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Singular fibers of Hitchin systems

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Abstract. In the recent years, Hitchin systems and Higgs bundle moduli spaces were intensively studied in mathematics and physics. Two major breakthroughs were the formulation of Langlands duality of Hitchin systems and the understanding of the asymptotics of the hyperkähler metric on Higgs bundle moduli space. Both results were considered for the regular locus of the Hitchin system and both results are conjectured to extend to the singular locus. In this work, we make the first steps towards generalizing these theorems to singular Hitchin fibers.

To that end, we develop spectral data for a certain class of singular fibers of the symplectic and odd orthogonal Hitchin system. These spectral data consist of abelian coordinates taking value in an abelian torsor and non-abelian coordinates parametrising local deformations of the Higgs bundles at the singularities of the spectral curve. First of all, these semi-abelian spectral data allow us to obtain a global description of singular Hitchin fibers. Moreover, we can construct solutions to the decoupled Hitchin equation on the singular locus of the Hitchin map. These are limits of solutions to the Hitchin equation along rays to the ends of the moduli space playing an important role in the analysis of the asymptotics of the hyperkähler metric. Finally, we can explicitly describe how Langlands duality extends to this class of singular Hitchin fibers. We discover a duality on the abelian part of the spectral data, similar to regular case. Instead, the non-abelian coordinates are symmetric under this Langlands correspondence.

Zusammenfassung. Hitchin-Systeme und Higgs-Bündel-Moduliräume erfuhren in den letzten Jahren ein wiedererstarktes Interesse von Seiten der Mathematik und Physik. Dies führte zu zwei großen Durchbrüchen: Zum einen zur Formulierung der Langlands-Dualität von Hitchin-Systemen und zum anderen zur Analyse der Asymptotik der hyperkählerschen Metrik. Beide Resultate wurden bisher für den regulären Lokus der Hitchin-Abbildung bewiesen. Es wird allerdings vermutet, dass sich beide Resultate auf den singulären Lokus fortsetzen lassen. Diese Arbeit will die ersten Schritte in diese Richtung gehen, indem gezeigt wird, wie sich Teilresultate auf singuläre Hitchin-Fasern erweitern lassen und welche neuen Herausforderungen sich ergeben.

Zu diesem Zweck werden Spektraldaten für eine bestimmte Klasse singulärer Fasern des symplektischen und ungerade orthogonalen Hitchin-Systems eingeführt. Diese Spektraldaten bestehen aus abelschen Koordinaten mit Werten in einem abelschen Torsor und nichtabelschen Koordinaten, die lokale Transformationen des Higgs-Bündels an den Singularitäten der Spektralkurve beschreiben. Zunächst vermitteln diese halbabelschen Spektraldaten ein globales Verständnis der Geometrie der singulären Hitchin-Fasern. Weiterhin können mit ihrer Hilfe Lösung zur entkoppelten Hitchin-Gleichung konstruiert werden. Die Lösungen dieser Gleichung haben beim Verständnis der Asymptotik der hyperkählerschen Metrik auf dem regulären Lokus eine wichtige Rolle gespielt. Schlussendlich, kann durch den direkten Vergleich der singulären Fasern eine Fortsetzung der Langlands-Korrespondenz auf den singulären Lokus formuliert werden. In den abelschen Koordinaten wird wie im regulären Fall eine Dualität beobachtet. Die nichtabelschen Koordinaten hingegen sind symmetrisch unter der Langlands-Korrespondenz.

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Contents

Introduction

> Functions, just like living beings, are characterised by their singularities
>
> *Paul Montel* [Mon32]
> *Vladimir I. Arnold* [Arn92]

For more than thirty years, the study of moduli spaces of Higgs bundles is a very active research area located at the crossroads of algebraic, complex and differential geometry with the theory of integrable systems and surface group representations. One major reason for the ongoing interest in these moduli spaces is their extremely rich geometry. They were introduced by Hitchin [Hit87b] as examples of non-compact hyperkähler spaces. They are homeomorphic to moduli spaces of flat G-bundles on X by the famous Non-Abelian Hodge Theory [Hit87b; Don87; Sim88; Cor88]. And most importantly for the present work, they have a dense subset carrying the structure of an algebraically completely integrable system - the so-called Hitchin system [Hit87a].

Hitchin systems. In physics, completely integrable systems are dynamical systems with sufficiently many independent conserved quantities to find explicit solutions for all times. Classical examples are the motion of a rigid body about its center of mass and the geodesic flow on an ellipsoid, a once important problem of geodesy .

In mathematical terms, a completely integrable system is a (complex) symplectic manifold of dimension $2n$ with a system of n independent Poisson commuting functions. If the level sets of these functions are compact and connected, it is a classical theorem of Liouville and Arnold [Arn78] that the Hamiltonian vector fields generate a simple transitive torus action.

By definition, the Higgs bundle moduli space $\mathcal{M}_G$ on a Riemann surface X associated to a complex linear group G is a moduli space of pairs (E, Φ). Here E is a holomorphic G-vector bundle on X and Φ is holomorphic one-form valued in $\mathfrak{g}$, called the Higgs field. $\mathcal{M}_G$ has a complex symplectic structure on its smooth points and a system of Poisson commuting functions is defined by the Hitchin map

$$\mathsf{Hit}_G : \mathcal{M}_G \to B_G.$$

This is a proper, surjective, holomorphic map to a complex vector space B_G of half the dimension of $\mathcal{M}_G$, referred to as the Hitchin base. And indeed, Hitchin showed for the classical groups [Hit87a] and Scognamillo for all complex reductive groups [Sco98], that on a dense subset $B_G^{\mathsf{reg}} \subset B_G$ the fibers of the Hitchin map are complex Lagrangian tori. Hence, the preimage of the regular locus B_G^{reg} under the Hitchin map is a completely integrable system, nowadays called Hitchin system.

In addition, the complex tori have the structure of an algebraic variety and are therefore abelian varieties. To identify the Hitchin fibers over the regular locus with abelian varieties one needs to introduce spectral data. The Hitchin map applied to a Higgs bundle (E, Φ) computes the eigenvalues of the Higgs field Φ. These eigenvalues are decoded in a complex curve covering the original Riemann surface X. Each sheet of this covering over a point $x \in X$ corresponds to an eigenvalue of Φ at x. This is the so-called spectral curve respectively spectral covering. Having fixed the eigenvalues, the eigenspaces determine a line bundle on the spectral curve. For a point in the regular locus B_G^{reg} the spectral curve is smooth. In this case, the moduli spaces of eigen line bundles are the classical examples of abelian varieties, most importantly Jacobians and Prym varieties (see Section 1.4). This gives the torus fibers the smoothly varying structure of abelian varieties turning the Hitchin system into an algebraically completely integrable system.

Langlands duality for Hitchin systems. The recent progress in the theory of Higgs bundle moduli spaces is highly stimulated by string theory. In string theory, spacetime is augmented by extra dimensions in certain compact Ricci-flat Kähler manifolds, so-called Calabi-Yau manifolds. Hyperkähler manifolds are Ricci-flat and, even though being non-compact, the physical framework of string theory was a driving force in the study of Higgs bundle moduli space. In the present work, we will be concerned with two instances of this recent progress: Firstly, the formulation of Langlands duality of Higgs bundle moduli spaces and, secondly, the study of the asymptotics of the hyperkähler metric at the ends of the moduli space.

Langlands duality of Higgs bundle moduli spaces is a reincarnation of mirror symmetry. Originally, mirror symmetry is a duality between different mathematical models of a certain string theory suggesting that Calabi-Yau manifolds come in pairs $(M, \hat{M})$: The symplectic geometry of M determines the complex geometry of $\hat{M}$ and vice versa. A geometric interpretation in terms of integrable systems is the Strominger-Yau-Zaslow (SYZ) conjecture [SYZ01]. It states that for a Calabi-Yau manifold M fibering over a base B by special Lagrangian tori one can obtain its mirror partner by dualizing the torus fibers.

For Hitchin systems, mirror symmetry is connected to another important duality in pure mathematics - the so-called Langlands duality. For a algebraic group G there exists a Langlands dual group G^L, such that conjecturally the representation theory of G is controlled by Galois representations into G^L.

Starting from the work of Hausel and Thaddeus [HT03] for $G = \mathsf{SL}(n, \mathbb{C})$, $G^L = \mathsf{PSL}(n, \mathbb{C})$ and Hitchin [Hit07] for $G = \mathsf{Sp}(2n, \mathbb{C})$, $G^L = \mathsf{SO}(2n+1, \mathbb{C})$ and $G = G^L = \mathsf{G}_2$, Donagi and Pantev [DP12] established the following formulation of Langlands duality of G-Hitchin systems for a complex semi-simple Lie group G.

i) The Hitchin bases B_G and B_{G^L} are isomorphic and the isomorphism restricts to the regular loci B_G^{reg} and $B_{G^L}^{\mathsf{reg}}$.
ii) The regular fibers over corresponding points $b \in B_G^{\mathsf{reg}}$ and $b' \in B_{G^L}^{\mathsf{reg}}$ are abelian torsors over dual abelian varieties.

Recall that an abelian torsor is an algebraic variety with a simple transitive algebraic group action by an abelian variety.

In terms of mirror symmetry, this suggests that $\mathcal{M}_G$ and $\mathcal{M}_{G_L}$ (or at least their regular loci) are mirror partners. And indeed recently it was proven, that the pair $(\mathcal{M}_{\mathsf{SL}(n,\mathbb{C})}, \mathcal{M}_{\mathsf{PSL}(n,\mathbb{C})})$ satisfies the Topological Mirror Symmetry Conjecture [GWZ17].

The general problem of the SYZ conjecture is that, for interesting Calabi-Yau manifolds, there can not be a global torus fibration. We rather find a map $M \to B$, such that the generic fiber is a complex Lagrangian torus. More explicitly, there exist points in the base B, over which the fiber is degenerate. This is the situation we met for the Hitchin system. It is a torus fibration on the regular locus B^{reg}, but over points in the complement the torus fibers degenerate.

It is still an active field of research to extend the SYZ conjecture to families of degenerating special Lagrangian tori [Gro09]. In Figure 1, we see a family of tori degenerating by pinching a curve. Such an example was consider in [Aur07] and it turns out that the singular fiber is self-mirror.

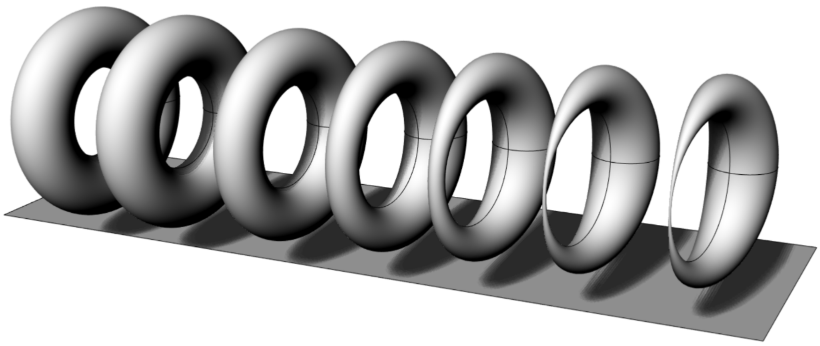

FIGURE 1. Degeneration to nodal torus[1]

For a global understanding of the Langlands duality of Higgs bundle moduli spaces again we are missing an extension to $B \setminus B^{\mathsf{reg}}$, the so-called singular locus. Donagi and Pantev state in [DP12]:

> "Our work deals with smooth cameral covers, establishing the Hitchin duality over the complement of the discriminant. A major step forward would be to formulate and prove the extension to the entire base."

[1]Illustration with permission by Menelaos Zikidis

Singular fibers of Hitchin systems. In the present work, we will do a first step in this direction. We will establish spectral data for a certain class of singular Hitchin fibers for the Langlands dual groups $\mathsf{Sp}(2n, \mathbb{C})$ and $\mathsf{SO}(2n+1, \mathbb{C})$. We will observe the relation between corresponding singular fibers over the same point in the Hitchin base extending Langlands duality to singular Hitchin fibers.

In general, the geometry of singular Hitchin fibers and their spectral curves is quite involved. The spectral curves can have several irreducible components and these components can be non-reduced. For example, consider the fiber over $0 \in B_G$, the so-called nilpotent cone. Here the spectral curve is a copy of the original Riemann surface of higher multiplicity. The nilpotent cone has itself many irreducible components and carries all the topological information about $\mathcal{M}_G$ (see [Hit87b] for $G = \mathsf{SL}(2, \mathbb{C})$). We will give a description of the irreducible components of the nilpotent cone for $\mathsf{SL}(3, \mathbb{C})$ in Chapter 6 of this work. This result impressively underlines the complexity of this singular Hitchin fiber. The intersection of the irreducible components is even more mysterious. For $\mathsf{SL}(2, \mathbb{C})$, it is subject of the recent work [ALS20].

In the main part of this work, we will analyse singular Hitchin fibers with irreducible and reduced spectral curve. Singular fibers of this kind were studied in [Sch98; Ngô10; GO13] mostly building on a theorem by Beauville-Narasimhan-Ramanan from the beginning of the history of Higgs bundles [BNR89]. It states that the Hitchin fibers with irreducible and reduced spectral curve can be identified with certain moduli spaces of torsion-free sheaves on the spectral curve. In [GO13], an analysis of these moduli spaces was used to prove connectedness of the singular Hitchin fibers for $\mathsf{SL}(2, \mathbb{C})$. However, this moduli spaces are themselves quite complicated objects in algebraic geometry. Moreover, it is hard to extract information about the Higgs bundle associated to a particular torsion-free sheaf under the Beauville-Narasimhan-Ramanan correspondence.

Stratification result. We take a more direct approach to the study of singular Hitchin fibers with irreducible and reduced spectral curve. The normalisation associates a smooth Riemann surface to the singular spectral curve. Similar to regular Hitchin fibers, the eigenspaces of the Higgs field define line bundles on the normalised spectral curve. However, these line bundles will live in different connected components of their moduli space depending on the local shape of the Higgs bundles at the singularities of the spectral curve. This yields a stratification of singular Hitchin fibers.

We will formulate this result for Hitchin fibers of $\mathfrak{sl}(2)$-type, a class of Hitchin fibers distinguished by the singularities of the spectral curve. For $G = \mathsf{SL}(2, \mathbb{C})$, all Hitchin fibers are of $\mathfrak{sl}(2)$-type.

THEOREM 1 (Theorem 4.2.13, 4.4.5). *Let $G = \mathsf{Sp}(2n, \mathbb{C})$ or $G = \mathsf{SO}(2n+1, \mathbb{C})$. Let $b \in B_G$ with irreducible and reduced spectral curve of $\mathfrak{sl}(2)$-type. Then there exists a stratification*

$$\mathsf{Hit}_G^{-1}(b) = \bigsqcup_{i \in I} \mathcal{S}_i$$

by finitely many locally closed subsets $\mathcal{S}_i$, such that every stratum $\mathcal{S}_i$ is a $(\mathbb{C}^*)^{r_i} \times \mathbb{C}^{s_i}$-bundle over a fixed abelian torsor.

The abelian torsor parametrises the eigen line bundles of $(E,\Phi) \in \mathsf{Hit}_G^{-1}(b)$ and will be referred to as the abelian part of the spectral data. The $(\mathbb{C}^*)^{r_i} \times \mathbb{C}^{s_i}$-fibers, the non-abelian part of the spectral data, decodes local deformations of the Higgs bundle at the singularities of the spectral curve by so-called Hecke transformations.

A Hecke transformation of a holomorphic vector bundle is the generalization of twisting a line bundle by a divisor. The work of Hwang-Ramanan [HR04] showed that Hecke transformations can be used to deform Higgs bundles along singular Hitchin fibers. We develop this approach and show that, depending on the singularities of the spectral curve, there is a certain family of Hecke transformations acting on the singular Hitchin fiber. Hecke transformations are parametrized by the directions, in which the holomorphic vector bundle is twisted and these parameters are the $(\mathbb{C}^*)^{r_i} \times \mathbb{C}^{s_i}$-fibers in Theorem 1.

