf{M}(x, y, \infty) = \mathsf{k}(v_y - v_x)$$

Proposition 3.2.15. *The Maslov index enjoys the following properties, for any pair-wise transverse triple $x, y, z, w \in \mathcal{S}$, $g \in G$, and $\sigma \in S_3$ a permutation:*

- *G–invariance :* $\mathsf{M}(gx, gy, gz) = \mathsf{M}(x, y, z)$.

- *Skew-symmetricity :* $\mathsf{M}(x_1, x_2, x_3) = \mathrm{sgn}(\sigma)\mathsf{M}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- *Cocycle identity :* $\mathsf{M}(y, z, w) - \mathsf{M}(x, z, w) + \mathsf{M}(x, y, w) - \mathsf{M}(x, y, z) = 0$.
- *Boundedness :* $|\mathsf{M}(x, y, z)| \leq \mathrm{rk}(X)$

These properties allow us to define a partial cyclic order on the Shilov boundary.

Proposition 3.2.16. *The relation $\overrightarrow{xyz}$ if and only if $\mathsf{M}(x, y, z) = \mathrm{rk}(X)$ defines a G–invariant partial cyclic order on $\mathcal{S}$.*

Proof. Since M is skew–symmetric, the relation automatically satisfies the first two axioms of a partial cyclic order. To prove the third axiom, assume $\mathsf{M}(x, y, z) = \mathsf{M}(x, z, w) = \mathrm{rk}(X)$. By the cocycle identity,

$$\mathsf{M}(y, z, w) - \mathsf{M}(x, z, w) + \mathsf{M}(x, y, w) - \mathsf{M}(x, y, z) = 0$$

and so

$$\mathsf{M}(y, z, w) + \mathsf{M}(x, y, w) = 2\mathrm{rk}(X)$$

which is only possible if $\mathsf{M}(y, z, w) = \mathsf{M}(x, y, w) = \mathrm{rk}(X)$. □

The partial cyclic order $\rightarrow$ is closely related with the causal structure on $\mathcal{S}$ introduced by Kaneyuki [Kan91]. Namely, whenever $\overrightarrow{xyz}$, there is a future-oriented timelike curve going through x, y, z in that order. Informally, y is in the intersection of the future of x and the past of z. The following two lemmas describe some immediate properties of cyclically ordered triples.

Lemma 3.2.17 ([Wie04], Lemma 5.5.4)**.** *Let $x, y, z \in \mathcal{S}$ with $\overrightarrow{xyz}$. Then x, y, z are pairwise transverse.*

Lemma 3.2.18. *Assume $x, y \in V$. Then, $\overrightarrow{xy\infty}$ if and only if $x <_\Omega y$.*

Proof. The cone Ω coincides with the region where $\mathsf{k}(v) = \mathrm{rk}(X)$. □

This lemma implies that $x, y \in \mathcal{S}$ are transverse if and only if they are comparable, i.e. there exists a third point z such that $\overrightarrow{xyz}$.

Remark 3.2.19. The interval topology on $\mathcal{S}$ is the same as the usual manifold topology. In particular, it is first–countable.

Proposition 3.2.20. *The PCO defined by $\rightarrow$ on $\mathcal{S}$ is increasing–complete, proper and regular.*

Proof. We first show that it is increasing–complete. Let $x_1, x_2, \dots$ be an increasing sequence in $\mathcal{S}$. Let $g \in G$ be such that $gx_2 = \infty$. Then, since we have $\overrightarrow{x_k x_{k+1} x_2}$ for all $k \geq 3$, the sequence $gx_3, gx_4, \dots$ is an increasing sequence transverse to ∞. Hence, there exist $v_3, v_4, \dots \in V$ with $p(v_k) = gx_k$.
This new sequence is increasing with respect to $<_\Omega$. Moreover, it is bounded since we have $\overrightarrow{(gx_k)(gx_1)(gx_2)}$ for all $k > 2$, so $v_k <_\Omega v_1$ where $p(v_1) = gx_1$. The tail of the

sequence is contained in $\overline{((v_3, v_1))}$ which is compact, so it has an accumulation point. If w, w' are two accumulations points of the sequence, let w_k, w'_k be subsequences converging respectively to each of them. Passing to subsequences if necessary, we can arrange so that $w_k <_\Omega w'_k$ for all k, and so $w'_k - w_k \in \Omega$. This implies $w' - w \in \overline{\Omega}$, and by the same argument we can also show $w - w' \in \overline{\Omega}$. Since Ω is a proper convex cone (in the sense of [FK94]), its closure does not contain any opposite pairs, so $w = w'$.

Now we turn to regularity of the PCO. Let $x \in \mathcal{S}$, $a_n \nearrow x$ an increasing and $b_n \searrow x$ a decreasing sequence. Furthermore, let $y \in \mathcal{S}$ be comparable to x. We have to show that $\bigcap ((a_n, b_n)) = \{x\}$ and $y \in \bigcup ((b_n, a_n))$.
Let $g \in G$ such that $gy = \infty$. After possibly deleting finitely many elements of the sequences, we may assume that a_n and b_n are transverse to y for all n. Then there are vectors $u, v_n, w_n \in V$ with $p(u) = gx, p(v_n) = ga_n, p(w_n) = gb_n$. By Lemma 3.1.16, $\overrightarrow{a_1 a_n x}$ for every $n > 1$, so

$$\mathsf{k}(v_n - v_1) + \mathsf{k}(u - v_n) + \mathsf{k}(v_1 - u) = \mathrm{rk}(X)$$

by definition of the generalized Maslov index. The three points a_1, a_n, x are pairwise transverse by Lemma 3.2.17, so $\det(v_n - v_1) \neq 0$, $\det(u - v_n) \neq 0$, $\det(v_1 - u) \neq 0$. Since the spectral decomposition depends continuously on the point ([FK94, Chapter 3]), this implies that $\mathsf{k}(v_n - v_1)$ and $\mathsf{k}(v_1 - u)$ cancel for n bigger than some constant n_0. Therefore, $\mathsf{k}(u - v_n)$ equals $\mathrm{rk}(X)$, so we have $\overrightarrow{\infty v_n u}$ and $u - v_n \in \Omega$ by Lemma 3.2.18. Analogously, we obtain $w_n - u \in \Omega$ for large n and consequently also $w_n - v_n \in \Omega$. Thus $\infty \in ((w_n, v_n))$, which was our first claim. Since $\mathsf{k}(v_n - w_n) = -\mathrm{rk}(X)$, the interval $((v_n, w_n))$ is the intersection of $v_n + \Omega$ and $w_n - \Omega$. In particular, this shows that $\bigcap ((v_n, w_n)) = \{u\}$.

Finally, we show that the PCO is proper. Let $(x_1, x_2, x_3, x_4) \in \mathcal{S}^4$ be a cycle. Using an element of G, we can assume that x_4 is ∞, so that $x_1, x_2, x_3 \in p(V)$. Let $v_i \in V$ be the vector such that $p(v_i) = x_i$ for $i = 1, 2, 3$. Now the cyclic relations $\overrightarrow{x_1 x_2 \infty}$ and $\overrightarrow{x_2 x_3 \infty}$ imply that both $v_2 - v_1$ and $v_3 - v_2$ lie in the cone Ω. The interval $((x_2, x_3))$ is therefore given by $p\left((v_2 + \Omega) \cap (v_3 - \Omega)\right)$. This implies the claim since $(v_2 + \Omega) \cap (v_3 - \Omega)$ is a relatively compact set in V whose closure is contained in $v_1 + \Omega$, which is mapped onto $((x_1, \infty))$ by p. $\square$

3.3 Complete oriented flags

We now consider the vector space $\mathbb{R}^n$, together with its standard basis and the induced orientation. Moreover, let $G = \mathrm{PSL}(n, \mathbb{R})$ and $B \subset G$ be the subgroup of upper triangular matrices.

Definition 3.3.1. A *complete flag* F in $\mathbb{R}^n$ is a sequence of nested subspaces

$$\{0\} \subset F^{(1)} \subset F^{(2)} \subset \ldots \subset F^{(n-1)} \subset \mathbb{R}^n,$$

where $\dim(F^{(i)}) = i$. For ease of notation, we sometimes include $F^{(0)} = \{0\}$ and $F^{(n)} = \mathbb{R}^n$.

The group G acts transitively on the space of complete flags. The stabilizer of the standard flag

$$< e_1 > \subset < e_1, e_2 > \subset \ \ldots \ \subset < e_1, \ldots, e_{n-1} > \tag{3.3.1}$$

is B, so the space of complete flags identifies with the homogeneous space G/B. We shall be interested in oriented flags. In terms of homogeneous spaces, this means that we consider the space G/B_0. At this point, the parity of the dimension n comes into play, since we have to quotient by -1 in even dimensions.

Definition 3.3.2. (i) A *complete oriented flag for* $\mathrm{SL}(n, \mathbb{R})$ is a complete flag in $\mathbb{R}^n$ together with a choice of orientation on each of the subspaces $F^{(i)}$, $1 \leq i \leq n-1$. The space of complete oriented flags for $\mathrm{SL}(n, \mathbb{R})$ will be denoted $\widehat{\mathcal{F}}_n$.

(ii) A *complete oriented flag for* $\mathrm{PSL}(n, \mathbb{R})$ is a complete flag in $\mathbb{R}^n$ together with a choice of orientation on each of the subspaces $F^{(i)}$, $1 \leq i \leq n-1$, up to simultaneously reversing all the odd-dimensional orientations if n is even. The space of complete oriented flags for $\mathrm{PSL}(n, \mathbb{R})$ will be denoted $\mathcal{F}_n$.

The extremal dimension $F^{(n)} = \mathbb{R}^n$ is always equipped with its standard orientation.

Remarks 3.3.3. (i) The reason we shall be working with $\mathrm{PSL}(n, \mathbb{R})$ instead of $\mathrm{SL}(n, \mathbb{R})$ in even dimension is not quite obvious at this point. It lies in the construction of the partial cyclic order later on. While it is possible to obtain a partial cyclic order on $\widehat{\mathcal{F}}_n$ using a similar definition, it has certain undesirable properties and is not suitable for constructing Schottky groups.

(ii) Our notation for oriented flag manifolds in Section 3.3 and Section 4.6 differs from the notation introduced in Section 2.2.1. $\mathcal{F}_n$ corresponds to $\mathcal{F}_{\{1\}}^{\mathrm{PSL}(n,\mathbb{R})}$, while $\widehat{\mathcal{F}}_n$ corresponds to $\mathcal{F}_{\{1\}}^{\mathrm{SL}(n,\mathbb{R})}$.

Again, it is easy to see that G acts transitively on $\mathcal{F}_n$. Letting $(e_1, \ldots, e_i)$ be a positive basis of the i-dimensional part, the standard flag (3.3.1) is a complete oriented flag. Its stabilizer is B_0, yielding the identification

$$\mathcal{F}_n = G/B_0.$$

The natural map $G \to G/B_0$ sends any element $g \in G$ to the image of the standard flag under g. In other words, $gB_0 \in G/B_0$ is the complete oriented flag F_g such that the first i columns of g form an oriented basis for $F_g^{(i)}$ (up to simultaneously changing all odd-dimensional orientations if n is even).

We will use matrices to denote elements of $\mathrm{PSL}(n, \mathbb{R})$ even though they are technically equivalence classes comprising two matrices if n is even.

3.3.1 Oriented transversality

The first notion we require before we can define the partial cyclic order on oriented flags is an oriented version of transversality for flags. This notion appears under the name *2–hyperconvexity* in [Gui05] and in the unoriented setting in [Lab06]. We will need direct sums of oriented subspaces, so we first fix some notation.

Definition 3.3.4. Let $V, W \subset \mathbb{R}^n$ be oriented subspaces.

- If V, W agree as oriented subspaces, we write $V \overset{\pm}{=} W$.
- $-V$ denotes the same subspace with the opposite orientation.
- If V and W are transverse, we interpret $V \oplus W$ as an oriented subspace by equipping it with the orientation induced by a positive basis of V followed by a positive basis of W.

Note that oriented direct sums depend on the ordering of the summands:

$$V \oplus W \overset{\pm}{=} (-1)^{\dim(V)\dim(W)} W \oplus V$$

Remark 3.3.5. We use negation to denote transformations of different spaces, and have to be careful not to confuse them: On a fixed oriented Grassmannian, it denotes the involution inverting all orientations. On the space $\widehat{\mathcal{F}}_n$ however, for even n, it denotes the induced action of -1 which inverts all odd-dimensional orientations.

Definition 3.3.6. Let $F_1, F_2 \in \mathcal{F}_n$ be complete oriented flags.

- If n is odd, the pair (F_1, F_2) is called *oriented transverse* if, for every $1 \leq i \leq n-1$, we have
$$F_1^{(i)} \oplus F_2^{(n-i)} \overset{\pm}{=} \mathbb{R}^n.$$
- If n is even, the pair (F_1, F_2) is called *oriented transverse* if there exist lifts $\widehat{F}_1, \widehat{F}_2 \in \widehat{\mathcal{F}}_n$ such that for every $1 \leq i \leq n-1$, we have
$$\widehat{F}_1^{(i)} \oplus \widehat{F}_2^{(n-i)} \overset{\pm}{=} \mathbb{R}^n.$$

  We then call the pair $(\widehat{F}_1, \widehat{F}_2)$ a *consistently oriented lift* of (F_1, F_2).

Clearly, the left action of $\mathrm{PSL}(n, \mathbb{R})$ preserves oriented transversality if n is odd. The same is true when n is even since -1 preserves the orientation of $\mathbb{R}^n$.We also note that if n is even and $F_1, F_2 \in \mathcal{F}_n$ is an oriented transverse pair, there are exactly two consistently oriented lifts: If $(\widehat{F}_1, \widehat{F}_2)$ is one such pair, $(-\widehat{F}_1, -\widehat{F}_2)$ is the other option. The following property of oriented transversality is immediate and will be important when defining the partial cyclic order.

Lemma 3.3.7. *Oriented transversality of pairs in $\mathcal{F}_n$ is symmetric.*

Proof. Let (F_1, F_2) be an oriented transverse pair in $\mathcal{F}_n$, and let $(\widehat{F}_1, \widehat{F}_2)$ be a consistently oriented lift to $\widehat{\mathcal{F}}_n$ (if n is odd, $\widehat{F}_i = F_i$). For each i, we have

$$\widehat{F}_1^{(i)} \oplus \widehat{F}_2^{(n-i)} \overset{+}{=} \mathbb{R}^n.$$

It follows that

$$\widehat{F}_2^{(n-i)} \oplus \widehat{F}_1^{(i)} \overset{+}{=} (-1)^{i(n-i)}\mathbb{R}^n.$$

If n is odd, $(-1)^{i(n-i)}$ is always equal to 1, so (F_2, F_1) is oriented transverse. If n is even, $(-1)^{i(n-i)}$ is equal to $(-1)^i$. Considering the lifts $-\widehat{F}_2, \widehat{F}_1$ thus shows that (F_2, F_1) is oriented transverse. □

An example of an oriented transverse pair of flags, which will serve as our standard example, is given by $F_e = eB_0$, the identity coset, and

$$F_{w_0} = w_0 B_0 = \begin{pmatrix} & & & & \iddots \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} B_0.$$

The notation w_0 comes from the fact that this matrix is a representative for w_0, the longest element of the Weyl group in $\mathrm{PSL}(n, \mathbb{R})$. We will consider w_0 to be an element of $\mathrm{PSL}(n, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{R})$, as needed. Oriented transversality follows from the fact that all minors of w_0 obtained using the last k rows and the first k columns are positive.

Lemma 3.3.8. *The left action of* $\mathrm{PSL}(n, \mathbb{R})$ *on oriented transverse pairs in* $\mathcal{F}_n$ *is transitive. The stabilizer of* (F_e, F_{w_0}) *is given by* A_0*, where* $A \subset G$ *is the subgroup of all diagonal matrices.*

Proof. Let (F_1, F_2) be an oriented transverse pair in $\mathcal{F}_n$. Since the action of $\mathrm{PSL}(n, \mathbb{R})$ on $\mathcal{F}_n$ is transitive, we may assume that $F_2 = F_{w_0}$. Then, by oriented transversality, we can write $F_1 = gB_0$ with

$$g = \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix}.$$

This lower triangular, unipotent representative is unique. The stabilizer of F_{w_0} under the left action of $\mathrm{PSL}(n, \mathbb{R})$ comprises all lower triangular matrices with positive diagonal entries (in even dimension, this should be interpreted as "diagonal entries with the same sign"). In particular, it contains an element mapping F_1 to F_e. Since the stabilizer of F_e under left multiplication is B_0, the stabilizer of the pair (F_e, F_{w_0}) is A_0, as claimed. □

Since our description of oriented transversality in even dimension is based on choosing lifts to $\widehat{\mathcal{F}}_n$, it will be useful to describe some basic properties of oriented transversality in $\widehat{\mathcal{F}}_n$.

Definition 3.3.9. Let $\widehat{F}_1, \widehat{F}_2 \in \widehat{\mathcal{F}}_n$ be complete oriented flags for $\mathrm{SL}(n,\mathbb{R})$. The pair $(\widehat{F}_1, \widehat{F}_2)$ is called *oriented transverse* if we have

$$\widehat{F}_1^{(i)} \oplus \widehat{F}_2^{(n-i)} \stackrel{\pm}{=} \mathbb{R}^n$$

for all $1 \leq i \leq n-1$.

Note that the elements of $\widehat{\mathcal{F}}_n$ represented by the identity matrix and w_0 are oriented transverse. They will be our standard oriented transverse pair in $\widehat{\mathcal{F}}_n$. The following lemma shows how symmetry of oriented transversality fails in $\widehat{\mathcal{F}}_n$.

Lemma 3.3.10. *Let n be even. If $(\widehat{F}_1, \widehat{F}_2)$ is an oriented transverse pair in $\widehat{\mathcal{F}}_n$, then $(-\widehat{F}_2, \widehat{F}_1)$ is oriented transverse, and $(-\widehat{F}_1, \widehat{F}_2)$ is not.*

Proof. For each i, we have

$$\widehat{F}_2^{(n-i)} \oplus \widehat{F}_1^{(i)} \stackrel{\pm}{=} (-1)^{i(n-i)} \widehat{F}_1^{(i)} \oplus \widehat{F}_2^{(n-i)} \stackrel{\pm}{=} (-1)^{i(n-i)} \mathbb{R}^n.$$

The sign is negative iff i or equivalently $n-i$ is odd. Therefore, $(-\widehat{F}_2, \widehat{F}_1)$ is oriented transverse.
To see that $(-\widehat{F}_1, \widehat{F}_2)$ is not oriented transverse, consider any splitting $\widehat{F}_1^{(i)} \oplus \widehat{F}_2^{(n-i)}$ where i is odd. □

Finally, by the same proof as for Lemma 3.3.8, we obtain

Lemma 3.3.11. *The left action of $\mathrm{SL}(n,\mathbb{R})$ on oriented transverse pairs in $\widehat{\mathcal{F}}_n$ is transitive. Let $\widehat{F}_e, \widehat{F}_{w_0} \in \widehat{\mathcal{F}}_n$ be represented by the identity matrix and w_0, respectively. Then the stabilizer of $(\widehat{F}_e, \widehat{F}_{w_0})$ consists of all diagonal matrices with positive diagonal entries.*

3.3.2 Oriented 3–hyperconvexity

The following property of triples of flags is the core of the partial cyclic order we are going to define. This is an oriented version of Fock–Goncharov triple positivity [FG06] and Labourie 3–hyperconvexity [Lab06].

Definition 3.3.12. Let (F_1, F_2, F_3) be a triple in $\mathcal{F}_n$.

- If n is odd, (F_1, F_2, F_3) is called *oriented* 3*–hyperconvex* if, for every triple of integers $0 \leq i_1, i_2, i_3 \leq n-1$ satisfying $i_1 + i_2 + i_3 = n$,

$$F_1^{(i_1)} \oplus F_2^{(i_2)} \oplus F_3^{(i_3)} \stackrel{\pm}{=} \mathbb{R}^n.$$

- If n is even, (F_1, F_2, F_3) is called *oriented* 3*–hyperconvex* if there exist lifts $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3 \in \widehat{\mathcal{F}}_n$ such that, for every triple of integers $0 \leq i_1, i_2, i_3 \leq n-1$ satisfying $i_1 + i_2 + i_3 = n$,

$$\widehat{F}_1^{(i_1)} \oplus \widehat{F}_2^{(i_2)} \oplus \widehat{F}_3^{(i_3)} \stackrel{\pm}{=} \mathbb{R}^n.$$

  We call the triple $(\widehat{F}_1, \widehat{F}_2, \widehat{F}_3)$ a *consistently oriented lift* of (F_1, F_2, F_3).

Since oriented 3–hyperconvexity for triples of oriented flags is the only notion of hyperconvexity appearing in this work, we will simply call such triples *hyperconvex.* Note that allowing one of the i_j to vanish automatically includes oriented transversality in the definition. Like oriented transversality, hyperconvexity is invariant under the action of $\mathrm{PSL}(n,\mathbb{R})$ on $\mathcal{F}_n$. Moreover, in even dimension, if (F_1,F_2,F_3) is a hyperconvex triple, it has exactly two consistently oriented lifts which are related by applying -1 to all its elements.
We can now describe the orbits of hyperconvex triples under $\mathrm{PSL}(n,\mathbb{R})$. For a triple (F_1,F_2,F_3) to be hyperconvex, the three pairs (F_1,F_2), (F_1,F_3) and (F_2,F_3) have to be oriented transverse. Therefore, we can normalize such that $(F_1,F_3)=(F_e,F_{w_0})$. Then, since (F_2,F_{w_0}) is an oriented transverse pair, we have $F_2=gB_0$ with the unique representative of the form

$$g=\begin{pmatrix}1 & & \\ & \ddots & \\ * & & 1\end{pmatrix}.$$

Hyperconvexity of the triple has a nice description in terms of this representative. It requires the notion of total positivity, so we quickly recap the definition and the notation we use.

Notation. *Let $M\in\mathrm{Mat}(n,\mathbb{R})$ be a $n\times n$-matrix. Then for any $1\le k\le n$ and indices $1\le i_1<\ldots<i_k\le n$, $1\le j_1<\ldots<j_k\le n$, we denote by*

$$M\left[\begin{smallmatrix} i_1\ldots i_k\\ j_1\ldots j_k\end{smallmatrix}\right]:=(M_{i_l j_m})_{\substack{1\le l\le k\\ 1\le m\le k}}$$

the submatrix determined by these indices, and by

$$M^{i_1\ldots i_k}_{j_1\ldots j_k}:=\det\left(M\left[\begin{smallmatrix} i_1\ldots i_k\\ j_1\ldots j_k\end{smallmatrix}\right]\right)$$

the corresponding minor. Indices will be separated by commas when necessary to avoid confusion.

Definition 3.3.13. Let $M\in\mathrm{Mat}(n,\mathbb{R})$ be a $(n\times n)$-matrix. Then M is *totally positive* if all minors of M are positive.
If M is either upper or lower triangular, we will call M *(triangular) totally positive* if all minors that do not vanish by triangularity are positive. Explicitly, if M is upper (resp. lower) triangular, the minors to consider are determined by indices $i_1,\ldots i_k,\ j_1,\ldots j_k$ such that $i_l\le j_l\ \forall l$ (resp. $i_l\ge j_l$).
An element of $\mathrm{PSL}(n,\mathbb{R})$ is called totally positive if it has a lift to $\mathrm{SL}(n,\mathbb{R})$ which is totally positive.

Lemma 3.3.14. *Let $F\in\mathcal{F}_n$ be a complete oriented flag such that (F_e,F,F_{w_0}) is a hyperconvex triple. Then F has a (unique) matrix representative which is unipotent, lower triangular and totally positive. Conversely, if F has such a representative, the triple is hyperconvex.*

Proof. Assume that (F_e, F, F_{w_0}) is hyperconvex. If n is even, let $\widehat{F}_e, \widehat{F}_{w_0} \in \widehat{\mathcal{F}}_n$ be the lifts defined by the identity matrix and w_0, and let $\widehat{F} \in \widehat{\mathcal{F}}_n$ be such that $(\widehat{F}_e, \widehat{F}, \widehat{F}_{w_0})$ is a consistently oriented lift (if n is odd, $\widehat{F} = F$). Let $M \in \mathrm{SL}(n, \mathbb{R})$ be a matrix representative for $\widehat{F}$. Then the conditions on M are as follows: Let i_1, i_2, i_3 be a triple of nonnegative integers satisfying $i_1 + i_2 + i_3 = n$. The oriented direct sum condition of Definition 3.3.12 means that the matrix composed of the first i_1 columns of the identity, the first i_2 columns of M and the first i_3 columns of w_0 (in that order) has positive determinant. We write I_j for the $j \times j$ identity matrix and

$$J_j = \begin{pmatrix} & & & \iddots \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

for the $j \times j$ antidiagonal matrix with alternating entries ± 1, starting with $+1$ in the lower left corner. The matrix we want to analyze has the form

$$\begin{pmatrix} I_{i_1} & * & \\ & M\left[\begin{smallmatrix} i_1+1 \dots i_1+i_2 \\ 1 \dots i_2 \end{smallmatrix}\right] & \\ & * & J_{i_3} \end{pmatrix},$$

where the stars are irrelevant for calculating the determinant. Since J_j has determinant 1 for any value of j, hyperconvexity of the triple is equivalent to

$$M_{1 \dots i_2}^{i_1+1 \dots i_1+i_2} > 0 \qquad \forall i_1 \geq 0,\ i_2 \geq 1,\ i_1 + i_2 \leq n. \tag{3.3.2}$$

We observed before that the representative M can be chosen uniquely to be lower triangular and unipotent. By [Pin10, Theorem 2.8], positivity of all minors using consecutive rows and the first columns is sufficient to conclude that this representative is (triangular) totally positive.
The converse direction follows immediately since total positivity of the representative implies (3.3.2). □

Corollary 3.3.15. *The stabilizer in* $\mathrm{PSL}(n, \mathbb{R})$ *of a positive triple is trivial.*

Proof. Since the action of $\mathrm{PSL}(n, \mathbb{R})$ on oriented transverse pairs is transitive, we can assume that the triple is of the form (F_e, F, F_{w_0}). We saw earlier that the stabilizer of the pair (F_e, F_{w_0}) is A_0, the set of diagonal matrices with positive entries (up to -1 if n is even). Furthermore, $F = gB_0$ where g is lower triangular, unipotent and totally positive. For $a \in A_0$, the unique such representative of the image flag aF is given by aga^{-1}. By total positivity of g, all entries below the diagonal are in particular nonzero. Therefore, $aga^{-1} = g$ iff a is the identity. □

We sometimes have to deal with matrix representatives for complete oriented flags F forming a hyperconvex triple (F_e, F, F_{w_0}) which are not lower triangular. For easy referencing, let us restate the general characterization we derived along the proof of Lemma 3.3.14.

Definition 3.3.16. Let $M \in \mathrm{SL}(n, \mathbb{R})$ be a matrix. A minor of M of size k is called *left–bound* if it uses the first k columns. It is called *connected* if it uses a set of k consecutive rows and k consecutive columns.

Lemma 3.3.17. *Let $F = gB_0 \in \mathcal{F}_n$ be a complete oriented flag. Then the following are equivalent:*

(i) (F_e, F, F_{w_0}) *is hyperconvex.*

(ii) $g \in \mathrm{PSL}(n, \mathbb{R})$ *has a lift* $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ *such that all left–bound connected minors of* $\widehat{g}$ *are positive.*

(iii) $g \in \mathrm{PSL}(n, \mathbb{R})$ *has a lift* $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ *such that all left–bound minors of* $\widehat{g}$ *are positive.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was observed in (3.3.2). To show that (i) implies (iii), note that the sign of a left–bound minor is independent of the choice of representative g for F:
If $b \in B_0$ and $g' = gb$ is another representative, let $\widehat{b} \in \mathrm{SL}(n, \mathbb{R})$ be the lift with positive diagonal entries. Right–multiplication by $\widehat{b}$ and by $\widehat{b}^{-1}$ multiplies each left–bound minor with a positive scalar, thus g satisfies (iii) if and only if g' does.
We saw in Lemma 3.3.14 that the representative can be chosen to be lower triangular, unipotent, totally positive. So in particular, all left–bound minors are positive. $\square$

3.3.3 Multiindex notation and the Cauchy–Binet formula

We now introduce some notation for multiindices which will make the statement of many formulas involving minors simpler and more readable.

Definition 3.3.18. Let $k, n \in \mathbb{N}$ be two integers. Then we write

$$\mathcal{I}(k, n) := \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < \ldots < i_k \leq n\}$$

for the set of multiindices with k entries in increasing order from $\{1, \ldots, n\}$.

Elements $\mathbf{i} \in \mathcal{I}(k, n)$ will be used to denote the rows or columns determining a minor: In combination with our previous notation, we can now write

$$M\left[\begin{smallmatrix} \mathbf{i} \\ \mathbf{j} \end{smallmatrix}\right] = M\left[\begin{smallmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{smallmatrix}\right]$$

for submatrices and

$$M^{\mathbf{i}}_{\mathbf{j}} = M^{i_1 \ldots i_k}_{j_1 \ldots j_k}$$

for minors.
The reason for using (ordered) multiindices instead of (unordered) k-subsets is that it makes them easier to compare.

Definition 3.3.19. Let $\mathbf{i} = (i_1, \dots, i_k),\ \mathbf{j} = (j_1, \dots, j_k) \in \mathcal{I}(k, n)$. There is a partial order on $\mathcal{I}(k, n)$, defined by

$$\mathbf{i} \leq \mathbf{j} \Leftrightarrow i_l \leq j_l\ \forall l.$$

The *absolute value* of a multiindex is the sum of its components,

$$|\mathbf{i}| = \sum_l i_l.$$

The partial order on multiindices is particularly useful when working with triangular matrices. As mentioned earlier, if a matrix M is upper (resp. lower) triangular, then all minors $M^{\mathbf{i}}_{\mathbf{j}}$ with $\mathbf{i} > \mathbf{j}$ (resp. $\mathbf{i} < \mathbf{j}$) vanish automatically, and we call M totally positive if $M^{\mathbf{i}}_{\mathbf{j}} > 0\ \forall \mathbf{i} \leq \mathbf{j}$ (resp. $\mathbf{i} \geq \mathbf{j}$).
As a first example of this notation in use, let us state the Cauchy–Binet formula. It describes how to calculate the determinant of a product of non–square matrices in terms of the minors of these matrices, and will play a central role later on. Using multiindices emphasizes the formal similarity to ordinary matrix multiplication (see for example [Tao12, (3.14)] for a proof).

Lemma 3.3.20 (Cauchy–Binet)**.** *Let M be a $(m \times r)$-matrix and N a $(r \times m)$-matrix. Then, we have*

$$\det(MN) = \sum_{\mathbf{k} \in \mathcal{I}(m,r)} M^{1\dots m}_{\mathbf{k}} N^{\mathbf{k}}_{1\dots m}.$$

Note that the formula includes the case $m > r$. Then $\det(MN)$ vanishes and, since $\mathcal{I}(m, r)$ is empty, the (empty) sum equals 0 as well.
Our use of the formula lies in the calculation of minors of the product of two matrices: If $k \leq m$ and $\mathbf{i}, \mathbf{j} \in \mathcal{I}(k, m)$ are multiindices, we obtain

$$(MN)^{\mathbf{i}}_{\mathbf{j}} = \sum_{\mathbf{k} \in \mathcal{I}(k,m)} M^{\mathbf{i}}_{\mathbf{k}} N^{\mathbf{k}}_{\mathbf{j}}. \tag{3.3.3}$$

As an immediate consequence of the Cauchy–Binet formula, one obtains the well–known fact that totally positive matrices form a semigroup.

Lemma 3.3.21. *Let $M, N \in \mathrm{Mat}(n, \mathbb{R})$ be totally positive. Then MN is totally positive as well. If both M and N are upper (resp. lower) triangular and totally positive, then MN is upper (resp. lower) triangular and totally positive.*

Proof. Let M, N be totally positive. Then equation (3.3.3) expresses any minor $(MN)^{\mathbf{i}}_{\mathbf{j}}$ as a sum of positive summands.
If M, N are both upper (resp. lower) triangular, then the same is true for MN, and for any two multiindices $\mathbf{i} \leq \mathbf{j} \in \mathcal{I}(k, n)$ (resp. $\mathbf{i} \geq \mathbf{j}$), we have

$$(MN)^{\mathbf{i}}_{\mathbf{j}} = \sum_{\mathbf{k} \in \mathcal{I}(k,n)} M^{\mathbf{i}}_{\mathbf{k}} N^{\mathbf{k}}_{\mathbf{j}} = \sum_{\substack{\mathbf{i} \leq \mathbf{k} \leq \mathbf{j} \\ \text{or } \mathbf{i} \geq \mathbf{k} \geq \mathbf{j}}} M^{\mathbf{i}}_{\mathbf{k}} N^{\mathbf{k}}_{\mathbf{j}}.$$

Since this sum is not empty, MN is totally positive as well. □

3.3.4 The partial cyclic order on complete oriented flags

It turns out that hyperconvexity satisfies the axioms of a partial cyclic order on $\mathcal{F}_n$. We will assume that $n > 1$, since the space $\mathcal{F}_1$ consists of only two points.

Proposition 3.3.22. *The relation $\mathcal{R} \subset (\mathcal{F}_n)^3$ defined by*

$$(F_1, F_2, F_3) \in \mathcal{R} \quad \Leftrightarrow \quad (F_1, F_2, F_3) \text{ is hyperconvex}$$

is a partial cyclic order.

Proof. Assume that $(F_1, F_2, F_3) \in \mathcal{R}$. If n is even, let $(\widehat{F}_1, \widehat{F}_2, \widehat{F}_3)$ denote a consistently oriented lift.
We first check that the relation is asymmetric. Let us start with the odd–dimensional case, since it does not involve choices of lifts. For any i_1, i_2, i_3, we have

$$F_1^{(i_1)} \oplus F_2^{(i_2)} \oplus F_3^{(i_3)} \stackrel{\pm}{=} \mathbb{R}^n$$

and therefore

$$F_3^{(i_3)} \oplus F_2^{(i_2)} \oplus F_1^{(i_1)} \stackrel{\pm}{=} (-1)^{i_1(n-i_1)+i_2 i_3}\mathbb{R}^n.$$

Whenever i_2 and i_3 are both odd, we get the negative sign, showing that $(F_3, F_2, F_1) \notin \mathcal{R}$.
If n is even, we want to show that the triple (F_3, F_2, F_1) does not admit a consistently oriented lift. Assume for the sake of contradiction that such a lift exists. Without loss of generality, it contains $\widehat{F}_1$. Then by Lemma 3.3.10, the pairs $(-\widehat{F}_2, \widehat{F}_1)$ and $(-\widehat{F}_3, \widehat{F}_1)$ are oriented transverse, so the lift of the triple must contain $-\widehat{F}_2$ and $-\widehat{F}_3$. But another application of Lemma 3.3.10 shows that $(-\widehat{F}_3, -\widehat{F}_2)$ is not oriented transverse, a contradiction.

Now we turn to cyclicity. Again, we first treat the case of odd dimension. Let i_1, i_2, i_3 be integers such that $i_1 + i_2 + i_3 = n$. Then we have

$$F_2^{(i_2)} \oplus F_3^{(i_3)} \oplus F_1^{(i_1)} \stackrel{\pm}{=} (-1)^{i_1(i_2+i_3)} F_1^{(i_1)} \oplus F_2^{(i_2)} \oplus F_3^{(i_3)} \stackrel{\pm}{=} (-1)^{i_1(n-i_1)}\mathbb{R}^n.$$

As n is odd, $i_1(n - i_1)$ is always even, and we conclude that $(F_2, F_3, F_1) \in \mathcal{R}$.
If n is even, the same calculation with $\widehat{F}_i$ instead of F_i yields a negative sign whenever i_1 is odd. Therefore, $(\widehat{F}_2, \widehat{F}_3, -\widehat{F}_1)$ is a consistently oriented lift of (F_2, F_3, F_1).

Finally, we prove transitivity by establishing a link with totally positive matrices and using their structure as a semigroup. Assume that we have a fourth flag $F_4 \in \mathcal{F}_n$ such that (F_1, F_3, F_4) is a hyperconvex triple. By Lemma 3.3.7 and Lemma 3.3.8, we can use the $\mathrm{PSL}(n, \mathbb{R})$ action to normalize $F_1 = F_{w_0}$ and $F_2 = F_e$. Then, by oriented transversality with F_{w_0}, we have $F_3 = g_3 B_0, F_4 = g_4 B_0$, where the representatives $g_3, g_4 \in \mathrm{PSL}(n, \mathbb{R})$ can be chosen to be unipotent and lower triangular. Cyclicity implies that the triple (F_e, F_3, F_{w_0}) is hyperconvex, so Lemma 3.3.14 shows that g_3 is totally positive. Now consider the left-action of g_3^{-1} on $\mathcal{F}_n$. It maps the triple (F_{w_0}, F_3, F_4) to $(F_{w_0}, F_e, g_3^{-1}F_4)$. $g_3^{-1}F_4$ is represented by $g_3^{-1}g_4$, thus another application of cyclicity and Lemma 3.3.14 shows that $g_3^{-1}g_4$ is totally positive. Since totally positive matrices form a semigroup, we conclude that $g_4 = g_3(g_3^{-1}g_4)$ is totally positive as well. The triple $(F_{w_0}, F_e, F_4) = (F_1, F_2, F_4)$ is therefore hyperconvex and the proof is complete. □

We will adopt the terminology and notation for partial cyclic orders introduced in Section 3.1: Positivity of triples will be denoted by $\overrightarrow{F_1F_2F_3}$. A k–tuple $(F_1, \dots, F_k)$ is a *cycle* or *increasing* if we have $\overrightarrow{F_iF_jF_k}$ for every subtriple with $i < j < k$. During the proof of the previous proposition, we obtained the following useful characterization of cycles.

Lemma 3.3.23. *Let (F_e, F_1, F_2, F_{w_0}) be a cycle, and let $F_1 = g_1B_0, F_2 = g_2B_0$ be represented by lower triangular, unipotent, totally positive matrices $g_1, g_2 \in \mathrm{PSL}(n,\mathbb{R})$. Then we have $g_2 = g_1h$, where $h \in \mathrm{PSL}(n,\mathbb{R})$ is lower triangular, unipotent and totally positive.*

If (F_1, F_3) is an oriented transverse pair, the interval between them is

$$(\!(F_1, F_3)\!) = \{F \in \mathcal{F}_n \mid \overrightarrow{F_1FF_3}\}.$$

In Lemma 3.3.14, we saw that the interval $(\!(F_e, F_{w_0})\!)$ is given by all lower triangular, unipotent, totally positive matrices. This gives a useful parametrization of the standard interval $(\!(F_e, F_{w_0})\!)$. For any matrix $\widehat{g} \in \mathrm{SL}(n,\mathbb{R})$, write

$$\chi_{k,i}(\widehat{g}) = \widehat{g}^{\,i\dots i+k-1}_{1\dots k}, \qquad k \le n, i \le n-k+1$$

for the size k connected left–bound minor starting at line i, and consider the function

$$\begin{aligned} \widehat{\chi}_k \colon\ \mathrm{SL}(n,\mathbb{R}) &\to \mathbb{S}(\mathbb{R}^{n-k+1}) \\ \widehat{g} &\mapsto [\chi_{k,1}(\widehat{g}) : \dots : \chi_{k,n-k+1}(\widehat{g})], \end{aligned}$$

where $\mathbb{S}$ denotes the spherical projectivization (modding out by positive scalars). It is invariant under right–multiplication with B_0, so it induces a map

$$\widehat{\chi}_k \colon \widehat{\mathcal{F}}_n \to \mathbb{S}(\mathbb{R}^{n-k+1}).$$

Multiplication by $-I$ inverts all coordinates if k is odd. For any value of k, we write

$$\chi_k \colon \mathcal{F}_n \to \mathbb{P}(\mathbb{R}^{n-k+1})$$

for the induced map with image in the projectivization.

Proposition 3.3.24. *The map*

$$\begin{aligned} \chi \colon (\!(F_e, F_{w_0})\!) &\to \prod_{k=1}^{n-1} \mathbb{P}(\mathbb{R}^{n-k+1}) \\ gB_0 &\mapsto (\chi_1(g), \dots, \chi_{n-1}(g)) \end{aligned}$$

is a smooth embedding. Its image is $\prod_{k=1}^{n-1} \mathbb{P}(\mathbb{R}_{>0}^{n-k+1})$*, where* $R_{>0}^m$ *is the cone with all coordinates positive.*

Proof. By Lemma 3.3.14, every flag $F = gB_0 \in ((F_e, F_{w_0}))$ has a unique lower triangular, unipotent, totally positive representative. The matrix topology on these representatives agrees with the topology on $\mathcal{F}_n = G/B_0$. All minors are polynomials in the matrix entries, so χ is smooth. Its image lies in $\prod_{k=1}^{n-1} \mathbb{P}(\mathbb{R}_{>0}^{n-k+1})$ since there are lower triangular, unipotent, totally positive representatives, and it is injective by uniqueness of these representatives. We claim that there is a smooth inverse

$$\varphi\colon \prod_{k=1}^{n-1} \mathbb{P}(\mathbb{R}_{>0}^{n-k+1}) \to ((F_e, F_{w_0})).$$

To see this, let $(x_1, \ldots, x_{n-1}) \in \prod_{k=1}^{n-1} \mathbb{P}(\mathbb{R}_{>0}^{n-k+1})$. We construct a lower triangular, unipotent matrix M as follows: The first set of homogeneous coordinates, $x_1 = [1 : x_{1,2} : \ldots : x_{1,n}]$, determines the first column of M. Then $x_2 = [1 : x_{2,2} : \ldots : x_{2,n-1}]$ determines the entries of the second column of M inductively by rational functions in $x_{1,i}$ and $x_{2,i}$. For example,

$$M_{32} = \frac{(M_{21}M_{32} - M_{31}) + M_{31}}{M_{21}} = \frac{x_{2,2} + x_{1,3}}{x_{1,2}}$$

and

$$M_{42} = \frac{x_{2,3} + x_{1,4}M_{32}}{x_{1,3}}.$$

In the same way, the remaining columns of M are determined by $x_3, \ldots, x_{n-1}$. The resulting matrix M is (lower triangular) totally positive by [Pin10, Theorem 2.8]. □

Remark 3.3.25. The coordinates given by left–bound connected minors are a special set of Plücker coordinates, which can be used to parametrize Grassmannians. We shortly review them in Section 4.6.1 and use them to define a metric on $((F_e, F_{w_0}))$.

In particular, this proposition implies that intervals are homeomorphic to open balls. The choice of lower triangular, unipotent, totally positive representatives corresponds to the affine parametrization where the first connected left–bound minor of each size is 1. The remaining minors are then coordinates for $((F_e, F_{w_0}))$ and identify it with the cone $\prod_{k=1}^{n-1} \mathbb{R}_{>0}^{n-k}$.

It will be useful later on to have a similar description of the opposite interval $((F_{w_0}, F_e))$. In order to obtain this description, we first associate an involution $\tau(F_1, F_2)$ to any oriented transverse pair (F_1, F_2). This involution will fix F_1 and F_2 and reverse the PCO, thereby providing a kind of symmetry for increasing and decreasing sequences. It is the analogue of the antisymplectic involution σ_{PQ} considered in Section 4.5.

Definition 3.3.26. Let $t \in \mathrm{PSL}(n,\mathbb{R})$ be the diagonal matrix with alternating ± 1 entries,

$$t = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

The involution $\tau(F_e, F_{w_0})$ is defined by

$$\begin{aligned} \tau(F_e, F_{w_0}) : \mathcal{F}_n &\to \mathcal{F}_n \\ gB_0 &\mapsto tgtB_0. \end{aligned}$$

If (F_1, F_2) is an oriented transverse pair, pick an element $h \in \mathrm{PSL}(n,\mathbb{R})$ mapping (F_e, F_{w_0}) to (F_1, F_2) and define

$$\tau(F_1, F_2) = h\left(\tau(F_e, F_{w_0})\right)h^{-1}.$$

First of all, we observe that $tB_0t = B_0$, so $\tau(F_e, F_{w_0})$ is well–defined. Furthermore, if $d \in A_0 = \mathrm{Stab}(F_e, F_{w_0})$,

$$d\tau(F_e, F_{w_0})d^{-1}(gB_0) = dtd^{-1}gtB_0 = tgtB_0.$$

Consequently, the involution $\tau(F_1, F_2)$ does not depend on the choice of h in the definition above, but only on the pair (F_1, F_2). Different involutions are related by

$$k\tau(F_1, F_2)k^{-1} = \tau(kF_1, kF_2)$$

for any element $k \in \mathrm{PSL}(n,\mathbb{R})$.

Lemma 3.3.27. *Let (F_1, F_2) be an oriented transverse pair in $\mathcal{F}_n$ and $F \in (\!(F_1, F_2)\!)$. Then $\tau(F_1, F_2)(F) \in (\!(F_2, F_1)\!)$.*

Proof. We first show that it is enough to consider $(F_1, F_2) = (F_e, F_{w_0})$. Let $h \in \mathrm{PSL}(n,\mathbb{R})$ be such that $(hF_e, hF_{w_0}) = (F_1, F_2)$. Then $h^{-1}(\!(F_1, F_2)\!) = (\!(F_e, F_{w_0})\!)$, $\tau(F_1, F_2) = h\tau(F_e, F_{w_0})h^{-1}$, and

$$\tau(F_1, F_2)(F) \in (\!(F_2, F_1)\!) \Leftrightarrow \tau(F_e, F_{w_0})(h^{-1}F) \in (\!(F_{w_0}, F_e)\!).$$

Next, we determine the set of conditions for a flag $F = gB_0$ to lie in $(\!(F_{w_0}, F_e)\!)$. This is completely analogous to Lemma 3.3.14, and we use the same notation.
First, we treat the case when n is odd. Then hyperconvexity of the triple (F_{w_0}, F, F_e) is equivalent to

$$\det \begin{pmatrix} & * & I_{i_3} \\ & g\left[\begin{smallmatrix} i_1+1\ldots i_1+i_2 \\ 1\ldots i_2 \end{smallmatrix}\right] & \\ J_{i_1} & * & \end{pmatrix} > 0$$

for any $i_1, i_2, i_3 \geq 0$ with $i_1 + i_2 + i_3 = n$. This is in turn equivalent to

$$(-1)^{i_1 i_2} g_{1\ldots i_2}^{i_1+1\ldots i_1+i_2} > 0. \tag{3.3.4}$$

Now observe that $\tau(F_e, F_{w_0})(gB_0) = tgtB_0$, and the representative tgt is obtained from g by negating all even columns and all even rows. Therefore, every minor changes by a sign determined by the number of even columns and even rows in the submatrix $g\left[\begin{smallmatrix} i_1+1\ldots i_1+i_2 \\ 1\ldots i_2 \end{smallmatrix}\right]$. If either of i_1 or i_2 is even, this submatrix has the same number of even columns and even rows. If both i_1 and i_2 are odd, there is one more even row. From this count, we obtain

$$(\tau(F_e, F_{w_0})(g))_{1\ldots i_2}^{i_1+1\ldots i_1+i_2} = (-1)^{i_1 i_2} g_{1\ldots i_2}^{i_1+1\ldots i_1+i_2}. \tag{3.3.5}$$

We can now combine (3.3.2), (3.3.4) and (3.3.5) to conclude

$$\begin{aligned} gB_0 \in (\!(F_e, F_{w_0})\!) &\Leftrightarrow g_{1\ldots i_2}^{i_1+1\ldots i_1+i_2} > 0 \qquad\qquad \forall i_1 \geq 0, i_2 \geq 1 \\ &\Leftrightarrow (-1)^{i_1 i_2} (tgt)_{1\ldots i_2}^{i_1+1\ldots i_1+i_2} > 0 \\ &\Leftrightarrow \tau(F_e, F_{w_0})(gB_0) \in (\!(F_{w_0}, F_e)\!). \end{aligned}$$

If n is even, let $\widehat{F}_{w_0}, \widehat{F}_e \in \widehat{\mathcal{F}}_n$ be the lifts determined by $w_0 \in \mathrm{SL}(n, \mathbb{R})$ and the identity matrix. Then $(\widehat{F}_{w_0}, -\widehat{F}_e)$ is an oriented transverse pair. Hyperconvexity of the triple (F_{w_0}, F, F_e) is equivalent to the existence of a lift $\widehat{F}$ such that $(\widehat{F}_{w_0}, \widehat{F}, -\widehat{F}_e)$ is a consistently oriented lift of the triple. Let $M \in \mathrm{SL}(n, \mathbb{R})$ be a matrix representative for $\widehat{F}$. Then we obtain

$$\det \begin{pmatrix} & * & -I_{i_3} \\ & M\left[\begin{smallmatrix} i_1+1\ldots i_1+i_2 \\ 1\ldots i_2 \end{smallmatrix}\right] & \\ J_{i_1} & * & \end{pmatrix} > 0$$

for any $i_1, i_2, i_3 \geq 0$ with $i_1 + i_2 + i_3 = n$. Simplifying a bit (and using the fact that n is even now) shows that this is again equivalent to

$$(-1)^{i_1 i_2} M_{1\ldots i_2}^{i_1+1\ldots i_1+i_2} > 0. \tag{3.3.6}$$

Now the same arguments as in the case of odd dimension apply to the representatives in $\mathrm{SL}(n, \mathbb{R})$. □

Corollary 3.3.28. *Let (F_1, F_2) be an oriented transverse pair in $\mathcal{F}_n$. Then $\tau(F_1, F_2)$ reverses the partial cyclic order.*

Proof. Let (F_3, F_4) be another oriented transverse pair. To improve readability, for the proof of this corollary, we set $\tau_{1,2} := \tau(F_1, F_2)$ and $\tau_{3,4} := \tau(F_3, F_4)$. We want to show that $\tau_{1,2}(\!(F_3, F_4)\!) = (\!(\tau_{1,2}F_4, \tau_{1,2}F_3)\!)$. By the previous lemma, we have

$$\tau_{1,2}(\!(F_3, F_4)\!) = \tau_{1,2}\tau_{3,4}(\!(F_4, F_3)\!).$$

The composition $\tau_{1,2}\tau_{3,4}$ is realized by the action of an element of $\mathrm{PSL}(n,\mathbb{R})$: If $h,k \in \mathrm{PSL}(n,\mathbb{R})$ are chosen such that $(hF_e, hF_{w_0}) = (F_1, F_2)$ and $(kF_e, kF_{w_0}) = (F_3, F_4)$, we know that

$$\tau_{1,2}\tau_{3,4}(gB_0) = ktk^{-1}hth^{-1}gttB_0 = (ktk^{-1}hth^{-1})gB_0.$$

Therefore,

$$\tau_{1,2}\tau_{3,4}(\!(F_4, F_3)\!) = (\!(\tau_{1,2}\tau_{3,4}F_4, \tau_{1,2}\tau_{3,4}F_3)\!) = (\!(\tau_{1,2}F_4, \tau_{1,2}F_3)\!).$$

□

Corollary 3.3.29. *Let $F \in \mathcal{F}_n$ be a complete oriented flag such that (F_{w_0}, F, F_e) is a hyperconvex triple. Then F has a (unique) matrix representative $M \in \mathrm{SL}(n,\mathbb{R})$ which is unipotent, lower triangular and satisfies*

$$(-1)^{|\mathbf{i}|+|\mathbf{j}|}M_{\mathbf{j}}^{\mathbf{i}} > 0 \qquad \forall \mathbf{i} \geq \mathbf{j} \in \mathcal{I}(k,n)\ \forall k \leq n$$

Conversely, if F has such a representative, the triple is hyperconvex.

Proof. Let F be as stated above. By Lemma 3.3.27 and Lemma 3.3.14, we know that $\tau(F_e, F_{w_0})(F) \in (\!(F_e, F_{w_0})\!)$ has a representative $M' \in \mathrm{SL}(n,\mathbb{R})$ which is lower triangular, unipotent, and totally positive. Interpreting t as an element of $\mathrm{GL}(n,\mathbb{R})$, $tM't \in \mathrm{SL}(n,\mathbb{R})$ is a representative for F. It is lower triangular, unipotent, and the sign of each minor $(tM't)_{\mathbf{j}}^{\mathbf{i}}$ is determined by the number of even rows and columns. Letting k denote the size of the minor,

$$\begin{aligned}(-1)^{|\{\text{even rows}\}|+|\{\text{even columns}\}|} &= (-1)^{k-|\{\text{odd rows}\}|+k-|\{\text{odd columns}\}|}\\ &= (-1)^{|\{\text{odd rows}\}|+|\{\text{odd columns}\}|}\\ &= (-1)^{|\mathbf{i}|+|\mathbf{j}|}.\end{aligned}$$

The converse direction follows immediately from the proof of Lemma 3.3.27: The inequalities (3.3.4) (or (3.3.6)) are special cases of the more general form given in this corollary. □

We now prove that the partial cyclic order on $\mathcal{F}_n$ satisfies some of the topological properties introduced in Section 3.1. The only one that remains, regularity, is satisfied as well, but the proof requires more work and is postponed to the next section.

Proposition 3.3.30. *The partial cyclic order on $\mathcal{F}_n$ determined by hyperconvexity is increasing–complete and proper.*

Proof. We first prove properness. Using the action of $\mathrm{PSL}(n,\mathbb{R})$, we can bring an arbitrary 4–cycle into the form (F_e, F, F', F_{w_0}). We want to show that $\overline{(\!(F, F')\!)} \subset (\!(F_e, F_{w_0})\!)$. Let $F = gB_0$ and $F' = g'B_0$, where the representatives $g, g' \in \mathrm{PSL}(n,\mathbb{R})$ are chosen to be lower triangular, unipotent, and totally positive (see Lemma 3.3.14). Let $F_m = g_mB_0 \in (\!(F, F')\!)$ be a sequence converging to a flag F_∞, with lower triangular, unipotent, totally positive representatives g_m. By transitivity of the PCO, we know that (F_e, F, F_m, F_{w_0}) and (F_e, F_m, F', F_{w_0}) are cycles. Therefore, Lemma 3.3.23

shows that $g_m = gh_m$ and $g' = g_m h'_m$ for some lower triangular, unipotent, totally positive matrices $h_m, h'_m \in \mathrm{PSL}(n,\mathbb{R})$. In the case of even dimension, let $\widehat{g} \in \mathrm{SL}(n,\mathbb{R})$ be the totally positive lift, and analogous for all the other representatives (if n is odd, $\widehat{g} = g$). Then it is not hard to see that minors of $\widehat{g}, \widehat{g}_n, \widehat{g}'$ are ordered the same way as the flags: For multiindices $\mathbf{i} \geq \mathbf{j} \in \mathcal{I}(k,n)$, we have

$$\widehat{g}^{\mathbf{i}}_{\mathbf{j}} \leq (\widehat{g}_n)^{\mathbf{i}}_{\mathbf{j}} \leq (\widehat{g}')^{\mathbf{i}}_{\mathbf{j}}$$

(see Lemma 3.3.31 for a proof). In fact, strict inequality holds unless $\mathbf{i} = \mathbf{j}$, in which case all three minors are equal to 1. Choosing singletons in the inequalities above bounds the matrix entries of $\widehat{g}_n$ between those of $\widehat{g}$ and $\widehat{g}'$. Since the sequence F_n was assumed to converge to F_∞, we conclude that the matrix entries of $\widehat{g}_n$ converge and F_∞ is represented by a lower triangular, unipotent matrix $\widehat{g}_\infty \in \mathrm{SL}(n,\mathbb{R})$. Furthermore, all minors $(\widehat{g}_\infty)^{\mathbf{i}}_{\mathbf{j}}$ lie between the corresponding minors of $\widehat{g}$ and $\widehat{g}'$, so $\widehat{g}_\infty$ is totally positive. This shows that $F_\infty \in (\!(F_e, F_{w_0})\!)$ and completes the proof of properness.

For increasing–completeness, assume that $F_1, F_2, \ldots$ is an increasing sequence. Let us normalize so that $F_2 = F_{w_0}$ and $F_3 = F_e$. Since we have $\overrightarrow{F_{w_0} F_e F_m}$ for every $n \neq 2, 3$, we can pick representatives $F_m = g_m B_0$ which are lower triangular, unipotent, and totally positive. Moreover, by Lemma 3.3.23, we have $g_{n+1} = g_m h_m$ for $m \geq 3$ and $g_1 = g_m h_{m,1}$, where h_m and $h_{m,1}$ are lower triangular, unipotent, totally positive matrices. Lifting to $\mathrm{SL}(n,\mathbb{R})$ as before, this implies that the entries below the diagonal of $\widehat{g}_m$, $m > 4$ are bounded between the corresponding entries of $\widehat{g}_4$ and $\widehat{g}_1$, and they are strictly increasing in m. Therefore, there exists a unique limit $F_\infty = \lim_{m\to\infty} F_m$. □

Lemma 3.3.31. *Let $A, B \in \mathrm{Mat}(n,\mathbb{R})$ be lower triangular, unipotent, totally positive matrices. Let $\mathbf{i} \geq \mathbf{j} \in \mathcal{I}(k,n)$ be two multiindices. Then we have $(AB)^{\mathbf{i}}_{\mathbf{j}} \geq \max(A^{\mathbf{i}}_{\mathbf{j}}, B^{\mathbf{i}}_{\mathbf{j}})$. The inequality is strict unless $\mathbf{i} = \mathbf{j}$.*

Proof. The Cauchy–Binet formula yields

$$(AB)^{\mathbf{i}}_{\mathbf{j}} = \sum_{\mathbf{k}\in\mathcal{I}(k,n)} A^{\mathbf{i}}_{\mathbf{k}} B^{\mathbf{k}}_{\mathbf{j}} = \sum_{\mathbf{i}\geq\mathbf{k}\geq\mathbf{j}} A^{\mathbf{i}}_{\mathbf{k}} B^{\mathbf{k}}_{\mathbf{j}}.$$

All summands are positive, hence we obtain the following lower bound by only considering the two summands where $\mathbf{k} = \mathbf{i}$ or $\mathbf{k} = \mathbf{j}$:

$$\begin{aligned} \text{If } \mathbf{i} = \mathbf{j}: &\quad (AB)^{\mathbf{i}}_{\mathbf{i}} = A^{\mathbf{i}}_{\mathbf{i}} = B^{\mathbf{i}}_{\mathbf{i}} = 1 \\ \text{If } \mathbf{i} \neq \mathbf{j}: &\quad (AB)^{\mathbf{i}}_{\mathbf{j}} \geq A^{\mathbf{i}}_{\mathbf{j}} + B^{\mathbf{i}}_{\mathbf{j}} \end{aligned}$$

This proves the claim. □

We finish this section with a characterization of totally positive and totally nonnegative matrices in terms of the image of the standard interval.

Lemma 3.3.32. *Let $g \in \mathrm{PSL}(n,\mathbb{R})$. Then g fixes F_{w_0} and maps F_e inside $(\!(F_e, F_{w_0})\!)$ if and only if g is lower triangular and totally positive. Similarly, g fixes F_e and maps F_{w_0} inside $(\!(F_e, F_{w_0})\!)$ if and only if g is upper triangular and totally positive.*

Proof. If g is lower triangular and totally positive, it fixes F_{w_0} and maps F_e into $(\!(F_e, F_{w_0})\!)$ by Lemma 3.3.14.
Conversely, assume that $g(F_{w_0}) = F_{w_0}$ and $g(F_e) \in (\!(F_e, F_{w_0})\!)$. Let $g' \in \mathrm{PSL}(n, \mathbb{R})$ be the lower triangular, unipotent, totally positive representative of $g(F_e)$. Then $g(F_e, F_{w_0}) = g'(F_e, F_{w_0})$, so g and g' differ only by right–multiplication with an element of $\mathrm{Stab}(F_e, F_{w_0})$, which preserves both triangularity and total positivity by Lemma 3.3.8.

If g is upper triangular and totally positive, it fixes F_e. Let $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ be the totally positive lift. Then

$$(\widehat{g}w_0)^{i+1\dots i+j}_{1\dots k} = \widehat{g}^{i+1\dots i+k}_{n-k+1\dots n} > 0$$

for any i, k with $i + k \leq n$. Since $\widehat{g}w_0$ is a representative for $g(F_{w_0})$, this implies that $g(F_{w_0}) \in (\!(F_e, F_{w_0})\!)$ by Lemma 3.3.17.
Conversely, assume that $g(F_e) = F_e$ and $g(F_{w_0}) \in (\!(F_e, F_{w_0})\!)$. Since $(F_e, g(F_{w_0}))$ is an oriented transverse pair, we can pick a representative in $\mathrm{SL}(n, \mathbb{R})$ for $g(F_{w_0})$ of the form

$$M = \begin{pmatrix} & * & & \iddots \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

with positive left–bound minors. Then $g'' = (-1)^{n+1} M w_0$ is upper triangular, unipotent, and has positive right–bound minors. Since reflecting at the antidiagonal preserves determinants, [Pin10, Theorem 2.8] shows that g'' is (upper triangular) totally positive. We have $g''(F_e, F_{w_0}) = g(F_e, F_{w_0})$, so as before, g must be upper triangular and totally positive as well. □

Proposition 3.3.33. *Let $g \in \mathrm{PSL}(n, \mathbb{R})$. Then g is totally nonnegative if and only if $g(\!(F_e, F_{w_0})\!) \subset (\!(F_e, F_{w_0})\!)$. Moreover, g is totally positive if and only if $\overline{g(\!(F_e, F_{w_0})\!)} \subset (\!(F_e, F_{w_0})\!)$.*

Proof. We first show how the inclusion statements imply total nonnegativity resp. total positivity. By Lemma 3.3.17, elements of $(\!(F_e, F_{w_0})\!)$ are characterized by the fact that they admit representatives in $\mathrm{SL}(n, \mathbb{R})$ such that all left–bound minors are positive. Let $F \in (\!(F_e, F_{w_0})\!)$ and $M \in \mathrm{SL}(n, \mathbb{R})$ be such a representative for F. Since $gF \in (\!(F_e, F_{w_0})\!)$, there is a lift $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ of g such that $\widehat{g}M$ has positive left–bound minors. As the interval $(\!(F_e, F_{w_0})\!)$ is connected (see Proposition 3.3.24), this lift is independent of the choice of F. By the Cauchy–Binet formula, we have

$$(\widehat{g}M)^{i+1\dots i+k}_{1\dots k} = \sum_{\mathbf{j} \in \mathcal{I}(k,n)} \widehat{g}^{i+1\dots i+k}_{\mathbf{j}} M^{\mathbf{j}}_{1\dots k}, \qquad i + k \leq n, \tag{3.3.7}$$

so the sum on the right hand side must be positive. We will show that for any fixed $\mathbf{j}_0 \in \mathcal{I}(k, n)$, there exists a totally positive M such that $M^{\mathbf{j}_0}_{1\dots k}$ is arbitrarily large compared to the other left–bound size k minors. A totally positive matrix in particular represents a flag in $(\!(F_e, F_{w_0})\!)$. Thus all minors $\widehat{g}_{\mathbf{j}}^{i+1\dots i+k}$ must be nonnegative by (3.3.7), which implies that $\widehat{g}$ is totally nonnegative by [Pin10, Proposition 2.7].

Observe that since totally positive matrices are dense in totally nonnegative matrices [Pin10, Theorem 2.6], it is enough to find a totally nonnegative matrix M with this property. Define the submatrix $M\left[\begin{smallmatrix}\mathbf{j}_0\\ 1\dots k\end{smallmatrix}\right]$ to be any totally positive $(k \times k)$–matrix and fill M up by zeroes. Then all other left–bound size k minors vanish and M is totally nonnegative, so it fits our criteria. This finishes the implication "inclusion $\Rightarrow$ totally nonnegative".