A global view on singular Hitchin fibers. The stratification of Theorem 1 contains a unique, open and dense stratum $\mathcal{S}_0 \subset \mathsf{Hit}_G^{-1}(b)$. This dense stratum is compactified by lower dimensional strata distinguished from $\mathcal{S}_0$ by a lower dimensional moduli space of Hecke parameters. For the unique closed stratum, this parameter space is a point and hence this stratum identifies with the abelian torsor. The collection of closed strata over certain subsets of $B_G \setminus B_G^{\mathsf{reg}}$ form the lower-dimensional integrable systems supported on the singular locus, that were described in Hitchin's recent work [Hit19].

To analyse how the strata glue together to form the singular Hitchin fiber we consider two examples in more detail. For $G = \mathsf{SL}(2,\mathbb{C})$, the Hitchin base is the vector space of quadratic differentials $H^0(X, K_X^2)$. In this setting, the examples we want to consider are Hitchin fibers over a quadratic differential $q \in H^0(X, K_X^2)$ with a single zero of order 2 or 3, such that all other zeroes are simple. For $\mathsf{Sp}(2n,\mathbb{C})$ and $\mathsf{SO}(2n+1,\mathbb{C})$, there are corresponding cases for all $n \in \mathbb{N}$.

Let us start wit the case of a single zero of order 3. Here we have two strata

$$\mathcal{S}_0 : (r_0 = 0, s_0 = 1), \quad \mathcal{S}_1 : (r_1 = 0, s_1 = 0).$$

In Figure 2, we sketched the situation by compressing the abelian part of the

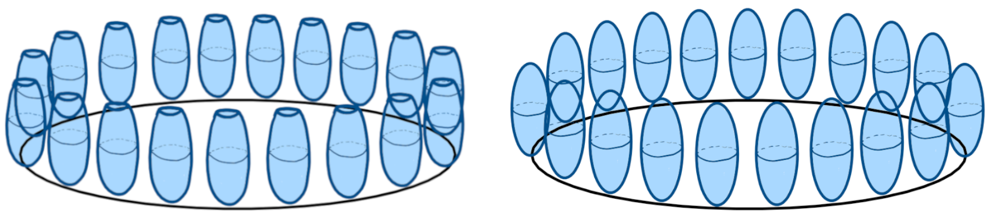

FIGURE 2. $\mathbb{P}^1$-bundle over abelian torsor

spectral data to a circle. On the left hand side, we see the open and dense stratum $\mathcal{S}_0$, where think of the $\mathbb{C}$-fiber as $\mathbb{P}^1 \setminus \{\infty\}$. When we glue in the lower dimensional stratum $\mathcal{S}_1$, we obtain the singular Hitchin fiber as a $\mathbb{P}^1$-bundle over an abelian torsor (see Section 2.6.5 for more details). Generalising this situation, we obtain the following theorem:

THEOREM 2 (Theorem 2.6.14, 4.2.14, 4.4.5). Let $b \in B_G$ of $\mathfrak{sl}(2)$-type, such that the spectral curve is reduced and locally irreducible, then the Hitchin fiber $\mathsf{Hit}_G^{-1}(b)$ is a holomorphic fiber bundle

$$\text{Compact moduli space of Hecke parameters } \to \mathsf{Hit}_G^{-1}(b) \to \text{Abelian Torsor.}$$

Here the compact moduli space of Hecke parameters is obtained by glueing the non-abelian spectral data of all strata. The fact that the projection map to the abelian torsor persists under this glueing is non-trivial, as we will see in the next example.

We consider the case of a quadratic differential $q \in H^0(X, K_X^2)$ with one double zero and all other zeroes simple. Again we have two strata

$$\mathcal{S}_0 : (r_0 = 1, s_0 = 0), \quad \mathcal{S}_1 : (r_1 = 0, s_1 = 0).$$

On the left hand side of Figure 3, we see a sketch of the open and dense stratum $\mathcal{S}_0$, where the $\mathbb{C}^*$-fibers are depicted by little tunnels. However, if we want to compactify the dense stratum $\mathcal{S}_0$ by $\mathcal{S}_1$, a new phenomena arises. The Higgs bundles corresponding to the points zero and infinity do not have the same eigen line bundle and hence do not correspond the same point on the abelian torsor. We obtain the singular Hitchin fiber by glueing the point at infinity to another point on the circle corresponding to the new eigen line bundle (see Example 2.7.3 for more details). This is sketched on the right hand side of Figure 3.

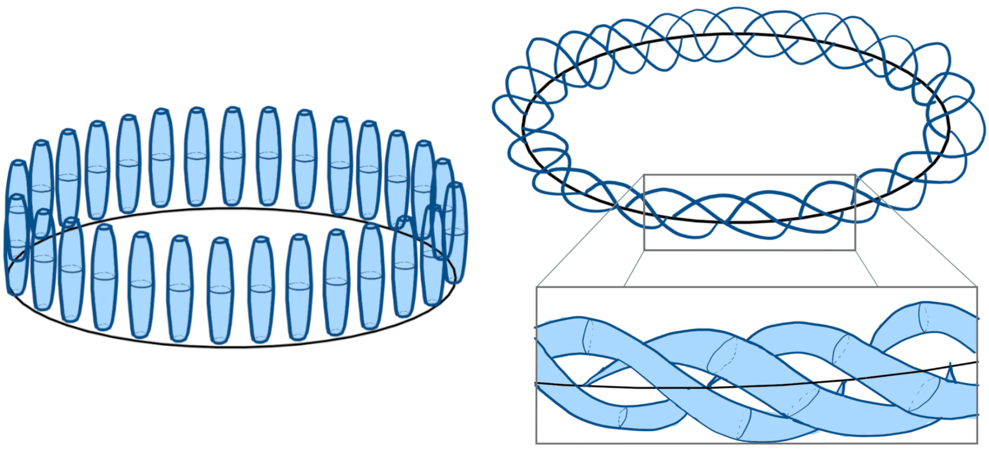

FIGURE 3. Twisted $\mathbb{P}^1$-bundle over an abelian torsor

In particular, we see that there can not be a well-defined map extending to $\mathsf{Hit}_G^{-1}(q)$ the projection of $\mathcal{S}_0$ to the abelian torsor. More generally, whenever the spectral curve is not locally irreducible, we are left with a surjective map from a fiber bundle $F_G(b)$

$$\text{Compact moduli of Hecke parameters } \to F_G(b) \to \text{Abelian Torsor}$$

to $\mathsf{Hit}_G^{-1}(b)$, such that the projection map from $F_G(b)$ to the abelian torsor does not factor. This was recognized before in [GO13; Hit19] in the $\mathsf{SL}(2,\mathbb{C})$-case.

Towards Langlands duality for singular Hitchin fibers. Concerning Langlands duality, we have to take a closer look at the abelian part of the spectral data. For $G = \mathsf{Sp}(2n,\mathbb{C})$ the spectral curve Σ has an involutive Deck transformation $\sigma : \Sigma \to \Sigma$. We can take its quotient and, together with the normalised spectral curve $\tilde{\Sigma}$, we obtain the commutative diagram of spectral curves in Figure 4.

$$\begin{tikzcd}
 & & \tilde{\Sigma} \arrow[dll] \arrow[dl] \arrow[ddl] \\
\Sigma/\sigma \arrow[dr] & \Sigma \arrow[l] \arrow[d] & \\
 & X &
\end{tikzcd}$$

FIGURE 4. Commutative diagram of spectral curves

By definition the spectral curve is of $\mathfrak{sl}(2)$-type if and only if Σ/σ is smooth. In this case, there is an abelian variety associated to the 2-sheeted branched covering of Riemann surfaces $\tilde{\Sigma} \to \Sigma/\sigma$, the so-called Prym variety. The abelian part of the spectral data for $G = \mathsf{Sp}(2n,\mathbb{C})$ is a torsor over this Prym variety.

For $G = \mathsf{SO}(2n+1,\mathbb{C})$, the abelian part of the spectral data is closely related. It is a union of torsors over a quotient of the Prym variety by the finite group $\mathbb{Z}_2^{2g}$, where g is the genus of X. This quotient can be identified with the dual abelian variety. We obtain the following formulation of Langlands duality for singular Hitchin fibers.

COROLLARY 3 (Corollary 4.4.7). Let $b \in B_{\mathsf{Sp}(2n,\mathbb{C})} = B_{\mathsf{SO}(2n+1,\mathbb{C})}$ of $\mathfrak{sl}(2)$-type, such that the spectral curve is irreducible and reduced. Then the Hitchin fibers $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(b)$ and $\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(b)$ are related as follows:

i) The abelian parts of the spectral data are unions of torsors over dual abelian varieties.
ii) The parameter spaces of Hecke transformations are isomorphic.

Recall that for $\mathsf{SL}(2,\mathbb{C})$, all Hitchin fibers are of $\mathfrak{sl}(2)$-type.

Going back to figures 2, 3, thinking of the abelian part as a circle of circumference C, we obtain the Langlands dual Hitchin fiber by changing the circumference to $\frac{1}{C}$ leaving the Hecke parameters unchanged.

Limiting Configurations for singular Hitchin fibers. Another recent development in the study of Higgs bundle moduli spaces is the analysis of the asymptotics of the hyperkähler metric at the ends of the moduli space. Evolving from an intriguing conjectural picture developed by Gaiotto, Moore and Neitzke [GMN13], it was shown that on the regular locus of the Hitchin map the asymptotics of the hyperkähler metric are described by a so-called semi-flat metric [Maz+19; Fre18b; Fre+20]. This is a hyperkähler metric defined on any algebraically completely integrable system by the theory of special Kähler manifolds [Fre99]. It does not extend over the singular locus, but Gaiotto, Moore and

Neitzke suggest that it can be modified by so-called instanton corrections to define a hyperkähler metric on $\mathcal{M}_G$. Recent progress in this direction can be found in [Tul19].

The first step in analysing the asymptotics of the hyperkähler metric, was finding limits of solutions to the Hitchin equation along rays to the ends of the moduli space [Maz+16; Moc16; Fre18a]. As shown in [Maz+14; Fre18a], these so-called limiting configurations satisfy a decoupled version of the Hitchin equation and are completely determined by spectral data. In Theorem 5.6, we will use the semi-abelian spectral data explained above to construct solutions to the decoupled Hitchin equation for $\mathfrak{sl}(2)$-type fibers of the symplectic Hitchin system. We conjecture them to be limiting configurations. For $\mathsf{SL}(2,\mathbb{C})$, this is a theorem by Mochizuki [Moc16].

Reader's guide

We will start with a short introduction to Higgs bundle moduli spaces, Hitchin systems and abelian varieties in Chapter 1, concentrating on the crucial aspects for the present work. In Chapter 2, the main results will be established for $G = \mathsf{SL}(2,\mathbb{C})$. We obtain the stratification result in Section 2.5 and analyze the global structure of the singular Hitchin fibers in sections 2.6 and 2.7. We will end this chapter with a description of $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles via semi-abelian spectral data (Section 2.8).

In Chapter 3, we review Hecke transformations of holomorphic vector bundles of arbitrary rank and analyse the pushforward of holomorphic vector bundles along branched coverings of Riemann surfaces.

This will be essential in Chapter 4. Here the main result is the identification of $\mathfrak{sl}(2)$-type Hitchin fibers for $G = \mathsf{Sp}(2n,\mathbb{C})$ and $G = \mathsf{SO}(2n+1,\mathbb{C})$ with fibers of an $\mathsf{Sp}(2,\mathbb{C})$- resp. $\mathsf{SO}(3,\mathbb{C})$-Hitchin system. This allows us to reduce the analysis of $\mathfrak{sl}(2)$-type Hitchin fibers to results of Chapter 2. Moreover, we obtain the description of Langlands duality for $\mathfrak{sl}(2)$-type Hitchin fibers by analysing the duality for $\mathrm{rk}\mathfrak{g} = 1$ (Section 4.4).

In Chapter 5, we will use semi-abelian spectral data to construct solutions to the decoupled Hitchin equation and give reason, why we conjecture them to be limiting configurations. Moreover, we use analytic techniques to show that the fiber bundles in Theorem 1 and 2 are smoothly trivial.

Chapter 6 is independent of the previous chapters. We develop a method to obtain spectral data for singular Hitchin fibers with non-reduced spectral curve. The main tool will be hypercohomology and we will shortly introduce it in the beginning of the chapter. We will apply this approach to certain strata of the nilpotent cone in $\mathsf{SL}(n,\mathbb{C})$ (Section 6.3).

In the last Chapter 7, we explain connections to other recent results in the field and give an outlook on future research projects evolving from the present work.

CHAPTER 1

Preliminaries

1.1. Notation

We will often consider a covering of Riemann surfaces $\pi : Y \to X$. We will call π a branched covering, if there exists at least one branch point. We call it unbranched, if there are no branch points. If we do not specify one of these options, the covering π can be branched or unbranched.

To avoid confusion, we will refer to points in Y, where different sheets meet or equivalently zeros of $\partial\pi$ as ramification points and to the images of these points under π as branch points. We denote by $R = \mathsf{div}(\partial\pi) \in \mathsf{Div}(Y)$ the ramification divisor and refer to its coefficient R_p at a ramification point $p \in Y$ as the ramification index. $B := \mathsf{Nm}(R) \in \mathsf{Div}(X)$ is referred to as branch divisor.

$\cdot^\vee$	Dual vector bundle or abelian variety.
$\mathcal{A}^{(i,j)}(\cdot)$	Smooth (i,j)-form valued sections of vector bundle.
B	Branch divisor.
$B_G(X, M)$	M-twisted G-Hitchin base on X.
$\mathsf{Div}(\cdot)$	Abelian group of divisors on a Riemann surface.
$\mathsf{Div}^+(\cdot)$	Effective divisors on a Riemann surface.
Hit_G	G-Hitchin map.
$\mathsf{Jac}(\cdot)$	Abelian group of holomorphic line bundles of degree 0.
$K = K_X$	Holomorphic line bundle of $(1,0)$-forms on a Riemann surface X, canonical bundle.
$\mathcal{M}_G(X, M)$	Moduli space of polystable M-twisted (linear) G-Higgs bundles on X.
ξ_x	Stalk of a sheaf ξ at x.
$\mathcal{O}_X$	Sheaf of holomorphic functions an a Riemann surface X, trivial holomorphic line bundle.
$\mathcal{O}(\cdot)$	Sheaf of holomorphic sections of a holomorphic vector bundle.
$\mathsf{Pic}(\cdot)$	Abelian group of holomorphic line bundles.
$\mathsf{Prym}_N(\cdot)$	Twisted Prym variety, see Section 1.4.2.
$\pi : \Sigma \to X$	Spectral cover.
$\tilde{\pi} : \tilde{\Sigma} \to X$	Normalised spectral cover.
R	Ramification divisor.
X	Riemann surface of genus $g \geq 2$.
$Z(\cdot)$	Zero set of holomorphic section of line bundle.

1.2. Higgs bundle moduli spaces

Let G be a complex reductive Lie group and $\mathfrak{g}$ its Lie algebra. Let M be a holomorphic line bundle on X.

DEFINITION 1.2.1. A M-twisted principal G-Higgs bundle is a pair (P, ϕ) of a holomorphic principal G-bundle P and a Higgs field $\phi \in H^0(X, (P \times_{\mathsf{Ad}} \mathfrak{g}) \otimes M)$.

Let $\mathcal{M}_G^P(X, M)$ denote the moduli space of polystable M-twisted principal G-Higgs bundles on X. This is a complex analytic space. It is an algebraic variety, whenever G is algebraic (see [GGR09] and references therein). Given a homomorphism of complex Lie groups $\rho : G \to \mathsf{GL}(n, \mathbb{C})$, we can associate a (linear) Higgs bundle to (P, ϕ) by

$$E := P \times_\rho \mathbb{C}^n, \quad \Phi =: d\rho(\phi) \in H^0(X, \mathrm{End}(E) \otimes M),$$

where

$$d\rho : P \times_{\mathsf{Ad}_G} \mathfrak{g} \to P \times_{\mathsf{Ad}_{\mathsf{GL}} \circ \rho} \mathsf{Mat}(n, \mathbb{C}) \cong \mathrm{End}(E)$$

is the induced map.

EXAMPLE 1.2.2 (linear $\mathsf{GL}(n, \mathbb{C})$-Higgs bundle). A M-twisted linear $\mathsf{GL}(n, \mathbb{C})$-Higgs bundle (E, Φ) is a holomorphic vector bundle E of rank n and a Higgs field $\Phi \in H^0(X, \mathrm{End}(E) \otimes M)$. We can recover P as the frame bundle of E.