Now assume that $\overline{g(\!(F_e, F_{w_0})\!)} \subset (\!(F_e, F_{w_0})\!)$. Then, since $gF_e, gF_{w_0} \in (\!(F_e, F_{w_0})\!)$, the minors

$$(\widehat{g}I)_{1\dots k}^{i+1\dots i+k} = \widehat{g}_{1\dots k}^{i+1\dots i+k} \qquad \text{and} \qquad (\widehat{g}w_0)_{1\dots k}^{i+1\dots i+k} = \widehat{g}_{n-k+1\dots n}^{i+1\dots i+k}$$

must be strictly positive. By [Pin10, Proposition 2.5], this is sufficient to conclude that $\widehat{g}$ is totally positive.

Conversely, let $g \in \mathrm{PSL}(n, \mathbb{R})$ be totally nonnegative and $F \in (\!(F_e, F_{w_0})\!)$. Let $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ be the totally nonnegative lift and $M \in \mathrm{SL}(n, \mathbb{R})$ a representative for F. Up to replacing M by $-M$, all of its left–bound minors are positive by Lemma 3.3.17. Among the minors $\widehat{g}_{\mathbf{j}}^{i+1\dots i+k}$, $\mathbf{j} \in \mathcal{I}(k, n)$, there must be one which is positive since $\widehat{g}$ is nonsingular. Therefore, (3.3.7) shows that left–bound connected minors of $\widehat{g}M$ are positive, so $gF \in (\!(F_e, F_{w_0})\!)$.

If g is totally positive, we use [Pin10, Theorem 2.10] to decompose it as $g = LDU$, where L is lower triangular, unipotent, totally positive, U is upper triangular, unipotent, totally positive and D is diagonal with positive diagonal entries (or rather entries of the same sign, since we work in $\mathrm{PSL}(n, \mathbb{R})$). Then U stabilizes F_e and sends F_{w_0} into $(\!(F_e, F_{w_0})\!)$ by Lemma 3.3.32. D stabilizes $(\!(F_e, F_{w_0})\!)$ by Cauchy–Binet. L sends F_e into $(\!(F_e, F_{w_0})\!)$ by Lemma 3.3.14, fixes F_{w_0} and thus sends $DU(F_{w_0})$ into $(\!(L(F_e), F_{w_0})\!)$. In particular, $(F_e, g(F_e), g(F_{w_0}), F_{w_0})$ is a cycle, so $\overline{g(\!(F_e, F_{w_0})\!)} \subset (\!(F_e, F_{w_0})\!)$ by properness. $\square$

Corollary 3.3.34. *Let $I = (\!(F, F')\!)$ be an interval. Then* $\mathrm{Stab}(I) = \mathrm{Stab}(F) \cap \mathrm{Stab}(F')$.

Proof. We may assume that $I = (\!(F_e, F_{w_0})\!)$. Let $g \in \mathrm{Stab}(I)$. We want to show that g is diagonal with all diagonal entries of the same sign. Both g and g^{-1} are totally nonnegative by Proposition 3.3.33. Let $\widehat{g} \in \mathrm{SL}(n, \mathbb{R})$ be the totally nonnegative lift, and $t \in \mathrm{SL}(n, \mathbb{R})$ the diagonal matrix with entries alternately 1 and -1. Then $t\widehat{g}^{-1}t$ is totally nonnegative by Jacobi's complementary minor formula (Lemma 3.3.38, see also the sign calculations in Corollary 3.3.29). Any nonsingular totally nonegative matrix has positive diagonal entries by [Pin10, Theorem 1.13]. This implies that $\widehat{g}^{-1}$ is the totally nonnegative lift of g^{-1}. Since both $\widehat{g}^{-1}$ and $t\widehat{g}^{-1}t$ are totally nonnegative, we have $(\widehat{g}^{-1})_{ij} = 0$ if $i + j$ is odd. Now observe that

$$0 \leq \widehat{g}_{i-1,i}^{i,j} = -\widehat{g}_{i-1,j}\widehat{g}_{ii}, \qquad 1 < i < j \leq n,$$

thus all entries below the diagonal vanish. Analogously, all entries above the diagonal vanish, so $\widehat{g}$ is diagonal with positive entries. □

The connection to cycles observed at the end of the proof of Proposition 3.3.33 does in fact give another characterization of totally positive matrices. Recall that $A \subset \mathrm{PSL}(n, \mathbb{R})$ denotes the subgroup of all diagonal matrices, and that A_0 is the stabilizer of the standard pair (F_e, F_{w_0}).

Lemma 3.3.35. *$g \in \mathrm{PSL}(n, \mathbb{R})$ is totally positive if and only if $(F_e, gF_e, gF_{w_0}, F_{w_0})$ is a cycle. The induced map*

$$\{g \in \mathrm{PSL}(n, \mathbb{R}) \mid g \text{ totally positive}\}/A_0 \longrightarrow \{(F, F') \in \mathcal{F}_n^2 \mid (F_e, F, F', F_{w_0}) \text{ is a cycle}\}$$

is a bijection.

Proof. If g is totally positive, we saw in the proof of the previous proposition that $(F_e, gF_e, gF_{w_0}, F_{w_0})$ is a cycle. Conversely, assume that $(F_e, gF_e, gF_{w_0}, F_{w_0})$ is a cycle. We will construct an element $g' \in \mathrm{PSL}(n, \mathbb{R})$ such that $g'(F_e) = g(F_e)$, $g'(F_{w_0}) = g(F_{w_0})$, and g' is totally positive. Then g and g' can only differ by right–multiplication with an element of $\mathrm{Stab}(F_e, F_{w_0}) = A_0$, which preserves total positivity.

Let $L \in \mathrm{PSL}(n, \mathbb{R})$ be a lower triangular, unipotent, totally positive representative for $g(F_e)$. Then $L(F_e) = g(F_e)$ and L fixes F_{w_0}. Since $g(F_{w_0}) \in (\!(L(F_e), F_{w_0})\!)$, we have $L^{-1}g(F_{w_0}) \in (\!(F_e, F_{w_0})\!)$. Let $U \in \mathrm{PSL}(n, \mathbb{R})$ be an upper triangular, unipotent, totally positive matrix mapping F_{w_0} to $L^{-1}g(F_{w_0})$. Then by the Cauchy–Binet formula, LU is totally positive and we have $LU(F_e, F_{w_0}) = (g(F_e), g(F_{w_0}))$. □

Corollary 3.3.36. *Let $I = (\!(F_2, F_3)\!)$ and $J = (\!(F_1, F_4)\!)$ be intervals in $\mathcal{F}_n$ such that $\overline{I} \subset J$. Then (F_1, F_2, F_3, F_4) is a cycle.*

Proof. Since the action of $\mathrm{PSL}(n, \mathbb{R})$ on oriented transverse pairs is transitive, we may assume that $J = (\!(F_e, F_{w_0})\!)$. Moreover, there exists $g \in \mathrm{PSL}(n, \mathbb{R})$ such that $gJ = I$. By Proposition 3.3.33, this g is totally positive, so Lemma 3.3.35 finishes the proof. □

3.3.5 Convergence of intervals

In this section, we study the limit of an interval and its opposite as the endpoints converge to one single point. To be more precise, our goal is to prove the following proposition, which in particular implies that the partial cyclic order on complete oriented flags is regular.

Recall that in a partially cyclically ordered set, the comparable set $\mathcal{C}(F)$ of an element F consists of all elements F' such that F, F' are part of a positive triple. In the case of $\mathcal{F}_n$, the comparable set $\mathcal{C}(F)$ consists of all flags F' such that (F, F') is oriented transverse. Using the notation of Chapter 2, it is the same as the set $C_{w_0}(F)$ of flags at relative position w_0 with respect to F.

Proposition 3.3.37. *Let $F \in \mathcal{F}_n$ and $A_m, B_m \in \mathcal{F}_n$ be two sequences such that the following conditions hold:*

- $\overrightarrow{A_1 F B_1}$
- $A_m \in ((A_1, F))$ *and* $B_m \in ((F, B_1))$ *for* $m > 1$
- $A_m \to F,\ B_m \to F$

Then

$$\bigcup_m ((B_m, A_m)) = C_{w_0}(F).$$

The proof of this proposition requires a few preparations.

Jacobi's complementary minor formula

The following classical formula is useful when computing with minors, and will allow us to set up an inductive argument for the proof of Proposition 3.3.37. For the reader's convenience, we provide a proof here.

Lemma 3.3.38 (Jacobi's complementary minor formula)**.** *Let M be an invertible $n \times n$ matrix, and let $\mathbf{i}, \mathbf{j} \in \mathcal{I}(k, n)$ be multiindices. We denote by $M_{\bar{\mathbf{j}}}^{\bar{\mathbf{i}}}$ the minor of M determined by removing all rows listed in $\mathbf{i}$ and all columns listed in $\mathbf{j}$. Then minors of M^{-1} are related to minors of M by the following formula:*

$$(M^{-1})_{\mathbf{j}}^{\mathbf{i}} \det(M) = (-1)^{|\mathbf{i}|+|\mathbf{j}|} M_{\bar{\mathbf{i}}}^{\bar{\mathbf{j}}} \tag{3.3.8}$$

Proof. Let us first assume that $\mathbf{i} = \mathbf{j} = (n-p+1, \dots, n)$. We write the matrix M in block form

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where X is a $k \times k$ block, so $\det(X)$ equals $M_{\bar{\mathbf{i}}}^{\bar{\mathbf{j}}}$. Let us also assume for now that X is invertible. Then we can apply block Gaussian elimination to write

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \\ Z & W \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \\ 0 & W - ZX^{-1}Y \end{pmatrix}.$$

This factorization has two consequences which are of special interest to us: First, taking determinants yields the *Schur determinant identity*

$$\det(M) = \det(X) \det(W - ZX^{-1}Y). \tag{3.3.9}$$

Second, the *Schur complement* $W - ZX^{-1}B$ has to be invertible and inverting both sides shows that

$$M^{-1} = \begin{pmatrix} * & * \\ * & (W - ZX^{-1}Y)^{-1} \end{pmatrix}. \tag{3.3.10}$$

Therefore, rewriting (3.3.9) as

$$\det(W - ZX^{-1}Y)^{-1}\det(M) = \det(X)$$

gives us the desired equality since $(-1)^{|\mathbf{i}|+|\mathbf{j}|} = 1$ in this case.

If the block X is singular, we approximate M by matrices M_ϵ such that the corresponding blocks X_ϵ are invertible and $M_\epsilon \xrightarrow{\epsilon\to 0} M$. Since all minors of M_ϵ and M_ϵ^{-1} are polynomials in the entries of M, we get the claim by continuity.

Finally, we deal with the case when $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{j} = (j_1, \ldots, j_k)$ are not the last k indices. We want to reduce it to the previous case by flipping columns and rows. Assume that $i_k \neq n$. Letting $\widehat{M}$ denote the matrix obtained by flipping columns i_k and n of M, we obtain $\det(\widehat{M}) = -\det(M)$ and

$$M_{\bar{\mathbf{i}}}^{\bar{\mathbf{j}}} = (-1)^{n-i_k-1}\widehat{M}_{\overline{i_1,\ldots,i_{k-1},n}}^{\bar{\mathbf{j}}}.$$

At the same time, $\widehat{M}^{-1}$ is obtained by flipping rows i_p and n of M^{-1}, so we have

$$(M^{-1})_{\mathbf{j}}^{\mathbf{i}} = (\widehat{M}^{-1})_{\mathbf{j}}^{i_1,\ldots,i_{k-1},n}$$

and (3.3.8) is seen to be equivalent to

$$(\widehat{M}^{-1})_{\mathbf{j}}^{i_1,\ldots,i_{k-1},n}\det(\widehat{M}) = (-1)^{n-i_k}(-1)^{|\mathbf{i}|+|\mathbf{j}|}\widehat{M}_{\overline{i_1,\ldots,i_{k-1},n}}^{\bar{\mathbf{j}}}.$$

Observe that this formula is also valid when $i_k = n$, so both cases can be treated the same. We now iterate this procedure, switching column i_{k-1} with column $n-1$ and picking up a factor of $(-1)^{(n-1)-i_{k-1}}$, and so on. The total sign change incurred after moving all columns amounts to

$$(-1)^{[n+(n-1)+\ldots+(n-k+1)]-(i_k+\ldots+i_1)}.$$

After changing rows in the same fashion, we observe that the total sign change is precisely

$$(-1)^{2[n+(n-1)+\ldots+(n-k+1)]-|\mathbf{i}|-|\mathbf{j}|} = (-1)^{|\mathbf{i}|+|\mathbf{j}|},$$

so we are done by the previous case. □

Corollary 3.3.39. *Let M be a $n \times n$ matrix. Then the following identity of minors holds (where we use the notation of the previous lemma):*

$$M_{\bar{1}}^{\bar{1}}M_{\bar{n}}^{\bar{n}} - M_{\bar{n}}^{\bar{1}}M_{\bar{1}}^{\bar{n}} = M_{\overline{1,n}}^{\overline{1,n}}\det(M)$$

Proof. Observe that the left hand side equals a specific minor of the adjugate matrix $\operatorname{adj}(M)$: Letting C_{ij} denote the entries of that matrix, it is precisely $\det\begin{pmatrix} C_{11} & C_{1n} \\ C_{n1} & C_{nn}\end{pmatrix}$.

Let us assume first that M is invertible. Then, since $\operatorname{adj}(M) = \det(M)M^{-1}$, applying Lemma 3.3.38 yields

$$\det\begin{pmatrix} C_{11} & C_{1n} \\ C_{n1} & C_{nn}\end{pmatrix} = \det(M)^2(-1)^{2+2n}\frac{M_{\overline{1,n}}^{\overline{1,n}}}{\det(M)} = M_{\overline{1,n}}^{\overline{1,n}}\det(M).$$

If the matrix M is singular, we get the equality by continuity: All minors are polynomials in the matrix entries, so we can approximate M by invertible matrices and see that the left hand side has to converge to 0. □

A Frenet type property for convergent sequences of flags

Lemma 3.3.40. *Let $F_m = g_m B_0 \in ((F_e, F_{w_0}))$ be a sequence of complete oriented flags. Then the following are equivalent:*

(i) F_m converges to F_{w_0}.

(ii) For all $1 \leq k \leq n-1$ and $1 \leq i \leq n-k$,

$$\frac{(g_m)_{1\dots k}^{i,n-k+2\dots n}}{(g_m)_{1\dots k}^{n-k+1\dots n}} \xrightarrow{m\to\infty} 0.$$

(Note that these quotients are well-defined for $g_m \in \mathrm{PSL}(n,\mathbb{R})$)

Proof. Observe that changing both minors by the same scalar will not change the quotient. Therefore, the quotient makes sense for $g_m \in \mathrm{PSL}(n,\mathbb{R})$ (by using any lift to $\mathrm{SL}(n,\mathbb{R})$) and does not depend on the choice of representative, but only on the flag F_m. By oriented transversality with F_e, the representatives g_m can be chosen to be of the form

$$\begin{pmatrix} & & & & \iddots \\ & * & & & \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

Convergence of F_m to F_{w_0} is then equivalent to all entries above the antidiagonal converging to 0. But the (i,k)-entry, for $i+k \leq n$, is given precisely by the quotient

$$\frac{(g_m)_{1\dots k}^{i,n-k+2\dots n}}{(g_m)_{1\dots k}^{n-k+1\dots n}}.$$ □

Lemma 3.3.41. *Let $F_m \in ((F_e, F_{w_0}))$ be a sequence of complete oriented flags converging to F_{w_0}. Choose lifts to $\widehat{\mathcal{F}}_n$ such that $(\widehat{F}_e, \widehat{F}_m, \widehat{F}_{w_0})$ is a consistently oriented* *lift for all m. ($\widehat{F}_x = F_x$ if n is odd). Furthermore, let $k_1, k_2 \geq 0$ with $k_1 + k_2 < n$. Then we have*

$$\widehat{F}_m^{(k_1)} \oplus \widehat{F}_{w_0}^{(k_2)} \xrightarrow{m\to\infty} \widehat{F}_{w_0}^{(k_1+k_2)} \tag{3.3.11}$$

in the oriented Grassmannian $\mathrm{Gr}^+_{k_1+k_2}(\mathbb{R}^n)$.

Proof. We may assume $\widehat{F}_e$ to be represented by the identity matrix and $\widehat{F}_{w_0}$ to be represented by $w_0 \in \mathrm{SL}(n, \mathbb{R})$. Then we can choose representatives $g_m \in \mathrm{SL}(n, \mathbb{R})$ for $\widehat{F}_m$ which are lower triangular, unipotent, and totally positive (we do not decorate g_m with a hat here to make the calculations more readable). Our first step will be to translate the convergence statement in (3.3.11) into a statement involving minors of g_m.

Let us start with the cases $k_1 = 1$ and $k_2 = 0, 1, \ldots, n-2$. Combined, they are equivalent to convergence of the flags $X_m = (\widehat{F}_m^{(1)}, \widehat{F}_m^{(1)} \oplus \widehat{F}_{w_0}^{(1)}, \ldots, \widehat{F}_m^{(1)} \oplus \widehat{F}_{w_0}^{(n-2)})$ to $\widehat{F}_{w_0}$. A matrix representative of X_m is given by

$$\begin{pmatrix} 1 & & & & \\ g_{21}(m) & & & & -1 \\ \vdots & & & \iddots & \\ & & 1 & & \\ & -1 & & & \\ g_{n1}(m) & 1 & & & \end{pmatrix}.$$

Lemma 3.3.40 now converts this convergence into a condition on minors of g_m. Evaluating on the above representative yields

$$\frac{g_{i1}(m)}{g_{j1}(m)} \xrightarrow{m\to\infty} 0 \quad \forall i < j.$$

Similarly, the cases $k_1 = 2$ and $k_2 = 0, 1, \ldots, n-3$ correspond to the flags represented by

$$\begin{pmatrix} 1 & & & & \\ g_{21}(m) & 1 & & & \\ \vdots & g_{32}(m) & & & 1 \\ & \vdots & & \iddots & \\ & & & -1 & \\ g_{n1}(m) & g_{n2}(m) & 1 & & \end{pmatrix}$$

converging to $\widehat{F}_{w_0}$. Applying Lemma 3.3.40 again, we get the additional conditions

$$\frac{(g_m)^{i,j+1}_{1,2}}{(g_m)^{j,j+1}_{1,2}} \xrightarrow{m\to\infty} 0 \quad \forall i < j \leq n-1.$$

Continuing in the same fashion, we see that (3.3.11) is equivalent to the following:

$$\forall k < n,\ i < j \leq n-k+1 : \quad \frac{(g_m)_{1\ldots k}^{i,j+1\ldots j+k-1}}{(g_m)_{1\ldots k}^{j\ldots j+k-1}} \xrightarrow{m\to\infty} 0 \tag{3.3.12}$$

Note that the case $j+k-1 = n$ is exactly the condition from Lemma 3.3.40 for the convergence $F_m \to F_{w_0}$ in $(\!(F_e, F_{w_0})\!)$. We will prove the remaining cases by an inductive argument, decreasing the number $j+k-1$.

The second step is therefore to prove (3.3.12) for $j+k-1 = n-1$. The remaining cases will then follow by the same argument.
The largest possible value of k in this case is $n-2$, so let any $k \leq n-2$ and $i < j \leq n-k$ be given, and assume that $j+k = n$. We want to show that

$$\frac{(g_m)_{1\ldots k}^{i,n-k+1\ldots n-1}}{(g_m)_{1\ldots k}^{n-k\ldots n-1}} \xrightarrow{m\to\infty} 0.$$

We pick a different representative $h_m \in \mathrm{SL}(n,\mathbb{R})$ for $\widehat{F}_m$ such that the i–th row is

$$(1, 0, \ldots, 0)$$

(total positivity guarantees that the first entry can be normalized to $+1$). Since all quotients in (3.3.12) are independent of the representative, it suffices to prove the convergence for h_m. Now we apply Corollary 3.3.39 to the $(k+1)\times(k+1)$ block in the lower left corner, which yields

$$\begin{aligned}&(h_m)_{1\ldots k}^{n-k\ldots n-1}(h_m)_{2\ldots k+1}^{n-k+1\ldots n} - (h_m)_{1\ldots k}^{n-k+1\ldots n}(h_m)_{2\ldots k+1}^{n-k\ldots n-1} \\ =&(h_m)_{1\ldots k+1}^{n-k\ldots n}(h_m)_{2\ldots k}^{n-k+1\ldots n-1}.\end{aligned} \tag{3.3.13}$$

By our choice of representative, we can write all appearing minors as leftmost minors by adding the i–th row and the first column:

$$\begin{aligned}(h_m)_{2\ldots k+1}^{n-k+1\ldots n} &= (h_m)_{1\ldots k+1}^{i,n-k+1\ldots n}\\ (h_m)_{2\ldots k+1}^{n-k\ldots n-1} &= (h_m)_{1\ldots k+1}^{i,n-k\ldots n-1}\\ (h_m)_{2\ldots k}^{n-k+1\ldots n-1} &= (h_m)_{1\ldots k}^{i,n-k+1\ldots n-1}\end{aligned}$$

Now leftmost minors of g_m are all positive, and those of h_m differ by positive scalars. Therefore, they are all positive as well, and we obtain

$$\begin{aligned}&(h_m)_{1\ldots k}^{n-k\ldots n-1}(h_m)_{1\ldots k+1}^{i,n-k+1\ldots n} > (h_m)_{1\ldots k+1}^{n-k\ldots n}(h_m)_{1\ldots k}^{i,n-k+1\ldots n-1}\\ \Rightarrow\quad &\frac{(h_m)_{1\ldots k}^{i,n-k+1\ldots n-1}}{(h_m)_{1\ldots k}^{n-k\ldots n-1}} < \frac{(h_m)_{1\ldots k+1}^{i,n-k+1\ldots n}}{(h_m)_{1\ldots k+1}^{n-k\ldots n}}.\end{aligned}$$

The quotient on the right hand side converges to 0 by the assumption that $F_m \to F_{w_0}$ and Lemma 3.3.40. The left hand side therefore has to converge to 0 as well. This proves (3.3.12) for $j+k = n$.

The third and last step is to iterate the previous step: Let $\widetilde{g}_m \in \mathrm{SL}(n,\mathbb{R})$ denote the matrix obtained from g_m by deleting the last column and row. It is again lower triangular, unipotent and totally positive. The flag $\widetilde{F}_m \in \mathcal{F}_{n-1}$ represented by $\widetilde{g}_m$ therefore lies in $(\!(F_e, F_{w_0})\!) \subset \mathcal{F}_{n-1}$. By the previous step and another application of Lemma 3.3.40, we have $\widetilde{F}_m \xrightarrow{m\to\infty} F_{w_0}$. Iterating the whole argument yields (3.3.12) for all smaller values of $j+k$. $\square$

Proof of Proposition 3.3.37. The inclusion "$\subset$" is clear: By transitivity of the PCO, we have $F \in (\!(A_m, B_m)\!)$ for all m. Therefore, any $X \in (\!(B_m, A_m)\!)$ satisfies $\overrightarrow{FB_mX}$.

For the reverse inclusion, fix a lift $\widehat{F}$ and let $(\widehat{A}_m, \widehat{F}, \widehat{B}_m)$ be a consistently oriented lift of the triple for all m (if n is odd, ignore all lifts). This automatically implies that $(\widehat{A}_1, \widehat{A}_m, \widehat{F})$ and $(\widehat{F}, \widehat{B}_m \widehat{B}_1)$ are consistently oriented lifts, so $\widehat{A}_m \to \widehat{F}$ and $\widehat{B}_m \to \widehat{F}$. Recall that the comparable set $C_{w_0}(F)$ consists of all flags $X \in \mathcal{F}_n$ admitting a lift $\widehat{X} \in \widehat{\mathcal{F}}_n$ such that

$$\widehat{F}^{(k)} \oplus \widehat{X}^{(n-k)} \stackrel{\pm}{=} \mathbb{R}^n \tag{3.3.14}$$

for all $k < n$. On the other hand, the intervals $(\!(B_m, A_m)\!)$ consist of all flags $X \in \mathcal{F}_n$ admitting a lift $\widehat{X} \in \widehat{\mathcal{F}}_n$ satisfying the following: For any k_1, k_2 with $k_1 + k_2 < n$, we must have

$$\widehat{B}_m^{(k_2)} \oplus \widehat{X}^{(n-k_1-k_2)} \oplus ((-1)^{n-1}\widehat{A}_m)^{(k_1)} \stackrel{\pm}{=} \mathbb{R}^n.$$

Here, the sign takes care of the fact that $(\widehat{B}_m, -\widehat{A}_m)$ is oriented transverse if n is even.
Independent of the parity of n, this equation is equivalent to

$$\widehat{A}_m^{(k_1)} \oplus \widehat{B}_m^{(k_2)} \oplus \widehat{X}^{(n-k_1-k_2)} \stackrel{\pm}{=} \mathbb{R}^n. \tag{3.3.15}$$

We will show that $\widehat{A}_m^{(k_1)} \oplus \widehat{B}_m^{(k_2)} \xrightarrow{m\to\infty} \widehat{F}^{(k_1+k_2)}$. Then, since the set of oriented transverse pairs is open in $\mathcal{F}_n \times \mathcal{F}_n$, any flag X satisfying (3.3.14) will also satisfy (3.3.15) for sufficiently large m and thus be contained in $(\!(B_m, A_m)\!)$.

In Lemma 3.3.41, we proved a similar type of convergence when the flags B_m are replaced by the constant flag F. We will reduce our claim to that statement. First, observe that

$$A_m \in (\!(A_1, F)\!) \subset (\!(A_1, B_m)\!) \subset (\!(A_1, B_1)\!).$$

Since $\widehat{B}_m \to \widehat{F}$, we can choose $g_m \in \mathrm{SL}(n,\mathbb{R})$ such that $g_m\widehat{B}_m = \widehat{F}$ and $g_m \to 1$. Then $g_m\widehat{A}_m \to \widehat{F}$, $(g_m\widehat{A}_m, g_m\widehat{B}_m) = (g_m\widehat{A}_m, \widehat{F})$ is oriented transverse, and we have the following inclusions in $\mathcal{F}_n$:

$$g_mA_m \in (\!(g_mA_1, g_mF)\!) \subset (\!(g_mA_1, g_mB_m)\!) = (\!(g_mA_1, F)\!)$$

Now pick any $A \in \mathcal{F}_n$ such that $A_1 \in (\!(A, F)\!)$. As g_mA_1 converges to A_1, it follows that $(\!(g_mA_1, F)\!) \subset (\!(A, F)\!)$ for sufficiently large m and therefore also $g_mA_m \in (\!(A, F)\!)$. We are now able to apply Lemma 3.3.41, which tells us that

$$(g_m\widehat{A}_m)^{(k_1)} \oplus \widehat{F}^{(k_2)} \xrightarrow{m\to\infty} \widehat{F}^{(k_1+k_2)}$$

for any k_1, k_2 with $k_1 + k_2 < n$. We can rewrite this as

$$g_m\left(\widehat{A}_m^{(k_1)} \oplus \widehat{B}_m^{(k_2)}\right) \xrightarrow{m\to\infty} \widehat{F}^{(k_1+k_2)}.$$

However, as $g_m \to 1$, this implies that

$$\widehat{A}_m^{(k_1)} \oplus \widehat{B}_m^{(k_2)} \xrightarrow{m\to\infty} \widehat{F}^{(k_1+k_2)}.$$ □

Recall that we call a partially cyclically ordered space regular if for every element F, every increasing sequence $A_m \nearrow F$ and decreasing sequence $B_m \searrow F$ satisfying $F \in ((A_1, B_1))$, we have $\bigcup((B_m, A_m)) = \mathcal{C}(F)$ and $\bigcap((A_m, B_m)) = \{F\}$.

Corollary 3.3.42. *$\mathcal{F}_n$ is regular.*

Proof. Let $F, A_m, B_m \in \mathcal{F}_n$ be as above. By Proposition 3.3.37, $\bigcup((B_m, A_m)) = \mathcal{C}(F)$, so we only have to verify that $\bigcap((A_m, B_m)) = \{F\}$. Normalize so that $(A_1, B_1) = (F_e, F_{w_0})$. Then for any $k \geq 2$, $(F_e, A_k, F, B_k, F_{w_0})$ is a cycle by transitivity. Let $g_k, h_k \in \mathrm{SL}(n,\mathbb{R})$ be the unique lower triangular, unipotent, totally positive representatives for A_k and B_k. Let $g \in \mathrm{SL}(n,\mathbb{R})$ represent a flag $F' \in ((A_k, B_k))$, where g is of the same form. Then by Lemma 3.3.23 and Lemma 3.3.31, the minors of g lie between the corresponding minors of g_k and h_k. Since A_k and B_k converge to F, this finishes the proof. □

4 Generalized Schottky groups

Throughout this chapter, C denotes a partially cyclically ordered set and $G = \mathrm{Aut}(C)$.

Let Σ be the interior of a compact, connected, oriented surface with boundary, of Euler characteristic $\chi < 0$. Then, the fundamental group $\pi_1(\Sigma)$ is free on $g = 1 - \chi$ generators. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be the holonomy of a (complete) hyperbolization of Σ. In this section, we construct free subgroups of G using Γ as a combinatorial model. We will be primarily interested in two cases:
One is when the hyperbolization has finite area, i.e. every end is a cusp. Generalized Schottky groups using such a combinatorial model turn out to be maximal representations if G is a Hermitian Lie group, and conversely every maximal representation admits a presentation as a generalized Schottky group (Section 4.3).
The second case is when every end is a funnel, i.e. each peripheral element is hyperbolic. This allows us to obtain contraction properties and thereby provides a link to Anosov representations, which will be discussed in Section 4.5 and Section 4.6.

4.1 Definition

It is well–known that there is a presentation for Γ of the following form : Γ is freely generated by $A_1, \dots, A_g \in \mathrm{PSL}(2,\mathbb{R})$ and there are $2g$ disjoint open intervals $I_1^+, \dots, I_g^+, I_1^-, \dots, I_g^- \subset \mathbb{RP}^1 \cong S^1$ such that $A_j(-I_j^-) = I_j^+$ (see [Mar07, Chapter 2.9] for instance, and recall that $-I$ denotes the opposite interval). If Γ is a finite area hyperbolization, $\bigcup_i \overline{I_i^+} \cup \bigcup_i \overline{I_i^-} = S^1$ (Figure 4.1). The cyclic ordering on S^1 gives a cyclic ordering to the intervals in the definition.

The following easy lemma, which is a reformulation of transitivity, motivates our definition of generalized Schottky groups in G.

Lemma 4.1.1. *Let $(a,b,c) \in C^3$ be a cycle. Then we have $(\!(b,c)\!) \subset (\!(b,a)\!)$. In particular, the intervals $(\!(a,b)\!)$ and $(\!(b,c)\!)$ are disjoint.*

Proof. Let $x \in (\!(b,c)\!)$, so we have $\overrightarrow{bxc}$. By transitivity, together with $\overrightarrow{bca}$, this implies $\overrightarrow{bxa}$. $\square$

We now define generalized Schottky subgroups of G by asking for a setup of intervals similar to the $\mathrm{PSL}(2,\mathbb{R})$ case and requiring generators to pair them the same way.

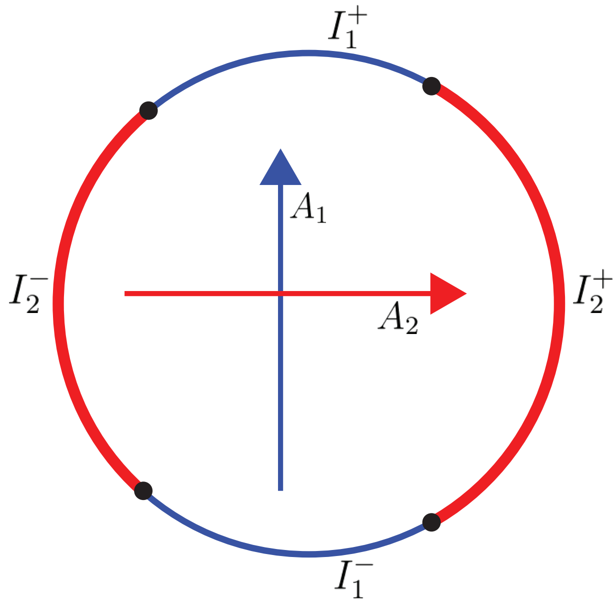


Figure 4.1: A combinatorial model for the once punctured torus.

Definition 4.1.2. Let ξ_0 be an increasing map from the set of endpoints of the intervals $I_1^+, \ldots, I_g^+, I_1^-, \ldots, I_g^-$ into C. Define the corresponding image intervals in C by $J_i^\pm = ((\xi_0(a_i^\pm), \xi_0(b_i^\pm)))$, where $I_i^\pm = ((a_i^\pm, b_i^\pm))$. Next, assume there exist $h_1, \ldots, h_g \in G$ which pair the endpoints of $J_i^\pm$ in the same way that the $A_i \in \mathrm{PSL}(2, \mathbb{R})$ pair the endpoints of $I_i^\pm$, so that $h_i(-J_i^-) = J_i^+$. We call the induced morphism $\Gamma \to G$ sending A_i to h_i a *generalized Schottky representation*, its image in G a *generalized Schottky group* and the intervals $J_i^\pm$ used to define it a set of *Schottky intervals* for this group.

Definition 4.1.3. Let $\rho\colon \Gamma \to G$ be a generalized Schottky representation. Then we call ρ (and its image in G) *purely hyperbolic* if no endpoints of the intervals $I_i^\pm$ used to define it coincide. We call ρ (and its image in G) *exhaustive* if Γ is a finite area hyperbolization of Σ.

Remarks 4.1.4.

(i) A generalized Schottky group will in general have many possible choices of a set of Schottky intervals. We will only use this term when a specific choice of both generators and intervals is fixed.

(ii) Since the cyclic ordering is a property of $\mathbb{RP}^1$ which is not shared by $\mathbb{CP}^1$, the Schottky groups defined here do *not* generalize the more well known $\mathbb{CP}^1$ Kleinian case.

(iii) If ρ is a purely hyperbolic generalized Schottky representation, the model Γ is necessarily a convex cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$, and every end of the quotient surface is a funnel. However, every such Γ also admits a choice of Schottky generators with contiguous Schottky intervals, so being purely hyperbolic depends on the choice of generators and intervals. The setup of intervals with disjoint closures gives rise to contraction properties (see Section 4.5 and Section 4.6).

(iv) If ρ is exhaustive, the union of the closures of the intervals $I_i^{\pm}$ is all of S^1. This case is more general than the purely hyperbolic one, and is adapted to our application towards maximal representations in Section 4.3.

(v) Our use of the term "Schottky" differs slightly from most references in that we allow for the closures of the ping pong subsets to intersect. This is sometimes called "Schottky-type".

We call a k–th order interval the image of any I_j^+(respectively I_j^-), $j \in \{1,\dots,g\}$, by a reduced word of length $k-1$ not ending in A_j^{-1}(respectively A_j) – that is, a word of the form

$$A_{i_1}^{\epsilon_1}\dots A_{i_{k-1}}^{\epsilon_{k-1}},$$

with $\epsilon_i \in \{-1,1\}$, such that no two consecutive letters cancel and $A_{i_{k-1}}^{\epsilon_{k-1}}$ is not equal to A_j^{-1} (respectively A_j). There are exactly $(2g)(2g-1)^{k-1}$ k–th order intervals. There is a natural bijection between words of length k and k–th order intervals: Generators are identified with their attractive intervals, and a reduced word

$$A_{i_1}^{\epsilon_1}\dots A_{i_{k-1}}^{\epsilon_{k-1}}A_{i_k}^{\pm 1}$$

is identified with the image of $I_{i_k}^{\pm}$ under $A_{i_1}^{\epsilon_1}\dots A_{i_{k-1}}^{\epsilon_{k-1}}$.
We use this bijection to index k–th order intervals : I_W is the interval corresponding to the word W. By construction, if W_1 and W_2 are two words, the associated intervals satisfy

$$W_1 I_{W_2} = I_{W_1 W_2}. \tag{4.1.1}$$

For any fixed k, the k–th order intervals are all pairwise disjoint, and so they are cyclically ordered. This induces a cyclic ordering on words of length k in Γ. Again, in the finite area case, the union of all closures of k–th order intervals is all of S^1.

With this setup, we can define k–th order intervals in C in the same way as above but starting with the intervals $J_i^{\pm}$ and their images under words in the h_i (see Figure 4.2). As above, denote by J_W the interval corresponding to W. Note that since ξ_0 is increasing, the k–th order intervals in C are also cyclically ordered, where the ordering is the same as the ordering of the corresponding intervals in S^1. Like classical Schottky groups, generalized Schottky groups are freely generated by the h_i. This fact is a direct consequence of the following version of the Ping–pong Lemma (see [Har00, section II.B] for this formulation).

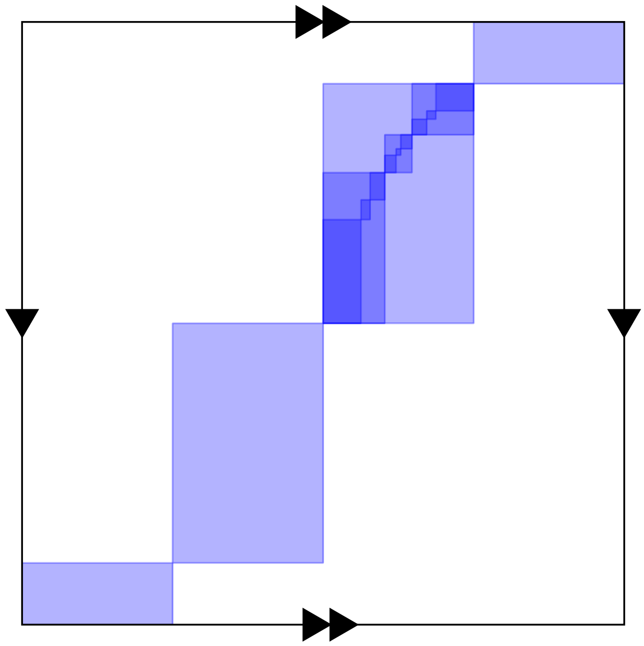


Figure 4.2: Some first, second and third order intervals for a generalized Schottky group acting on $S^1 \times S^1$.

Lemma 4.1.5 (Ping–pong Lemma)**.** *Let G be a group acting on a set X, Γ_1 and Γ_2 two subgroups of G and Γ the group generated by the Γ_i. Assume that Γ_1 has order at least 3, and there exist nonempty subsets X_1, X_2 of X, with X_2 not contained in X_1, such that the following holds:*

$$\begin{aligned} \gamma(X_2) &\subset X_1 \quad \forall 1 \neq \gamma \in \Gamma_1 \\ \gamma(X_1) &\subset X_2 \quad \forall 1 \neq \gamma \in \Gamma_2. \end{aligned}$$

Then Γ is isomorphic to the free product of Γ_1 and Γ_2.

Proposition 4.1.6. *The group generated by $h_1 \dots h_g$ is free on those generators.*

Proof. Define $J_i = J_i^+ \cup J_i^-$. Note that $J_i \cap J_j = \emptyset$ whenever $i \neq j$. Moreover, for any $n \neq 0$, $h_i^n(J_j) \subset J_i$ and so the proposition follows from the Ping–pong Lemma. □

The endpoints of k–th order intervals in C satisfy the same cyclic order relations as the corresponding endpoints in S^1, and we can extend ξ_0 to an increasing equivariant map defined on the orbit of its domain, the set of all endpoints of intervals of any order in S^1. We denote this countable subset of S^1 by S^1_Γ. It accumulates on the limit set $\Lambda_\Gamma \subset S^1$ of the group Γ. For a finite area hyperbolization, this is the whole circle, whereas it is a Cantor set if any of the ends are funnels.

4.2 Limit curves

We now come to a construction of limit maps for exhaustive and purely hyperbolic generalized Schottky representations. In the exhaustive case, this will allow us to establish the link to maximal representations, while the limit map in the purely hyperbolic case provides a link to Anosov representations. First, we prove an easy lemma which will help with some technical details in the construction.

Lemma 4.2.1. *Let $P_n \nearrow P$ be an increasing sequence in a proper, increasing–complete, PCO set C. Assume Q_n is another sequence with $Q_n \in \overline{((P_n, P_{n+1}))}$ for all n. Then Q_n converges to P and is 3-increasing in the following sense: whenever $i+2 < j < k-2$, we have $\overrightarrow{Q_i Q_j Q_k}$.*

Proof. For every $n \geq 2$, $Q_n \in ((P_{n-1}, P_{n+2}))$ by properness, which already implies that Q_n is 3-increasing. Now, consider the following sequence:

$$P_1, Q_2, P_4, Q_5, \ldots, P_{3n+1}, Q_{3n+2}, \ldots$$

It is increasing, and admits a subsequence which is also a subsequence of P_n. Since increasing sequences have unique limits, this sequence must converge to P. The increasing subsequence Q_{3n+2} therefore converges to P. Using the same argument, we see that Q_{3n+1} and Q_{3n} also converge to P, so in fact the sequence Q_n converges to P. □

The boundary map we construct for an exhaustive Schottyk group will be defined on S^1 and will be left–continuous as a map to some first–countable topological space C. To avoid confusion, let us fix the definition here: In a small neighborhood U of a point $a \in S^1$, the cyclic order induces a linear order. A sequence $a_n \in U$ converges to a from the left if $a_n < a$ and $a_n \xrightarrow{n\to\infty} a$. The function f is left–continuous at a if $f(a_n) \to f(a)$ for all sequences a_n converging to a from the left.
It is worth noting that to check for left–continuity at a point a, it is in fact sufficient to check the convergence of $f(a_n)$ for increasing sequences a_n converging to a. The reason is the following: Assume a_n is a sequence converging to a from the left such that $f(a_n)$ does not converge to $f(a)$. Then it has a subsequence such that $f(a_{n_k})$ stays bounded away from $f(a)$. But since $a_{n_k} \to a$ from the left, we can pick a further subsequence which is increasing and still a counterexample to left–continuity.

Theorem 4.2.2. *Let $\rho\colon \Gamma \to G = \mathrm{Aut}(C)$ be an exhaustive generalized Schottky representation, and assume that C is first–countable, increasing–complete and proper. Then there is a left–continuous, equivariant, increasing boundary map $\xi : S^1 \to C$.*

Proof. Recall that $S^1_\Gamma \subset S^1$ denotes the domain of ξ_0, the dense Γ–orbit of the endpoints of the Schottky intervals. We construct the map ξ as follows: For $x \in S^1$, pick any increasing sequence $x_n \in S^1_\Gamma$ converging to x and set

$$\xi(x) = \lim_{n\to\infty} \xi_0(x_n).$$

First of all, let us show that this value is well–defined. Since x_n is an increasing sequence, the increasing map ξ_0 maps it to an increasing sequence in C which therefore has a unique limit. Furthermore, this limit does not depend on the choice of x_n: Let y_m be another increasing sequence converging to x. Then the two sequences $\xi_0(x_n)$ and $\xi_0(y_m)$ are compatible, so they have the same limit by Lemma 3.1.12.

We then verify that ξ is equivariant. Let $x \in S^1$, $\gamma \in \Gamma$, and $x_n \nearrow x$ an increasing sequence, so we have $\xi(x) = \lim_{n\to\infty} \xi_0(x_n)$. Then $\gamma(x_n)$ is an increasing sequence converging to $\gamma(x)$, so by continuity of $\rho(\gamma)$ and equivariance of ξ_0, we have the following equalities:

$$\rho(\gamma)(\xi(x)) = \lim_{n\to\infty} \rho(\gamma)(\xi_0(x_n)) = \lim_{n\to\infty} \xi_0(\gamma(x_n)) = \xi(\gamma(x)).$$

Next, we show that it is left–continuous. Assume $x_n \in S^1$ is a sequence converging to x from the left. As explained above, without loss of generality we can take x_n to be an increasing sequence. We pick points $y_n \in S^1_\Gamma$ such that $y_n \in (\!(x_{n-1}, x_n)\!)$. Then y_n is increasing and we have $x_n \in (\!(y_n, y_{n+1})\!)$. Furthermore, y_n also converges to x, hence

$$\xi(x) = \lim \xi_0(y_n). \tag{4.2.1}$$

Now, for each n, let $\{a_k(n)\}_{k\in\mathbb{N}} \subset S^1_\Gamma$ be an increasing sequence converging to x_n, so

$$\xi(x_n) = \lim_{k\to\infty} \xi_0(a_k(n)). \tag{4.2.2}$$

Then $a_k(n) \in (\!(y_n, y_{n+1})\!)$ for large k, so

$$\lim_{k\to\infty} \xi_0(a_k(n)) \in \overline{(\!(\xi_0(y_n), \xi_0(y_{n+1}))\!)} \tag{4.2.3}$$

because ξ_0 is increasing. By (4.2.2) and (4.2.3), Lemma 4.2.1 now applies to the sequences $P_n = \xi_0(y_n)$, $Q_n = \xi(x_n)$ and, combined with (4.2.1), tells us that $\xi(x_n)$ converges to $\xi(x)$.

The final property we need to check is that ξ is increasing. Assume that we have $\overrightarrow{xyz}$ for points $x, y, z \in S^1$. By density of S^1_Γ, we can find a cycle

$$(a, a_1, a_2, b, b_1, b_2, c, c_1, c_2) \in (S^1_\Gamma)^9$$

such that $x \in (a_1, a_2)$, $y \in (b_1, b_2)$, $z \in (c_1, c_2)$. As in the proof of left–continuity, this implies that $\xi(x) \in \overline{(\!(\xi_0(a_1), \xi_0(a_2))\!)} \subset (\!(\xi_0(a), \xi_0(b))\!)$, and similar for the other two points. Using transitivity, we conclude $\overrightarrow{\xi(x)\xi(y)\xi(z)}$. $\square$

In the setting of purely hyperbolic generalized Schottky representations, we can obtain a continuous limit map, given an additional contraction assumption.

Theorem 4.2.3. *Let $\rho\colon \Gamma \to G = \mathrm{Aut}(C)$ be a purely hyperbolic generalized Schottky representation. Assume that C is first–countable, increasing–complete and proper. Furthermore, assume there is a constant $c < 1$ and every Schottky interval $J_i^\pm \subset C$ comes equipped with a complete metric, compatible with the topology of C, such that the following holds. For any generator h_i (resp. h_i^{-1}), the map $h_i\colon J \to J_i^+$ (resp. $h_i^{-1}\colon J \to J_i^-$) it induces on any Schottky interval $J \neq J_i^-$ (resp. $J \neq J_i^+$) is c–Lipschitz.*
Then there is a continuous, equivariant, increasing boundary map $\xi\colon \partial_\infty\Gamma \cong \Lambda_\Gamma \to C$.

Proof. Let $x \in \partial_\infty\Gamma$ be a boundary point. Then x corresponds to a unique infinite sequence in the generators $A_i \in \mathrm{PSL}(2,\mathbb{R})$ and their inverses, which is reduced in the sense that no letter is followed by its inverse. This correspondence goes via the bijection between words of length k in Γ and k–th order intervals that we described in Section 4.1: A boundary point is the intersection of a unique nested sequence $I^{(1)} \supset I^{(2)} \supset \ldots$, where $I^{(j)}$ is a j–th order interval. Each such interval $I^{(j)}$ then corresponds to a word $x^{(j)}$ of length j, $I^{(j)} = I_{x^{(j)}}$, and $x^{(j+1)}$ is obtained from $x^{(j)}$ by adding a letter on the right. The infinite sequence x is the limit of these words.
Let $J_{x^{(1)}} \supset J_{x^{(2)}} \supset \ldots$ be the nested sequence of intervals in C associated to the intervals $I_{x^{(j)}}$. Since $J_{x^{(1)}}$ is one of the initial Schottky intervals, it is equipped with a metric. We will show that the Lipschitz assumption we imposed on the generators $h_i^\pm \in G$ implies the existence of a constant $M > 0$ such that

$$\mathrm{diam}_{J_{x^{(1)}}}(J_{x^{(j)}}) \leq Mc^{j-2}. \tag{4.2.4}$$

Then, using completeness of the metric, the intersection of those intervals yields a unique point,

$$\bigcap_{j\geq 1} J_{x^{(j)}} = \{a\},$$

allowing us to define the map ξ by

$$\xi(x) = a.$$

So let us prove (4.2.4). Since ρ is purely hyperbolic, second order intervals are strictly nested inside first order intervals, thus there is a constant $M > 0$ such that

$$\mathrm{diam}_{J_{x^{(1)}}}(J_{x^{(2)}}) \leq M$$

for any second order interval inside a first order interval. Any third order interval is the image of a second order interval under a generator whose repelling interval does not contain the second order interval. Therefore, the Lipschitz assumption implies that

$$\mathrm{diam}_{J_{x^{(1)}}}(J_{x^{(3)}}) \leq Mc$$

for any third order interval inside a first order interval. An inductive argument completes the proof of (4.2.4).

We now show that the boundary map ξ constructed in this way is continuous. Let $x_n \to x$ be a sequence in $\partial_\infty\Gamma$ converging to x. This implies that for any $N \in \mathbb{N}$, we can find an index n_0 such that for all $n \geq n_0$, the first N letters of x_n and x agree. Then $\xi(x_n)$ and $\xi(x)$ lie in the same N-th order interval, and we saw in (4.2.4) that its diameter is bounded by Mc^{N-2} (in the first order interval determined by the first letter of x). Thus ξ is continuous.

Next, we prove equivariance. Let $\gamma \in \Gamma$ be some element, expressed as a reduced word of length l in the generators A_i and their inverses. Then $\gamma x \in \partial_\infty\Gamma$, as an infinite sequence in the generators, is simply the concatenation of the finite word γ and the infinite word x. The corresponding nested sequence of intervals is therefore $I_{(\gamma x)^{(j)}}$, $j \in \mathbb{N}$, and ξ maps γx to the unique point in $\bigcap_j J_{(\gamma x)^{(j)}}$. By (4.1.1), we have $\rho(\gamma)J_{x^{(j)}} = J_{\gamma x^{(j)}}$, so this intersection point agrees with $\rho(\gamma)\xi(x)$.

Finally, we show that ξ is increasing. Let $x, y, z \in \partial_\infty\Gamma \cong \Lambda_\Gamma$ be three points such that $\overrightarrow{xyz}$. Then there are indices $K, L, M \in \mathbb{N}$ such that the intervals $I_{x^{(K)}}, I_{y^{(L)}}, I_{z^{(M)}}$ are in increasing configuration. Their image intervals $J_{x^{(K)}}, J_{y^{(L)}}, J_{z^{(M)}}$ satisfy the same cyclic relations and contain the points $\xi(x), \xi(y), \xi(z)$ respectively, so we conclude $\overrightarrow{\xi(x)\xi(y)\xi(z)}$. $\square$

The very general construction described in this section applies to many examples. We encountered two classes of homogeneous spaces carrying partial cyclic orders in Chapter 3: The Shilov boundary of a Hermitian symmetric space admits a PCO which is increasing–complete, proper and regular. We will give a more explicit description of the case of $\mathrm{Lag}(\mathbb{R}^{2n})$ in Section 4.5, and show that it also satisfies the contraction assumption of Theorem 4.2.3. The second class of examples, which also satisfies all the required assumptions, is complete oriented flags in $\mathbb{R}^n$. The corresponding Schottky groups are subgroups of $\mathrm{PSL}(n, \mathbb{R})$, and will be the subject of Section 4.6.

4.3 Relation to maximal representations

In Section 3.2 we defined a PCO on the Shilov boundary $\mathcal{S}$ of a Hermitian symmetric space of tube type on which the group of holomorphic isometries G acts by order–preserving diffeomorphisms. We recall that this action is transitive on transverse pairs. The Schottky construction described in Section 4.1 therefore gives representations $\rho : \Gamma \to G$, where Γ is the fundamental group of a surface with boundary.
Maximal representations are a class of geometrically interesting representations and we will show in this section that in the case of surfaces with boundary, they correspond to generalized Schottky representations. They are defined by associating a natural invariant to the representation and requiring it to attain its maximal possible value. While the study of this invariant was originally restricted to closed surfaces ([Tol79],[DT87],[Tol89]), the definition was extended to surfaces with boundary in [BIW10].

Let X be a Hermitian symmetric space and ω be the Kähler form on X. Then, ω defines a continuous, bounded cohomology class $\kappa_G^b \in H^2_{cb}(G, \mathbb{R})$ called the Kähler class. If $\rho : \pi_1(\Sigma) \to G$ is a representation, the pullback $\rho^*\kappa_G^b$ is a bounded cohomology class in $H^2_b(\pi_1(\Sigma), \mathbb{R}) \cong H^2_b(\Sigma, \mathbb{R})$. In order to get an invariant out of this class, we use the isomorphism $j : H^2_b(\Sigma, \partial\Sigma, \mathbb{R}) \to H^2_b(\Sigma, \mathbb{R})$ (see [BIW10] for details).

Definition 4.3.1. The *Toledo invariant* is the real number

$$\mathsf{T}(\rho) = \langle j^{-1}\rho^*\kappa_G^b, [\Sigma, \partial\Sigma]\rangle$$

where $[\Sigma, \partial\Sigma]$ is the relative fundamental class.

The Toledo invariant satisfies a sharp bound known as the *Milnor-Wood-inequality*:

$$|\mathsf{T}(\rho)| \leq |\chi(\Sigma)| \mathrm{rk}(X).$$

A representation ρ is called *maximal* whenever equality is attained. The key to our analysis is the following characterization:

Theorem 4.3.2 ([BIW10, Theorem 8]). *Let $h : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ be a complete finite area hyperbolization of the interior of Σ and $\rho : \Gamma \to G$ a representation into a group of Hermitian type. Then ρ is maximal if and only if there exists a left continuous, equivariant, increasing map*

$$\xi : \partial\mathbb{H}^2 \cong S^1 \to \mathcal{S}$$

where $\mathcal{S}$ is the Shilov boundary of the bounded symmetric domain associated to G.

Using this characterization and our earlier construction of a boundary map for generalized Schottky representations, we see that the two notions agree:

Theorem 4.3.3. *The representation $\rho : \Gamma \to G$ is maximal if and only if it is an exhaustive Schottky representation.*

Proof. Assume ρ has an exhaustive Schottky presentation. Proposition 3.2.20 states that all the prerequisites of Theorem 4.2.2 are fulfilled. Therefore, there exists a boundary map ξ satisfying the conditions of the characterization above, so ρ is maximal.

Conversely, if ρ is maximal, then we have such a map ξ. Choosing a Schottky presentation for the hyperbolisation h, we get a Schottky presentation for ρ by using the intervals $(\!(\xi(a), \xi(b))\!)$, where $(\!(a, b)\!)$ is some Schottky interval in the presentation for h. Equivariance and positivity of ξ ensure that these intervals fit our definition of generalized Schottky groups. □

Theorem 4.3.3, as stated, assumes that G is of tube type. However, this assumption is not necessary. This is because of the following observations. Let X be a Hermitian symmetric space, and $\mathcal{S}$ its Shilov boundary. Then, in the same way as for tube type, the generalized Maslov index defines a partial cyclic order on $\mathcal{S}$. Let $x, y \in \mathcal{S}$ be transverse. Then, x, y are contained in the Shilov boundary of a unique maximal tube type subdomain of X [Wie04, Lemma 4.4.2]. Moreover, this is also true of any increasing triple in $\mathcal{S}$ [Wie04, Proposition 5.1.4]. This means that any increasing subset of $\mathcal{S}$ is contained in the Shilov boundary of a tube type subdomain, and so the proofs of this section generalize to arbitrary Hermitian symmetric spaces.

4.4 Relation to Anosov representations

Anosov representations constitute another class of geometrically interesting representations which were already discussed extensively in Chapter 2. One important property of an Anosov representation is the existence of a continuous boundary map into the appropriate flag manifold. We already constructed such boundary maps for purely hyperbolic generalized Schottky representations in Section 4.2. We will now show that they are in fact Anosov, given a suitable target group and an additional regularity property of the partial cyclic order.

To do so, we will make use of one of the various equivalent characterizations of Anosov representations given in [KLP17]. It requires the notions of P_θ–contraction and $\mathcal{F}_\theta$–limit set. Recall that for a flag $F \in \mathcal{F}_\theta$, $C(F) \subset \mathcal{F}_{\iota(\theta)}$ denotes the set of flags transverse to F.

Definition 4.4.1 ([KLP17, Definition 4.1])**.** Let $P_\theta, P_{\iota(\theta)}$ be opposite parabolic subgroups. A sequence $g_n \in G$ is P_θ*–contracting* if there exist flags $F_+ \in \mathcal{F}_\theta, F_- \in \mathcal{F}_{\iota(\theta)}$ such that

$$g_n|_{C(F_-)} \xrightarrow{n\to\infty} F_+$$

locally uniformly.

Definition 4.4.2 ([KLP17, Definition 4.32])**.** Let $H < G$ be a subgroup. Then the $\mathcal{F}_\theta$*–limit set* consists of all flags F_+ as in the previous definition for all contracting sequences $g_n \in H$.

The following is part of [KLP17, Theorem 1.1].

Theorem 4.4.3. *Let G be a connected semi–simple Lie group with finite center and P_θ a parabolic subgroup determined by a subset $\theta \subset \Delta$ of the simple restricted roots satisfying $\iota(\theta) = \theta$. Furthermore, let Γ be a word hyperbolic group and $\rho\colon \Gamma \to G$ a representation. Then ρ is P_θ–Anosov if and only if:*

(i) There is a ρ–equivariant embedding

$$\xi\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$$

whose image is the $\mathcal{F}_\theta$–limit set of Γ such that for any $x \neq y \in \partial_\infty\Gamma$, $\xi(x)$ and $\xi(y)$ are transverse.

(ii) Every diverging sequence sequence $\gamma_n \to \infty$ in Γ has a P_θ–contracting subsequence.

The following lemma will be useful to find contracting subsequences in generalized Schottky groups.

Lemma 4.4.4. *Let $H < G$ be a generalized Schottky group with Schottky generators $g_1, \ldots, g_m$ and associated Schottky intervals $I_1^\pm, \ldots, I_m^\pm$. Let w be a reduced word of length at least $2k$ in the generators, let a_k be the subword consisting of the first k letters and b_k the subword consisting of the last k letters. Then w maps $-I_{b_k^{-1}}$ into I_{a_k}.*

Proof. We write
$$w = g_{i_1}^{\epsilon_1} \ldots g_{i_\ell}^{\epsilon_\ell},$$
where ℓ is the word length of w in the generating set $g_1, \ldots, g_m$, $i_j \in \{1, \ldots, m\}$ and $\epsilon_j \in \{\pm 1\}$. Then a_k maps $-I_{i_k}^{-\epsilon_k}$ to I_{a_k}. Analogously, b_k^{-1} maps $-I_{i_{\ell-k+1}}^{\epsilon_{\ell-k+1}}$ to $I_{b_k^{-1}}$, so b_k maps $-I_{b_k^{-1}}$ to $I_{i_{\ell-k+1}}^{\epsilon_{\ell-k+1}}$. Since w is reduced, the image of $I_{i_{\ell-k+1}}^{\epsilon_{\ell-k+1}}$ under the middle $\ell - 2k$ letters is contained in $-I_{i_k}^{-\epsilon_k}$, so the result follows. □

Theorem 4.4.5. *Let G be a semi–simple, connected, linear Lie group and $\theta \subset \Delta$ with $\iota(\theta) = \theta$. Let $\rho\colon \Gamma \to G$ be a purely hyperbolic generalized Schottky representation and let $P_R \subset G$, $R = \langle r(\theta), E\rangle$ be an oriented parabolic subgroup such that $\mathcal{F}_R = G/P_R$ is equipped with a (nontrivial) G–invariant, increasing–complete, proper and regular partial cyclic order. Assume that any comparable pair (F_1, F_2) in $\mathcal{F}_R$ is transverse (Definition 2.2.25). Furthermore, assume that the contraction condition from Theorem 4.2.3 is satisfied.*
Then ρ is P_R–Anosov.

Proof. By Theorem 4.2.3, there exists a continuous, ρ–equivariant, increasing map $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$. To show that ρ is P_R–Anosov, we need to show that it is P_θ–Anosov, which we will do using Theorem 4.4.3. Consider the projection $\xi = \pi \circ \widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_\theta$. It is ρ–equivariant, continuous, and any pair $\xi(x), \xi(y)$, $x \neq y$ is transverse since $\widehat{\xi}(x), \widehat{\xi}(y)$ are comparable. As $\partial_\infty\Gamma$ is compact, it is an embedding, so the only property of the boundary map we have to check that its image is the $\mathcal{F}_\theta$–limit set of $\rho(\Gamma)$. We first prove (ii) from Theorem 4.4.3.

Let $\rho(\gamma_n) \in \rho(\Gamma)$ be a diverging sequence, and write each $\rho(\gamma_n)$ as a word in a set $\{g_1, \ldots, g_m\} \subset \rho(\Gamma)$ of Schottky generators. Let $a_k(n)$ be the subword consisting of the first k letters of $\rho(\gamma_n)$, and let $b_k(n)$ be the subword consisting of the last k letters of $\rho(\gamma_n)$. After taking a subsequence, we may assume that for $n \geq k$, both $a_k(n)$ and $b_k(n)$ are constant and do not overlap. We will write a_k and b_k for these stable subwords. Recall from Section 4.1 that we associate an interval $I_w \subset \mathcal{F}_R$ to every word in the Schottky generators. By Lemma 4.4.4, for any $n \geq k$, $\rho(\gamma_n)$ maps the opposite interval $-I_{b_k^{-1}}$ into I_{a_k}. Note that the intervals I_{a_k} (resp. $I_{b_k^{-1}}$) are nested for increasing k, since a_{k+1} (resp. b_{k+1}^{-1}) is obtained from a_k (resp. b_k^{-1}) by adding a letter on the right. By construction of the boundary map $\widehat{\xi}\colon \partial_\infty\Gamma \to \mathcal{F}_R$ in Theorem 4.2.3, we have
$$\bigcap_k I_{a_k} = \{\widehat{\xi}(\gamma^+)\} \quad \text{and} \quad \bigcap_k I_{b_k^{-1}} = \{\widehat{\xi}(\gamma^-)\}$$

for points $\gamma^+, \gamma^- \in \partial_\infty \Gamma$. Their endpoints therefore form increasing and decreasing sequences converging to $\widehat{\xi}(\gamma^+)$ (resp. $\widehat{\xi}(\gamma^-)$), so regularity of the partial cyclic order implies that $\bigcup -I_{b_k^{-1}} = \mathcal{C}(\widehat{\xi}(\gamma^-))$. By G–invariance and nontriviality of the partial cyclic order, this comparable set contains the cell $C_{[\![w_0]\!]}(\widehat{\xi}(\gamma^-))$ of (partial oriented) flags at relative position $[\![w_0]\!]$ to $\widehat{\xi}(\gamma^-)$ for some transverse position $[\![w_0]\!]$. It projects to the set $C(\xi(\gamma^-))$ of (partial unoriented) flags transverse to $\xi(\gamma^-)$. Thus $\pi(-I_{b_k^{-1}})$ is an increasing sequence of open sets covering $C(\xi(\gamma^-))$, and $\rho(\gamma_n)\pi(-I_{b_k^{-1}}) \subset \pi(I_{a_k})$ for $n \geq k$. This implies that $\rho(\gamma_n)|_{C(\xi(\gamma^-))} \xrightarrow{n\to\infty} \xi(\gamma^+)$ locally uniformly.