DEFINITION 1.2.3. Let $G \subset \mathsf{GL}(n, \mathbb{C})$ a complex reductive linear group. A (linear) G-Higgs bundle is a $\mathsf{GL}(n, \mathbb{C})$-Higgs bundle (E, Φ) with a reduction of structure group to G, i. e. there exist G-transition function for E, such that

$$\Phi \in H^0(X, \mathfrak{g}(E) \otimes M) \subset H^0(X, \mathrm{End}(E) \otimes M),$$

where $\mathfrak{g}(E) =: E \times_{\mathsf{Ad}_G} \mathfrak{g}$.

Let $G \subset \mathsf{GL}(n, \mathbb{C})$ a complex reductive linear group. Denote by $\mathcal{M}_G(X, M)$ the moduli space of polystable (linear) M-twisted G-Higgs bundles on X.

LEMMA 1.2.4. *Let $(E, \Phi) \in \mathcal{M}_G(X, M)$ be stable and simple. Then there is an exact sequence*

$$\begin{aligned} 0 \to \mathfrak{z}(\mathfrak{g}) \to H^0(X, \mathfrak{g}(E)) &\xrightarrow{\mathsf{ad}(\Phi)} H^0(X, \mathfrak{g}(E) \otimes M) \to T_{(E,\Phi)} \mathcal{M}_G(X, M) \\ \to H^1(X, \mathfrak{g}(E)) &\xrightarrow{\mathsf{ad}(\Phi)} H^1(X, \mathfrak{g}(E) \otimes M). \end{aligned}$$

PROOF. If a G-Higgs bundle is stable and simple, it is a smooth point of the moduli space (see [GO17] Proposition 3.12). Furthermore, from stability

$$\mathsf{aut}(E, \Phi) = \{\psi \in H^0(X, \mathfrak{g}(E)) \mid \mathsf{ad}(\Phi)(\psi) = 0\} = \mathfrak{z}(\mathfrak{g}).$$

Let $\bar{\partial}_E$ the Dolbeault operator on the underlying smooth vector bundle. Then it easy to see that the tangent space $T_{(E,\Phi)} \mathcal{M}_G(X, M)$ can be identify as

$$\frac{\{(\beta, \psi) \in \mathcal{A}^{0,1}(\mathfrak{g}(E)) \oplus \mathcal{A}^0(\mathfrak{g}(E) \otimes M) \mid \bar{\partial}_E^{\mathrm{End}} \psi + [\beta, \Phi] = 0\}}{\{(\bar{\partial}_E^{\mathrm{End}} \alpha, [\Phi, \alpha]) \mid \alpha \in \mathcal{A}^0(\mathfrak{g}(E))\}}.$$

Hence, we can define a map $T_{(E,\Phi)} \mathcal{M}_G \to H^{0,1}(X, \mathfrak{g}(E))$ by projecting to the Dolbeault cohomology class of $\beta \in \mathcal{A}^{0,1}(\mathfrak{g}(E))$. Then

$$\mathsf{ad}(\Phi)(\beta) = \bar{\partial}_E^{\mathrm{End}} \psi$$

if and only if β lies in the image of this projection. Hence the map is exact at $H^{0,1}(X, \mathfrak{g}(E))$.

To the left side, we can clearly map a section $\psi \in H^0(X, \mathfrak{g}(E) \otimes M)$ onto $(0, \psi) \in T_{(E,\Phi)}\mathcal{M}_G$ and this lies in the kernel of the projection to $H^{0,1}(X, \mathfrak{g}(E))$. On the other hand, if $\beta = \bar{\partial}_E^{\mathrm{End}}\alpha$, then $\psi + [\alpha, \psi] \in H^0(X, \mathfrak{g}(E) \otimes M)$ and

$$(\beta, \psi) - (\bar{\partial}_E^{\mathrm{End}}\alpha, [\psi, \alpha]) = (0, \psi + [\alpha, \psi]).$$

$\psi \in H^0(X, \mathfrak{g}(E) \otimes M)$ is map on the tangent space to the gauge orbit if and only if it is in the image of $\mathsf{ad}(\Phi)$. $\square$

Proposition 1.2.5 ([Nit91; Mar94; Bot95]). *The dimension of $\mathcal{M}_G(X, M)$ is given by the following table*

Condition	*Dimension*
$\deg(M) > 2g-2$	$\deg(M)\dim\mathfrak{g} + \dim\mathfrak{z}(\mathfrak{g})$
$\deg(M) = 2g-2$ *and* $M \not\cong K$	$(2g-2)\dim\mathfrak{g} + \dim\mathfrak{z}(\mathfrak{g})$
$M = K$	$(2g-2)\dim\mathfrak{g} + 2\dim\mathfrak{z}(\mathfrak{g})$

Furthermore, $\mathcal{M}_G(X, K)$ has the structure of a complex symplectic manifold, i. e. there exists a holomorphic symplectic form $\omega \in \mathcal{A}^{2,0}(X)$. More generally, if there exists a holomorphic section of MK^{-1}, then $\mathcal{M}_G(X, M)$ is a Poisson manifold.

Proof. Let $(E, \Phi) \in \mathcal{M}_G(X, M)$ be stable and simple. To use the exact sequence of the previous lemma, we have to compute

$$\mathsf{coker}\left(\mathsf{ad}(\Phi) : H^1(X, \mathfrak{g}(E)) \to H^1(X, \mathfrak{g}(E) \otimes M)\right).$$

Combing Serre duality with a non-degenerate Ad-invariant bilinear form on $\mathfrak{g}$, it is dual to the kernel of

$$\mathsf{ad}(\Phi) : H^0(X, \mathfrak{g}(E) \otimes M^{-1}K) \to H^0(X, \mathfrak{g}(E) \otimes K).$$

An element $\psi \in H^0(X, \mathfrak{g}(E) \otimes M^{-1}K)$ in the kernel has generically full rank from the stability of (E, Φ). Hence, the kernel is $\{0\}$ in the first two cases and $\mathfrak{z}(\mathfrak{g})$ for $M = K$.

Now using the exact sequence of the previous lemma together with Riemann-Roch, we obtain

$$\dim T_{(E,\Phi)}\mathcal{M}_G(X, M) = \deg(M)\dim\mathfrak{g} + \mathfrak{z}(\mathfrak{g})(1 + \dim H^1(X, M)).$$

For $M = K$, combining Serre duality with a non-degenerate Ad-invariant bilinear form on $\mathfrak{g}$, yields a non-degenerate pairing

$$H^0(X, \mathfrak{g}(E) \otimes K) \times H^1(X, \mathfrak{g}(E)) \to \mathbb{C}.$$

inducing a well-defined holomorphic symplectic form on $\mathcal{M}_G(X, K)$. The proof of the last assertion can be found in [Mar94; Bot95]. $\square$

1.3. Hitchin systems and spectral data

DEFINITION 1.3.1 (Algebraically completely integrable system). An algebraically completely integrable system is a complex symplectic manifold M together with a holomorphic map $p : M \to B$, such that

i) the fibers are complex Lagrangian tori and,
ii) for all $b \in B$, there exists a cohomology class $\rho_b \in H^{1,1}(p^{-1}(b)) \cap H^2(p^{-1}(b), \mathbb{Z})$ smoothly varying in b, such that the induced hermitian metric on $p^{-1}(b)$ is positive definite (cf. Theorem 1.4.3).

Let $\mathbb{C}[\mathfrak{g}]^G$ denote the algebra of $\mathsf{Ad}(G)$-invariant polynomials on $\mathfrak{g}$. Let $a_1, \dots, a_{\mathrm{rk}(\mathfrak{g})}$ be homogeneous generators for $\mathbb{C}[\mathfrak{g}]^G$, then the Hitchin map is defined by

$$\begin{aligned} \mathcal{M}_G(X, M) \to B_G(X, M) &:= \bigoplus_{i=1}^{\mathrm{rk}(\mathfrak{g})} H^0(X, M^{\deg(a_i)}), \\ (E, \Phi) \mapsto \quad &\left(a_1(\Phi), \dots, a_{\mathrm{rk}(\mathfrak{g})}(\Phi)\right). \end{aligned}$$

The Hitchin map is a proper, flat, surjective, holomorphic map (see [Sim95]). The collection of integers $\deg(a_i) - 1$ are an invariant of the Lie algebra, the so-called exponents of $\mathfrak{g}$. Let us first compute the dimension of the Hitchin base B_G.

PROPOSITION 1.3.2. *For* $\deg(M) > 2g - 2$ *the dimension of the Hitchin base is given by*

$$\dim B_G(X, M) = \tfrac{1}{2} \deg(M)(\dim \mathfrak{g} + \mathrm{rk}(\mathfrak{g})) + \mathrm{rk}\mathfrak{g}(1 - g).$$

If $M = K$*, we have*

$$\dim B_G(X, K) = (g - 1) \dim \mathfrak{g} + \dim \mathfrak{z}(\mathfrak{g}).$$

PROOF. For a semi-simple Lie algebra $\mathfrak{h}$, Varadarajan [Var68] proved

$$\sum_{i=1}^{\mathrm{rk}(\mathfrak{h})} \deg(a_i) = \tfrac{1}{2}(\dim \mathfrak{h} + \mathrm{rk}(\mathfrak{h})).$$

The reductive Lie algebra $\mathfrak{g}$ has a Levi decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}^{ss}$, where $\mathfrak{g}^{ss}$ is semi-simple. We claim that the number of exponents equal to 1 is the dimension of $\mathfrak{z}(\mathfrak{g})$. If $\alpha \in \mathbb{C}[\mathfrak{g}]^{\mathsf{Ad}G}$ is of degree 1, then it factors through

$$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}/[\mathfrak{g}^{ss}, \mathfrak{g}^{ss}] = \mathfrak{z}(\mathfrak{g}).$$

This proves the claim. Hence, for a complex reductive Lie group G, we have

$$\begin{aligned} \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \deg(a_i) &= \sum_{i=1}^{\mathrm{rk}(\mathfrak{g}^{ss})} \deg(a_i^{ss}) + \dim \mathfrak{z}(\mathfrak{g}) = \tfrac{1}{2}(\dim \mathfrak{g}^{ss} + \mathrm{rk}(\mathfrak{g}^{ss})) + \dim \mathfrak{z}(\mathfrak{g}) \\ &= \tfrac{1}{2}(\dim \mathfrak{g} + \mathrm{rk}(\mathfrak{g})). \end{aligned}$$

With Riemann-Roch, we compute for $\deg(M) > 2g-2$

$$\begin{aligned}\dim B_G &= \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} \left(\deg(M)a_i + 1 - g + \dim H^1(X, M^{a_i})\right) \\ &= \tfrac{1}{2}\deg(M)(\dim \mathfrak{g} + \mathrm{rk}(\mathfrak{g})) + \mathrm{rk}\mathfrak{g}(1-g).\end{aligned}$$

In the same way, we obtain the formula for $M = K$. □

Theorem 1.3.3. *Let $f_1, f_2 \in B_G(X,K)^\vee$, then the compositions with the Hitchin map*

$$F_i = f_i \circ \mathsf{Hit} : \mathcal{M}_G(X,K) \to \mathbb{C}$$

Poisson-commute with respect to the complex symplectic structure of $\mathcal{M}_G(X,K)$.

Proof. This was proven by Hitchin [Hit87a] for the cotangent bundle

$$T^*N_G \subset \mathcal{M}_G(X,K),$$

where N_G is the moduli space of stable holomorphic G-bundles, using the construction of this moduli space by symplectic reduction. A detailed exposition of the proof of the statement on $\mathcal{M}_{\mathsf{GL}(n,\mathbb{C})}(X,K)$ can be found in [Ste15]. It generalizes verbatim to complex reductive linear groups $G \subset \mathsf{GL}(n,\mathbb{C})$. □

Theorem 1.3.4 ([Hit87a; Sco98]). *There exists a dense subset of the Hitchin base $B_G^{\mathsf{reg}} \subset B_G(X,K)$, such that the Hitchin map restricted to B_G^{reg}*

$$\mathsf{Hit}_G : \mathsf{Hit}_G^{-1}(B^{\mathsf{reg}}) \to B^{\mathsf{reg}}$$

defines an algebraically completely integrable system.

Remark 1.3.5. This is proven by identifying the fibers of the Hitchin map with so-called spectral data showing that the fibers are abelian varieties with a smoothly varying polarization. For the Lie groups $\mathsf{GL}(n,\mathbb{C})$, $\mathsf{SL}(n,\mathbb{C})$, $\mathsf{Sp}(2n,\mathbb{C})$, $\mathsf{SO}(2n+1,\mathbb{C})$, G_2, this was proven case by case in [Hit87a; Hit07]. Indeed for $\mathsf{Sp}(2n,\mathbb{C})$ and $\mathsf{SO}(2n+1,\mathbb{C})$, we will reprove this result in Theorem 4.2.13 and 4.4.5. To prove it for a general complex reductive Lie group one needs to use the language of cameral data introduced in [Sco98].

Notice that for $\mathsf{SL}(2,\mathbb{C})$, Hitchin has directly determined the critical locus of the Hitchin map in his recent work [Hit19]. Then the complete integrability follows from the Liouville-Arnold Theorem [Arn78].

Spectral data is obtained by a spectral analysis of the Higgs bundle. The coefficients of the characteristic polynomial of an element in $\mathfrak{g}$ are polynomials in $\mathbb{C}[\mathfrak{g}]^G$ (but in general no generators). Hence, the Hitchin map determines the characteristic polynomial of $(E,\Phi) \in \mathcal{M}_G(X,M)$. For every point in the Hitchin base the characteristic polynomial defines an analytic curve $\Sigma \subset \mathsf{Tot}(M)$ determing the eigenvalues of the Higgs bundles in the corresponding Hitchin fiber. If the spectral curve is irreducible and reduced, these Higgs bundles are almost everywhere on X locally diagonalizable with distinct eigenvalues. At all these points the natural projection $\Sigma \to X$ is a unbranched covering of Riemann surfaces. At the points, where the characteristic equation has zeroes of higher multiplicity, different sheets of this covering meet. Such points can be smooth ramification points, but also singularities of the spectral curve.

Before making this more precise in an example, let us prove the following lemma that will be useful to compute the canonical bundle of the spectral curve.

LEMMA 1.3.6. *Let $p : V \to X$ be a holomorphic vector bundle on a Riemann surface. Let $Y = \mathsf{Tot}(V)$, then $K_Y \cong p^*(K_X \otimes \det(V)^{-1})$.*

PROOF. Let $(U, z) \subset X$ open coordinate chart and $s_1, \dots, s_r$ a frame of V_U. As V is a vector bundle the vertical tangent bundle is identified with V_U. Hence,

$$\mathrm{d}z \wedge s_1^\vee \wedge \dots \wedge s_r^\vee$$

defines a local frame of $K_Y = \bigwedge^{r+1} T^\vee Y$. Now, it is easy to see that, when changing the coordinate chart and the trivialisation of V, this section transforms like a section of $p^*(K_X \otimes \det(V)^{-1})$. ☐

EXAMPLE 1.3.7 ($\mathsf{GL}(n, \mathbb{C})$-spectral data). Let $G = \mathsf{GL}(n, \mathbb{C})$. In this case, the coefficients $(a_1, \dots, a_n)$ of the characteristic polynomial define generators of $\mathbb{C}[\mathfrak{g}]^G$ and the Hitchin map is given by

$$\begin{aligned} \mathcal{M}_G(X, K) \to B_{\mathsf{GL}(n,\mathbb{C})}(X, K) &= \bigoplus_{i=1}^n H^0(X, K^i), \\ (E, \Phi) \mapsto \quad &(a_1(\Phi), \dots, a_n(\Phi)) \,. \end{aligned}$$

Fix $\underline{a} = (a_1, \dots, a_n) \in B_{\mathsf{GL}(n,\mathbb{C})}(X, K)$, then the characteristic equation of (E, Φ) $\in \mathsf{Hit}^{-1}_{\mathsf{GL}(n,\mathbb{C})}(\underline{a})$ is given by

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0.$$

Let $p_K : K \to X$ the bundle map and $\eta : K \to p_K^*(K)$ the tautological section. Then the spectral curve is the divisor

$$\Sigma := Z\left(\eta^n + (p_K^* a_1)\eta^{n-1} + \dots + (p_K^* a_{n-1})\eta + p_K^* a_n\right) \subset \mathsf{Tot}(K).$$

The projection map p_K restricts to an n-sheeted covering $\pi : \Sigma \to X$. Define the regular locus by

$$B^{\mathsf{reg}}_{\mathsf{GL}(n,\mathbb{C})} = \{\underline{a} \in B_{\mathsf{GL}(n,\mathbb{C})}(X, K) \mid \Sigma(\underline{a}) \text{ is smooth } \}.$$

The regular locus is dense in the Hitchin base by the discriminant criterion. The discriminant of the characteristic polynomial defines a section

$$\mathsf{disc}(\underline{a}) \in H^0(X, K^{n(n-1)}).$$

Generically, the discriminant has simple zeroes. It is easy to see that in this case, all points of the spectral curve, where different sheets meet, are smooth ramification points of index one (cf. Proposition 4.2.3). In particular, the spectral curve is smooth. We can compute its canonical bundle by the adjunction formula using Lemma 1.3.6. We have

$$K_\Sigma = \left(K_{\mathsf{Tot}(K)} \otimes p_K^* K_X^n)\right) |_\Sigma = \pi^* K_X^n.$$

and hence its genus is given by $g(\Sigma) = n^2(g-1) + 1$.