Now we return to the boundary map. Let $F \in \mathcal{F}_\theta$ be a point in the image of ξ. By construction of $\widehat{\xi}$, F is the projection of a point in $\mathcal{F}_R$ corresponding to an infinite reduced word a in the Schottky generators (see Theorem 4.2.3). Let $a_k \in \rho(\Gamma)$ consist of its first k letters, and let $b_k \in \rho(\Gamma)$ be the first k letters of a different infinite reduced word. Then by the description above, $a_k(b_k)^{-1}$ is a contracting sequence with attracting limit F.
Conversely, let $F \in \mathcal{F}_\theta$ be an element of the $\mathcal{F}_\theta$–limit set of $\rho(\Gamma)$ and $\rho(\gamma_n)$ a contracting sequence with F as its attracting limit. After taking a subsequence, for all k, the subword $a_k(n)$ consisting of the first k letters of $\rho(\gamma_n)$ (in the Schottky generators) is constant for all $n \geq k$. Let a be the infinite word obtained as limit of the a_k. As we saw before, a corresponds to the attracting limit of this contracting sequence which is therefore contained in the image of $\xi = \pi \circ \widehat{\xi}$. $\square$

4.5 Schottky groups in $\mathrm{Sp}(2n, \mathbb{R})$

In this section, we consider the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, acting on $\mathbb{R}^{2n}$ equipped with a symplectic form ω. We will describe the Maslov index and intervals in $\mathrm{Lag}(\mathbb{R}^{2n})$ in detail, showing in particular how intervals can be identified with symmetric spaces and thus equipped with Riemannian metrics. By associating a "halfspace" in $\mathbb{RP}^{2n-1}$ to every interval, we will exhibit a fundamental domain in $\mathbb{RP}^{2n-1}$ for the action of a generalized Schottky group in $\mathrm{Sp}(2n,\mathbb{R})$. In the purely hyperbolic case, the domain of discontinuity which is the orbit of this fundamental domain admits a nice description (Theorem 4.5.15). Using a contraction property for symplectic maps sending one interval inside another, we will also show that purely hyperbolic generalized Schottky representations are Anosov with respect to the stabilizer of a Lagrangian.

4.5.1 The Maslov index in $\mathrm{Sp}(2n,\mathbb{R})$

Definition 4.5.1. Let P, Q be transverse Lagrangians in $\mathbb{R}^{2n}$. We associate to them an antisymplectic involution σ_{PQ} defined using the splitting $\mathbb{R}^{2n} = P \oplus Q$:

$$\begin{aligned}\sigma_{PQ} : P \oplus Q &\to P \oplus Q \\ (v, w) &\mapsto (-v, w)\end{aligned}$$

We call this antisymplectic involution the *reflection* in the pair P, Q. Changing the order in the splitting yields the negative,

$$\sigma_{QP} = -\sigma_{PQ}.$$

This generalizes the projective reflection in $\mathbb{RP}^1$.

We will sometimes abuse notation and use σ_{PQ} to denote the induced transformation on Grassmannians.

Using this involution, we associate a symmetric bilinear form to the pair P, Q :

Definition 4.5.2.

$$\mathcal{B}_{PQ}(v, w) := \omega(v, \sigma_{PQ}(w))$$

This bilinear form is nondegenerate and has signature (n, n). Since the assignment $(P, Q) \mapsto \sigma_{PQ}$ is antisymmetric, the same is true for this bilinear form: $\mathcal{B}_{QP} = -\mathcal{B}_{PQ}$.

Definition 4.5.3. Let P, Q, R be pairwise transverse Lagrangians in $\mathbb{R}^{2n}$. The *Maslov index* of the triple (P, Q, R) is the index of the restriction of $\mathcal{B}_{PR}$ to Q. We denote it by $\mathsf{M}(P, Q, R)$.

Since $\mathrm{Lag}(\mathbb{R}^{2n})$ is the Shilov boundary for the bounded domain realization of the symmetric space of $\mathrm{Sp}(2n,\mathbb{R})$, it is an example of the general construction in Section 3.2. In fact, the Maslov index we just defined agrees with the more general version that we introduced before. Hence, the relation defined by $\overrightarrow{PQR}$ whenever $\mathsf{M}(P, Q, R) = n$ is a partial cyclic order on $\mathrm{Lag}(\mathbb{R}^{2n})$, enabling us to apply the constructions and results from the previous sections.
We also remark that the definition makes sense for any isotropic subspace Q, not only the maximal isotropic ones.

The following property of the Maslov index is well–known.

Proposition 4.5.4. *The Maslov index classifies orbits of triples of pairwise transverse Lagrangians, i.e. the map*

$$(P, Q, R) \mapsto \mathsf{M}(P, Q, R)$$

induces a bijection from orbits of pairwise transverse Lagrangians under $\mathrm{Sp}(2n,\mathbb{R})$ *to the set* $\{-n, -n + 2, \ldots, n\}$.

The Maslov index and the reflection in a pair of Lagrangians are related in the following way:

Proposition 4.5.5.
$$\mathsf{M}(P, \sigma_{PQ}(V), Q) = -\mathsf{M}(P, V, Q).$$

Proof. $\mathcal{B}_{PQ}\big(\sigma_{PQ}(u), \sigma_{PQ}(v)\big) = \omega(\sigma_{PQ}(u), v) = -\omega(u, \sigma_{PQ}(v)) = -\mathcal{B}_{PQ}(u, v)$. □

The proposition above means that reflections reverse the partial cyclic order.

4.5.2 Positive halfspaces and fundamental domains in $\mathbb{RP}^{2n-1}$

We now associate a "halfspace" in $\mathbb{RP}^{2n-1}$ to each interval in $\mathrm{Lag}(\mathbb{R}^{2n})$ and explain how to construct the fundamental domain for a generalized Schottky group.

Definition 4.5.6. Let P, Q be an ordered pair of transverse Lagrangians. We define the *positive halfspace* $\mathcal{P}(P, Q)$ as the subset

$$\mathcal{P}(P, Q) := \{\ell \in \mathbb{RP}^{2n-1} \mid \mathcal{B}_{PQ}|_{\ell \times \ell} > 0\}.$$

It is the set of positive lines for the form $\mathcal{B}_{PQ}$.

The positive halfspace $\mathcal{P}(P, Q)$ is bounded by the conic defined by $\mathcal{B}_{PQ} = 0$. This type of bounding hypersurface was introduced by Guichard and Wienhard in order to describe Anosov representations of closed surfaces into $\mathrm{Sp}(2n, \mathbb{R})$. They are also the boundaries of $\mathbb{R}$–tubes defined in [BP15]. A symplectic linear transformation $T \in \mathrm{Sp}(2n, \mathbb{R})$ acts on positive halfspaces in the following way : $T\mathcal{P}(P, Q) = \mathcal{P}(TP, TQ)$. If S is an antisymplectic transformation, on the other hand, we have $S\mathcal{P}(P, Q) = \overline{\mathcal{P}(SP, SQ)}^c = \mathcal{P}(SQ, SP)$.

Proposition 4.5.7. *Let P, Q be an ordered pair of Lagrangians. Then,*

$$\mathcal{P}(Q, P) = \overline{\mathcal{P}(P, Q)}^c = \sigma_{PQ}(\mathcal{P}(P, Q))$$

Proof. For the first equality,

$$\mathcal{B}_{QP}(v, w) = \omega(v, \sigma_{QP}(w)) = \omega(v, -\sigma_{PQ}(w)) = -\mathcal{B}_{PQ}(v, w).$$

For the second equality, notice that $\mathcal{B}_{PQ}(\sigma_{PQ}(v), \sigma_{PQ}(w)) = -\mathcal{B}_{PQ}(v, w)$. □

Proposition 4.5.8. *A positive halfspace is the projectivization of an interval, that is,*

$$\mathcal{P}(P, Q) = \bigcup_{L \in ((P, Q))} \mathbb{P}(L)$$

Proof. If $\ell \subset L$ for some $L \in ((P, Q))$, then

$$\mathcal{B}_{PQ}|_{\ell \times \ell} > 0$$

and so $\ell \in \mathcal{P}(P, Q)$.

Conversely, if $\ell \in \mathcal{P}(P, Q)$, then we wish to find a Lagrangian $L \supset \ell$ with $\mathsf{M}(P, L, Q) = n$. Consider the subspace $V = \ell \oplus \sigma_{PQ}(\ell)$. The form $\mathcal{B}_{PQ}$ has signature $(1, 1)$ on that subspace, and so its orthogonal has signature $(n-1, n-1)$. Moreover, the form ω is nondegenerate on V so $V^{\perp_\omega}$ is a symplectic subspace. Notice that, since $\mathcal{B}_{PQ}(v, w) = \omega(v, \sigma_{PQ}(w))$, the equality

$$(\sigma_{PQ}(U))^{\perp_\omega} = U^{\perp_\mathcal{B}}$$

holds for any subspace U. As a consequence, we obtain

$$V^{\perp_\mathcal{B}} = (\ell \oplus \sigma_{PQ}(\ell))^{\perp_\mathcal{B}} = \ell^{\perp_\mathcal{B}} \cap (\sigma_{PQ}(\ell))^{\perp_\mathcal{B}} = (\sigma_{PQ}(\ell))^{\perp_\omega} \cap \ell^{\perp_\omega} = V^{\perp_\omega}.$$

Therefore we can pick a positive definite Lagrangian $L' \subset V^\perp$ which will be orthogonal to ℓ for both ω and $\mathcal{B}_{PQ}$, so $L = L' \oplus \ell$ is a positive definite Lagrangian containing ℓ. □

Now we can prove the disjointness criterion for positive halfspaces.

Proposition 4.5.9. *If (P, Q, R, S) is a cycle in $\mathrm{Lag}(\mathbb{R}^{2n})$, then $\overline{\mathcal{P}(P, Q)}$ is disjoint from $\overline{\mathcal{P}(R, S)}$. Moreover, $\mathcal{P}(P, Q)$ and $\mathcal{P}(Q, R)$ are disjoint and we have $\overline{\mathcal{P}(P, Q)} \cap \overline{\mathcal{P}(Q, R)} = \mathbb{P}(Q)$.*

Proof. We first examine $\mathcal{P}(P, Q)$ and $\mathcal{P}(Q, R)$. Let $\ell \in \mathcal{P}(P, Q)$. By Proposition 4.5.8, $\ell \subset L$ for some Lagrangian L with $\mathsf{M}(P, L, Q) = n$, or equivalently $\overrightarrow{PLQ}$. Transitivity of the PCO yields $\overrightarrow{LQR}$, so $\mathsf{M}(Q, L, R) = -n$ by skew–symmetricity of M. Therefore, $\mathcal{B}_{QR}|_{\ell \times \ell} < 0$, so $\mathcal{P}(P, Q)$ and $\mathcal{P}(Q, R)$ are disjoint. In fact, this argument shows that $\mathcal{P}(P, Q)$ and $\overline{\mathcal{P}(Q, R)}$ are disjoint, and completely analogous reasoning shows that $\overline{\mathcal{P}(P, Q)}$ and $\mathcal{P}(Q, R)$ are disjoint as well.

Now assume that $\ell \in \overline{\mathcal{P}(P, Q)} \cap \overline{\mathcal{P}(Q, R)}$, and let $0 \neq v \in \ell$. We just saw that we must have

$$\mathcal{B}_{PQ}(v, v) = \omega(v, \sigma_{PQ}(v)) = 0 \tag{4.5.1}$$

and

$$\mathcal{B}_{QR}(v, v) = \omega(v, \sigma_{QR}(v)) = 0. \tag{4.5.2}$$

Let $v = p + q$ be the expression of v in the splitting $P \oplus Q$. Similarly, let $v = q' + r$ be the splitting according to $Q \oplus R$. Rewriting (4.5.1) and (4.5.2),

$$0 = \omega(p + q, p - q) = 2\omega(q, p)$$

and

$$0 = \omega(q' + r, q' - r) = 2\omega(r, q').$$

Using these, we can deduce that $\omega(p,r)=0$, since

$$0=\omega(v,v)=\omega(p+q,q'+r)=\omega(p,q')+\omega(p,r)+\omega(q,r),$$

and

$$\begin{aligned}\omega(p,q')&=\omega(p+q,q')=\omega(q'+r,q')=\omega(r,q')=0,\\ \omega(q,r)&=\omega(q,q'+r)=\omega(q,p+q)=\omega(q,p)=0.\end{aligned}$$

But since $v=p+q=q'+r$,

$$q-q'=r-p$$

is the splitting of $q-q'\in Q$ according to $P\oplus R$, and by $\overrightarrow{PQR}$, it should satisfy

$$0<\omega(-p+r,-p-r)=2\omega(p,r)$$

unless $r-p=0$. So we deduce that r=p, and by transversality of P and R, $p=r=0$ and so $v=q\in Q$. We conclude

$$\overline{\mathcal{P}(P,Q)}\cap\overline{\mathcal{P}(Q,R)}\subset\mathbb{P}(Q).$$

The reverse inclusion is clear from the definition of $\mathcal{B}_{PQ}$ and $\mathcal{B}_{QR}$.

Finally, in order to show that $\overline{\mathcal{P}(P,Q)}$ and $\overline{\mathcal{P}(R,S)}$ are disjoint, let us refine the cycle. Pick Lagrangians $A\in(\!(S,P)\!)$ and $B\in(\!(Q,R)\!)$, so that (A,P,Q,B,R,S) is a cycle. For any $\ell\in\overline{\mathcal{P}(P,Q)}$, we can pick a sequence $\ell_n\in\mathcal{P}(P,Q)$ converging to ℓ. Using Proposition 4.5.8 again, we find Lagrangians $L_n\supset\ell_n$ with $\overrightarrow{PL_nQ}$. After taking a subsequence, L_n converges to $L\in\overline{(\!(P,Q)\!)}\subset(\!(A,B)\!)$, and we have $\ell\subset L$. Now it follows as before that $\mathsf{M}(R,L,S)=-n$, so $\mathcal{B}_{RS}|_{\ell\times\ell}<0$ and ℓ cannot be in the closure of $\mathcal{P}(R,S)$. $\square$

For any generalized Schottky group, we can use the previous proposition to construct a fundamental domain. If the defining intervals for the Schottky group are

$$(\!(a_1^\pm,b_1^\pm)\!),\dots,(\!(a_g^\pm,b_g^\pm)\!)\subset\mathrm{Lag}(\mathbb{R}^{2n}),$$

let

$$D=\bigcap_{j=1}^{g}\left(\mathcal{P}(a_j^+,b_j^+)\cup\mathcal{P}(a_j^-,b_j^-)\right)^c.$$

That is, D is the closed subset of $\mathbb{RP}^{2n-1}$ which is the complement of the union of all positive halfspaces defined by the intervals. The interiors of the translates of D are all disjoint by the two previous propositions and the boundary components are identified pairwise, so D is a fundamental domain for its orbit Ω (Figure 4.4). Ω is in general hard to describe, but if the generalized Schottky group is purely hyperbolic, we can identify it precisely. To do so, we first need to equip intervals with Riemannian metrics by identifying them with symmetric spaces. This will allow us to make use of a contraction property proven in [Bou93].

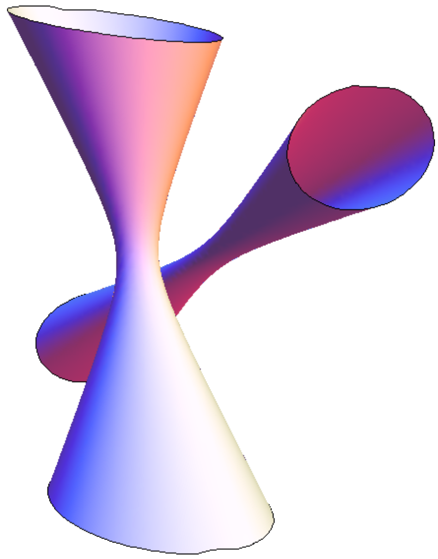


Figure 4.3: A pair of disjoint positive halfspaces in $\mathbb{RP}^3$

4.5.3 Intervals as symmetric spaces

We will now describe how to identify an interval in $\mathrm{Lag}(\mathbb{R}^{2n})$ with the symmetric space associated with $\mathrm{GL}(n, \mathbb{R})$, endowing any interval with a canonical Riemannian metric.

Let $P, Q \in \mathrm{Lag}(\mathbb{R}^{2n})$ be two transverse Lagrangians. As we saw in Lemma 3.2.17, all Lagrangians in the interval $(\!(P, Q)\!)$ have to be transverse to Q, so they are graphs of linear maps $f : P \to Q$. The isotropy condition on f is given by

$$\omega(v + f(v), v' + f(v')) = \omega(v, f(v')) + \omega(f(v), v') = 0 \quad \forall v, v' \in P.$$

Now we recall from our discussion of the Maslov index that we can associate the bilinear form

$$\begin{aligned} \mathcal{B}_{PQ} : P \oplus Q &\to \mathbb{R} \\ (v, w) &\mapsto \omega(v, \sigma_{PQ}(w)) \end{aligned}$$

to the splitting given by P and Q, and the index of its restriction to $\mathrm{graph}(f)$ is the Maslov index $\mathsf{M}(P, \mathrm{graph}(f), Q)$. We observe that this restriction is given by

$$\mathcal{B}_{PQ}(v + f(v), v' + f(v')) = \omega(v, f(v')) - \omega(f(v), v') = 2\omega(v, f(v')),$$

where the last equation follows from the isotropy condition on f. This bilinear form on $\mathrm{graph}(f)$ can also be seen as a symmetric bilinear form on P. Maximality of the Maslov index then translates to this form being positive definite.

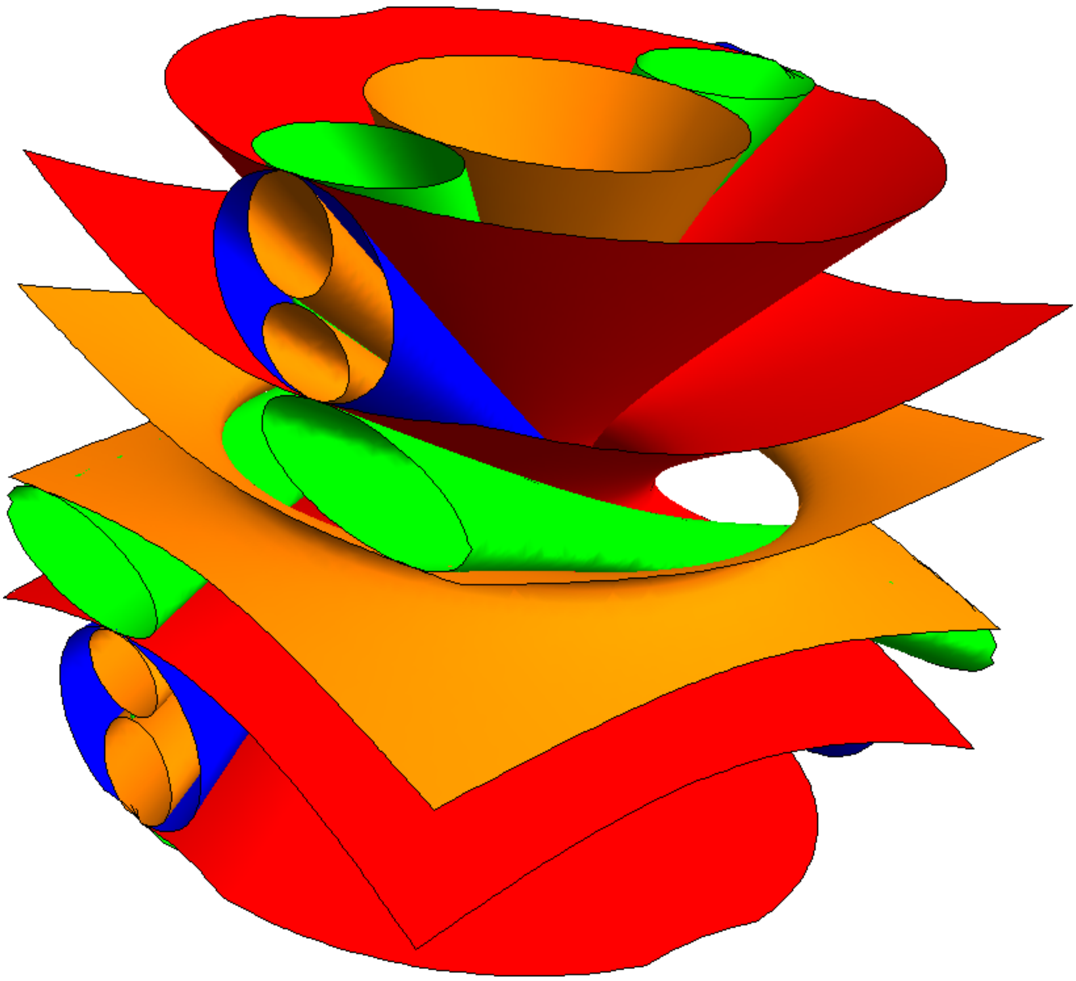


Figure 4.4: The first two generations of positive halfspaces for a two-generator Schottky group in $\mathrm{Sp}(4, \mathbb{R})$. First order (boundaries of) halfspaces are in red and blue, second order halfspaces are in orange and green.

Conversely, given a symmetric bilinear form b on P, we obtain, for any $v' \in P$, a linear functional

$$\left(v \mapsto \frac{1}{2} b(v, v')\right) \in P^*.$$

Using the isomorphism

$$\begin{aligned} Q &\to P^* \\ w &\mapsto \omega(\cdot, w), \end{aligned}$$

we see that there is a unique vector $f(v') \in Q$ such that $b(v, v') = 2\omega(v, f(v'))\ \forall v \in P$. This uniquely defines a linear map $f : P \to Q$, and we have

$$2\left(\omega(v, f(v')) + \omega(f(v), v')\right) = b(v, v') - b(v', v) = 0,$$

so graph(f) is Lagrangian. The Maslov index $\mathsf{M}(P, \mathrm{graph}(f), Q)$ is maximal if and only if b is positive definite. This gives an identification of $(\!(P, Q)\!)$ with the space of positive definite symmetric bilinear forms on P, which is the symmetric space of $\mathrm{GL}(P)$.

The stabilizer in $\mathrm{Sp}(2n, \mathbb{R})$ of the pair $(\!(P, Q)\!)$ can be identified with $\mathrm{GL}(P)$ since any element $A \in \mathrm{GL}(P)$ uniquely extends to a symplectomorphism of $\mathbb{R}^{2n}$ preserving Q: The linear forms $v \mapsto \omega(A(v), w)$ on P, for $w \in Q$, give rise to a unique automorphism $A^* : Q \to Q$ such that

$$\omega(A(v), w) = \omega(v, A^*(w)) \quad \forall v \in P, w \in Q.$$

Then $A \oplus (A^*)^{-1}$ is the unique symplectic extension of A preserving Q. It acts on linear maps $f : P \to Q$ by

$$f \mapsto (A^*)^{-1} f A^{-1},$$

on bilinear forms on P by

$$(A \cdot b)(v, v') = b(A^{-1}v, A^{-1}v'),$$

and the identification of graphs and bilinear forms is equivariant with respect to these actions. In particular, $\mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(\!(P, Q)\!)$ is identified with the isometry group of the symmetric space $(\!(P, Q)\!)$.

4.5.4 The Riemannian metric on intervals

The identification of an interval $(\!(P, Q)\!)$ with the space of positive definite symmetric bilinear forms on P yields the following Riemannian metric on $(\!(P, Q)\!)$: The tangent space at any point b is naturally identified with the vector space of symmetric bilinear forms on P. A vector $\alpha \in \mathrm{T}_b(\!(P, Q)\!)$ then corresponds to a b–self–adjoint endomorphism X_α of P via

$$\alpha(v, w) = b(v, X_\alpha w) \qquad \forall v, w \in P.$$

The Riemannian metric is given by

$$\langle \alpha, \beta \rangle_b = \operatorname{tr}(X_\alpha X_\beta).$$

There is also a simple formula for the Riemannian distance between two points in the interval $(\!(P,Q)\!)$ (see, for example, [Maa71, Theorem p.27]):

Definition 4.5.10. Let f, g be linear maps from P to Q whose graphs are elements of $(\!(P,Q)\!)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the automorphism fg^{-1}. Then, define

$$d_{PQ}(f,g) = \sqrt{\sum_{i=1}^{n} \log(\lambda_i)^2}.$$

Both descriptions readily imply that an element $T \in \mathrm{Sp}(2n, \mathbb{R})$ maps any interval isometrically onto its image interval.

Proposition 4.5.11. *Let $T \in \mathrm{Sp}(2n,\mathbb{R})$ be a symplectic transformation. Then for any transverse Lagrangians P and Q, T induces an isometry*

$$T : (\!(P,Q)\!) \to (\!(TP,TQ)\!).$$

Proof. T maps a linear map $f : P \to Q$ to $(A^*)^{-1} f A^{-1}$, where $A : P \to TP$ is the restriction of T to P and $(A^*)^{-1} : Q \to TQ$ is its restriction to Q. The automorphism fg^{-1} is therefore mapped to $(A^*)^{-1} f g^{-1} A^*$ and its eigenvalues remain unchanged. □

The following useful proposition is proved in [Bou93].

Proposition 4.5.12. *Let $T \in \mathrm{Sp}(2n,\mathbb{R})$ such that $\overline{T(\!(P,Q)\!)} \subset (\!(P,Q)\!)$. Then T is a Lipschitz contraction for the distance d_{PQ}.*

Combining the previous two propositions yields the version of the contraction statement that we want to use:

Corollary 4.5.13. *Let $T \in \mathrm{Sp}(2n,\mathbb{R})$ such that $\overline{T(\!(P,Q)\!)} \subset (\!(R,S)\!)$. Then T is a Lipschitz contraction with respect to the distances d_{PQ} and d_{RS}.*

4.5.5 The domain of discontinuity in $\mathbb{RP}^{2n-1}$

Let $\rho\colon \Gamma \to \mathrm{Sp}(2n,\mathbb{R})$ be a purely hyperbolic generalized Schottky representation. We will now analyze the orbit $\Omega = \rho(\Gamma) \cdot D$ of the fundamental domain $D \subset \mathbb{RP}^{2n-1}$ we defined in Section 4.5.2. Corollary 4.5.13 implies that the prerequisites of Theorem 4.2.3 are satisfied, so we obtain a continuous, equivariant, increasing boundary map

$$\xi\colon \partial_\infty \Gamma \to \mathrm{Lag}(\mathbb{R}^{2n}).$$

Recall that we constructed this boundary map by using nested contracting sequences of intervals. The following lemma shows that when a nested sequence of intervals in $\mathrm{Lag}(\mathbb{R}^{2n})$ collapses to a single Lagrangian L, the corresponding halfspaces in $\mathbb{RP}^{2n-1}$ collapse to the projectivization of L.

Lemma 4.5.14. *Let L_1^k, L_2^k be sequences of Lagrangians such that $L_1^k \to L$ and $L_2^k \to L$ with $\overrightarrow{L_1^k L L_2^k}$ for all k. Then,*

$$\bigcap_{k=1}^{\infty} \overline{\mathcal{P}(L_1^k, L_2^k)} = \bigcap_{k=1}^{\infty} \mathcal{P}(L_1^k, L_2^k) = \mathbb{P}(L).$$

Proof. Assume $\mathcal{B}_{L_1^k L_2^k}(v, v) \geq 0$ for all k. Then we can find $v_k \xrightarrow{k\to\infty} v$ such that $\mathcal{B}_{L_1^k L_2^k}(v_k, v_k) > 0$ for all k. Now, by Proposition 4.5.8, v_k can be completed to a Lagrangian L^k with $\mathsf{M}(L_1^k, L^k, L_2^k) = n$, so $L^k \in (\!(L_1^k, L_2^k)\!)$ for all k, which implies $L^k \to L$, and so $v \in L$. $\square$

Now we are ready to describe the orbit Ω.

The union of D with the positive halfspaces associated to the defining Schottky intervals is all of $\mathbb{RP}^{2n-1}$, by definition of D. Denote by Γ_ℓ the set of words in Γ of length up to ℓ. Then, the union of $\rho(\Gamma_\ell)D$ with the projectivizations (positive halfspaces) of all ℓ–th order intervals again covers all of $\mathbb{RP}^{2n-1}$. Thus, when taking words of arbitrary length in Γ, these two pieces become respectively the full orbit $\rho(\Gamma)D$ and limits of nested positive halfspaces. By Corollary 4.5.13, ℓ–th order intervals collapse to single Lagrangians as $\ell \to \infty$ (see the proof of Theorem 4.2.3 for more details), so Lemma 4.5.14 shows that every such sequence of nested halfspaces collapses to the projectivization of a single Lagrangian. From this, we conclude:

Theorem 4.5.15. *The domain $\Omega = \rho(\Gamma)D$ is the complement of a Cantor set of projectivized Lagrangian n–planes in $\mathbb{RP}^{2n-1}$. This Cantor set is exactly the projectivization of the increasing set of Lagrangians defined by the boundary map ξ.*

Remark 4.5.16. The symplectic structure on $\mathbb{R}^{2n}$ induces a contact structure on $\mathbb{RP}^{2n-1}$ preserved by the symplectic group. The projectivizations of Lagrangian subspaces correspond to *Legendrian* $(n-1)$–dimensional planes in $\mathbb{RP}^{2n-1}$.

4.5.6 The Anosov property

By combining results of previous sections, we are now able to show that purely hyperbolic Schottky representations are Anosov.

Theorem 4.5.17. *Let $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ be a purely hyperbolic generalized Schottky representation. Then ρ is Anosov with respect to $P_{\{\alpha_n\}}$, the stabilizer of a Lagrangian.*

Proof. We just have to verify that all the hypotheses of Theorem 4.4.5 are satisfied. We saw that the partial cyclic order determined by the Maslov index on $\mathrm{Lag}(\mathbb{R}^{2n})$ is increasing–complete, proper and regular in Proposition 3.2.20. Moreover, two Lagrangians are comparable if and only if they are transverse. This follows from Lemma 3.2.17 and Lemma 3.2.18 together with transitivity of the G–action on transverse pairs. Finally, by Corollary 4.5.13, the contraction condition is satisfied as well. □

4.6 Schottky groups in $\mathrm{PSL}(n, \mathbb{R})$

4.6.1 The Riemannian metric on intervals

We will now equip each interval in $\mathcal{F}_n$ with a canonical Riemannian metric. Analogous to the $\mathrm{Sp}(2n, \mathbb{R})$ case, this will allow us to derive a contraction property for maps sending one interval strictly inside another. The metric is constructed by using Plücker coordinates to embed flags into a product of projective spaces and using Hilbert metrics on those.

By considering all subspaces contained in a flag separately, we can map a flag into a product of oriented Grassmannians. If n is odd, this is an embedding

$$\mathcal{F}_n \longrightarrow \mathrm{Gr}_1^+(\mathbb{R}^n) \times \ldots \times \mathrm{Gr}_{n-1}^+(\mathbb{R}^n).$$

If n is even, recall that we quotient out the action of $-I$. We get the embedding

$$\mathcal{F}_n \longrightarrow \left(\mathrm{Gr}_1^+(\mathbb{R}^n) \times \ldots \times \mathrm{Gr}_{n-1}^+(\mathbb{R}^n)\right)/(-I),$$

where the action of $-I$ inverts the orientation on all odd–dimensional oriented Grassmannians simultaneously.

Let us briefly review the Plücker embedding. Consider the canonical map

$$\begin{aligned} f\colon (\mathbb{R}^n)^k &\longrightarrow \bigwedge^k \mathbb{R}^n \\ (v_1, \ldots, v_k) &\longmapsto v_1 \wedge \ldots \wedge v_k. \end{aligned}$$

Interpreting an element of $(\mathbb{R}^n)^k$ as a $(n \times k)$–matrix M, the coefficients of $f(M)$ with respect to the standard basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$$

of $\bigwedge^k \mathbb{R}^n$ are given by the minors $M^{i_1 \ldots i_k}_{1 \ldots k}$.

Applying f to (arbitrarily chosen) bases of k–planes in $\mathbb{R}^n$ gives rise to the Plücker embedding

$$\mathrm{Gr}_k(\mathbb{R}^n) \longrightarrow \mathbb{P}(\bigwedge^k \mathbb{R}^n).$$

In the same way, applying f to positive bases of oriented k–planes yields the embedding

$$\iota_k \colon \mathrm{Gr}_k^+(\mathbb{R}^n) \longrightarrow \mathbb{S}(\bigwedge^k \mathbb{R}^n),$$

where $\mathbb{S}$ denotes the spherical projectivization (modding out by positive scalars). As before, when representing an element $P \in \mathrm{Gr}_k^+(\mathbb{R}^n)$ by a $(n \times k)$–matrix M, the spherical equivalence class of the vector of minors $M^{i_1 \dots i_k}_{1 \dots k}$ gives the standard homogeneous coordinates for $\phi(P) \in \mathbb{S}(\bigwedge^k \mathbb{R}^n)$. They are called the Plücker coordinates of P.

Now we combine the two embeddings. Assume for now that n is odd. Let $F \in ((F_e, F_{w_0})) \subset \mathcal{F}_n$ be a complete oriented flag, and consider its image under the composition

$$\mathrm{Pl}_l \colon \mathcal{F}_n \longrightarrow \prod_k \mathrm{Gr}_k^+(\mathbb{R}^n) \longrightarrow \prod_k \mathbb{S}(\bigwedge^k \mathbb{R}^n) \xrightarrow{\pi_l} \mathbb{S}(\bigwedge^l \mathbb{R}^n),$$

where π_l is the projection to the l–th factor. Its Plücker coordinates are all positive since F has a lower triangular, unipotent, totally positive representative. Therefore, $\mathrm{Pl}_l(F)$ lies in the interior of the standard (convex) simplex $\Delta \subset \mathbb{S}(\bigwedge^l \mathbb{R}^n)$ which carries the Hilbert metric d_Δ. Explicitly, this metric is given by

$$d_\Delta([x_1 : \dots : x_r], [y_1 : \dots : y_r]) = \max_{i,j} \left| \log \frac{\frac{x_i}{x_j}}{\frac{y_i}{y_j}} \right| = \log \frac{\max_i \left| \frac{x_i}{y_i} \right|}{\min_i \left| \frac{x_i}{y_i} \right|}, \tag{4.6.1}$$

where $r = \binom{n}{l}$.

If n is even and l is odd, Pl_l maps to $\mathbb{P}(\bigwedge^l \mathbb{R}^n)$. This does not change anything for the Hilbert metric (the projection $\mathbb{S}(\mathbb{R}^m) \to \mathbb{P}(\mathbb{R}^m)$ restricts to a diffeomorphism on Δ)

Definition 4.6.1. Let $F_1, F_2 \in ((F_e, F_{w_0}))$ be complete oriented flags. Then the *interval distance* of F_1, F_2 is given by

$$d_{((F_e, F_{w_0}))}(F_1, F_2) := \max_{1 \le k \le n-1} d_\Delta \left(\mathrm{Pl}_k(F_1), \mathrm{Pl}_k(F_2) \right).$$

Lemma 4.6.2. *The metric $d_{((F_e, F_{w_0}))}$ is invariant under the action of $A_0 = \mathrm{Stab}(F_e) \cap \mathrm{Stab}(F_{w_0})$.*

Proof. The stabilizer consists of diagonal matrices with positive diagonal entries λ_i. The induced action on $\bigwedge^l \mathbb{R}^n$ multiplies the basis vector $e_{i_1} \wedge \dots \wedge e_{i_l}$ with the scalar $\lambda_{i_1} \dots \lambda_{i_l}$. By (4.6.1), this action is an isometry for each Hilbert metric, so it is an isometry for the interval metric. □

By this lemma and since $\mathrm{Stab}((F_e, F_{w_0})) = \mathrm{Stab}(F_e) \cap \mathrm{Stab}(F_{w_0})$ by Corollary 3.3.34, we obtain a unique Riemannian metric d_I on every interval $I \subset \mathcal{F}_n$. If $gI = J$ for $g \in \mathrm{PSL}(n, \mathbb{R})$, then $g : I \to J$ is an isometry.

A special case of a result proved in [Bir57] shows that linear maps sending the simplex Δ strictly inside itself act as contractions for the Hilbert metric.

Lemma 4.6.3 ([Bir57, Lemma 4.1])**.** *Let $\Delta \subset \mathbb{P}(\mathbb{R}^n)$ be the standard simplex and $g \in \mathrm{PSL}(n,\mathbb{R})$ a projective linear transformation such that $\overline{g(\Delta)} \subset \mathring{\Delta}$. Then $g|_\Delta$ is a contraction for the metric d_Δ.*

Corollary 4.6.4. *Let $I, J \subset \mathcal{F}_n$ be intervals and $g \in \mathrm{PSL}(n,\mathbb{R})$ such that $\overline{gI} \subset J$. Then g, considered as a map $I \to J$, is a contraction with respect to the metrics d_I and d_J.*

Proof. We first assume that $I = J = (\!(F_e, F_{w_0})\!)$. Then by Proposition 3.3.33, g is totally positive. Let $\widehat{g} \in \mathrm{SL}(n\mathbb{R})$ be the totally positive lift. For any matrix $M \in \mathrm{SL}(n,\mathbb{R})$, the Cauchy–Binet formula yields

$$(\widehat{g}M)^{\mathbf{i}}_{1\dots k} = \sum_{\mathbf{j}\in\mathcal{I}(k,n)} \widehat{g}^{\mathbf{i}}_{\mathbf{j}} M^{\mathbf{j}}_{1\dots k}.$$

In other words, the matrix coefficients of the induced action of $\widehat{g}$ on $\bigwedge^k \mathbb{R}^n$ are exactly the minors $\widehat{g}^{\mathbf{i}}_{\mathbf{j}}$. Since all of them are positive, this implies $\overline{g(\Delta)} \subset \Delta$ and the claim follows from Lemma 4.6.3.

For arbitrary I and J, choose $f_1, f_2 \in \mathrm{PSL}(n,\mathbb{R})$ satisfying $f_1(\!(F_e, F_{w_0})\!) = I$ and $f_2(\!(F_e, F_{w_0})\!) = J$. We observed before that f_1, f_2 are isometries between $(\!(F_e, F_{w_0})\!)$ and I resp. J. Then $f_2^{-1} g f_1 \colon (\!(F_e, F_{w_0})\!) \to (\!(F_e, F_{w_0})\!)$ is a contraction, and so is $g\colon I \to J$. ☐

4.6.2 The Anosov property

We now have all the necessary ingredients to show that Schottky representations into $\mathrm{PSL}(n,\mathbb{R})$ are Anosov (see Definition 2.3.3 for the setting of oriented parabolic subgroups).

Theorem 4.6.5. *Let $\rho\colon \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a purely hyperbolic generalized Schottky representation. Then ρ is B_0–Anosov with transversality type*

$$w_0 = \begin{pmatrix} & & & & \iddots \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

Proof. As in the case of $\mathrm{Sp}(2n,\mathbb{R})$, we collect results from previous sections to show that the hypotheses of Theorem 4.4.5 are satisfied. The partial cyclic order on complete oriented flags is increasing–complete and proper by Proposition 3.3.30, and it is regular by Corollary 3.3.42. Two complete oriented flags $F_1, F_2 \in \mathcal{F}_n$ are comparable if and only if F_2 is contained in the cell $C_{w_0}(F_1)$ of flags at relative position w_0 to F_1 (see Section 3.3.2). In particular, F_1 and F_2 are transverse. The contraction condition is satisfied by Corollary 4.6.4. □

As we mentioned in Section 2.6, this theorem implies that all w_0–balanced ideals in $\widetilde{W}^{\mathrm{PSL}(n,\mathbb{R})}$ describe cocompact domains of discontinuity for purely hyperbolic generalized Schottky representations. In particular, this applies to the ideals in Section 2.5.5. If $n = 4k+3$, the corresponding domains in S^{4k+2} are obtained by removing the spherical projectivizations of all positive halves of $(2k+2)$–dimensional parts of the oriented flag limit curve. If the dimension $n = 2k$ is even, there is a cocompact domain in $\mathbb{RP}^{2k-1}$, obtained by removing the projectivizations of all k–dimensional parts of the limit curve.

These cocompact domains admit nice fundamental domains, similar to the case of $\mathrm{Sp}(2n,\mathbb{R})$ (see Section 4.5.2). There is a notion of halfspace in S^{4k+2} if $n = 4k+3$ or in $\mathbb{RP}^{2k-1}$ if $n = 2k$, obtained by (spherically) projectivizing all the (positive halves of) $(2k+2)$– resp. k–dimensional parts of complete oriented flags contained in an interval. Properties of these halfspaces are linked to variation diminishing properties of totally positive matrices. This will be treated in more detail in [BT18].

5 Proper affine actions

In this chapter, we will deal with both linear and affine representations into $\mathrm{SO}(p,q)$ and $\mathrm{SO}(p,q) \ltimes \mathbb{R}^{p+q}$ respectively. We will use ϱ for linear representations and ρ for affine representations. The homomorphism taking the linear part will be denoted by

$$\begin{aligned} L\colon\ \mathrm{SO}(p,q) \ltimes \mathbb{R}^{p+q} &\to \mathrm{SO}(p,q) \\ (a,v) &\mapsto a. \end{aligned}$$

5.1 Preliminaries

5.1.1 On maximal isotropic and maximal definite subspaces with respect to an indefinite form

In this section, we state some results about indefinite orthogonal groups and orientations on certain subspaces of $\mathbb{R}^{p,q}$ which will prove useful later on.

Let $p \leq q$, and let $\mathbb{R}^{p,q}$ denote the vector space $\mathbb{R}^{p+q}$, equipped with an indefinite symmetric bilinear form $b_{p,q}$ of signature (p,q). Furthermore, let π_+ and π_- denote the two projections corresponding to an orthogonal splitting

$$\mathbb{R}^{p+q} = V_+ \oplus V_-$$

such that $b_{p,q}|_{V_+}$ is positive definite and $b_{p,q}|_{V_-}$ is negative definite. The stabilizer of V_+ (and equivalently the pair (V_+,V_-)) in $\mathrm{SO}(p,q)$ is isomorphic to $\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$. We consider the space

$$X_{p,q} := \{V \subset \mathbb{R}^{p+q} \mid \dim(V) = p,\ b_{p,q}|_{V\times V} \text{ is positive definite}\}.$$

It is a model for the symmetric space associated to $\mathrm{SO}(p,q)$ and can be identified with $\mathrm{SO}(p,q)/\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$. It is simply connected, which we can in fact see directly by the following argument.

Lemma 5.1.1. *The space $X_{p,q}$ is contractible.*

Proof. We construct a deformation retraction

$$f\colon X_{p,q} \times [0,1] \longrightarrow X_{p,q}$$

onto the point V_+, where $f(\cdot,0) = \mathrm{Id}$ and $f(\cdot,1)$ is the constant map with image V_+. Decompose any vector $v \in \mathbb{R}^{p+q}$ as $v = v_+ + v_-$, where $v_\pm = \pi_\pm(v)$, and consider the map

$$\begin{aligned} g : \mathbb{R}^{p+q} \times [0,1] &\to \mathbb{R}^{p+q} \\ (v_+ + v_-, t) &\mapsto v_+ + (1-t)v_-. \end{aligned}$$

We observe the following:

- If $b_{p,q}(v,v) > 0$, then $b_{p,q}(g(v,t), g(v,t)) > 0 \; \forall t$.
- For any $V \in X_{p,q}$, the projection π_+ restricts to an isomorphism $V \xrightarrow{\cong} V_+$: Otherwise V would have to be contained in the subspace $\pi_+^{-1}(\pi_+(V))$ of signature (p',q) with $p' < p$, a contradiction.

Therefore, g induces the desired map f. □

Using this lemma, we can describe the two connected components of $\mathrm{SO}(p,q)$. By simple connectivity, it is possible to choose an orientation on each subspace $V \in X_{p,q}$ in a continuous way (in other words, the twofold orientation cover of $X_{p,q}$ is disconnected). An element $A \in \mathrm{SO}(p,q)$ can then either preserve or reverse orientations on the elements of $X_{p,q}$, and a short discussion shows that this distinguishes the two components:
Any element A can be deformed to one that fixes V_+. To do this, choose a path between $A(V_+)$ and V_+ (e.g. the one described in the previous lemma), then choose a corresponding path A_t in $\mathrm{SO}_0(p,q)$ such that $A_0 = A$ and A_1 fixes V_+. If A preserves orientations on $X_{p,q}$, we thus obtain a transformation in $\mathrm{SO}(p) \times \mathrm{SO}(q)$, which is connected, so A lies in $\mathrm{SO}_0(p,q)$. On the other hand, if A reverses orientations on $X_{p,q}$, it cannot lie in the identity component by continuity of these orientations. We can then deform A to a fixed standard representative in $\mathrm{O}_-(p) \times \mathrm{O}_-(q)$ (where $\mathrm{O}_-(p)$ denotes the nonidentity component of $\mathrm{O}(p)$).

In the following sections, our main interest lies with the space

$$\mathrm{Is}_p(\mathbb{R}^{p,q}) = \{V \subset \mathbb{R}^{p+q} \mid \dim(V) = p, \; b_{p,q}|_{V \times V} \equiv 0\}$$

of maximal isotropic subspaces of $\mathbb{R}^{p,q}$, as well as the maximal parabolic subgroups of $\mathrm{SO}_0(p,q)$ stabilizing such a subspace.

We observe that the above choice of orientations for elements of $X_{p,q}$ induces a consistent choice of orientations for $\mathrm{Is}_p(\mathbb{R}^{p,q})$ as well:
As above, let $\mathbb{R}^{p,q} = V_+ \oplus (V_+)^\perp = V_+ \oplus V_-$ be any orthogonal splitting into a positive definite and a negative definite subspace, and let $\pi_\pm$ denote the corresponding projections. By the same argument as used in the previous lemma for positive

definite subspaces, the restriction of π_+ induces an isomorphism $L \xrightarrow{\cong} V_+$ for any $L \in \mathrm{Is}_p(\mathbb{R}^{p,q})$. We use this isomorphism and the orientation on V_+ to define an orientation on L. Since $X_{p,q}$ is connected and the orientations vary continuously, the induced orientation on L does not depend on the choice of V_+. Similarly, this choice of orientations on elements of $\mathrm{Is}_p(\mathbb{R}^{p,q})$ is continuous (note that $\mathrm{Is}_p(\mathbb{R}^{p,q})$ has two connected components if $p = q$, see Proposition 5.1.4).
The description of the two connected components of $\mathrm{SO}(p,q)$ now applies in the same way to the action on $\mathrm{Is}_p(\mathbb{R}^{p,q})$: For $A \in \mathrm{SO}(p,q)$, let $V'_+ = A(V_+), V'_- = (V'_+)^\perp = A(V_-)$ and $\pi'_+ = A\pi_+A^{-1}$ be the corresponding projection. Then the diagram

$$\begin{tikzcd} L \arrow[r, "A"] \arrow[d, "\pi_+"] & A(L) \arrow[d, "\pi'_+"] \\ V_+ \arrow[r, "A"] & V'_+ \end{tikzcd}$$

commutes. Both projections preserve orientation by definition, and the map $A : V_+ \to V'_+$ preserves orientation iff $A \in \mathrm{SO}_0(p,q)$, therefore the same is true for the restriction $A : L \to A(L)$.

We summarize this discussion in the following two propositions:

Proposition 5.1.2. *The orientation covers $X^+_{p,q}$ and $\mathrm{Is}^+_p(\mathbb{R}^{p,q})$ are trivial twofold covers. Elements of $X_{p,q}$ and $\mathrm{Is}_p(\mathbb{R}^{p,q})$ can therefore be equipped with orientations in a continuous way.*

Whenever relevant, the orientations in the previous proposition will be assumed to be compatible in the following way: For any $L \in \mathrm{Is}_p(\mathbb{R}^{p,q})$ and $V_+ \in X_{p,q}$, the projection

$$\pi_+ : V_+ \oplus (V_+)^\perp \to V_+$$

induces an orientation–preserving isomorphism $L \xrightarrow{\cong} V_+$. Equivalently, for the induced map between orientation covers, the preimage of the “positive” component of $X^+_{p,q}$ is the positive part of $\mathrm{Is}^+_p(\mathbb{R}^{p,q})$.

Proposition 5.1.3. *A transformation $A \in \mathrm{SO}(p,q)$ belongs to the identity component $\mathrm{SO}_0(p,q)$ if and only if it preserves orientations on (elements of) $X_{p,q}$ and $\mathrm{Is}_p(\mathbb{R}^{p,q})$. Equivalently, if and only if it preserves the two copies in $X^+_{p,q}$ and $\mathrm{Is}^+_p(\mathbb{R}^{p,q})$.*

In view of the following proposition, it is interesting to note that the above construction of orientations on elements of $\mathrm{Is}_p(\mathbb{R}^{p,q})$ does not depend on whether p and q are equal or not.

Proposition 5.1.4. $\mathrm{SO}_0(p,q)$ *acts transitively on* $\mathrm{Is}_p(\mathbb{R}^{p,q})$ *if $p < q$. In particular,* $\mathrm{Is}_p(\mathbb{R}^{p,q})$ *is connected in that case. On the other hand,* $\mathrm{Is}_p(\mathbb{R}^{p,p})$ *has two connected components which are preserved by* $\mathrm{SO}(p,p)$.

Proof. Let b be the (p,q)–bilinear form corresponding to the matrix

$$\begin{pmatrix} & & I_p \\ & -I_{q-p} & \\ I_p & & \end{pmatrix},$$

and let

$$\mathbb{R}^{p,q} = V_+ \oplus V_- = V_+ \oplus (V_+)^\perp \tag{5.1.1}$$

be an orthogonal splitting into a positive definite and a negative definite subspace, with associated projection $\pi_+\colon \mathbb{R}^{p,q} \to V_+$. By Proposition 5.1.2, we can consistently equip maximal isotropic subspaces and positive definite subspaces with orientations. Let $(v_1, \dots, v_p)$ be a positive orthonormal basis of V_+.

We first treat the case $p < q$. Let $E \in \mathrm{Is}_p(\mathbb{R}^{p,q})$ be arbitrary. The isomorphism

$$\pi_+|_E\colon E \xrightarrow{\cong} V_+$$

defines the positive basis (e_i) of E, where $e_i = \pi_+|_E^{-1}(v_i)$. Let $e_i = v_i + w_i$ be the decomposition according to (5.1.1), and define

$$g_i := \frac{1}{2}(v_i - w_i).$$

One verifies easily that $b(g_i, g_j) = 0$ and $b(e_i, g_j) = \delta_{ij}$. Setting $G = \mathrm{span}(g_1, \dots, g_p)$, we thus have a direct sum decomposition

$$\mathbb{R}^{p,q} = E \oplus (E^\perp \cap G^\perp) \oplus G,$$

where $F = (E^\perp \cap G^\perp)$ is negative definite and of dimension $q - p$. Choose an orthonormal basis $(f_1, \dots, f_{q-p})$ of F such that

$$(e_1, \dots, e_p, f_1, \dots, f_{q-p}, g_1, \dots, g_p) \tag{5.1.2}$$

is a positive basis of $\mathbb{R}^{p+q}$. Let M be the matrix with these vectors as columns (with respect to the standard basis). Then $M \in \mathrm{SO}_0(p,q)$ since all scalar products between basis vectors are the same, (5.1.2) is a positive basis and $\mathbb{R}^p \times \{0\}$ is mapped to E in an orientation–preserving way.

We now turn to the case $p = q$. Again, let $E \in \mathrm{Is}_p(\mathbb{R}^{p,p})$ be arbitrary, and construct the basis $(e_1, \dots, e_p, g_1, \dots, g_p)$ as before. The value

$$d(E) := \mathrm{sgn}(\det(e_1, \dots, e_p, g_1, \dots, g_p)) \in \{\pm 1\}$$

only depends on E because a change of basis corresponds to right–multiplication of $(e_1, \dots, e_p)$ with $A \in \mathrm{GL}(p)$ and right–multiplication of $(g_1, \dots, g_p)$ with $(A^t)^{-1}$ ($A \in \mathrm{GL}_+(p)$ if both bases are positive). It is constant on connected components by continuity, and as in the case $p < q$, we see that $\mathrm{SO}_0(p,q)$ acts transitively on $\{E \mid d(E) = 1\}$. One element of $\mathrm{SO}(p,q) \setminus \mathrm{SO}_0(p,q)$ is given by negating e_1 and g_1, which preserves the two connected components of $\mathrm{Is}_p(\mathbb{R}^{p,p})$ since it preserves E. Thus all of $\mathrm{SO}(p,q)$ preserves the connected components of $\mathrm{Is}_p(\mathbb{R}^{p,p})$. Both components $\{E \mid d(E) = \pm 1\}$ are indeed nonempty since switching e_p and g_p in a basis as above inverts $d(E)$. $\square$

5.1.2 The Gromov geodesic flow

Let Γ be a word hyperbolic group and let $\partial_\infty\Gamma$ be its Gromov boundary. The natural action of Γ on its boundary has the following North–South dynamics:

Proposition 5.1.5 ([KB02, Proposition 4.2 & Theorem 4.3])**.** *Every element $\gamma \in \Gamma$ of infinite order has exactly two fixed points γ_+, γ_- in $\partial_\infty\Gamma$. For any open sets $U, V \subset \partial_\infty\Gamma$ such that $\gamma_+ \in U$, $\gamma_- \in V$, there exists $n_0 \in \mathbb{N}$ such that $\gamma^n(\partial_\infty\Gamma - V) \subset U$ $\forall n \geq n_0$.*

The action of Γ on $\partial_\infty\Gamma$ extends to a diagonal action of Γ on

$$\partial_\infty\Gamma^{(2)} := \partial_\infty\Gamma \times \partial_\infty\Gamma \setminus \{(x,x) \mid x \in \partial_\infty\Gamma\}.$$

We denote $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ by $\widetilde{\mathsf{U}_0\Gamma}$ and define the flow

$$\begin{aligned} \psi \colon \widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R} &\to \widetilde{\mathsf{U}_0\Gamma} \\ ((x,y,s),t) &\mapsto (x,y,s+t) \end{aligned}$$

for all $(x,y) \in \partial_\infty\Gamma^{(2)}$ and $s,t \in \mathbb{R}$. Gromov showed that there exists a proper cocompact action of Γ on $\widetilde{\mathsf{U}_0\Gamma}$ which commutes with the flow $\{\psi_t\}_{t\in\mathbb{R}}$ [Gro87, Theorem 8.3C]. The induced action on $\partial_\infty\Gamma^{(2)}$ is the diagonal action. The $\mathbb{R}$–action of the flow extends to a $\mathbb{R} \rtimes \mathbb{Z}_2$–action that commutes with Γ (here, $\mathbb{Z}_2$ switches the two points in $\partial_\infty\Gamma^{(2)}$ and also changes the parameter in $\mathbb{R}$). Moreover, there exists a metric on $\widetilde{\mathsf{U}_0\Gamma}$, well–defined up to Hölder equivalence, such that the $\Gamma \times \mathbb{Z}_2$–action is isometric, the flow ψ_t acts by Lipschitz homeomorphisms and every orbit of the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is a quasi–isometric embedding of $\mathbb{R}$. This metric has the following structure: The visual metric on $\partial_\infty\Gamma$ is well–defined up to Hölder equivalence ([KB02, Theorem 2.18]), inducing the product metric on $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ up to Hölder equivalence. The metric on $\widetilde{\mathsf{U}_0\Gamma}$ satisfying the above properties is bi–Lipschitz with the product metric [Gro87, Corollary 8.3H].

The flow ψ_t on $\widetilde{\mathsf{U}_0\Gamma}$ descends to the *Gromov geodesic flow* on the compact quotient

$$\mathsf{U}_0\Gamma := \Gamma \backslash \left(\partial_\infty\Gamma^{(2)} \times \mathbb{R}\right)$$

which we call the *flow space* of Γ. We denote it by ψ_t as well.

Further details about this construction were worked out by Champetier [Cha94] and Mineyev [Min05]. In particular, the flow space has the following properties which will be important to us later:

Proposition 5.1.6 ([Min05, Theorem 60])**.**

(i) The flow space $\widetilde{\mathsf{U}_0\Gamma}$ is a proper metric space.

(ii) To every element $\gamma \in \Gamma$ of infinite order, we associate its translation length

$$l(\gamma) = \lim_{n\to\infty} \frac{d(\gamma^n x, x)}{n},$$

where $x \in \widetilde{\mathsf{U}_0\Gamma}$ is any point. Then we have

$$l(\gamma) = \inf_{y \in \widetilde{\mathsf{U}_0\Gamma}} d(y, \gamma y)$$

and this infimum is realized on the axis $\{(\gamma_-, \gamma_+, t), t \in \mathbb{R}\}$.

We will also need the following result, which follows from the proof of Lemma 1.3 of [GLM09], using Proposition 4.2 and Theorem 4.3 of [KB02]. We give the proof here for the reader's convenience.

Lemma 5.1.7. *The space $\mathsf{U}_0\Gamma$ is connected.*

Proof. By Proposition 4.2 (1) and (3) in [KB02], every infinite order element $\gamma \in \Gamma$ has exactly two fixed points $\gamma^{\pm} \in \partial_\infty\Gamma$, and the set

$$\{\gamma^- \mid \gamma \in \Gamma \text{ of infinite order }\} \subset \partial_\infty\Gamma$$

is dense. Fix one such element γ and consider the set

$$U = \Gamma \setminus \{(\gamma^-, y, t) \mid y \neq \gamma^-, t \in \mathbb{R}\} \subset \mathsf{U}_0\Gamma.$$

We will show that it is connected. Assume that $W_1, W_2 \subset \mathsf{U}_0\Gamma$ are open sets such that $U = (W_1 \cap U) \sqcup (W_2 \cap U)$, and that W_1 contains a point of $[\{(\gamma^-, \gamma^+, t), t \in \mathbb{R}\}]$. Denoting by $\widetilde{W_i}$ the preimages in $\widetilde{\mathsf{U}_0\Gamma}$, we see that $\{(\gamma^-, \gamma^+, t), t \in \mathbb{R}\} =: \overrightarrow{\gamma^-\gamma^+}$ has to be contained in $\widetilde{W_1}$ since it is connected. Now for any $\gamma^- \neq y \in \partial_\infty\Gamma$, consider the set $\overrightarrow{\gamma^- y}$. We have

$$\lim_{n\to\infty} \gamma^n \cdot \overrightarrow{\gamma^- y} = \lim_{n\to\infty} \overrightarrow{\gamma^-(\gamma^n y)} = \overrightarrow{\gamma^-\gamma^+},$$

so by openness of $\widetilde{W_1}$, the orbit $\Gamma \cdot \overrightarrow{\gamma^- y}$ has to be contained in $\widetilde{W_1}$. Therefore, U is entirely contained in W_1.
By Proposition 4.2 (2) in [KB02], the orbit $\Gamma \cdot \gamma^-$ is dense in $\partial_\infty\Gamma$, so U is a dense connected subset of $\mathsf{U}_0\Gamma$, which is thus connected as well. □

5.1.3 Anosov representations via the geodesic flow

In this section, we recall the general definition of an Anosov representation and explain how to obtain a modified contraction/expansion property in our setting that we will need later on. The setup used here is very close to the one in [GW10] and [GW12], which in turn is a generalization of the original definition in [Lab06]. It is rather different from the more recent equivalent definitions avoiding the geodesic flow (which is rather involved in the case of a general word–hyperbolic group). These definitions appeared in [KLP14b] and [GGKW17], and we used one of them in an earlier chapter (Definition 2.3.2).

Let G be a semi–simple Lie group, Γ a word hyperbolic group and $\varrho : \Gamma \to G$ a homomorphism. Furthermore, let (P^+, P^-) be a pair of opposite parabolic subgroups of G and

$$\mathcal{X} \subset G/P^+ \times G/P^-$$

the unique open G–orbit.

Next, we need the geodesic flow. We will use the flow space $\mathsf{U}_0\Gamma$ together with the flow ψ_t introduced in Section 5.1.2. It induces a flow ϕ_t on the trivial bundle $\widetilde{\mathsf{U}_0\Gamma} \times \mathcal{X}$ by acting as the identity on fibers. This flow then descends to a flow ϕ_t on the bundle

$$\mathsf{P}_\varrho = \Gamma \setminus \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathcal{X}\right)$$

over $\mathsf{U}_0\Gamma$, where Γ acts on $\widetilde{\mathsf{U}_0\Gamma}$ as described in Section 5.1.2 and on $\mathcal{X}$ via ϱ. The product structure of $\mathcal{X}$ implies that it comes equipped with two distributions X^+ and X^-, where $(X^+)_{(gP^+,gP^-)} := \mathrm{T}_{gP^+}G/P^+$, and $(X^-)_{(gP^+,gP^-)} := \mathrm{T}_{gP^-}G/P^-$. Since these distributions are G–invariant, they are in particular Γ–invariant and determine vector bundles over P_ϱ, which we will also denote by X^+ and X^-. The flow ϕ_t on P_ϱ preserves the product structure of $\mathcal{X}$. We can thus extend it to a flow on the vector bundles $X^\pm$ by using the derivative of ϕ_t in fiber directions.
Now we are ready to state the definition of an Anosov representation.

Definition 5.1.8. A representation $\varrho : \Gamma \to G$ is (P^+, P^-)–Anosov if the bundle P_ϱ admits an Anosov section σ, i.e. a section $\sigma : \mathsf{U}_0\Gamma \to \mathsf{P}_\varrho$ such that

- σ is parallel (or locally constant) along flow lines of the geodesic flow, with respect to the locally flat structure on P_ϱ
- The flow ϕ_t is contracting on the bundle σ^*X^+ and dilating on the bundle σ^*X^-.

Remarks 5.1.9. (i) The flow ϕ_t used in the definition is again an induced flow: Since σ is parallel along flow lines, the flow on $X^\pm$ induces a flow on the pullback bundles.

(ii) The contraction/dilation condition in the definition means the following: Pick any continuous norm $(\|\cdot\|_v)_{v\in\mathsf{U}_0\Gamma}$ on the bundles σ^*X^+ and σ^*X^-. Then there exist constants $c, C > 0$ such that, for any $w \in \mathsf{U}_0\Gamma$ and $x \in (\sigma^*X^+)_w$, we have

$$\|\phi_t(x)\|_{\psi_t(w)} < C\exp(-ct)\|x\|_w$$

for all $t > 0$, and similarly for any $y \in (\sigma^*X^-)_w$,

$$\|\phi_{-t}(y)\|_{\psi_{-t}(w)} < C\exp(-ct)\|y\|_w.$$

By compactness of the base, the choice of norm does not matter.

(iii) It is sometimes easier in terms of notation to lift σ to a Γ–equivariant section of the trivial bundle $\widetilde{\mathsf{U}_0\Gamma} \times \mathcal{X}$. We will write

$$\widetilde{\sigma} : \widetilde{\mathsf{U}_0\Gamma} \to \mathcal{X}$$

for the Γ–equivariant map corresponding to this section. It is constant along flow lines.

We now turn to the case $G = \mathrm{SO}_0(n+1,n), P^+ = \mathrm{Stab}_G(E)$ for some maximal isotropic subspace $E \in \mathrm{Is}_n(\mathbb{R}^{n+1,n})$. Then P^+ is conjugate to its opposite parabolic P^- and the unique open G–orbit $\mathcal{X}$ is identified with the space of transverse pairs $(E,F) \in (\mathrm{Is}_n(\mathbb{R}^{n+1,n}))^2$. Transversality is equivalent to having a direct sum splitting $\mathbb{R}^{n+1,n} = E \oplus F^\perp$ in this case. Our goal for the remainder of this section will be to prove a contraction property that is slightly different from the one in Definition 5.1.8.