The spectral covering is the first part of the spectral data. The second part decodes the eigen spaces of (E, Φ). Let $\lambda = \eta|_\Sigma : \Sigma \to \pi^* K$, then we can define an eigen sheaf

$$\mathsf{ker}\left(\pi^*\Phi - \lambda \mathrm{id}_{\mathcal{O}(\pi^* E)}\right) \subset \mathcal{O}(\pi^* E).$$

This defines a point in the moduli space of holomorphic line bundles of degree $d = \deg(E) + (n - n^2)(g - 1)$ on Σ. For $\underline{a} \in B^{\mathsf{reg}}_{\mathsf{GL}(n,\mathbb{C})}$, this identifies

$$\mathsf{Hit}^{-1}_{\mathsf{GL}(n,\mathbb{C})}(\underline{a}) \cong \mathsf{Pic}^d(\Sigma).$$

We have a simple transitive group action of the abelian group $\mathsf{Jac}(\Sigma)$ of line bundles of degree 0 on Y

$$\mathsf{Jac}(Y) \times \mathsf{Pic}^d(\Sigma) \to \mathsf{Pic}^d(\Sigma), \quad (L, N) \mapsto (L \otimes N).$$

As we will see below $\mathsf{Jac}(\Sigma)$ is an abelian variety of dimension $\mathfrak{g}(\Sigma)$ and we just proved that $\mathsf{Pic}^d(\Sigma)$ is a torsor over it.

Summing up, for $\underline{a} \in B^{\mathsf{reg}}_{\mathsf{GL}(n,\mathbb{C})}$ the Hitchin fiber $\mathsf{Hit}^{-1}_{\mathsf{GL}(n,\mathbb{C})}(\underline{a})$ is a complex torus and hence the Hitchin system is completely integrable on the regular locus. As we will see below, having a smoothly varying structure of an abelian variety is equivalent to the existence of a smoothly varying polarization $\rho_{\underline{a}}$ as demanded in Definition 1.3.1 (see Theorem 1.4.3). Hence, the $\mathsf{GL}(n,\mathbb{C})$-Hitchin system is an algebraically completely integrable system.

Remark 1.3.8. Spectral curves of Higgs bundles a priori defined in the analytic category are algebraic. Let X a Riemann surface, then there exists a projective algebraic curve $\mathcal{X}$, such that the underlying analytic space $\mathcal{X}_{\mathrm{an}} \cong X$. By the GAGA principle [Ser56], the induced maps

$$H^0(\mathcal{X}, K^i_{\mathcal{X}}) \to H^0(X, K^i_X)$$

are isomorphism for all $i \geq 1$. Hence, every point in the Hitchin base corresponds to a collection of regular sections of some power of the canonical of $\mathcal{X}$. With such choice of sections the spectral equation is defined in the algebraic category and hence the zero divisor $\Sigma \subset \mathsf{Tot}(M)$ defines an algebraic curve.

In the following, we will mainly consider singular Hitchin fibers with irreducible and reduced spectral curve. This has the big advantage that we do not have to worry about stability conditions.

Lemma 1.3.9. *Let $(E, \Phi) \in \mathcal{M}_{\mathsf{GL}(n,\mathbb{C})}(X, M)$ with irreducible and reduced spectral curve. Then (E, Φ) is stable.*

Proof. If there is a Φ-invariant subbundle $F \subsetneq E$, then the characteristic polynomial of $\Phi|_F$ divides the characteristic polynomial of Φ. Hence, the spectral curve can not be irreducible and reduced. □

The $\mathfrak{g}$-discriminant. In Example 1.3.7, we saw that the discriminant of the characteristic equation can be used to detect the regular locus. However, for other complex Lie groups the discriminant of the characteristic has generically higher order zeroes (cf. Example 4.1.4 for $\mathsf{Sp}(4,\mathbb{C})$-case). The $\mathfrak{g}$-discriminant is a generalization for complex semi-simple Lie groups detecting the regular locus of B_G.

Let G be a semisimple connected Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\Delta \subset \mathfrak{h}^\vee$ the associated set of roots. Let W denote the

Weyl group permuting the roots and $\mathbb{C}[\mathfrak{h}]^W$ the W-invariant polynomials on $\mathfrak{h}$. Then

$$\prod_{\alpha \in \Delta} \alpha \in \mathbb{C}[\mathfrak{h}]^W$$

is of degree $|\Delta|$ and defines an $\mathsf{Ad}(G)$-invariant polynomial $\mathsf{disc}_{\mathfrak{g}}$ on $\mathfrak{g}$ by the Chevalley restriction isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W, f \mapsto f|_{\mathfrak{h}}.$$

We refer to $\mathsf{disc}_{\mathfrak{g}}$ as $\mathfrak{g}$-discriminant. Let $(E, \Phi) \in \mathcal{M}_G(X, M)$, then $\mathsf{disc}_{\mathfrak{g}}(\Phi) \in H^0(X, M^{|\Delta|})$. Being an invariant polynomial, it factors through the Hitchin map. For $\underline{a} \in B_G$, we will write $\mathsf{disc}_{\mathfrak{g}}(\underline{a})$ for the holomorphic section computed in this terms.

Theorem 1.3.10 ([Sco98]). *Let $\underline{a} \in B_G$, such that $\mathsf{disc}_{\mathfrak{g}}(\underline{a})$ has simple zeroes, then the Hitchin fiber $\mathsf{Hit}_G^{-1}(\underline{a})$ is a union of abelian torsors.*

This is proven using so-called cameral data, a way to formulate spectral data without specifying a linear representation of G. If the discriminant has simple zeroes, the cameral curve is smooth and the Hitchin fiber can be identified with an abelian variety. We will see in Lemma 4.1.3 and Lemma 4.3.2, that for $G = \mathsf{Sp}(2n, \mathbb{C})$ and $G = \mathsf{SO}(2n+1, \mathbb{C})$ the spectral curve Σ is smooth, when the discriminant has simple zeroes. In this way, we recover the result for these Lie groups.

1.4. Abelian varieties

In this section, we want to collect some basic facts about abelian varieties. The regular fibers of Hitchin systems are torsors over abelian varieties and also for the singular Hitchin fibers considered below, one part of the spectral data will be defined in this way. We will mainly focus on Prym varieties, which will play a major role in the remainder of this work. In the end of the section, we will compute the dual variety of the Prym variety - the cornerstone for the Langlands duality of the $\mathsf{Sp}(2n, \mathbb{C})$- and $\mathsf{SO}(2n+1, \mathbb{C})$-Hitchin system. A beautiful introduction to the topic is Mumford's book [Mum74a]. Prym varieties of double covers are intensively studied in [Mum74b].

Definition 1.4.1. An abelian variety is a complex torus that is also an algebraic variety.

Example 1.4.2. Consider a complex torus $T = \mathbb{C}/\Lambda$ of dimension 1, where $\Lambda \subset \mathbb{C}$ is a lattice. Choosing generators for Λ, the Weierstraß elliptic function $\wp$ defines a meromorphic function on $\mathbb{C}$ invariant by Λ and hence a meromorphic function on T. We can define an projective embedding $T \to \mathbb{P}^2$ by the extension of

$$z \mapsto (1 : \wp(z) : \wp'(z)).$$

Hence, all complex tori of dimension 1 are abelian varieties by Chow's Theorem [Cho49].

THEOREM 1.4.3 ([Mum74a] Section I.3). *Let V a complex vector space of dimension g and $\Lambda \subset V$ a lattice. Let $A = V/\Lambda$. The following are equivalent:*

i) *A is an abelian variety.*
ii) *A is a projective complex torus.*
iii) *There exist g algebraically independent meromorphic functions on A.*
iv) *There exists an alternating bilinear form $\rho \in H^{1,1}(V) \cap \bigwedge^2 \mathsf{Hom}(\Lambda, \mathbb{Z})$, such that the associated hermitian metric on V is positive definite.*

A bilinear form $\rho \in \bigwedge^2 \mathsf{Hom}(\Lambda, \mathbb{Z})$ as in iv) is called a polarization on A.

EXAMPLE 1.4.4. The basic example is the Jacobian $\mathsf{Jac}(X)$ of a Riemann surface X, the abelian group of holomorphic line bundles of degree 0. From the long exact sequence associated to the exponential sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0,$$

we obtain

$$\mathsf{Jac}(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{Z}^{2g}.$$

An ample line bundle on $\mathsf{Jac}(X)$ is given by the theta-divisor

$$\Theta : \mathrm{Sym}^{g-1}(X) \to \mathsf{Jac}(X), \qquad (p_1, \dots, p_{g-1}) \mapsto \mathcal{O}(\sum_{i=1}^{g-1}(p_i - p_0)),$$

where $p_0 \in X$ is fixed.

REMARK 1.4.5. In contrast to Example 1.4.2, almost no complex torus V/Λ of dimension ≥ 2 is an abelian variety. One can show that on almost every torus

$$H^{1,1}(V) \cap \bigwedge^2 \mathsf{Hom}(U, \mathbb{Z}) = \{0\}$$

(see [Mum74a] Section I.3).

1.4.1. Prym varieties. Consider a n-sheeted branched covering of Riemann surfaces $\pi : Y \to X$. Define the norm map

$$\mathsf{Nm} : \mathsf{Div}(Y) \to \mathsf{Div}(X), \quad \sum_{y \in Y} a_y y \mapsto \sum_{x \in X} \left(\sum_{y \in \pi^{-1}(x)} a_y \right) x.$$

One way to show that it descends to divisor classes is the formula

$$\det(\pi_* \mathcal{O}(D)) = \mathcal{O}_X(\mathsf{Nm}(D)) \otimes \det(\pi_* \mathcal{O}_Y), \quad D \in \mathsf{Div}(Y)$$

proved in Lemma 3.2.3. The left hand side does only depend on the divisor class and hence does $\mathcal{O}_X(\mathsf{Nm}(D))$. We obtain a surjective morphism of abelian varieties

$$\mathsf{Nm} : \mathsf{Pic}(Y) \to \mathsf{Pic}(X)$$

with $\deg(\mathsf{Nm}(L)) = \deg(L) \in \mathbb{Z}$.

DEFINITION 1.4.6. Let $\pi : Y \to X$ be a covering of Riemann surfaces, then the associated Prym variety is defined by

$$\mathsf{Prym}(\pi : Y \to X) = \mathsf{ker}(\mathsf{Nm} : \mathsf{Jac}(Y) \to \mathsf{Jac}(X)).$$

By definition, the Prym variety is an abelian variety of dimension

$$g(Y) - g(X).$$

We have the defining exact sequence

$$0 \to \mathsf{Prym}(\pi : Y \to X) \to \mathsf{Jac}(Y) \xrightarrow{\mathsf{Nm}} \mathsf{Jac}(X) \to 0.$$

In general, $\mathsf{Prym}(\pi : Y \to X)$ is not connected. In a very general setup, the connected components were studied in [HP12]. Let us take a closer look at Prym varieties of two-sheeted coverings of Riemann surfaces, which play an important role in the remainder of this work. Similar considerations can be found in [Mum71; AC19].

LEMMA 1.4.7. *Let $\pi : Y \to X$ a two-sheeted covering of Riemann surfaces. Let $L \in \mathsf{Prym}(\pi : Y \to X)$, then there exists a divisor $D \in \mathsf{Div}(Y)$, such that $\mathcal{O}(D) \cong L$ and $D + \sigma^* D = 0$.*

PROOF. Let $L \in \mathsf{Prym} = \ker(\mathsf{Nm})$. Choose $D \in \mathsf{Div}(Y)$, such that $\mathcal{O}(D) = L$. Then $C =: \mathsf{Nm}(D)$ is the divisor of a meromorphic function on X. By Tsen's theorem [Lan52], there exists a divisor $C' \in \mathsf{Div}(Y)$ of a meromorphic function on Y, such that $C' + \sigma^* C' = \pi^* C$. Hence, $D' = D - C$ is a divisor of L with $0 = (\pi^* \circ \mathsf{Nm})(D') = D' + \sigma^* D'$. □

PROPOSITION 1.4.8. *Let $\pi : Y \to X$ be a two-sheeted branched covering of Riemann surfaces, then $\mathsf{Prym}(\pi : Y \to X)$ is connected and is given by*

$$\mathsf{Prym}(\pi : Y \to X) = \{L \in \mathsf{Jac}(Y) \mid L \otimes \sigma^* L = \mathcal{O}_X\}.$$

PROOF. In this case, the pullback $\pi^* : \mathsf{Jac}(X) \to \mathsf{Jac}(Y)$ is injective. Hence, $L \in \mathsf{Prym}$ if and only if $\mathcal{O}_X = (\pi^* \circ \mathsf{Nm})(L) = L \otimes \sigma^* L$. To prove connectedness, consider the map

$$\Psi : \mathsf{Jac}(Y) \to \mathsf{Jac}(Y), \quad L \mapsto L \otimes \sigma^* L^{-1}.$$

We want to show that $\mathsf{Im}(\Psi) = \mathsf{Prym}(\pi : Y \to X)$. Clearly, $\mathsf{Im}(\psi) \subset \mathsf{Prym}$. For the converse, let $L \in \mathsf{Prym}$. By the previous lemma, there exists $D \in \mathsf{Div}(Y)$, such that $\mathcal{O}(D) = L$ and $D + \sigma^* D = 0$. There exists an effective divisor $C \in \mathsf{Div}^+(Y)$, such that $D = C - \sigma^* C$. Let $p \in Y$ a ramification point, then $C + kp$, for $k \in \mathbb{Z}$, has the same property. Choosing k, such that $\deg(C + kp) = 0$, this proves $\mathsf{Im}(\Psi) = \mathsf{Prym}$. In particular, the Prym variety is connected. □

PROPOSITION 1.4.9. *Let $\pi : Y \to X$ be a two-sheeted unbranched covering of Riemann surfaces, then $\mathsf{Prym}(\pi : Y \to X)$ has two connected components.*

PROOF. Let us again define a map

$$\Psi : \mathsf{Pic}(Y) \to \mathsf{Pic}(Y), \quad L \mapsto L \otimes \sigma^* L^{-1}.$$

We claim that the following sequence is exact.

$$0 \to \mathbb{Z}_2 \to \mathsf{Pic}(X) \xrightarrow{\pi^*} \mathsf{Pic}(Y) \xrightarrow{\Psi} \mathsf{Pic}(Y) \xrightarrow{\mathsf{Nm}} \mathsf{Jac}(Y) \to 0.$$

Let us start with the exactness at $\mathsf{Pic}(X)$. As π is unbranched, we can consider Y as a $\mathbb{Z}_2$-bundle $\check{H}^1(X, \mathbb{Z}_2) \subset \check{H}^1(X, \mathcal{O}_X^*)$. The associated holomorphic line bundle I pulls back to the trivial bundle on Y. Clearly, $\mathcal{O}_X = (\mathsf{Nm} \circ \pi^*) I = I^2$.

Hence, $\mathsf{ker}(p^*) = \{\mathcal{O}_X, I\} \cong \mathbb{Z}_2$. Furthermore, $L \in \mathsf{ker}(\Psi)$ if and only if $L \cong \sigma^* L$. These are exactly the pullbacks $\pi^* M$ of line bundles $M \in \mathsf{Pic}(X)$. This shows exactness at the third term.

Let $D \in \mathsf{Div}(Y)$, then $\mathsf{Nm}(D - \sigma^* D) = 0$, hence $\mathsf{Im}(\Psi) \subset \mathsf{ker}(\mathsf{Nm})$. For the converse, let $L \in \mathsf{ker}(\mathsf{Nm})$. By Lemma 1.4.7, there exists $D \in \mathsf{Div}(Y)$, such that $\mathcal{O}(D) = L$ and $D + \sigma^* D = 0$. There exists a unique effective divisor $C \in \mathsf{Div}^+(Y)$, such that $C - \sigma^* C = D$. Hence, $\Psi(\mathcal{O}(C)) = L$ and $\mathsf{Im}(\Psi) = \mathsf{ker}(\mathsf{Nm})$. This proves the claim.

Finally, the elements of $\pi^*\mathsf{Pic}(X)$ have even degree. Hence, Ψ maps the subsets

$$P^i = \{L \in \mathsf{Pic}(Y) \mid \deg(L) \equiv i \mod 2\},$$

for $i = 0, 1$, onto two connected components of $\mathsf{Prym}(\pi : Y \to X)$. $\square$

1.4.2. Abelian Torsors over Prym varieties.

Definition 1.4.10. A analytic space X is called a torsor over an abelian variety A, if there exists a free and transitive analytic group action of A on X.

Lemma 1.4.11. *Let $\pi : Y \to X$ a two-sheeted branched covering of Riemann surfaces. Let $N \in \mathsf{Pic}(Y)$, then*

$$\mathsf{Prym}_N(\pi : Y \to X) := \{L \in \mathsf{Pic} \mid L \otimes \sigma^* L \otimes N = \mathcal{O}_X\}.$$

is a torsor over $\mathsf{Prym}(\pi : Y \to X)$, whenever it is non-empty.

Proof. It is easy to check that the analytic group action

$$\mathsf{Prym} \times \mathsf{Prym}_N \to \mathsf{Prym}_N, \quad (L, M) \mapsto L \otimes M$$

is free and transitive. $\square$

Lemma 1.4.12. *Let $\pi : Y \to X$ a two-sheeted unbranched covering of Riemann surfaces. Let $N \in \mathsf{Pic}(X)$ with $\deg(N) \equiv 0 \mod 2$, then*

$$\mathsf{Prym}_N(\pi : Y \to X) := \mathsf{Nm}^{-1}(N^{-1}).$$

is a torsor over $\mathsf{Prym}(\pi : Y \to X)$.

Proof. Again it is easy to check, that the tensor product defines a free and transitive, analytic action. For every square root $N^{\frac{1}{2}}$, we have $\pi^* N^{\frac{1}{2}} \in \mathsf{Prym}_N$. Hence, Prym_N is non-empty. $\square$

We will refer to these torsors as twisted Prym varieties. To simplify the notation, we will mostly write $\mathsf{Prym}_N(Y)$ instead of $\mathsf{Prym}_N(\pi : Y \to X)$, when the covering map is clear from the context. Furthermore, for a divisor D, we will write $\mathsf{Prym}_D(Y)$ to mean $\mathsf{Prym}_{\mathcal{O}(D)}(Y)$.

1.4.3. Dual abelian varieties. The dual $A^\vee$ of an abelian variety A is defined to be the moduli spaces of holomorphic line bundles of degree 0 on A. It is itself an abelian variety. $A^\vee$ is the dual of A in the sense that the double dual is A. Furthermore, if $A = V_1/U_1$ and $A^\vee = V_2/U_2$, one can find a non-degenerate bilinear pairing $B : V_1 \otimes V_2 \to \mathbb{C}$, such that U_1 and U_2 are dual lattices under its imaginary part $\mathsf{Im}(B)$ (see [Mum74a] Section II.9).

Let $\mathcal{L} \to A$ be an ample line bundle, we obtain an surjective morphism of abelian varieties
$$\phi_{\mathcal{L}} : A \to A^{\vee}, \quad t_a^*\mathcal{L} \otimes \mathcal{L}^{-1},$$
where $t_a : A \to A$ denotes the translation by a (see [Mum74a] II.6 Application 1). The kernel of such a morphism of abelian varieties is a finite cyclic group. In particular, every abelian variety is a finite covering of its dual.

Let $A = \mathsf{Jac}(X)$ and $\mathcal{L}$ the line bundle associated to the theta-divisor, then $\phi_{\mathcal{L}}$ is an isomorphism. In particular, $\mathsf{Jac}(X)$ is self-dual.

THEOREM 1.4.13 ([HT03] Lemma 2.3). *Let $\pi : Y \to X$ an n-sheeted branched covering of Riemann surfaces. Then*
$$\mathsf{Prym}(Y)^{\vee} = \mathsf{Prym}(Y)/\mathsf{Jac}(X)[n],$$
where $\mathsf{Jac}(X)[n]$ is the group of n-torsion points of $\mathsf{Jac}(X)$ acting on $\mathsf{Prym}(Y)$ by
$$\mathsf{Jac}(X)[n] \times \mathsf{Prym}(Y) \to \mathsf{Prym}(Y), \quad (N, L) \mapsto \pi^*N \otimes L.$$

PROOF. The dual of the norm map is given by the pullback (see [Mum74b]). Dualizing the defining sequence of the Prym variety
$$0 \to \mathsf{Prym}(Y) \to \mathsf{Jac}(Y) \xrightarrow{\mathsf{Nm}} \mathsf{Jac}(X) \to 0$$
results in
$$0 \to \ker(\pi^*) \to \mathsf{Jac}(X) \xrightarrow{\pi^*} \mathsf{Jac}(Y) \to \mathsf{Prym}(Y)^{\vee} \to 0.$$
We obtain
$$\mathsf{Prym}(Y)^{\vee} \cong \mathsf{Jac}(Y)/\pi^*\mathsf{Jac}(X).$$
Define the morphism of abelian varieties
$$\mathsf{Prym}(Y) \times \mathsf{Jac}(X) \to \mathsf{Jac}(Y), \quad (L, M) \mapsto L \otimes \pi^*M^{-1}.$$
A pair (L, M) is in the kernel, if and only if $L = \pi^*M$. Hence $(\mathsf{Nm} \circ \pi^*)M = M^n = \mathcal{O}_X$. In particular, $M \in \mathsf{Jac}(X)[n]$. On the other hand, let $M \in \mathsf{Jac}(X)[n]$ then $(\mathsf{Nm} \circ \pi^*)M = \mathcal{O}_X$ and (π^*M, M) is contained in the kernel. Furthermore, it is a surjective morphism as the kernel is finite. Hence, there is an isomorphism of abelian varieties
$$(\mathsf{Prym}(Y) \times \mathsf{Jac}(X))\,/\mathsf{Jac}(X)[n] \to \mathsf{Jac}(Y).$$
Quotienting out $\mathsf{Jac}(X)$ we obtain
$$\mathsf{Prym}(Y)^{\vee} \cong \mathsf{Jac}(Y)/\pi^*\mathsf{Jac}(Y) \cong \mathsf{Prym}(Y)/\mathsf{Jac}(X)[n].$$
$\square$

CHAPTER 2

Semi-abelian spectral data for singular fibers of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin system

In this chapter, we will study singular fibers of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin systems with irreducible and reduced spectral curve. This will lay the ground for the results about symplectic and orthogonal Hitchin systems in Chapters 4 and 5.

As a first result, we will stratify the singular Hitchin fibers by semi-abelian spectral data in Section 2.4. The abelian part of the spectral data will be a torsor over the Prym variety of the normalised spectral curve and parametrises the eigen line bundles. The non-abelian part of the spectral data is a product $(\mathbb{C}^*)^r \times \mathbb{C}^s$ parametrizing manipulations of the Higgs field at the zeroes of the quadratic differential by Hecke transformations.

In the second part of this chapter (Sections 2.6 and 2.7), we will study how the strata fit together to form the singular Hitchin fiber. Here again the interpretation of the non-abelian part of the spectral data in terms of Hecke parameters proves to be very useful. This allows us to study the irreducible components of singular Hitchin fibers and to give an explicit description of the first degenerations (Sect 2.6.5). We will develop these results for the M-twisted $\mathsf{SL}(2,\mathbb{C})$-Hitchin system, which will be crucial for the analysis of $\mathfrak{sl}(2)$-type Hitchin fibers in Chapter 4.

Finally in Section 2.8, we will study, how the $\mathsf{SL}(2,\mathbb{R})$-points in singular Hitchin fibers are parametrised in terms of these semi-abelian spectral data.

2.1. The $\mathsf{SL}(2,\mathbb{C})$-Hitchin system

Let X be a Riemann surface of genus $g \geq 2$. Let M a holomorphic line bundle over X.

Definition 2.1.1. A M-twisted $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle is a pair (E,Φ) of a holomorphic vector bundle E of rank two with trivial determinant and a Higgs fields $\Phi \in H^0(X, \mathrm{End}(E) \otimes M)$, such that $\mathrm{tr}(\Phi) = 0$.

(E,Φ) is called stable, if for all Φ-invariant subbundles $L \subset E$, $\deg(L) < 0$. (E,Φ) is called polystable, if for all Φ-invariant $L \subset E$, $\deg(L) \leq 0$ and, in case of equality, there is a splitting $(E,\Phi) = (L \oplus L^{-1}, \mathrm{diag}(\lambda, -\lambda))$.

The Hitchin map is given by

$$\mathsf{Hit}_{\mathsf{SL}(2,\mathbb{C})} : \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X,M) \to H^0(X,M^2), \quad (E,\Phi) \mapsto \det(\Phi)$$

For $M = K$, the Hitchin base is the $3g-3$-dimensional vector space of quadratic differentials. In this case, the Hitchin map defines an algebraically completely integrable system on the dense subset of quadratic differentials with simple zeroes ([Hit87b], [Hit87a]).

Let $a_2 \in H^0(X, M^2)$. The Hitchin map computes the coefficients of the characteristic polynomial of $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$. It is given by

$$\eta^2 + a_2.$$

Let $p_M : M \to X$ the bundle map and $\eta : M \to p_M^* M$ the tautological section. The spectral curve is the complex analytic curve

$$\Sigma := Z_M(\eta^2 + p_M^* a_2) \subset \mathsf{Tot}(M).$$

The projection p_M restricts to a two-sheeted branched covering $\pi : \Sigma \to X$ with branch points at the zeroes of a_2. The spectral curve Σ is smooth besides the ramification points. It is smooth at a ramification point if and only if the corresponding zero of the quadratic differential q_2 is of order one. Due to the specific type of characteristic equation the spectral curve comes with an involutive automorphism $\sigma : \Sigma \to \Sigma$ interchanging the sheets.

For $M = K$, the subset of quadratic differentials with simple zeroes is an open and dense subset of $H^0(X, K^2)$, which we refer to as the regular locus. Its compliment will be referred to as the singular locus. For $a_2 \in H^0(X, M^2)$, we will refer to $\mathsf{Hit}^{-1}(a_2)$ as regular Hitchin fiber, if a_2 has simple zeroes and as singular Hitchin fiber, if not. The regular $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers are abelian torsors over Prym varieties.

THEOREM 2.1.2 (Abelian Spectral Data [Hit87b]). *Let $a_2 \in H^0(X, M^2)$, such that all zeroes are simple. Then $\mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$ is a torsor over the Prym variety $\mathsf{Prym}(\pi : \Sigma \to X)$ of dimension* $\deg(M) + g - 1$.

This will be a special case of the description of $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers with irreducible and reduced spectral curve given below. We want to sketch the classical construction for context.

PROOF. Let $\lambda = \eta|_\Sigma$ and $\Lambda = \mathsf{div}(\lambda)$. Let $(E, \Phi) \in \mathsf{Hit}^{-1}(a_2)$, then λ is an eigensection of $\pi^*\Phi$ and the line bundle of eigen vectors

$$\mathcal{O}(L) = \mathsf{ker}(\pi^*\Phi - \lambda \mathrm{id}_{\mathcal{O}(\pi^* E)})$$

is an element of the twisted Prym variety $\mathsf{Prym}_\Lambda(\Sigma)$. By Lemma 1.4.11, $\mathsf{Prym}_\Lambda(\Sigma)$ is a torsor over $\mathsf{Prym}(\pi : \Sigma \to X)$. The eigenline bundle uniquely determines the Higgs bundle by the algebraic pushforward $(E, \Phi) = \pi_*(L \otimes \pi^* K, \lambda)$ (cf. 2.3.10). □

In this chapter, we study Hitchin fibers with irreducible and reduced spectral curve. The spectral curve is irreducible and reduced if and only if a_2 has no global square root on X, i. e. there exists no $\lambda \in H^0(X, M)$, such that $\lambda^2 = a_2$. In this case, there is a covering of Riemann surfaces associated to the characteristic equation. It is the unique two-sheeted branched covering of Riemann surfaces $\tilde{\pi} : \tilde{\Sigma} \to X$, such that there exists $\lambda \in H^0(\tilde{\Sigma}, \tilde{\pi}^* M)$ solving

$$\lambda^2 + \tilde{\pi}^* a_2 = 0.$$

From a algebro-geometric perspective $\tilde{\Sigma}$ is the normalisation of Σ and we will refer to $\tilde{\Sigma}$ as the normalised spectral curve. The geometry of this covering can be easily understood. The restriction

$$\pi : \Sigma \setminus \pi^{-1}(Z(a_2)) \to X \setminus Z(a_2)$$

is a unbranched covering of Riemann surfaces and there is a unique way to extend it in a smooth way. Whenever the local polynomial equation for Σ in a neighbourhood of $p \in \pi^{-1}(Z(a_2))$ is irreducible, or equivalently the corresponding zero of a_2 is of odd order, we glue in a disc, such that the covering map locally extends to $\tilde{\pi} : z \mapsto z^2$. If instead the local polynomial is reducible, or equivalently the zero of a_2 is of even order, we glue in two discs separating the two sheets. Hence, the branch points of $\tilde{\pi} : \tilde{\Sigma} \to X$ are the zeroes of a_2 of odd order. By the Riemann-Hurwitz formula, the genus of $\tilde{\Sigma}$ is given by

$$g(\tilde{\Sigma}) = 2g - 1 + \frac{n_{\text{odd}}}{2},$$

where n_{odd} denotes the number of odd zeroes of a_2 (without multiplicity).

2.2. σ-invariant Higgs bundles on the normalised spectral curve

2.2.1. The Pullback. Let $p : Y \to X$ be a two-sheeted covering of Riemann surfaces and σ the involutive biholomorphism changing the sheets.

Definition 2.2.1. A σ-invariant holomorphic vector bundle $(E, \hat{\sigma})$ on Y is holomorphic vector bundle E on Y with a lift

$$\begin{tikzcd} E \arrow[r, "\hat{\sigma}"] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "\sigma"] & Y \end{tikzcd}$$

such that

i) $\hat{\sigma}^2 = \mathrm{id}_E$, and
ii) $\hat{\sigma}|_y = \mathrm{id}_{E_y}$ for all ramification points $y \in Y$.

Let $(M, \hat{\sigma}_M)$ be σ-invariant holomorphic line bundle on Y. A σ-invariant $(M, \hat{\sigma}_M)$-twisted Higgs bundle $(E, \Phi, \hat{\sigma}_E)$ on Y is a M-twisted Higgs bundle (E, Φ) on Y, such that $(E, \hat{\sigma}_E)$ is σ-invariant holomorphic vector bundle and

iii) $(\hat{\sigma}_E \otimes \hat{\sigma}_M) \circ \Phi = \Phi \circ \hat{\sigma}_E$.