We start by giving a more explicit description of the bundles σ^*X^+ and σ^*X^-. For any $(V^+,V^-) \in \mathcal{X}$, a chart for $G/P^+ = \mathrm{Is}_n(\mathbb{R}^{n+1,n})$ containing the point V^+ is given by

$$\{f \in \mathrm{Hom}(V^+,(V^-)^\perp) \mid \forall v,w \in V^+ : b(v+f(v), w+f(w)) = 0\}, \tag{5.1.3}$$

where b denotes the symmetric bilinear form of signature $(n+1,n)$. Therefore, the subspace defined by the first distribution,

$$(X^+)_{(V^+,V^-)} = \mathrm{T}_{V^+}\,\mathrm{Is}_n(\mathbb{R}^{n+1,n}),$$

is given by

$$\{g \in \mathrm{Hom}(V^+,(V^-)^\perp) \mid \forall v,w \in V^+ : b(v,g(w)) + b(g(v),w) = 0\}.$$

The section σ now allows us to convert this pointwise description into a description of the associated bundle

$$\mathsf{R}_\varrho = \Gamma \setminus \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{n+1,n}\right).$$

More precisely, $\widetilde{\sigma}$ defines a Γ–invariant splitting

$$\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{n+1,n} = \mathcal{V}^+ \oplus \mathcal{L} \oplus \mathcal{V}^-$$

by choosing, for $\widetilde{\sigma}(v) = (V^+,V^-)$,

$$\mathcal{V}^+_v = V^+, \quad \mathcal{L}_v = (V^+)^\perp \cap (V^-)^\perp, \quad \mathcal{V}^-_v = V^-.$$

Here, orthogonal complements are taken with respect to the bilinear form b. The flow action extends to this (trivial) bundle as well by acting trivially on the fiber component. We remark that b is preserved by the flow, which will be useful later on. The flow, the bilinear form and the splitting then descend to give a flow–invariant splitting of R_ϱ, which we denote by

$$\mathsf{R}_\varrho = \mathsf{V}^+ \oplus \mathsf{L} \oplus \mathsf{V}^-. \tag{5.1.4}$$

The bundle $\sigma^* X^+$ is now identified with the bundle

$$\mathrm{Hom}_{b-\mathrm{skew}}(\mathsf{V}^+, \mathsf{L} \oplus \mathsf{V}^-) = \mathrm{Hom}(\mathsf{V}^+, \mathsf{L}) \oplus \mathrm{Hom}_{b-\mathrm{skew}}(\mathsf{V}^+, \mathsf{V}^-),$$

while

$$\sigma^* X^- = \mathrm{Hom}(\mathsf{V}^-, \mathsf{L}) \oplus \mathrm{Hom}_{b-\mathrm{skew}}(\mathsf{V}^-, \mathsf{V}^+).$$

The flow ϕ_t acts on $\sigma^* X^\pm$ by

$$(\phi_t \alpha)(x) = \phi_t(\alpha(\phi_{-t} x)), \quad \text{for } \alpha \in (\sigma^* X^\pm)_p,\ x \in \mathsf{V}^\pm_{\psi_t p}.$$

The Anosov property tells us that this action is contracting on $\sigma^* X^+$ and dilating on $\sigma^* X^-$. Since this holds true for any choice of norm, let us first pick an auxiliary positive definite quadratic form e on R_ϱ such that the splitting above is orthogonal and e agrees with b on L (this is possible since the fibers of L are spacelike for b). The induced operator norms are our norms of choice for $\sigma^* X^+$ and $\sigma^* X^-$.

After this somewhat lengthy setup, we are finally ready to conclude. All norms in the following statements are induced by e.

Lemma 5.1.10. *Let $p \in \mathsf{U}_0\Gamma$ and $v \in \mathsf{V}^+_p$ be arbitrary. Then there exists $\alpha \in \mathrm{Hom}(\mathsf{V}^+_p, \mathsf{L}_p)$ such that $0 \neq \alpha(v) = l \in \mathsf{L}_p$ and $\|\alpha\| = \frac{\|l\|}{\|v\|}$. Analogously, for $w \in \mathsf{V}^-_p$, we find $\beta \in \mathrm{Hom}(\mathsf{V}^-_p, \mathsf{L}_p)$ such that $\|\beta\| = \frac{\|l\|}{\|w\|}$.*

Proof. Complete v to an e–orthogonal basis of V^+_p, map v to $l \in \mathsf{L}_p$ and map all other basis vectors to 0. □

Corollary 5.1.11. *The bundle V^+ is dilated by the flow ϕ_t. The bundle V^- is contracted by the flow ϕ_t.*

Proof. Let $p \in \mathsf{U}_0\Gamma$ and $v \in \mathsf{V}^+_p$ be arbitrary. We saw earlier that

$$(\sigma^* X^+)_p = \mathrm{Hom}(\mathsf{V}^+_p, \mathsf{L}_p) \oplus \mathrm{Hom}_{b-\mathrm{skew}}(\mathsf{V}^+_p, \mathsf{V}^-_p).$$

Using the previous lemma, we can therefore pick $\alpha \in (\sigma^* X^+)_p$ such that $\alpha(v) = l$ and $\|\alpha\| = \frac{\|l\|}{\|v\|}$ for some $0 \neq l \in \mathsf{L}_p$ (by picking it in the first summand). Then we have

$$\frac{\|l\|}{\|\phi_t(v)\|} = \frac{\|\phi_t(l)\|}{\|\phi_t(v)\|} \leq \|\phi_t(\alpha)\| < C\exp(-ct)\|\alpha\| = C\exp(-ct)\frac{\|l\|}{\|v\|},$$

where we used the fact that b is preserved by the flow and agrees with e on L to get the first equality.
The proof for V^- follows in the same way. □

Note that contraction/dilation is reversed for the bundles $\mathsf{V}^\pm$. This is consistent because the Anosov property gives contraction of $\sigma^* X^+$, which we identified with a subbundle of $\mathrm{Hom}(\mathsf{V}^+, (\mathsf{V}^-)^\perp) = (\mathsf{V}^+)^* \otimes (\mathsf{V}^-)^\perp$.

In the more modern setup avoiding the use of flows, the fact that contraction on the bundle $\mathrm{Hom}_{b-\mathrm{skew}}(\mathsf{V}^+, (\mathsf{V}^-)^\perp)$ implies dilation/contraction on $\mathsf{V}^\pm$ corresponds to the fact that singular values for $\mathrm{SO}_0(n+1,n)$ which are not equal to 1 come in inverse pairs. To see this, we recall the following equivalent characterization of an Anosov representation. For simplicity, we state it only for representations into $\mathrm{GL}(n,\mathbb{R})$. For $a \in \mathrm{GL}(n,\mathbb{R})$, denote by $\sigma_p(a)$ the p–th singular value of a, in decreasing order.

Theorem 5.1.12 ([KLP14a],[BPS16])**.** *Let Γ be a finitely generated group and $\varrho\colon \Gamma \to \mathrm{GL}(n,\mathbb{R})$ a representation. Let $|\cdot|$ be the word metric on Γ with respect to some finite generating set. Assume that there exist constants $C, \lambda > 0$ such that*

$$\frac{\sigma_{p+1}(\varrho(\gamma))}{\sigma_p(\varrho(\gamma))} \leq C\mathrm{e}^{-\lambda|\gamma|} \qquad \forall \gamma \in \Gamma. \tag{5.1.5}$$

Then Γ is word hyperbolic and ϱ is Anosov with respect to the stabilizer of a partial flag consisting of a p–dimensional and an $(n-p)$–dimensional subspace.

Conversely, if Γ is word hyperbolic and ϱ is $(p, n-p)$–Anosov, then (5.1.5) *is satisfied.*

For the general setup, the exponential gap condition on singular values is replaced by uniform P_θ–divergence (a linear rate of divergence in Definition 2.3.1(i)). As $\sigma_p(a) = \sigma_{n-p+1}(a^{-1})^{-1}$, (5.1.5) implies the same inequality for the quotient $\frac{\sigma_{n-p}}{\sigma_{n-p+1}}$. Since in our case ϱ is Anosov with respect to the stabilizer of a maximal isotropic in $\mathbb{R}^{n+1,n}$, it has a uniform singular value gap between the n–th and $(n+1)$–th as well as between the $(n+1)$–th and $(n+2)$–th singular value. In $\mathrm{SO}_0(n+1,n)$, if a singular value is not equal to 1, then its inverse is also a singular value [1]. Thus 1 has to be a singular value, the first n singular values are strictly bigger than 1 and the last n singular values are strictly smaller than 1.

5.1.4 (AMS)–Proximality

In this section, we explain a useful property that Anosov representations satisfy.

Let G be a semi–simple Lie group and (P^+, P^-) a pair of opposite parabolic subgroups of G. Recall that $C(x^-) \subset G/P^+$ is the open Schubert cell of elements transverse to x^-. An element $g \in G$ is called *proximal* relative to G/P^+ if g has two transverse fixed points $x^\pm \in G/P^\pm$ and the following holds:

$$\lim_{n\to\infty} \gamma^n x = x^+ \quad \text{for all } x \in C(x^-)$$

Moreover, a subgroup $H < G$ containing a proximal element is also called *proximal.*

[1] If $a \in \mathrm{SO}(n+1,n)$, then $a^t a \in \mathrm{SO}(n+1,n)$ as well. A nonzero eigenvalue of $a^t a$ must correspond to an isotropic eigendirection, so there must be another eigendirection transverse to the orthgonal complement of the first. Then the corresponding eigenvalues must be inverses of each other.

We now turn to a quantitative version of proximality. For any $x^- \in G/P^-$, we define

$$\mathrm{nt}(x^-) := \{x \in G/P^+ \mid x \text{ not transverse to } x^-\}.$$

It is the complement of $C(x^-)$. Let d be a Riemannian distance on G/P^+ and let $x^\pm \in G/P^\pm$. We fix constants $r, \epsilon > 0$ and consider the neighborhoods

$$N_\epsilon(x^+) = \{x \in G/P^+ \mid d(x, x^+) < \epsilon\}$$

and

$$N_\epsilon(\mathrm{nt}(x^-)) = \{x \in G/P^+ \mid d(x, \mathrm{nt}(x^-)) < \epsilon\}.$$

An element $g \in G$ is called (r,ϵ)*–proximal* relative to G/P^+ if it has two transverse fixed points $x^\pm \in G/P^\pm$ satisfying

$$d(x^+, \mathrm{nt}(x^-)) \geq r$$

and

$$g(N_\epsilon(\mathrm{nt}(x^-))^c) \subset N_\epsilon(x^+).$$

A subgroup H of G is called *(AMS)–proximal* relative to G/P^+ if there exist constants $r > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$, there exists a finite set $S = S(r,\epsilon) \subset H$ satisfying the following: For any $g \in H$, there exists $s \in S$ such that sg is (r,ϵ)–proximal.

Finally, a representation $\varrho : \Gamma \to G$ is called *(AMS)–proximal* if $\ker(\varrho)$ is finite and $\varrho(\Gamma)$ is (AMS)–proximal.

This definition was introduced by Abels–Margulis–Soifer in [AMS95], where they proved a more general version of the following result:

Theorem 5.1.13 ([AMS95], Theorem 4.1)**.** *Let $H < \mathrm{SL}(n,\mathbb{R})$ be a strongly irreducible subgroup, i.e. all finite index subgroups of H act irreducibly on $\mathbb{R}^n$. Assume that H contains a proximal element. Then H is (AMS)–proximal relative to $\mathbb{RP}^{n-1}$.*

Subsequently, Guichard–Wienhard used this result to prove (AMS)–proximality for Anosov representations, which we use in Section 5.2.5:

Theorem 5.1.14 ([GW12], Theorem 1.7)**.** *Let Γ be a word hyperbolic group and $\varrho : \Gamma \to G$ Anosov with respect to $P^\pm$. Then ϱ is (AMS)–proximal with respect to $G/P^\pm$.*

5.1.5 The Margulis invariant, classical version

Let $G = \mathrm{SO}_0(n+1,n)$, $P^{\pm}$ be the stabilizers of transverse elements of $\mathrm{Is}_n(\mathbb{R}^{n+1,n})$ and b be the corresponding $(n+1,n)$–bilinear form.

Assume that $a \in G$ admits an invariant transverse pair $(V^+, V^-) \in \mathrm{Is}_n(\mathbb{R}^{n+1,n})^2$ of an attracting and a repelling maximal isotropic subspace, i.e. $a^k|_{C(V^-)}$ converges uniformly to V^+ as $k \to \infty$. Then, since a acts as an orientation–preserving transformation on both V^+ and V^- (see Proposition 5.1.2), it fixes the spacelike line $\ell = (V^+)^{\perp} \cap (V^-)^{\perp}$ pointwise. Let

$$\mathsf{v}^1(a) \in \ell$$

be the unique vector such that $b(v^1(a), v^1(a)) = 1$ and

$$V^+ \oplus \langle \mathsf{v}^1(a) \rangle \oplus V^- \stackrel{\pm}{=} \mathbb{R}^{2n+1}$$

is an oriented direct sum. Observe that

$$\mathsf{v}^1(a^{-1}) = (-1)^n \mathsf{v}^1(a). \tag{5.1.6}$$

The following definition of the Margulis invariant for $G \ltimes \mathbb{R}^{2n+1}$ can be found in [AMS02, Section 4].

Definition 5.1.15. Let $g \in G \ltimes \mathbb{R}^{2n+1}$ such that the linear part $L(g)$ admits an invariant attracting/repelling transverse pair $(V^+, V^-) \in \mathrm{Is}_n(\mathbb{R}^{n+1,n})^2$. Then the *Margulis invariant* of g is

$$\alpha(g) = b\left(gx - x, \mathsf{v}^1(L(g))\right),$$

where $x \in \mathbb{R}^{2n+1}$ is any point.

Since $\mathsf{v}^1(L(g))$ is a spacelike fixed vector of the linear part, $\alpha(g)$ is simply the signed ℓ–component of the translational part of g. Equivalently, it is the signed distance that g translates along the unique affine line it stabilizes. In particular, it does not depend on the choice of x. We include elements of finite order in the domain of definition of α by setting $\alpha(g) = 0$ in that case. The Margulis invariant satisfies the following properties:

(i) α is invariant under conjugation.

(ii) $\alpha(g^k) = k\alpha(g)$ for $k > 0$.

(iii) $\alpha(g^k) = (-1)^{n+1}|k|\alpha(g)$ for $k < 0$.

(iv) $\alpha(g) = 0 \Leftrightarrow g$ fixes a point in $\mathbb{R}^{2n+1}$.

If $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ is a representation of a group Γ such that α is defined on $\varrho(\gamma) = L(\rho(\gamma))$ for every $\gamma \in \Gamma$ (for example, if ϱ is Anosov with respect to $P^{\pm}$), we write

$$\begin{aligned} \alpha_\rho\colon \Gamma &\to \mathbb{R} \\ \gamma &\mapsto \alpha(\rho(\gamma)). \end{aligned}$$

By a result of Fried and Goldman ([FG83]), if a discrete subgroup $H \subset \mathrm{Aff}(\mathbb{R}^3)$ acts properly discontinuously and freely on $\mathbb{R}^3$, its linear part is a discrete subgroup of a conjugate of $\mathrm{SO}_0(2,1)$. Margulis showed that in this case, the sign of α must be uniformly positive or uniformly negative ([Mar83], [Mar84]).

Lemma 5.1.16 (Opposite Sign Lemma)**.** *Let $H \subset \mathrm{Aff}(\mathbb{R}^3)$ be a discrete subgroup acting properly and freely on $\mathbb{R}^3$. Then either $\alpha(h) > 0$ for all $h \in H$ or $\alpha(h) < 0$ for all $1 \neq h \in H$.*

This lemma is proved by showing that the group generated by two elements g_1, g_2 with hyperbolic linear parts in general position does not act properly if $\alpha(g_1)\alpha(g_2) \leq 0$. Basically the same proof works for subgroups of $\mathrm{Aff}(\mathbb{R}^{2n+1})$ whose linear part is Anosov with respect to $P^{\pm}$ (see [AMS02, Proposition 4.9]). In the same reference, this is used to show that because $\alpha(g^{-1}) = (-1)^{n+1}\alpha(g)$, for n even, there are no subgroups of $\mathrm{Aff}(\mathbb{R}^{2n+1})$ acting properly discontinuously and whose linear part is Zariski dense in $\mathrm{SO}(n+1,n)$. This fact has a direct analogue in our setting of affine Anosov representations, as we will see in Lemma 5.2.8.

Labourie introduced a continuous version of the Margulis invariant which plays a major role in our results. However, since it requires several objects which were not yet introduced, we defer the discussion of this generalization to Section 5.2.2.

5.2 Affine Anosov representations

In this section, we define the notion of affine Anosov representations of a word hyperbolic group Γ into the semidirect product $\mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$. We will make use of the well–developed theory of linear Anosov representations via the homomorphism

$$L\colon\ \mathrm{SO}(p,q) \ltimes \mathbb{R}^{p+q} \to \mathrm{SO}(p,q)$$

which maps an element to its linear part. Abusing notation slightly, the map sending any affine subspace of $\mathbb{R}^{2n+1}$ to its underlying linear subspace will also be denoted L.

Let us fix some notation first. We write $G = \mathrm{SO}_0(n+1,n) = \mathrm{SO}_0(b)$, where b is the $(n+1,n)$–bilinear form given by the matrix

$$J := \begin{pmatrix} & & I_n \\ & 1 & \\ I_n & & \end{pmatrix}.$$

Here, I_n denotes the $n \times n$ identity matrix. In particular, $V^+ = \mathbb{R}^n \times \{0\} = \operatorname{span}(e_1, \ldots, e_n)$ and $V^- = \{0\} \times \mathbb{R}^n = \operatorname{span}(e_{n+2}, \ldots, e_{2n+1})$ are transverse maximal isotropic subspaces:

$$\mathbb{R}^{2n+1} = (\mathbb{R}^n \times \{0\}) \oplus (\{0\} \times \mathbb{R}^n)^{\perp},$$

where both summands are elements of

$$\mathrm{Is}_n\left(\mathbb{R}^{2n+1}\right) = \left\{V \subset \mathbb{R}^{2n+1} \mid \dim(V) = n,\ b|_{V \times V} \equiv 0\right\}.$$

We denote the corresponding transverse parabolic subgroups in G by P^+ and P^-:

$$P^+ = \mathrm{Stab}_G(V^+), \qquad P^- = \mathrm{Stab}_G(V^-).$$

Then $G/(P^+ \cap P^-)$ identifies with transverse pairs $L_1, L_2 \in \mathrm{Is}_n(\mathbb{R}^{n+1,n})$. The intersection $P^+ \cap P^-$ is the reductive group $\mathrm{GL}_+(n, \mathbb{R})$:

Lemma 5.2.1. *With the above notation, $P^+ \cap P^-$ identifies with* $\mathrm{GL}_+(V^+)$.

Proof. Since any element X of $P^+ \cap P^-$ stabilizes both $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$, it has to be of block form

$$X = \begin{pmatrix} A_1 & B_1 & 0 \\ 0 & C & 0 \\ 0 & B_2 & A_2 \end{pmatrix},$$

where A_i are $n \times n$ matrices, B_i are $n \times 1$ and C is 1×1. The equation $JXJ = (X^t)^{-1}$ reduces this further to the form

$$X = \begin{pmatrix} A & & \\ & C & \\ & & (A^t)^{-1} \end{pmatrix},$$

where $A \in \mathrm{GL}(n, \mathbb{R})$ and $C = \pm 1$. Now since X preserves orientation on $\mathbb{R}^{2n+1}$, C has to be $+1$. Moreover, we saw in Section 5.1.1 that we can consistently choose orientations on all elements of $\mathrm{Is}_n(\mathbb{R}^{n+1,n})$, and an element $g \in \mathrm{SO}(p, q)$ preserves these orientations iff it lies in $\mathrm{SO}_0(p, q)$. We conclude that $A \in \mathrm{GL}_+(n, \mathbb{R})$. □

5.2.1 Setup and definition

Pseudoparabolics in $G \ltimes \mathbb{R}^{2n+1}$

In order to define what an Anosov representation into the affine group should be, we will require a class of subgroups corresponding to parabolic subgroups in reductive Lie groups. To that end, let $E = E^{2n+1}$ denote the affine space modeled on $\mathbb{R}^{n+1,n}$, and $\mathrm{Is}_n(E)$ the set of affine isotropic subspaces. By this we mean all affine subspaces whose underlying linear subspace is n–dimensional and isotropic.

In the linear case, we can interchangeably speak about either maximal isotropic subspaces or $(n+1)$–dimensional subspaces of signature $(n,1,0)$ – here, the first number denotes degenerate directions and the second number denotes positive directions. Taking orthogonal complements allows to switch between the two sets, and any element $g \in G$ fixing a maximal isotropic subspace also fixes its orthogonal complement. However, this is no longer true in the affine case. Since there is no natural basepoint, there is no canonical way of choosing an orthogonal complement of an affine subspace of type $(n,1,0)$. Our construction will make use of these $(n+1)$–dimensional affine subspaces instead of affine maximal isotropic subspaces.

Definition 5.2.2. Let $F \subset E$ be an affine subspace of type $(n,1,0)$. Then we call the subgroup

$$P_{\mathrm{aff}} = \mathrm{Stab}_{G \ltimes \mathbb{R}^{2n+1}} F$$

a *pseudoparabolic*.
Two affine subspaces A_1, A_2 of type $(n,1,0)$ will be called *transverse* if their underlying vector subspaces W_1, W_2 satisfy $\mathbb{R}^{2n+1} = W_1 \oplus (W_2)^{\perp}$. Two pseudoparabolics will be called *transverse* if they are stabilizers of transverse affine subspaces.

Remark 5.2.3. Since $G \ltimes \mathbb{R}^{2n+1}$ acts transitively on affine subspaces of type $(n,1,0)$ (see Proposition 5.1.4), all pseudoparabolic subgroups are isomorphic and can be identified (albeit not canonically) with $P \ltimes \mathbb{R}^{n+1}$, where $P < G$ is the stabilizer of some fixed maximal isotropic subspace of $\mathbb{R}^{n+1,n}$, and the group of translations along the orthogonal complement of the maximal isotropic is identified with $\mathbb{R}^{n+1}$.

Recall that, in the linear case, we defined the space $\mathcal{X} = G/(P^+ \cap P^-)$ for transverse parabolic subgroups $P^{\pm}$. When $P^{\pm}$ are stabilizers of maximal isotropic subspaces, it identifies with the space of transverse pairs of maximal isotropic subspaces. Analogously, for transverse pseudoparabolics $P^{\pm}_{\mathrm{aff}}$, the quotient $(G \ltimes \mathbb{R}^{2n+1})/(P^+_{\mathrm{aff}} \cap P^-_{\mathrm{aff}}) =:$ $\mathcal{X}_{\mathrm{aff}}$ can be identified with the space of transverse pairs of affine subspaces of type $(n,1,0)$, so we can view it as a subset

$$\mathcal{X}_{\mathrm{aff}} \subset \left((G \ltimes \mathbb{R}^{2n+1})/P^+_{\mathrm{aff}} \times (G \ltimes \mathbb{R}^{2n+1})/P^-_{\mathrm{aff}}\right).$$

It is the unique open $(G \ltimes \mathbb{R}^{2n+1})$–orbit in the space of all pairs of affine subspaces of type $(n,1,0)$.

We fix a choice of transverse pseudoparabolics $P^{\pm}_{\mathrm{aff}}$ such that $L(P^{\pm}_{\mathrm{aff}}) = P^{\pm}$ is our standard choice of transverse parabolics.

Affine bundles

We have to adjust the setup of bundles and flows to the affine case. Recall that the flow space of the hyperbolic group Γ is defined as

$$\mathsf{U}_0\Gamma = \Gamma\backslash\widetilde{\mathsf{U}_0\Gamma} = \Gamma\backslash(\partial_\infty\Gamma^{(2)} \times \mathbb{R}).$$

We will make use of several bundles over the flow space $\mathsf{U}_0\Gamma$. They are defined in terms of a given representation $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ with linear part $\mathrm{L}(\rho) = \varrho\colon \Gamma \to G$. The first one is the affine equivalent of the bundle P_ϱ,

$$\mathsf{P}_\rho = \Gamma \setminus \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathcal{X}_{\mathrm{aff}}\right),$$

whose fiber is the space of transverse pairs of affine subspaces of type $(n, 1, 0)$. The homomorphism L extends to a bundle map

$$\mathsf{P}_\rho \xrightarrow{L} \mathsf{P}_\varrho.$$

Next, we need the bundle

$$\mathsf{R}_\rho = \Gamma \setminus \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{2n+1}\right),$$

sections of which plays the role of a basepoint in affine space. In both cases, Γ acts diagonally via its natural action on $\widetilde{\mathsf{U}_0\Gamma}$ and via the representation ρ on the second factor. Finally, there is the linear version of the latter bundle,

$$\mathsf{R}_\varrho = \Gamma \setminus \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{2n+1}\right),$$

where the action on the second factor is given by the linear part ϱ. Since ϱ preserves the $(n+1, n)$–form b on $\mathbb{R}^{2n+1}$, this bundle comes equipped with a $(n+1, n)$–form.

Tangent distributions and covariant derivatives along flow lines

As in the linear case, there are two distributions $X^{\pm}_{\mathrm{aff}}$ on $\mathcal{X}_{\mathrm{aff}}$ coming from its product structure, given by

$$(X^{\pm}_{\mathrm{aff}})_{(gP^+_{\mathrm{aff}}, gP^-_{\mathrm{aff}})} = \mathrm{T}_{gP^{\pm}_{\mathrm{aff}}}\left(G \ltimes \mathbb{R}^{2n+1}/P^{\pm}_{\mathrm{aff}}\right)$$

for any $g \in G \ltimes \mathbb{R}^{2n+1}$. Observe that these tangent spaces can be identified with the sum of the tangent space to the linear part and transverse translations: Let (A^+, A^-) be a transverse pair of affine $(n, 1, 0)$ subspaces, and let (W^+, W^-) be their linear parts. Then we can write

$$(A^+, A^-) = (A^+ \cap A^- \cap [(W^+)^\perp \oplus (W^-)^\perp]) + (W^+, W^-),$$

where we chose a common base point for both A^+ and A^-. From this, it follows that we can identify

$$\mathrm{T}_{(A^+, A^-)}\mathcal{X}_{\mathrm{aff}} = \mathrm{T}_{((W^+)^\perp, (W^-)^\perp)}\mathcal{X} \oplus (W^+)^\perp \oplus (W^-)^\perp,$$

and the tangent space splits into the two components

$$\mathrm{T}_{A^{\pm}}\left(G \ltimes \mathbb{R}^{2n+1}/P^{\pm}_{\mathrm{aff}}\right) = \mathrm{T}_{(W^{\pm})^\perp}(G/P^{\pm}) \oplus (W^{\mp})^\perp. \tag{5.2.1}$$

The two distributions $X_{\text{aff}}^{\pm}$ are $G \ltimes \mathbb{R}^{2n+1}$–invariant and we will see them as vector bundles over P_ρ.

We also have the flow ϕ_t acting on the bundles P_ρ and R_ρ as Gromov geodesic flow on the base and via parallel transport (with respect to the locally flat structure) on the fibers. Using the derivative of the flow on P_ρ in fiber directions gives an induced flow on the bundles $X_{\text{aff}}^{\pm}$.

The flow ϕ_t gives rise to a covariant derivative in flow direction.

Definition 5.2.4. Let $\tau\colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ be a section. We call τ *(continuously) differentiable along flow lines* if the derivative

$$\nabla_\phi \tau(z) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t}(\tau(\psi_t z))$$

exists for every $z \in \mathsf{U}_0\Gamma$ and defines a continuous section $\nabla_\phi\tau\colon \mathsf{U}_0\Gamma \to \mathsf{R}_\varrho$.

We could equivalently lift τ to obtain a ρ–equivariant map

$$\widetilde{\tau} : \widetilde{\mathsf{U}_0\Gamma} \to \mathbb{R}^{2n+1}.$$

Then, the covariant derivative in flow direction is simply given by

$$\nabla_\phi \widetilde{\tau}(x, y, t_0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{\tau}(x, y, t_0 + t).$$

The neutral section

There is a natural map which takes two transverse subspaces as above and returns a (spacelike) vector in the linear part of their intersection, chosen to be normalized and according to an orientation convention. The following is a version of the neutral section defined in [GLM09] that is adapted to our setting.

Definition 5.2.5 (Neutral section)**.**

(i) Let

$$\nu : \mathcal{X} \to \mathbb{R}^{2n+1}$$

be the map assigning to the pair (V_1, V_2) the vector $v \in (V_1)^\perp \cap (V_2)^\perp$ with $b(v, v) = 1$ and satisfying the same orientation convention as the vector $\mathsf{v}^1(a)$ defined earlier for elements $a \in \mathrm{SO}_0(n+1, n)$: From Section 5.1.1, we know that we can consistently choose orientations on elements of $\mathrm{Is}_n(\mathbb{R}^{n+1,n})$. Then v is chosen such that

$$V_1 \oplus \langle v \rangle \oplus V_2 \stackrel{\pm}{=} \mathbb{R}^{2n+1}$$

is an oriented direct sum.
Since ν is G–equivariant, it induces a bundle map

$$\nu\colon \mathsf{P}_\varrho \to \mathsf{R}_\varrho$$

for any representation $\varrho\colon \Gamma \to G$.

We also write ν for the composition

$$\mathcal{X}_{\text{aff}} \xrightarrow{L} \mathcal{X} \xrightarrow{\nu} \mathbb{R}^{2n+1},$$

which takes the linear parts of the two affine subspaces and applies the previous definition. It induces a bundle map

$$\mathsf{P}_\rho \xrightarrow{L} \mathsf{P}_\varrho \xrightarrow{\nu} \mathsf{R}_\varrho$$

which we denote by ν as well.

(ii) Let $\varrho\colon \Gamma \to G$ be Anosov with respect to $P^\pm$, and $\sigma_\varrho\colon \mathsf{U}_0\Gamma \to \mathsf{P}_\varrho$ its Anosov section. The *neutral section* is the section

$$\nu_\varrho\colon \mathsf{U}_0\Gamma \xrightarrow{\sigma_\varrho} \mathsf{P}_\varrho \xrightarrow{\nu} \mathsf{R}_\varrho.$$

For ϱ Anosov, σ_ϱ is parallel, thus the same holds for ν_ϱ. Moreover, 1 is a simple eigenvalue of $\varrho(\gamma)$ for any element γ of infinite order (see Section 5.1.5). By definition, the neutral section is the diffuse analogue of the normalized choice of eigenvector $\mathsf{v}^1(\varrho(\gamma))$:

Lemma 5.2.6. *Let $\varrho\colon \Gamma \to G$ be Anosov with respect to $P^\pm$, $\sigma_\varrho\colon \mathsf{U}_0\Gamma \to \mathsf{P}_\varrho$ its Anosov section and $\widetilde{\sigma}_\varrho\colon \widetilde{\mathsf{U}_0\Gamma} \to \mathcal{X}$ its ϱ–equivariant lift. Let $\gamma \in \Gamma$ be any element of infinite order and $x \in \operatorname{axis}(\gamma) \subset \widetilde{\mathsf{U}_0\Gamma}$. Then*

$$\nu \circ \widetilde{\sigma}_\varrho(x) = \mathsf{v}^1(\varrho(\gamma)).$$

We now have all the ingredients needed to state the definition of an affine Anosov representation.

Definition 5.2.7. Let Γ be a word hyperbolic group and let

$$\rho : \Gamma \mapsto G \ltimes \mathbb{R}^{2n+1}$$

be a homomorphism. Furthermore, let $P^+_{\text{aff}}, P^-_{\text{aff}}$ be two transverse pseudoparabolic subgroups. Then ρ is called *affine Anosov* (with respect to $P^\pm_{\text{aff}}$) if and only if

(i) The bundle P_ρ admits an *affine Anosov section* σ, i.e. a section $\sigma : \mathsf{U}_0\Gamma \to \mathsf{P}_\rho$ such that:

- σ is parallel (locally flat) along flow lines of the geodesic flow on $\mathsf{U}_0\Gamma$.

- The (induced) flow ϕ_t is contracting on the bundle $\sigma^* X_{\mathrm{aff}}^+$ and dilating on the bundle $\sigma^* X_{\mathrm{aff}}^-$.

(ii) There exists a Hölder section τ of the bundle R_ρ which is differentiable along flow lines such that $b(\nabla_\phi \tau, \nu \circ \sigma)$ has no zero on $\mathsf{U}_0\Gamma$.

Using (5.2.1), we see that the bundles $\sigma^* X_{\mathrm{aff}}^\pm$ split in the following way: Taking linear parts gives the section

$$L \circ \sigma \colon \mathsf{U}_0\Gamma \to \mathsf{P}_\varrho,$$

and we have the decomposition

$$\sigma^* X_{\mathrm{aff}}^\pm = (L \circ \sigma)^* X^\pm \oplus \mathsf{V}^\mp, \tag{5.2.2}$$

where $X^\pm$ and $\mathsf{V}^\mp$ are the corresponding vector bundles in the linear case (see Section 5.1.3).