Lemma 2.2.2. *Let $(E, \Phi, \hat{\sigma}_E)$ be a σ-invariant $(M, \hat{\sigma}_M)$-twisted Higgs bundle and $g \in \mathcal{A}^0(\mathsf{SL}(E))$ an element of the gauge group. Then $(gE, g\Phi g^{-1}, g \circ \hat{\sigma} \circ g^{-1})$ is a σ-invariant $(M, \hat{\sigma}_M)$-twisted Higgs bundle.*

Let $(M, \hat{\sigma}_M)$ a σ-invariant holomorphic line bundle on Y. Define

$$\mathcal{M}^\sigma(Y, M, \hat{\sigma}_M) = \left\{ (E, \Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(Y, M) \;\middle|\; \exists \hat{\sigma} : \begin{array}{l} (E, \Phi, \hat{\sigma}) \ \sigma\text{-invariant} \\ (M, \hat{\sigma}_M)\text{-twisted} \end{array} \right\}.$$

Proposition 2.2.3.

i) *Let E be a holomorphic vector bundle on X. Then p^*E has a induced lift $\hat{\sigma}_{p^*E}$, such that $(p^*E, \hat{\sigma}_{p^*E})$ is a σ-invariant holomorphic vector bundle.*
ii) *We have a natural map*

$$p^* : \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X, M) \to \mathcal{M}^\sigma(Y, p^*M, \hat{\sigma}_{p^*M}).$$

Proof. i) Let $U \subset X$ open, such that $E|_U \cong U \times \mathbb{C}^r$. The trivialisation induces a trivialisation $p^*E\,|_{p^{-1}(U)} \cong p^{-1}U \times \mathbb{C}^r$. If $x \in U$ is not a branch point, i. e. $p^{-1}(x) = \{y, \sigma(y)\}$, such trivialisation induces a identification of the fibers $p^*E_y \cong p^*E_{\sigma(y)}$. This defines a lift $\hat{\sigma}_{p^*E} : p^*E \to p^*E$ away from the ramification points. This lift extends over the ramification points by the identity. Therefore, $(p^*E, \hat{\sigma}_{p^*E})$ is a σ-invariant holomorphic vector bundle.

ii) Clearly, $(p^*E, p^*\Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(Y, p^*M)$ and by i) $(p^*E, \hat{\sigma}_{p^*E})$ is a σ-invariant holomorphic vector bundle. Property iii) of Definition 2.2.1 becomes clear in a trivialisation as in the proof of i).

□

In the sequel, a pullback will always carry the induced lift $\hat{\sigma}$ and we will omit it in the notation.

2.2.2. The σ-invariant Pushforward.

Definition 2.2.4. Let ξ be an analytic sheaf on Y. A lift $\hat{\sigma} : \xi \to \xi$ of σ is a family of involutive homomorphisms of abelian groups

$$\hat{\sigma}_V : H^0(V, \xi) \to H^0(\sigma(V), \xi)$$

commuting with restriction maps, such that for all $f \in \mathcal{O}_V$ and $s \in H^0(V, \xi)$

$$\hat{\sigma}(fs) = (\sigma^* f)\hat{\sigma}(s).$$

The pair $(\xi, \hat{\sigma})$ is called an analytic σ-sheaf.

Definition 2.2.5. Let $(\xi, \hat{\sigma})$ be an analytic σ-sheaf on Y, then the σ-invariant pushforward $p_*(\xi, \hat{\sigma})$ is the analytic sheaf on X defined through

$$H^0(U, p_*(\xi, \hat{\sigma})) = H^0(p^{-1}U, \xi)^{\hat{\sigma}}$$

for open sets $U \subset X$. Here $H^0(p^{-1}U, \xi)^{\hat{\sigma}}$ denotes the $\hat{\sigma}$-invariant sections of $(\xi, \hat{\sigma})$.

Lemma 2.2.6. i) *Let $(\xi, \hat{\sigma})$ be a locally free σ-sheaf of rank r on Y, such that for every ramification point $y \in Y$ there exists an open, σ-invariant neighbourhood $V \subset Y$ of y and an isomorphism $H^0(V, \xi) \cong \mathcal{O}_V^r$, such that*

$$\hat{\sigma}\,|_V : \mathcal{O}_V^r \to \mathcal{O}_V^r, \quad f \mapsto f \circ \sigma.$$

Then $p_*(\xi, \hat{\sigma})$ is locally free of rank r.

ii) *Let $(E, \hat{\sigma})$ be a σ-invariant holomorphic vector bundle of rank r, then $(\mathcal{O}(E), \hat{\sigma})$ satisfies the assumption in i). In particular, the pushforward $p_*(\mathcal{O}(E), \hat{\sigma})$ is locally free of rank r.*

Proof. i) Let $U \subset X$ an open subset trivializing the covering. Let $p^{-1}(U) = U_1 \cup U_2$. A section in $H^0(p^{-1}U, \xi)^{\hat{\sigma}}$ is fixed by its values on U_1. Hence $H^0(p^{-1}U, \xi)^{\hat{\sigma}} \cong \mathcal{O}_{U_1}^r \cong O_U^r$. Let $x \in X$ a branch point. By assumption there exists a neighbourhood $U \subset X$, such that

$$H^0(p^{-1}U, \xi)^{\hat{\sigma}} \cong \{f \in \mathcal{O}_{p^{-1}U}^r \mid f = \sigma^* f\} \cong p^{-1}\mathcal{O}_U^r \cong \mathcal{O}_U^r.$$

ii) Clearly, a lift $\hat{\sigma}$ on E induces a lift on the sheaf of sections $\hat{\sigma} : \mathcal{O}(E) \to \mathcal{O}(E)$ satisfying Definition 2.2.4. To check the extra assumption in i), let $y \in Y$ be a ramification point. Assumption ii) of Definition 2.2.1 guarantees the existence of a local frame of σ-invariant sections in a σ-invariant neighbourhood V of y. Take a local basis for E_y and extend it to a holomorphic frame $s_1, \ldots, s_r$ of E_V. Then a σ-invariant frame is given by $s_1 + \hat{\sigma}s_1, \ldots, s_r + \hat{\sigma}s_r$ for a small enough neighbourhood V of y. A σ-invariant frame induces an isomorphism $\mathcal{O}(E)_V \cong \mathcal{O}_V^r$ such that $\hat{\sigma}|_V$ has the desired form.

□

Definition 2.2.7. Let $(E, \hat{\sigma})$ be a σ-invariant vector bundle. We define the σ-invariant pushforward $p_*(E, \hat{\sigma})$ to be the vector bundle corresponding to the locally free sheaf $p_*(\mathcal{O}(E), \hat{\sigma})$.

Lemma 2.2.8. *Let E be a holomorphic vector bundle on X and $(p^*E, \hat{\sigma}_{p^*E})$ the corresponding σ-invariant holomorphic vector bundle on Y, then*

$$p_*(p^*E, \hat{\sigma}_{p^*E}) = E.$$

Example 2.2.9. Let $p : Y \to X$ be a unbranched 2-covering of Riemann surfaces. Let L be a line bundle on X and $(p^*L, \hat{\sigma})$ the induced σ-invariant line bundle on Y. Then $-\hat{\sigma}$ is another lift of σ on L. However, $p_*(p^*L, -\hat{\sigma}) \ncong L$. We have

$$p_*(p^*L, -\hat{\sigma}) \cong L \otimes I,$$

where $I = p_*(O_Y, -\mathrm{id}_{O_Y})$ is the unique non trivial line bundle on X, which pulls back to the trivial bundle on Y. $p^*(I^2) \cong O_Y$ and the induced lift $\hat{\sigma}_{p^*(I^2)}$ is the identity. Hence, $I^2 = O_X$. I is the holomorphic line bundle defined by regarding the unbranched covering as a $\mathbb{Z}_2$-bundle in $\check{H}^1(X, \mathbb{Z}_2) \subset \check{H}^1(X, \mathcal{O}_X^*)$.

2.2.3. Pullback and Pushforward of singular Hitchin fibers. Let $a_2 \in H^0(X, M^2)$ with no global square root on X. Let $\tilde{\pi} : \tilde{\Sigma} \to X$ be the covering by the normalized spectral curve and $\sigma : \tilde{\Sigma} \to \tilde{\Sigma}$ the involution changing the sheets. We want to parametrize the singular fibers by parametrizing their pullback to $\tilde{\Sigma}$. However, the pullback

$$\pi^* : \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2) \to \mathcal{M}^\sigma(\Sigma, \pi^*K)$$

in general is not injective, as there can be multiple lifts of σ.

Example 2.2.10. Let $a_2 \in H^0(X, M^2)$ with only double zeroes, which has no global square root on X. Then $\tilde{\Sigma} \to X$ is a 2-sheeted unbranched covering of Riemann surfaces. We just saw that there exists a non-trivial line bundle I with $\tilde{\pi}^*(I) \cong \mathcal{O}_{\tilde{\Sigma}}$ and $I^2 = \mathcal{O}_X$. For $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$, also $(E \otimes I, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$. We clearly have

$$\tilde{\pi}^*(E, \Phi) \cong \tilde{\pi}^*(E \otimes I, \Phi).$$

Proposition 2.2.11. *Let $a_2 \in H^0(X, M^2)$ with no global square root. Let*

$$(E, \Phi) \in \tilde{\pi}^*\mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2) \subset \mathcal{M}^\sigma(\tilde{\Sigma}, \tilde{\pi}^*K).$$

i) *If a_2 has at least one zero of odd order, then there is a unique lift $\hat{\sigma}$ such that $(E, \Phi, \hat{\sigma})$ is a σ-invariant Higgs bundle.*
ii) *If a_2 has only zeroes of even order, then there are two such lifts $\pm\hat{\sigma}$.*

PROOF. Let $(E, \Phi) \in \tilde{\pi}^*\mathsf{Hit}^{-1}(a_2)$. Assume that there a two lifts $\hat{\sigma}_1, \hat{\sigma}_2$, such that $(E, \Phi, \hat{\sigma}_i)$ is a σ-invariant Higgs bundle. Then $\hat{\sigma}_1 \circ \hat{\sigma}_2 \in \mathsf{Aut}(E, \Phi)$. If (E, Φ) is stable, this implies that $\hat{\sigma}_1 = \pm\hat{\sigma}_2$. If in addition, a has only even zeroes, the spectral covering $\tilde{\pi}$ is unbranched and this gives the two possible lifts. If (E, Φ) is stable, and a_2 has at least on zero of odd order, then $\tilde{\pi} : \tilde{\Sigma} \to X$ has at least one ramification point $p \in Y$. In particular, $(\hat{\sigma}_1)_p = (\hat{\sigma}_2)_p = \mathrm{id}_{E_p}$ and therefore $\hat{\sigma}_1 = \hat{\sigma}_2$.

$(E, \Phi) \in \tilde{\pi}^*\mathsf{Hit}^{-1}(a_2)$ is strictly polystable if and only if

$$(E, \Phi) = \left(L \oplus L^{-1}, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right).$$

with $\deg(L) = 0$. Hence, a_2 has only even zeroes. Then $\hat{\sigma}_1 = g\hat{\sigma}_2$ with

$$g \in \mathsf{Aut}(E, \Phi) = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \},$$

such that $g^2 = \mathrm{id}_E$. Hence $g = \pm\mathrm{id}_E$. □

PROPOSITION 2.2.12. *Let $a_2 \in H^0(X, M^2)$ with no global square root. The pullback*

$$\tilde{\pi}^* : \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2) \to \mathcal{M}^\sigma(\tilde{\Sigma}, \tilde{\pi}^*M)$$

i) *is injective, if a_2 has at least one zero of odd order, and*
ii) *is generically two-to-one, if a_2 has only even zeroes.*

Let I be the unique non-trivial line bundle with $\tilde{\pi}^*I = O_Y$. The non-injectivity in ii) is due to the identification of the pullback of (E, Φ) and $(E \otimes I, \Phi)$.

PROOF. We already saw in Lemma 2.2.11 that in the first case there is a unique lift $\hat{\sigma}$. Hence the injectivity follows from Lemma 2.2.8. In the second case, we saw that there are two possible lifts $\pm\hat{\sigma}$. From Example 2.2.9 this implies

$$\pi_*(E, \hat{\sigma}) = (\pi_*(E, -\hat{\sigma})) \otimes I.$$

Together with Lemma 2.2.8, this gives the result in case ii). □

EXAMPLE 2.2.13. In case ii) branching exists. The section $\lambda : \tilde{\Sigma} \to \tilde{\pi}^*M$ has the property $\sigma^*\lambda = -\lambda$. Hence, it descends to a section $\alpha \in H^0(X, KI)$. Then

$$(E, \Phi) = \left(I^{\frac{1}{2}} \oplus I^{-\frac{1}{2}}, \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \right)$$

defines a Higgs bundle in $\mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$, such that $E \otimes I \cong E$.

2.3. Hecke transformations

In Section 2.4, we will stratify singular Hitchin fibers by fiber bundles over twisted Prym varieties. The twisted Prym variety will parametrize the eigenline bundles of the Higgs bundles in the stratum. The fibers of these bundles parametrize the manipulation of Higgs bundles by Hecke transformations. In this section, we recall the definition of Hecke transformation (see [HR04]) and adapt it to our purpose. In this section, we will only treat the case of holomorphic vector bundles of rank two. The general definition will be given in Section 3.1.

Let us first recall the rank 1 analogue. The Hecke transformation of a line bundle L at $p \in X$ is the line bundle $L(-p)$. We have an exact sequence

$$0 \to \mathcal{O}(L(-p)) \xrightarrow{s_p} \mathcal{O}(L) \to \mathcal{T}_X(p) \to 0,$$

where s_p is a canonical section of $\mathcal{O}(p)$ and $\mathcal{T}_X(p)$ is the torsion sheaf of length 1 at p.

Definition 2.3.1 ([HR04]). Let E be a holomorphic vector bundle of rank 2 on a Riemann surface X. Let $p \in X$ and $\alpha \in E_p^\vee \setminus \{0\}$, the dual fiber at p. The Hecke transformation $\hat{E}^{(p,\alpha)}$ of E is defined through the exact sequence of coherent sheaves

$$0 \to \mathcal{O}(\hat{E}^{(p,\alpha)}) \to \mathcal{O}(E) \xrightarrow{\alpha} \mathcal{T}_X(p) \to 0.$$

A coherent subsheaf of a locally free sheaf on a Riemann surface is locally free [Gun67] Theorem 3. Hence, $\hat{E}^{(p,\alpha)}$ is well-defined.

For a more concrete description of Hecke transformations, we want to describe it on the level of transition functions. Let $\mathcal{GL}(n)$ denote the sheaf of holomorphic $\mathsf{GL}(n,\mathbb{C})$-valued functions on X. Let $\mathcal{U} = \{U_i\}_{i=1}^m$ a covering of X by contractible open sets, such that $p \in U_i$ if and only if $i = 1$. Let $\{\psi_{ij}\} \in \check{H}^1(\mathcal{U}, \mathcal{GL}(2))$ transition functions for E. Choose a holomorphic frame s_1, s_2 of $E|_{U_1}$, such that $\alpha = (s_2)_p^\vee$. Define a covering $\mathcal{V} = \{V_i\}_{i=0}^m$ by $V_0 = U_1$, $V_1 = U_1 \setminus \{p\}$ and $V_i = U_i$ for $i \geq 2$. Define transition functions $\{\hat{\psi}_{ij}\} \in \check{H}^1(\mathcal{V}, \mathcal{GL}(2))$ by

$$\begin{aligned} \hat{\psi}_{01} : V_0 \cap V_1 \times \mathbb{C}^2 &\to V_0 \cap V_1 \times \mathbb{C}^2, \\ (z, x_1, x_2) &\mapsto (z, x_1, z x_2) \end{aligned} \tag{1}$$

respective the frame s_1, s_2,

$$\hat{\psi}_{0j} = \psi_{1j} \circ \hat{\psi}_{01}, \quad \hat{\psi}_{j0} = \hat{\psi}_{0j}^{-1} \quad \text{for} \quad j \geq 1, \quad \text{and} \quad \hat{\psi}_{ij} = \psi_{ij} \quad \text{for} \quad i, j \geq 1. \tag{2}$$

Lemma 2.3.2. *The holomorphic vector bundle associated to the transition functions $\{\hat{\psi}_{ij}\} \in \check{H}^1(\mathcal{V}, \mathcal{GL}(2))$ is the Hecke transformation $\hat{E}^{p,\alpha}$ of E.*

Proof. By definition of the transition function $\hat{\psi}_{01}$, the associated vector bundle fits into an exact sequence as in Definition 2.3.1. $\square$

We generalize this concept by allowing higher order twists. Let $D \in \mathsf{Div}^+(X)$ and E a holomorphic vector bundle on X. The Hecke transformations at D will be parametrised by polynomial germs on D. Define

$$H^0(D, E) := \bigoplus_{p \in \mathsf{supp} D} \mathcal{O}(E)_p / \sim,$$

where $[s_1] \sim [s_2]$ if and only if $\mathsf{ord}_p([s_1]-[s_2]) \geq D_p$, for all $p \in \mathsf{supp}D$. Furthermore, denote by $H^0(D,E)^* \subset H^0(D,E)$ the equivalence classes of germs, such that for all $p \in \mathsf{supp}D$ the evaluation at p is non-zero.

DEFINITION 2.3.3. Let E be a holomorphic vector bundle of rank 2. Let $D \in \mathsf{Div}^+(X)$ and $\alpha \in H^0(D,E^\vee)^*$. Then the Hecke transformation $\hat{E}^{(D,\alpha)}$ of E at D in direction α is defined by the exact sequence of locally free sheaves

$$0 \to \mathcal{O}(\hat{E}^{(D,\alpha)}) \to \mathcal{O}(E) \xrightarrow{\alpha} \mathcal{T}_X(D) \to 0,$$

where $\mathcal{T}_X(D)$ is the torsion sheaf of length D_p at $p \in \mathsf{supp}D$.

LEMMA 2.3.4. *Let $D \in \mathsf{Div}^+(X)$ and $\alpha \in H^0(D,E^\vee)^*$, then* $\det(\hat{E}^{(D,\alpha)}) = \det(E)(-D)$.

PROOF. By definition, $\det(\mathcal{T}_X(D) \cong \mathcal{O}(D)$. □

For our purposes, it will be more convenient to use the dual version of this concept.

DEFINITION 2.3.5. Let $D \in \mathsf{Div}^+(X)$ and $\alpha \in H^0(D,E)^*$ then the (dual) Hecke transformations $\hat{E}^{(D,\alpha)}$ of E at D in direction α is defined by the exact sequence of locally free sheaves

$$0 \to \mathcal{O}((\hat{E}^{(D,\alpha)})^\vee) \to \mathcal{O}(E^\vee) \xrightarrow{\alpha} \mathcal{T}_X(D) \to 0.$$

LEMMA 2.3.6. *Let $\{\psi_{ij}\} \in \check{H}^1(\mathcal{U},\mathcal{GL}(2))$ transition functions of E as above. For $p \in X, l \in \mathbb{N}$, let $D := lp \in \mathsf{Div}^+(X)$. Let further $\alpha \in H^0(D,E)^*$. The Hecke transformation $\hat{E}^{(D,\alpha)}$ is the holomorphic vector bundle associated to the transition functions $\{\hat{\psi}_{ij}\} \in \check{H}^1(\mathcal{V},\mathcal{GL}(2))$ defined as in (1),(2), where the frame s_1, s_2 is chosen, such that*

$$[(s_2)_p] = \alpha \in H^0(D,E)$$

and

$$\begin{aligned} \hat{\psi}_{01} : V_0 \cap V_1 \times \mathbb{C}^2 &\to V_0 \cap V_1 \times \mathbb{C}^2 \\ (z,x_1,x_2) &\mapsto (z,x_1,z^{-l}x_2). \end{aligned}$$

More generally, for $D \in \mathsf{Div}^+(X)$ and $\alpha \in H^0(D,E)^*$, we obtain transition functions of $\hat{E}^{(D,\alpha)}$ by introducing a new transition function like this for all $p \in \mathsf{supp}(D)$.

LEMMA 2.3.7. *Let $D \in \mathsf{Div}^+(X)$ and $\alpha \in H^0(D,E)^*$, then* $\det(\hat{E}^{(D,\alpha)}) = \det(E)(D)$.

2.3.1. Parameters of Hecke transformations.

LEMMA 2.3.8. *Let $D \in \mathsf{Div}^+(X), \alpha \in H^0(D,E)^*$ and $\phi \in H^0(D,\mathcal{O}_X)^*$. Then*

$$E^{(D,\alpha)} \cong E^{(D,\phi\alpha)}.$$

An equivalence class in the quotient $H^0(D,E)/H^0(D,\mathcal{O}_X)^*$ is referred to as a Hecke parameter.

PROPOSITION 2.3.9. *$H^0(D,\mathcal{O}_X)^*$ is a complex solvable Lie group with respect to the multiplication of germs of non-vanishing holomorphic functions. Let $D = lp$ with $l \in \mathbb{N}$ and $p \in X$. Then*

$$H^0(D,\mathcal{O}_X)^* \cong \left\{ \begin{pmatrix} x_0 & x_1 & \dots & x_{l-1} \\ & \ddots & \ddots & \vdots \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix} \mid x_0 \in \mathbb{C}^*, x_i \in \mathbb{C} \right\} \cong \mathbb{C}^* \times \mathbb{C}^{l-1}.$$

For $D \in \mathsf{Div}^+(X)$, $H^0(D,\mathcal{O}_X)^*$ is isomorphic to a Cartesian product of such groups.

2.3.2. Leading example. As a leading example, we show how the algebraic pushforward of a line bundle along a two-sheeted covering of Riemann surfaces can be recovered using Hecke transformations and the σ-invariant pushforward defined in Section 2.2.

Let $p : Y \to X$ be a two-sheeted covering of Riemann surfaces and $\sigma : Y \to Y$ the holomorphic involution changing the sheets. Denote by $R \subset Y$ the ramification divisor. Let $L \in \mathsf{Pic}(Y)$, then $E = L \oplus \sigma^* L$ has a natural lift $\hat{\sigma} : E \to E$ induced by pullback along σ. At a ramification point, we can choose a frame, such that $\hat{\sigma}$ is locally given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, E is no σ-invariant holomorphic vector bundle (cf. Definition 2.2.1). This can be corrected by applying a Hecke transformation.

Choose a neighbourhood U of $\mathsf{Fix}(\sigma)$ separating all ramification points and a frame $s \in H^0(U, L)$. Then

$$s_1 = s \oplus \sigma^* s, \quad s_2 = s \oplus -\sigma^* s$$

is a frame of E diagonalizing $\hat{\sigma}$. Let

$$\alpha = [s_2]_R^\vee \in H^0(R, E^\vee)^*.$$

For $y \in \mathsf{supp}(R)$ choose a coordinate z, such that the involution is given by $\sigma : z \mapsto -z$. We saw above that $\hat{E}^{(R,\alpha)}$ is obtained form E by introducing new transition functions of the form

$$\begin{aligned} \hat{\psi}_{01} : V_0 \cap V_1 \times \mathbb{C}^2 &\to V_0 \cap V_1 \times \mathbb{C}^2 \\ (z, x_1, x_2) &\mapsto (z, x_1, z x_2) \end{aligned}$$

at every point $y \in \mathsf{supp}(R) = \mathsf{Fix}(\sigma)$. $\hat{\sigma}$ induces a lift of σ on $\hat{E}^{(R,\alpha)}$, that we keep calling $\hat{\sigma}$. The frame $s_1, z s_2$ extends to a $\hat{\sigma}$-invariant frame s_1^σ, s_2^σ of $\hat{E}^{(R,\alpha)}$. Hence, $(\hat{E}^{(R,\alpha)}, \hat{\sigma})$ is a σ-invariant holomorphic vector bundle and $p_*(\hat{E}^{(R,\alpha)}, \hat{\sigma})$ defines a holomorphic vector bundle of rank 2 on X.

LEMMA 2.3.10. *$p_*(\hat{E}^{(R,\alpha)}, \hat{\sigma}) = p_* L$.*

PROOF. Let $U_1 \subset X$ be open, contractible subset trivializing the covering p, i. e. $p^{-1} U_1 = V^+ \sqcup V^-$. $\mathcal{O}(p_* L)$ is a free of rank 2 over O_{U_1}. This is apparent from decomposing

$$H^0(U_1, p_*L) = H^0(p^{-1}U_1, L) = H^0(V^+, L) \oplus H^0(V^-, L).$$

Hence, we have a natural isomorphism

$$H^0(U_1, p_*L) \cong H^0(U_1, p_*(\hat{E}^{(D,\alpha)}, \hat{\sigma})) = H^0(p^{-1}U_1, L \oplus \sigma^*L)^{\hat{\sigma}}. \tag{3}$$

Let $U_2 \subset X$ be open, contractible neighbourhood of a branch point $x \in X$. Choose a coordinate on $p^{-1}(U_2)$, such that $\sigma\,|_{p^{-1}(U_2)} : z \mapsto -z$. Let $s \in H^0(p^{-1}U_2, L)$ a local frame and $\phi \in H^0(p^{-1}U_2, L)$. Then there exist $\phi_1, \phi_2 \in \mathcal{O}_{U_2}$, such that

$$\phi(z) = \phi_1(z^2)s + \phi_2(z^2)zs. \tag{4}$$

Hence, $p_*L\,|_{p^{-1}(U_2)}$ is free over $\mathcal{O}_{U_2}$ of rank 2 with generators s, zs. Let s_1, s_2 be the σ-invariant frame of $\hat{E}^{(D,\alpha)}$ defined above, then we define an isomorphism

$$H^0(p^{-1}U_2, L) \to H^0(p^{-1}U_2, \hat{E}^{(D,\alpha)})^{\hat{\sigma}}, \qquad \phi \mapsto \phi_1 s_1 + \phi_2 s_2. \tag{5}$$

We claim that (3) and (5) define an isomorphism of locally free sheaves, i. e. they commute with the restriction functions.

Let $U_1, U_2 \subset X$ as above, such that $U_1 \subset U_2$. Choosing a coordinate w on U_1 we can identify the two branches $V^{\pm}$ with the square roots $\pm\sqrt{w}$. Let $\phi \in H^0(U_2, p_*L) = H^0(p^{-1}U_2, L)$. From (4) we obtain

$$\phi\,|_{V^+} = (\phi_1(z^2) + \phi_2(z^2)z)s\,|_{V^+} = (\phi_1(w) + \phi_2(w)\sqrt{w})s\,|_{V^+},$$
$$\phi\,|_{V^-} = (\phi_1(z^2) + \phi_2(z^2)z)s\,|_{V^-} = (\phi_1(w) - \phi_2(w)\sqrt{w})s\,|_{V^-}.$$

So the restriction map is given by

$$r_{U_2U_1} = \begin{pmatrix} 1 & \sqrt{w} \\ 1 & -\sqrt{w} \end{pmatrix}.$$

This agrees with the restriction map of $p_*(\hat{E}^{(D,\alpha)}, \hat{\sigma})$ by construction. □

COROLLARY 2.3.11. *Consider a two-sheeted covering of Riemann surfaces $p : Y \to X$, then*

$$p_*\mathcal{O}_Y = \mathcal{O}_X \oplus J.$$

If p is a branched covering, then $J \in \mathsf{Pic}(X)$ is the unique line bundle, such that $p^*J = \mathcal{O}(-R)$, where R is the ramification divisor of p. If p is unbranched, then $J \in \mathsf{Jac}(X)$ is the unique non-trivial line bundle, such that $p^*J = \mathcal{O}_X$.

PROOF. Let $L = \mathcal{O}_Y$ in the construction above. So, $E = O_Y \oplus O_Y$ and

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The diagonalizing frame for $\hat{\sigma}$ defines a global splitting

$$E = \mathcal{O}_Y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{O}_Y \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{with} \quad \hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If p is a branched covering, we apply a Hecke transformation and obtain

$$\hat{E}^{(R,\alpha)} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-R) = p^*O_X \oplus p^*J.$$

The uniqueness of J follows from the injectivity of the pullback along branched coverings. If p is unbranched, E is a σ-invariant vector bundle with the lifted σ-action on the second factor being $-\mathrm{id}_{\mathcal{O}_Y}$. Hence, the second factor descends to the line bundle J. In both cases, Lemma 2.3.10 gives the result. $\square$

2.4. Moduli of σ-invariant Higgs bundles

After identifying the Hitchin fibers with certain moduli spaces of σ-invariant Higgs bundles on the normalised spectral curve in Section 2.2, we will now prove the stratification result for these moduli spaces. Thereafter, we will identify these strata as fiber bundles over Prym varieties.

2.4.1. The Stratification. Let $p : Y \to X$ be a two-sheeted branched covering of Riemann surfaces. Let $\sigma : Y \to Y$ the involution changing the sheets. Let M be a line bundle on X with a non-zero section $\lambda : Y \to p^*M$, such that $\hat{\sigma}\lambda = -\lambda$. Here p^*M is regarded as a σ-invariant holomorphic line bundle with the lift $\hat{\sigma}$ induced by pullback (cf. Proposition 2.2.3). In particular, λ has a zero of odd order at all ramification points. Let $\Lambda = \mathsf{div}(\lambda)$. In this section, we parametrize

$$\mathcal{M}^\sigma_\lambda = \mathcal{M}^\sigma(Y, p^*M, \lambda) := \mathcal{M}^\sigma(Y, p^*M) \cap \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(-\lambda^2),$$

the polystable σ-invariant p^*M-twisted $\mathsf{SL}(2,\mathbb{C})$-Higgs bundles on Y with characteristic equation

$$T^2 - \lambda^2 = 0.$$

By assumption, $-\lambda^2$ is a σ-invariant section of p^*M^2 and hence descends to $a \in H^0(X, M^2)$. $\mathcal{M}^\sigma_\lambda$ is identified with the image of $p^* : \mathsf{Hit}^{-1}_M(a) \to \mathcal{M}(Y, p^*M)$ by Proposition 2.2.3 and is therefore an analytic subset.

Lemma 2.4.1. *Let $(E, \Phi) \in \mathcal{M}^\sigma_\lambda$ and $y \in Y$. There exists a coordinate chart (U, z) centred at y, a local frame $m \in H^0(U, \pi^*M)$ and a local frame of $E|_U$, such that the Higgs field is given by*

$$\Phi = z^{D_y} \begin{pmatrix} 0 & 1 \\ z^{2\Lambda_y - 2D_y} & 0 \end{pmatrix} \otimes m.$$

Proof. Choose a coordinate disc (U, z) centred at y, such that the determinant $\det(\Phi) = -z^{2\Lambda_y} m^2$. There exists a non vanishing section $\phi \in H^0(U, \mathrm{End}(E))$, such that

$$\Phi(z) = z^{D_y} \phi(z) \otimes m.$$

There are two possible Jordan forms of ϕ at y. If $D_y < \Lambda_y$ there is one Jordan block of size 2, if $D_y = \Lambda_y$, ϕ is diagonalizable with eigenvalues ± 1. Thus, after a constant gauge transformation we can assume

$$\phi(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} \quad \text{with} \quad \phi(0) = \begin{pmatrix} 0 & 1 \\ * & 0 \end{pmatrix}.$$

Hence,

$$g = \frac{1}{\sqrt{b(z)}} \begin{pmatrix} b(z) & 0 \\ -a(z) & 1 \end{pmatrix} \in \mathsf{Aut}(E)|_U,$$

is a well-defined gauge, such that

$$g^{-1}\phi g = \begin{pmatrix} 0 & 1 \\ -\det(\phi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z^{2\Lambda_y - 2D_y} & 0 \end{pmatrix}.$$

□

For $(E,\Phi) \in \mathcal{M}^\sigma_\lambda$, we denote by $\mathsf{div}(\Phi)$ the vanishing divisor of Φ. In the notation of the previous lemma $\mathsf{div}(\Phi)_y = D_y$, for all $y \in Y$. The properties of vanishing divisors of Higgs fields are summarized in the following definition.

Definition 2.4.2. An effective divisor $D \in \mathsf{Div}(Y)$ is called σ-Higgs divisor on (Y,σ,λ) if $0 \leq D \leq \Lambda$, $\sigma^* D = D$ and $D_y \equiv 0 \mod 2$, for all $y \in \mathsf{Fix}(\sigma)$.

Theorem 2.4.3. *There exists a stratification*

$$\mathcal{M}^\sigma(Y, p^*M, \lambda) = \bigsqcup_D \mathcal{S}_D$$

by locally closed analytic subsets

$$\mathcal{S}_D = \{(E,\Phi) \in \mathcal{M}^\sigma(Y, p^*M, \lambda) \mid \mathsf{div}(\Phi) = D\}$$

indexed by σ-Higgs divisors $D \in \mathsf{Div}(Y)$.