In the definition of an affine Anosov representation, (ii) is designed for the application to proper actions on $\mathbb{R}^{2n+1}$ later on. Basic properties of the neutral section imply that it cannot be satisfied for n even: Let $x \in \operatorname{axis}(\gamma), \iota(x) \in \operatorname{axis}(\gamma^{-1})$, where $\iota\colon \widetilde{\mathsf{U}_0\Gamma} \to \widetilde{\mathsf{U}_0\Gamma}$ is the involution given by the $\mathbb{Z}_2$–action on $\widetilde{\mathsf{U}_0\Gamma}$. Then (5.1.6) translates to

$$\nu \circ \widetilde{\sigma}_\varrho(\iota(x)) = (-1)^n \nu \circ \widetilde{\sigma}_\varrho(x).$$

Lemma 5.2.8. *If n is even, there are no affine Anosov representations $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$.*

Proof. Let $\varrho\colon \Gamma \to G$ be Anosov with respect to $P^\pm$. Let $p \in \mathsf{U}_0\Gamma$ be a point in the projection of axis(γ) for some infinite order element $\gamma \in \Gamma$. Then the function $b(\nabla_\phi \tau, \nu \circ \sigma_\varrho)$ has different signs or vanishes at p and $\iota(p)$. It thus has a zero by Lemma 5.1.7. □

5.2.2 The Margulis invariant, continuous version

We briefly return to the continuous version of the Margulis invariant mentioned earlier. It was introduced in [Lab01] and further studied in [GLM09].

A *geodesic current* on $\mathsf{U}_0\Gamma$ is a ψ_t–invariant Borel probability measure on $\mathsf{U}_0\Gamma$. The space of geodesic currents on $\mathsf{U}_0\Gamma$ will be denoted by $\mathscr{C}(\mathsf{U}_0\Gamma)$. Every infinite order element $\gamma \in \Gamma$ has an associated geodesic current: Its axis $\{(\gamma^-, \gamma^+, t) \mid t \in \mathbb{R}\} \subset \widetilde{\mathsf{U}_0\Gamma}$ projects to a closed quasigeodesic in $\mathsf{U}_0\Gamma$. Let

$$\begin{aligned} f\colon [0, l(\gamma)] &\to \mathsf{U}_0\Gamma \\ t &\mapsto \psi(x, t) \end{aligned}$$

be a parametrization by the flow, where x is any point on this closed curve. Then

$$\mu_\gamma = f_* \lambda_{[0, l(\gamma)]}$$

is the push–forward of the normalized Lebesgue measure on $[0, l(\gamma)]$.

The notion of a geodesic current was first considered in [Bon86]. Our definition is a slight variation, following the terminology of [GLM09] and extending it from fundamental groups of hyperbolic surfaces to arbitrary hyperbolic groups.

Let $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a representation such that $\varrho = L(\rho)$ is Anosov with respect to $P^{\pm}$. Let $\tau\colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ be a section that is differentiable along flow lines. Then, define

$$\begin{aligned} \Psi_\rho\colon \mathscr{C}(\mathsf{U}_0\Gamma) &\to \mathbb{R} \\ \mu &\mapsto \int_{\mathsf{U}_0\Gamma} b(\nabla_\phi \tau, \nu_\varrho)\,\mathrm{d}\mu. \end{aligned}$$

By [GLM09, Lemma 6.1] (replacing the unit tangent bundle by the flow space), the value $\Psi_\rho(\mu)$ does not depend on the choice of τ. It is continuous with respect to the weak*–topology on $\mathscr{C}(\mathsf{U}_0\Gamma)$, and its relation to the Margulis invariant is described by the following lemma, which is a direct generalization of [Lab01, Proposition 4.2] and [GLM09, Proposition 6.2].

Proposition 5.2.9. *Let $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism whose linear part ϱ is Anosov with respect to $P^{\pm}$. Moreover, let $\gamma \in \Gamma$ be of infinite order and $l(\gamma)$ its translation length. Then for all sections $\tau : \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ which are differentiable along flow lines, we have*

$$\alpha_\rho(\gamma) = l(\gamma)\Psi_\rho(\mu_\gamma),$$

where μ_γ denotes the geodesic current on $\mathsf{U}_0\Gamma$ corresponding to γ.

Proof. Let $p \in \mathsf{U}_0\Gamma$ be a point on the projection of the axis of γ. Then we have

$$l(\gamma)\int b(\nabla_\phi\tau, \nu_\varrho)\,\mathrm{d}\mu_\gamma = \int_0^{l(\gamma)} b(\nabla_\phi\tau, \nu_\varrho)(\psi_t p)\,\mathrm{d}t.$$

Since ν_ϱ is a parallel section of R_ϱ of constant norm 1 and $\psi_{l(\gamma)}p = p$, this integral can be rewritten as

$$\begin{aligned} \int_0^{l(\gamma)} b(\nabla_\phi\tau, \nu_\varrho)(\psi_t p)\,\mathrm{d}t &= b\left(\tau(\psi_{l(\gamma)}p) - \phi_{l(\gamma)}\tau(p), \nu_\varrho(\psi_{l(\gamma)}p)\right) \\ &= b\left(\tau(p) - \phi_{l(\gamma)}\tau(p), \nu_\varrho(p)\right). \end{aligned}$$

Let $\widetilde{\tau}\colon \widetilde{\mathsf{U}_0\Gamma} \to \mathbb{R}^{2n+1}$ be the ρ–equivariant map corresponding to τ and $\widetilde{\sigma}_\varrho\colon \widetilde{\mathsf{U}_0\Gamma} \to \mathcal{X}$ the ϱ–equivariant map corresponding to σ_ϱ. Moreover, let $\widetilde{p} \in \widetilde{\mathsf{U}_0\Gamma}$ be a lift of p. Then evaluating in the fiber over $\psi_{l(\gamma)}\widetilde{p}$ yields

$$\begin{aligned} b\left(\tau(p) - \phi_{l(\gamma)}\tau(p), \nu_\varrho(p)\right) &= b\left(\widetilde{\tau}(\psi_{l(\gamma)}\widetilde{p}) - \widetilde{\tau}(\widetilde{p}), \nu\circ\widetilde{\sigma}_\varrho(\widetilde{p})\right) \\ &= b\left(\rho(\gamma)\widetilde{\tau}(\widetilde{p}) - \widetilde{\tau}(\widetilde{p}), \nu\circ\widetilde{\sigma}_\varrho(\widetilde{p})\right) = \alpha_\rho(\gamma). \qquad \square \end{aligned}$$

Thus Ψ_ρ is a continuous extension of the normalized Margulis invariant $\frac{\alpha_\rho}{l}$. Using this extension, Goldman–Labourie–Margulis showed that the converse of the Opposite Sign Lemma holds in the following setting.

Let Σ be a compact surface with nonempty boundary. We call a representation $\varrho\colon \pi_1(\Sigma) \to \mathrm{SL}(k,\mathbb{R})$ *Fuchsian* if it is the composition of a discrete and faithful representation $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ and the irreducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(k,\mathbb{R})$ (see Section 2.6.1 for a description of the irreducible representation).

Theorem 5.2.10 ([GLM09, Theorem (Introduction)])**.** *Let $\rho\colon \pi_1(\Sigma) \to G \ltimes \mathbb{R}^{2n+1}$ be a representation whose linear part is Fuchsian and without unipotent elements. Then ρ defines a properly discontinuous action of $\pi_1(\Sigma)$ on $\mathbb{R}^{2n+1}$ if and only if $\Psi_\rho(\mu) \neq 0$ for every $\mu \in \mathscr{C}(\mathrm{T}^1(\Sigma))$.*

Theorem 5.2.18 extends this result to representations of any word hyperbolic group whose linear part is Anosov with respect to $P^\pm$.

5.2.3 Sections of the affine bundle R_ρ

We now explain how to construct sections of the affine bundle R_ρ which are differentiable along flow lines. This will be useful in the next section. The construction is based on a partition of unity argument, making sure that the bump functions are differentiable along flow lines. We have to be careful defining "nice" neighborhoods since the action of Γ on $\widetilde{\mathsf{U}_0\Gamma}$ may have fixed points. Moreover, the bump functions should be adapted to the product structure of $\widetilde{\mathsf{U}_0\Gamma}$ because the metric on $\partial_\infty\Gamma$ is only well–defined up to Hölder equivalence, so differentiability is not defined there.

Recall from Section 5.1.2 that $\widetilde{\mathsf{U}_0\Gamma}$ is equipped with a metric which is unique up to Hölder equivalence. This metric is bi–Lipschitz equivalent to the product metric of the visual metric on $\partial_\infty\Gamma$ and the standard metric on $\mathbb{R}$. For $x = (a,b,t) \in \widetilde{\mathsf{U}_0\Gamma}$ and $\epsilon < \frac{d(a,b)}{2}$, we define

$$U_x^\epsilon := B_\epsilon(a,b) \times (t-\epsilon, t+\epsilon) \subset \partial_\infty\Gamma^{(2)} \times \mathbb{R},$$

where B_ϵ denotes the ϵ-ball in $\partial_\infty\Gamma^2$.

As Γ acts properly on $\widetilde{\mathsf{U}_0\Gamma}$, stabilizers of points in $\widetilde{\mathsf{U}_0\Gamma}$ are finite. It also allows us to find a good set of neighborhoods: Since

$$\begin{aligned} f\colon \Gamma \times \widetilde{\mathsf{U}_0\Gamma} &\to \widetilde{\mathsf{U}_0\Gamma} \times \widetilde{\mathsf{U}_0\Gamma} \\ (\gamma, x) &\mapsto (\gamma x, x) \end{aligned}$$

is proper and Γ is discrete, for any compact neighborhood $x \in K$ of a point $x \in \widetilde{\mathsf{U}_0\Gamma}$,

$$\pi_1(f^{-1}(K \times K)) = \{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\} =: \Gamma_K$$

is finite. We can therefore assume that $\Gamma_K = \Gamma_x$ by shrinking the neighborhood if necessary. Pick $\epsilon > 0$ small enough such that $U_x^\epsilon \subset K$. We distinguish two cases, depending on whether Γ_x is trivial or not.

(i) Assume first that $\Gamma_x = \{1\}$. Write $x = (a, b, t)$. Let $\alpha : \partial_\infty \Gamma^{(2)} \to \mathbb{R}$ be a Hölder continuous bump function which is positive on $B_\epsilon(a, b)$ and zero elsewhere, and $\beta : \mathbb{R} \to \mathbb{R}$ a smooth bump function which is positive on $(t - \epsilon, t + \epsilon)$ and zero elsewhere. Then

$$\begin{aligned} \theta : \widetilde{\mathsf{U}_0\Gamma} &\to \mathbb{R} \\ (c, d, s) &\mapsto \alpha(c, d)\beta(s) \end{aligned}$$

is a bump function at $x = (a, b, t)$ which is positive on U_x^ϵ and zero elsewhere, and which is smooth along flow lines. Moreover, the derivative along flow lines is again Hölder continuous and smooth along flow lines. Since ϵ is chosen such that $\Gamma_{U_x^\epsilon} = \{1\}$, it projects to a bump function at $\pi(x)$ on $\mathsf{U}_0\Gamma$ with the same properties.

(ii) Assume now that $\Gamma_x \neq \{1\}$. Let

$$V_x := \bigcup_{\gamma \in \Gamma_x} \gamma U_x^\epsilon,$$

and observe that $\Gamma_{V_x} = \Gamma_x$. Define θ as before to be a bump function which is positive on U_x^ϵ and zero elsewhere, and set

$$\vartheta := \sum_{\gamma \in \Gamma_x} \theta \circ \gamma.$$

Then ϑ is Hölder continuous, smooth along flow lines, positive on V_x and zero elsewhere. Since $\Gamma_{V_x} = \Gamma_x$ and ϑ is invariant under Γ_x, it projects to a bump function on $\mathsf{U}_0\Gamma$ with the same properties. Note that V_x gets arbitrarily small as ϵ approaches 0.

We can use these bump functions to construct sections of the affine bundle

$$\mathsf{R}_\rho = \Gamma \backslash \left(\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{2n+1} \right).$$

For every point $z \in \mathsf{U}_0\Gamma$, pick a neighborhood U_z such that $\mathsf{R}_\rho|_{U_z}$ is trivial and the above construction yields a bump function $\vartheta_z : \mathsf{U}_0\Gamma \to \mathbb{R}$ which is positive on U_z and zero elsewhere. By compactness of $\mathsf{U}_0\Gamma$, finitely many such neighborhoods U_z cover $\mathsf{U}_0\Gamma$. Denote them by U_{z_i}, $1 \leq i \leq k$. After normalizing, we may assume that $\sum_i \vartheta_{z_i} = 1$. Letting $s_{z_i} : U_{z_i} \to \mathsf{R}_\rho|_{U_{z_i}}$ denote a constant section (with respect to a local trivialization), observe that the affine combination

$$s = \sum_i \vartheta_{z_i} s_{z_i}$$

is a well–defined section of the affine bundle R_ρ which is Hölder continuous and smooth along flow lines.

5.2.4 Affine deformations of Anosov representations

Let Γ be a word hyperbolic group, $G = \mathrm{SO}_0(n+1, n)$ and $P^\pm$ stabilizers of transverse elements of $\mathrm{Is}_n(\mathbb{R}^{n+1,n})$. Moreover, let $\rho : \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism with linear part $\varrho = L(\rho)$ Anosov with respect to $P^\pm$. We denote the Anosov section of ϱ by σ_ϱ and write $\widetilde{\sigma}_\varrho : \widetilde{\mathsf{U}_0\Gamma} \to \mathcal{X}$ for the ϱ–equivariant map corresponding to it.

Definition 5.2.11 (Neutralized section)**.** A *neutralized section* is a Hölder continuous section

$$f \colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$$

which is differentiable along flow lines and satisfies

$$\nabla_\phi f(p) \in \mathbb{R}\nu_\varrho(p)$$

for all $p \in \mathsf{U}_0\Gamma$.

We now show that neutralized sections always exist. The arguments in the proof are the same ones used in [GLM09, Lemma 8.4].

Proposition 5.2.12. *Let ρ be as above. Then the bundle R_ρ admits a neutralized section.*

Proof. By a partition of unity argument, There exists a Hölder continuous section $s \colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ which is differentiable along flow lines (see Section 5.2.3). We want to modify the section s in such a way that it varies only in the direction of the neutral section as we follow any flow line in $\mathsf{U}_0\Gamma$.
Recall that we defined the splitting

$$\mathsf{R}_\varrho = \mathsf{V}^+ \oplus \mathsf{L} \oplus \mathsf{V}^-$$

in (5.1.4), where L is the line bundle spanned by the neutral section. Let $\nabla^+_\phi s$ and $\nabla^-_\phi s$ denote the components of $\nabla_\phi s$ in V^+ and V^-, so that

$$\nabla_\phi s(p) - \nabla^+_\phi s(p) - \nabla^-_\phi s(p) \in \mathsf{L}_p$$

for all $p \in \mathsf{U}_0\Gamma$.

Let $\|\cdot\|_e$ denote a Euclidean norm on the bundle R_ϱ. Then Corollary 5.1.11 implies that

$$\left\| \phi_{-t}\left(\nabla^+_\phi s(p)\right)\right\|_e \leq C\mathrm{e}^{-ct}\left\|\nabla^+_\phi s(p)\right\|_e$$

and

$$\left\| \phi_{t}\left(\nabla^-_\phi s(p)\right)\right\|_e \leq C\mathrm{e}^{-ct}\left\|\nabla^-_\phi s(p)\right\|_e$$

for some constants $C, c \in \mathbb{R}$ and every $p \in \mathsf{U}_0\Gamma$. Since $\mathsf{U}_0\Gamma$ is compact, the functions $\|\nabla^\pm_\phi s\|$ are bounded by some constant B. The improper integral

$$\int_0^\infty \phi_{-t}\left(\nabla^+_\phi(\psi_t p)\right)\mathrm{d}t$$

therefore converges by the following inequality:

$$\left\| \int_0^\infty \phi_{-t} \left(\nabla^+_\phi s(\psi_t p) \right) \mathrm{d}t \right\|_e \leq \int_0^\infty \left\| \phi_{-t} \left(\nabla^+_\phi s(\psi_t p) \right) \right\|_e \mathrm{d}t$$
$$\leq \int_0^\infty C \mathrm{e}^{-ct} \left\| \nabla^+_\phi s(\psi_t p) \right\|_e \mathrm{d}t \leq \int_0^\infty C \mathrm{e}^{-ct} B \, \mathrm{d}t < \infty$$

An analogous argument for the second improper integral shows that

$$f(p) := s(p) - \int_0^\infty \phi_t \left(\nabla^-_\phi s(\psi_{-t} p) \right) \mathrm{d}t + \int_0^\infty \phi_{-t} \left(\nabla^+_\phi s(\psi_t p) \right) \mathrm{d}t$$

is a well–defined section of the affine bundle R_ρ. Its derivative in flow direction is

$$\begin{aligned}
\nabla_\phi f(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t} f(\psi_t p) \\
&= \nabla_\phi s(p) - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t} \int_0^\infty \phi_u \left(\nabla^-_\phi s(\psi_{t-u} p) \right) \mathrm{d}u \\
&\quad + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t} \int_0^\infty \phi_{-u} \left(\nabla^+_\phi s(\psi_{t+u} p) \right) \mathrm{d}u \\
&= \nabla_\phi s(p) - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{-t}^\infty \phi_{-t}\phi_{u+t} \left(\nabla^-_\phi s(\psi_{-u} p) \right) \mathrm{d}u \\
&\quad + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_t^\infty \phi_{-t}\phi_{-u+t} \left(\nabla^+_\phi s(\psi_u p) \right) \mathrm{d}u \\
&= \nabla_\phi s(p) - \nabla^-_\phi s(p) - \nabla^+_\phi s(p),
\end{aligned}$$

where we used that the bounded linear operator $\phi_{-t} \colon (\mathsf{R}_\rho)_{\psi_t p} \to (\mathsf{R}_\rho)_p$ commutes with the integral. So f is a neutralized section. □

Proposition 5.2.13. *Let $\rho \colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism whose linear part ϱ is Anosov with respect to $P^\pm$. Then P_ρ admits an affine Anosov section.*

Proof. Let $f \colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ be a neutralized section and let $\sigma_\varrho \colon \mathsf{U}_0\Gamma \to \mathsf{P}_\varrho$ be the Anosov section for ϱ. Recall that the fiber of P_ϱ is the space of transverse pairs (V_1, V_2) in $\mathrm{Is}_n(\mathbb{R}^{n+1,n})$. By taking orthogonal complements, we can interpret these as pairs of $(n+1)$–dimensional subspaces of $\mathbb{R}^{n+1,n}$ of type $(n, 1, 0)$. Let

$$\sigma := f + \sigma_\varrho \colon \mathsf{U}_0\Gamma \to \mathsf{P}_\rho$$

denote the section obtained by using f as base point and σ_ϱ as linear parts of a pair of transverse affine subspaces of type $(n, 1, 0)$. The section σ_ϱ is parallel along flow lines, and since f is a neutralized section, we have $\nabla_\phi f(p) \in \mathbb{R}\nu_\varrho(p)$. Lifting to ϱ–equivariant maps, we have $\nu \circ \widetilde{\sigma}_\varrho(\widetilde{p}) \in (V_1)^\perp \cap (V_2)^\perp$, where $\widetilde{\sigma}_\varrho(\widetilde{p}) = (V_1, V_2) \in \mathrm{Is}_n(\mathbb{R}^{n+1,n})$. Thus σ is parallel along flow lines.

It remains to show that the bundle $\sigma^* X^+_{\text{aff}}$ is contracted by the flow and $\sigma^* X^-_{\text{aff}}$ is dilated. Recall from (5.2.2) that we have a splitting $\sigma^* X^\pm_{\text{aff}} = (L \circ \sigma)^* X^\pm \oplus \mathsf{V}^\mp$. By construction, the linear part $L \circ \sigma$ is the linear Anosov section σ_ϱ, so the bundles $(L \circ \sigma)^* X^+$ resp. $(L \circ \sigma)^* X^-$ are contracted resp. dilated by the flow. By Corollary 5.1.11, the bundles V^- resp. V^+ are also contracted resp. dilated by the flow, so the result follows. $\square$

The following corollary connects the affine Anosov property with the existence of special geodesic currents on $\mathsf{U}_0\Gamma$. It will provide a link to proper actions in the next section.

Corollary 5.2.14. *Let $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism whose linear part ϱ is Anosov with respect to $P^\pm$, and let $\tau\colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ be a section that is differentiable along flow lines. Assume that ρ is not affine Anosov with respect to $P^\pm_{\text{aff}}$. Then there exists a geodesic current μ_τ on $\mathsf{U}_0\Gamma$ such that*

$$\int b(\nabla_\phi \tau, \nu_\varrho) \, \mathrm{d}\mu_\tau = 0.$$

Proof. By Proposition 5.2.13, ρ admits an affine Anosov section, hence the second part of Definition 5.2.7 must fail: For any Hölder section τ' of R_ρ, the function $b(\nabla_\phi \tau', \nu_\varrho)$ must have a zero.

Now assume that no such measure μ_τ exists. Since the space of ψ–invariant probability measures is connected (convex combination are again ψ–invariant probability measures), we can assume that

$$\int b(\nabla_\phi \tau, \nu_\varrho) \, \mathrm{d}\mu > 0$$

for all ψ–invariant probability measures μ on $\mathsf{U}_0\Gamma$ (the proof of the negative case is the same). Then by [GL12, Lemma 3], the function $b(\nabla_\phi \tau, \nu_\varrho)$ is Livšic cohomologous to some positive function f,

$$(b(\nabla_\phi \tau, \nu_\varrho) - f)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \circ \psi_t(p) \qquad \forall p \in \mathsf{U}_0\Gamma$$

for a function $g\colon \mathsf{U}_0\Gamma \to \mathbb{R}$. So the Hölder section $\tau' := \tau - g\nu_\varrho$ satisfies

$$b(\nabla_\phi \tau', \nu_\varrho) = f > 0,$$

contradicting our hypothesis. $\square$

Corollary 5.2.15. *Let $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism whose linear part ϱ is Anosov with respect to $P^{\pm}$. Then ρ is affine Anosov with respect to $P^{\pm}_{\mathrm{aff}}$ if and only if*

$$\Psi_\rho(\mu) \neq 0 \quad \forall \mu \in \mathscr{C}(\mathsf{U}_0\Gamma).$$

Proof. Assume that ρ is affine Anosov. Then there exists a Hölder section τ of R_ρ such that $b(\nabla_\phi \tau, \nu_\varrho)$ is either positive everywhere or negative everywhere, thus the same holds for

$$\Psi_\rho(\mu) = \int_{\mathsf{U}_0\Gamma} b(\nabla_\phi \tau, \nu_\varrho)\,\mathrm{d}\mu$$

for all $\mu \in \mathscr{C}(\mathsf{U}_0\Gamma)$.

Conversely, if ρ is not affine Anosov, there exists a geodesic current μ such that $\Psi_\rho(\mu) = 0$ by Corollary 5.2.14. □

5.2.5 Relation to proper actions

We now prove that affine Anosov representations into $G \ltimes \mathbb{R}^{2n+1}$ correspond to proper actions on $\mathbb{R}^{2n+1}$.

Using Corollary 5.2.14 and with some adjustments, the proof of [GLM09, Proposition 8.1] extends to the case of word hyperbolic groups and any Anosov linear parts.

Theorem 5.2.16. *Let Γ be a word hyperbolic group and $\rho\colon \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ a homomorphism whose linear part $L(\rho) = \varrho$ is Anosov with respect to $P^{\pm}$. Assume that the induced Γ–action on $\mathbb{R}^{2n+1}$ is proper. Then ρ is affine Anosov with respect to $P^{\pm}_{\mathrm{aff}}$.*

Proof. As Γ acts properly on $\mathbb{R}^{2n+1}$, it also acts properly on

$$\partial_\infty \Gamma^{(2)} \times \mathbb{R}^{2n+1} \cong (\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{2n+1})/\mathbb{R},$$

where the action of $\mathbb{R}$ on $\mathbb{R}^{2n+1}$ is trivial (this is the action of the flow ϕ_t). By [GLM09, Lemma 5.2], this implies that $\mathbb{R}$ acts properly on

$$\Gamma \backslash (\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^{2n+1}) = \mathsf{R}_\rho,$$

where Γ acts on $\mathbb{R}^{2n+1}$ via ρ.

Now assume that ϱ is Anosov with respect to $P^{\pm}$ but ρ is not affine Anosov with respect to $P^{\pm}_{\mathrm{aff}}$. By Proposition 5.2.12, there exists a neutralized section τ, and by Corollary 5.2.14, there exists a geodesic current μ_τ on $\mathsf{U}_0\Gamma$ such that

$$\int b(\nabla_\phi \tau, \nu_\varrho)\,\mathrm{d}\mu_\tau = 0.$$

Let $t > 0$ and $p \in \mathsf{U}_0\Gamma$. We define $\tau_t \colon \mathsf{U}_0\Gamma \to \mathbb{R}$ by

$$\tau_t(p) := \int_0^t b(\nabla_\phi \tau, \nu_\varrho)(\psi_s p)\, \mathrm{d}s.$$

As τ is neutralized and ν_ϱ is parallel along flow lines and of constant norm 1, we have

$$\tau(\psi_t p) = \phi_t(\tau(p)) + \tau_t(p)\nu_\varrho(\psi_t p) \tag{5.2.3}$$

for every $p \in \mathsf{U}_0\Gamma$ and $t \in \mathbb{R}$. Moreover, $\int \tau_t d\mu_\tau = 0$ by Fubini's theorem. Therefore, since $\mathsf{U}_0\Gamma$ is connected (Lemma 5.1.7), for every $t > 0$, there exists $p_t^\tau \in \mathsf{U}_0\Gamma$ such that

$$\tau_t(p_t^\tau) = 0.$$

We conclude that

$$\tau(\psi_t p_t^\tau) = \phi_t(\tau(p_t^\tau))$$

for every $t \in \mathbb{R}$. This implies that

$$\phi_t(\tau(\mathsf{U}_0\Gamma)) \cap \tau(\mathsf{U}_0\Gamma) \neq \emptyset$$

for every $t \in \mathbb{R}$, which is a contradiction to properness of the $\mathbb{R}$–action on R_ρ since $\tau(\mathsf{U}_0\Gamma)$ is compact. □

Theorem 5.2.17. *Let Γ be a word hyperbolic group and let $\rho : \Gamma \to G \ltimes \mathbb{R}^{2n+1}$ be a homomorphism which is affine Anosov with respect to $P^\pm_{\mathrm{aff}}$. Then Γ acts properly on $\mathbb{R}^{2n+1}$ and the linear part ϱ is Anosov with respect to $P^\pm$.*

Proof. Let σ_ρ be the affine Anosov section of ρ. Its linear part

$$\sigma_\varrho = L \circ \sigma_\rho$$

is a section of the bundle P_ϱ that is parallel along flow lines. Since the bundles $\sigma_\rho^* X^\pm_{\mathrm{aff}}$ are contracted/dilated by the flow ϕ_t, so are $\sigma_\varrho^* X^\pm$ (see (5.2.2)). Thus ϱ is Anosov with respect to $P^\pm$.

Now assume that Γ does not act properly on $\mathbb{R}^{2n+1}$. Then, there exists a sequence $\gamma_m \in \Gamma$ with $\gamma_m \to \infty$ and a converging sequence $x_m \to x \in \mathbb{R}^{2n+1}$ with $\rho(\gamma_m)x_m \to y \in \mathbb{R}^{2n+1}$.

First of all, we show that we may assume without loss of generality that

- γ_m has infinite order,
- the endpoints $\gamma_m^\pm \in \partial_\infty\Gamma$ of the axis of γ_m have distinct limits $a^\pm$,
- $l(\gamma_m) \to \infty$.

By [GW12, Theorem 1.7], the image $\varrho(\Gamma)$ is (AMS)–proximal (see Section 5.1.4 for a short discussion). This implies that for any $r > 0, \epsilon > 0$, there exists a finite set $S \subset \varrho(\Gamma)$ with the following property: For every m, there is an element $s(m) \in S$ such that $s(m)\varrho(\gamma_m)$ is (r,ϵ)–proximal. After taking a subsequence, we may assume that $s(m) = s$ is constant. Let $s' \in \Gamma$ be a preimage of s and set $\gamma'_m = s'\gamma_m$. Then $\gamma'_m, \varrho(\gamma'_m), \rho(\gamma'_m)$ all have infinite order, we still have $\gamma'_m \to \infty$ and $\rho(\gamma'_m)x_m = \rho(s')\rho(\gamma_m)x_m \to \rho(s')y$. Taking a subsequence again, we can assume that $\gamma'^{\pm}_m \in \partial_\infty\Gamma$ converge to $a^\pm$. By (r,ϵ)–proximality, the attracting and repelling maximal isotropics of $\varrho(\gamma'_m)$ have transverse limits, so we must have $a^+ \neq a^-$. Therefore, $\mathrm{axis}(\gamma'_m)$ converges as a subset of $\widetilde{\mathsf{U}_0\Gamma}$ [2], which implies $l(\gamma'_m) \to \infty$ since Γ acts properly on $\widetilde{\mathsf{U}_0\Gamma}$.

We assume from now on that γ_m has the properties listed above. Let $\widetilde{\sigma}_\varrho \colon \widetilde{\mathsf{U}_0\Gamma} \to \mathcal{X}$ be the ϱ–equivariant map corresponding to σ_ϱ. Since the subsets $\mathrm{axis}(\gamma_m) \subset \widetilde{\mathsf{U}_0\Gamma}$ converge, we can pick a convergent sequence $p_m \in \mathrm{axis}(\gamma_m),\ p_m \to p$. It follows that

$$\alpha_\rho(\gamma_m) = b\big(\rho(\gamma_m)x_m - x_m, \nu \circ \widetilde{\sigma}_\varrho(p_m)\big) \xrightarrow{m\to\infty} b\big(y - x, \nu \circ \widetilde{\sigma}_\varrho(p)\big),$$

so in particular $\alpha_\rho(\gamma_m)$ stays bounded. We now show that this is a contradiction, finishing the proof.

Because ρ is affine Anosov, there is a Hölder section $\tau \colon \mathsf{U}_0\Gamma \to \mathsf{R}_\rho$ such that $b(\nabla_\phi\tau, \nu_\varrho)$ is positive everywhere or negative everywhere on $\mathsf{U}_0\Gamma$. We assume that $b(\nabla_\phi\tau, \nu_\varrho) > 0$. Since $\mathsf{U}_0\Gamma$ is compact, $b(\nabla_\phi\tau, \nu_\varrho)$ is bounded from below by a constant $M > 0$, thus

$$\int b(\nabla_\phi\tau, \nu_\varrho)\,\mathrm{d}\mu_{\gamma_m} \geq M,$$

where μ_{γ_m} is the geodesic current corresponding to γ_m. Proposition 5.2.9 therefore implies that $\alpha_\rho(\gamma_m) \to \infty$. □

Combining Theorem 5.2.16, Theorem 5.2.17 and Corollary 5.2.15 now yields our main result, generalizing the corresponding result in [GLM09].

Theorem 5.2.18. *Let Γ be a word hyperbolic group, $\rho \colon \Gamma \to \mathrm{SO}_0(n+1,n) \ltimes \mathbb{R}^{2n+1}$ a homomorphism and $\varrho = L(\rho)$ its linear part. Then the following are equivalent.*

(i) ϱ is Anosov with respect to $P^\pm$ and Γ acts properly on $\mathbb{R}^{2n+1}$ via ρ.

(ii) ρ is affine Anosov with respect to $P^\pm_{\mathrm{aff}}$.

(iii) ϱ is Anosov with respect to $P^\pm$ and $\Psi_\rho(\mu) > 0\ \forall \mu \in \mathscr{C}(\mathsf{U}_0\Gamma)$.

[2] with respect to the pointed Hausdorff topology on the set of closed subsets of the metric space $\widetilde{\mathsf{U}_0\Gamma}$. A basis of this topology is given by sets of the form

$$N_{K,\epsilon}(A) = \{B \subset \widetilde{\mathsf{U}_0\Gamma} \text{ closed } \mid d_H(B \cap K, A \cap K) < \epsilon\},$$

where $K \subset \widetilde{\mathsf{U}_0\Gamma}$ is a compact set, $A \subset \widetilde{\mathsf{U}_0\Gamma}$ is a closed set and d_H is the Hausdorff metric on closed subsets of the compact set K.

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