Proof. First, it is easy to see that for $(E,\Phi) \in \mathcal{M}^\sigma(Y, p^*M, \lambda)$ the vanishing divisor $\mathsf{div}(\Phi)$ is a σ-Higgs divisor. These divisors form a lower semi-continuous invariant on $\mathcal{M}^\sigma_\lambda$ (cf. Lemma 2.4.1). In particular, for a fixed σ-Higgs divisor D

$$\bigcup_{D' \geq D} \mathcal{S}_{D'} \quad \text{is closed} \quad \text{and} \quad \bigcup_{D' \leq D} \mathcal{S}_{D'} \quad \text{is open.}$$

Hence, $\mathcal{S}_D$ is locally closed. To see that the closed subset is an analytic subset, we need to identify it with the pullback of a Hitchin fiber of $\mathsf{SL}(2,\mathbb{C})$-Higgs bundles with a different twist. Fix a σ-Higgs divisor D and let s_D be the canonical section of $\mathcal{O}(D)$, which is σ-invariant. Then $(p^*M)(-D)$ is the pullback of $M(-\frac{1}{2}\mathsf{Nm}D)$ and $\frac{\lambda}{s_D} \in H^0(Y, p^*M(-D))$ satisfies $\sigma^*(\lambda/s_D) = -\lambda/s_D$. So

$$\mathcal{M}^\sigma(Y, p^*M(-D), \lambda/s_D)$$

defines another moduli space of σ-invariant Higgs bundles. This is the pullback of a Hitchin fiber in the moduli space of $M(-\frac{1}{2}\mathsf{Nm}D)$-twisted $\mathsf{SL}(2,\mathbb{C})$-Higgs bundles on X and hence an analytic space. There is a holomorphic bijective map

$$\begin{aligned} \mathcal{M}^\sigma(Y, p^*M(-D), \lambda/s_D) &\to \bigcup_{D' \geq D} \mathcal{S}_{D'} \subset \mathcal{M}^\sigma(Y, p^*M, \lambda), \\ (E,\Phi) &\mapsto (E, s_D\Phi). \end{aligned}$$

Therefore, its image is an analytic subspace of $\mathcal{M}^\sigma(Y, p^*M, \lambda)$ (see [Gra+94] I.10.13). □

2.4.2. The abelian part of the spectral data. Recall from Lemma 1.4.11, the definition of the twisted Prym variety

$$\mathsf{Prym}_D(Y) = \{L \in \mathsf{Pic}(Y) \mid L \otimes \sigma^* L = \mathcal{O}_X(D)^{-1}\}.$$

THEOREM 2.4.4. *Consider $\mathcal{M}^\sigma_\lambda$ as above. For every stratum $\mathcal{S}_D$, there exists a holomorphic map*

$$\mathsf{Eig}_D : \mathcal{S}_D \to \mathsf{Prym}_{\Lambda - D}(Y), \qquad (E, \Phi) \mapsto \mathsf{ker}(\Phi - \lambda \mathrm{id}_{\mathcal{O}(E)}).$$

PROOF. Let $(E, \Phi) \in \mathcal{S}_D$ and let $\mathcal{O}(L) = \mathsf{ker}(\Phi - \lambda \mathrm{id}_{\mathcal{O}(E)})$ the sheaf-theoretical kernel. Then $\mathcal{O}(\sigma^* L) = \mathsf{ker}(\Phi + \lambda \mathrm{id}_{\mathcal{O}(E)})$. The inclusions $\mathcal{O}(L) \to \mathcal{O}(E)$, $\mathcal{O}(\sigma^* L) \to \mathcal{O}(E)$ define an exact sequence of coherent analytic sheaves

$$0 \to \mathcal{O}(L) \oplus \mathcal{O}(\sigma^* L) \to \mathcal{O}(E) \to \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a torsion sheaf supported at $Z(\lambda)$. $\mathcal{T}$ can be explicitly constructed using the local description of Φ in Lemma 2.4.1. In particular,

$$O_Y = \det(E) = L \otimes \sigma^* L \otimes \det(\mathcal{T}) = L \otimes \sigma^* L \otimes \mathcal{O}(\Lambda - D).$$

$\square$

2.4.3. Hecke parameters - the non-abelian part of the spectral data. The fibers of each stratum $\mathcal{S}_D$ over $\mathsf{Prym}_{\Lambda-D}(Y)$ can be identified as Hecke parameters (cf. Section 3). Consider $\mathcal{M}^\sigma_\lambda$ as above, D a σ-Higgs divisor and $L \in \mathsf{Prym}_{\Lambda-D}(Y)$. Let

$$(E_L, \Phi_L) = \left(L \oplus \sigma^* L, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right).$$

E_L has a natural lift $\hat{\sigma} : E_L \to E_L$ of σ induced by the pullback. Choose a local frame s of L at a branch point $p \in Y$, then

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

respective the frame $s, \sigma^* s$ of E_L. Fix the diagonalizing frame

$$s_+ = s + \sigma^* s, \qquad s_- = s - \sigma^* s. \tag{6}$$

LEMMA 2.4.5. *Let $\alpha \in H^0(\Lambda - D, E_L)^*$. Then $\hat{E}_L^{(\Lambda-D,\alpha)}$ is a σ-invariant holomorphic vector bundle, if*

i) *for all $p \in \mathsf{Fix}(\sigma)$, $[\hat{\sigma}\alpha]_p = -[\alpha]_p$, and*
ii) *for all $p \in \mathsf{supp}(\Lambda - D) \setminus \mathsf{Fix}(\sigma)$, $[\hat{\sigma}\alpha]_p = [\alpha]_{\sigma p}$.*

PROOF. Let $p \in \mathsf{supp}(\Lambda - D) \setminus \mathsf{Fix}(\sigma)$. If ii) is satisfied, the lift $\hat{\sigma}$ induces a lift of σ on $\hat{E}_L^{(\Lambda-D,\alpha)}$ at p . Let $p \in \mathsf{Fix}(\sigma)$. α can be extended to a local section s_α around p, such that s_+, s_α is a frame of E_L. Respective this frame, the Hecke transformation is equivalent to introducing the new transition function

$$\psi_{01} = \begin{pmatrix} 1 & 0 \\ 0 & z^{\Lambda_p - D_p} \end{pmatrix}$$

(with notation as in 1,2). Because $(\Lambda - D)_p \equiv 1 \mod 2$, we conclude that the induced frame of $\hat{E}_L^{(\Lambda-D,\alpha)}$ at p is σ-invariant. In conclusion, $\hat{E}_L^{(\Lambda-D,\alpha)}$ is a σ-invariant holomorphic bundle (cf. Definition 2.2.1). $\square$

Let $D \in \mathsf{Div}^+(Y)$ a σ-Higgs divisor. Define

$$H^0(\Lambda - D, (E_L, \hat{\sigma})) := \{\alpha \in H^0(\Lambda - D, E_L) \mid \alpha \text{ satisfies i) and ii) } \}$$
$$H^0(\Lambda - D, (E_L, \hat{\sigma}))^* := \{\alpha \in H^0(\Lambda - D, E_L)^* \mid \alpha \text{ satisfies i) and ii) } \}$$
$$G_{\Lambda - D} := \{\phi \in H^0(\Lambda - D, \mathcal{O}_Y)^* \mid \sigma^* \phi = \phi\}.$$

Proposition 2.4.6. *Let $Z(\lambda) = \mathsf{Fix}(\sigma)$, D a σ-Higgs divisor and $N := \Lambda - D \in \mathsf{Div}^+(Y)$. Let (U, z) be a union of coordinate neighbourhoods (U_p, z_p) around $p \in \mathsf{Fix}(\sigma)$ disconnecting $\mathsf{Fix}(\sigma)$, such that $\sigma : z_p \mapsto -z_p$. For all frames $s \in H^0(U, L)$, there is a holomorphic surjective map*

$$u_s : H^0(N, (E_L, \hat{\sigma}))^* \to \bigoplus_{p \in \mathsf{Fix}(\sigma)} \mathbb{C} z_p + \mathbb{C} z_p^3 + \cdots + \mathbb{C} z_p^{N_p - 2},$$

which factors through the action of G_N, inducing a bijection on the quotient. Let $s' = fs$ with $f = f_e + f_o \in H^0(U, \mathcal{O}_Y^*)$ and f_e, f_o the even and odd part respective σ, then

$$u_{s'} = \frac{f_e u_s - f_o}{f_e - f_o u_s} \mod z^N.$$

Proof. A choice of s induces a frame $s_+ = s + \sigma^* s,\ s_- = s - \sigma^* s$ of E_L. Respective such frame, we can explicitly parametrize

$$H^0(N, (E_L, \hat{\sigma}))^* = \{[a s_+ + b s_-] \mid a, b \in H^0(N, \mathcal{O}_Y) : \sigma^* a = -a, \sigma^* b = b\}.$$

Let $N = lp$ with $l \in \mathbb{N}$, $p \in Y$. We define the isomorphism

$$\begin{array}{rcl} u_s : H^0(N, (E_L, \hat{\sigma}))^* & \to & \{u \in H^0(N, \mathcal{O}_Y) \mid \sigma^* u = -u\} \\ {[a s_+ + b s_-]} & \mapsto & \frac{a}{b} \mod z^{N_p}. \end{array}$$

This map clearly factors through G_N and separates the orbits. The right side is a $\mathbb{C}$ vector space over the basis $z, z^3, \ldots, z^{N_p - 2}$. Let now $s' = (f_e + f_o)s$, then

$$s'_+ = f_e s_+ + f_o s_-, \quad s'_- = f_e s_- + f_o s_+$$

and

$$\alpha = \frac{1}{f_e^2 - f_o^2} \left((a f_e - b f_o) s'_+ + (-a f_o + f_e b) s'_- \right).$$

Applying $u_{s'}$ gives the result. □

Proposition 2.4.7. *Let $Z(\lambda) = \mathsf{Fix}(\Sigma)$, $D \in \mathsf{Div}^+(Y)$ a σ-Higgs divisor and $N = \Lambda - D$. Then*

$$F_D = \left\{ H^0(N, (E_L, \hat{\sigma}))^* / G_N \mid L \in \mathsf{Prym}_N(Y) \right\} \to \mathsf{Prym}_N(Y)$$

is a holomorphic vector bundle of rank

$$r = \sum_{y \in Z(\lambda)} \tfrac{1}{2}(\Lambda_y - D_y - 1) = \tfrac{1}{2}\left(\deg(\Lambda) - \deg(D) - \#\mathsf{Fix}(\sigma)\right).$$

Proof. There exists a universal line bundle

$$\mathcal{L} \to Y \times \mathsf{Prym}_N.$$

Let U be a disconnecting neighbourhood of $\mathsf{Fix}(\sigma)$ as above. A local trivialization of $\mathcal{L}$ over $U \times V \subset Y \times \mathsf{Prym}_N$ is equivalent to choosing a local frame $s \in H^0(U, L)$ over $V \in \mathsf{Prym}_N$ in a coherent way. This defines a u-coordinate on

V, in other words, a local trivialisation $F_D|_V \cong V \times \mathbb{C}^r$. Changing the trivialisation corresponds to choosing a different holomorphic frame $s' \in H^0(U, L)$. The corresponding transformation of u-coordinates is holomorphic by the previous proposition. $\square$

THEOREM 2.4.8. *Let $Z(\lambda) = \mathsf{Fix}(\sigma)$ and D a σ-Higgs divisor. Then there is an isomorphism $F_D \to S_D$ making the following diagram commute:*

$$\begin{tikzcd} F_D \arrow[r] \arrow[d] & \mathcal{S}_D \arrow[d, "\mathsf{Eig}_D"'] \\ \mathsf{Prym}_{\Lambda-D} \arrow[r, "\mathrm{id}"] & \mathsf{Prym}_{\Lambda-D} \end{tikzcd}$$

In particular,

$$\dim \mathcal{S}_D = \deg(M) - \tfrac{1}{2}\deg(D) + g(X) - 1$$

PROOF. Let $N = \Lambda - D$. Let $L \in \mathsf{Prym}_N(Y)$. Let (U, z) be a union of coordinate neighbourhood of $\mathsf{Fix}(\sigma)$ disconnecting $\mathsf{Fix}(\sigma)$, such that $\sigma : z \mapsto -z$. Let $t \in H^0(U, p^*M)$ and $s \in H^0(U, L)$ local frames. We will show in i) how to produce Higgs bundles in $\mathcal{S}_D$ by applying Hecke transformation to (E_L, Φ_L). This defines the map $F_D \to \mathcal{S}_D$. To see that it is an isomorphism, we will show in ii) how to recover the u-coordinate from $(E, \Phi) \in \mathcal{S}_D$.

i) Let $\alpha \in H^0(N, (E_L, \hat{\sigma}))^*$, we saw in Lemma 2.4.5 that $\hat{E}^{(N,\alpha)}$ with the induced lift $\hat{\sigma}$ is a σ-invariant holomorphic vector bundle. Furthermore,

$$\det(\hat{E}^{(N,\alpha)}) = \det(E_L)(N) = L \otimes \sigma^* L \otimes \mathcal{O}(\Lambda - D) = \mathcal{O}_Y.$$

The Higgs field Φ_L induces a Higgs field on $\hat{E}^{(N,\alpha)}$. From Lemma 2.3.6 it is easy to see that the Hecke transformation of E_L in direction $\alpha = us_+ + s_-$ is given by introducing a new transition function

$$\hat{\psi}_{01} = \begin{pmatrix} 1 & -uz^{-N} \\ 0 & z^{-N} \end{pmatrix}$$

respective s_+, s_- at every $p \in Z(\lambda)$ (with notation as in 1,2). In particular, the induced Higgs field is given by

$$\hat{\Phi}_L^{(N,\alpha)} = \psi_{01}^{-1} \Phi_L \psi_{01} = \begin{pmatrix} -uz^{\Lambda} & z^D(1-u^2) \\ z^{2\Lambda - D} & uz^{\Lambda} \end{pmatrix} t. \tag{7}$$

This is a well-defined Higgs field on $\hat{E}^{(N,\alpha)}$ with $\det(\hat{\Phi}_L^{(N,\alpha)}) = -\lambda^2$ and vanishing divisor D. In conclusion,

$$(\hat{E}^{(N,\alpha)}, \hat{\Phi}_L^{(N,\alpha)}) \in \mathsf{Eig}_D^{-1}(L) \subset \mathcal{S}_D.$$

ii) Let $(E, \Phi) \in \mathsf{Eig}_D^{-1}(L) \subset \mathcal{S}_D$. Fix inclusions

$$i_+ : L \to E, \quad i_- : \sigma^* L \to E$$

onto the corresponding subbundles. (There are $\mathbb{C}^*$-many such inclusions, but the coordinate will not depend on this choice.) Let

$$s_+ := i_+(s) + i_-(\sigma^*(s)), \quad s_- := i_+(s) - i_-(\sigma^*(s)) \in H^0(U, E).$$

As σ is fixing the fiber over $y \in \mathsf{Fix}(\sigma)$, s_+ is non-vanishing. Instead, s_- has a zero of odd order at y. We augment s_+ to a σ^*-invariant frame s_+, s_a of E. Respective this frame $s_- = -us_+ + u's_a$ with holomorphic odd functions $u, u' \in H^0(U, \mathcal{O}_Y)$. After acting on this frame by gauge transformations of the form

$$\begin{pmatrix} 1 & \phi_1 \\ 0 & \phi_2 \end{pmatrix}$$

with $\phi_i \in H^0(U, \mathcal{O}_Y)$, such that $\sigma^*\phi_i = \phi_i$, we may assume that $u' = z^n$ and u is of polynomial degree $< n$. The σ-invariant section s_a is uniquely defined by these conditions. By definition $\Phi s_+ = \lambda s_-$ and $\Phi s_- = \lambda s_+$. Hence, the Higgs field is given by

$$\Phi = \begin{pmatrix} -uz^{\Lambda_p} & z^{\Lambda_p - n}(1-u^2) \\ z^{\Lambda_p + n} & uz^{\Lambda_p} \end{pmatrix} t \tag{8}$$

respective the invariant frame s_+, s_a. Computing the vanishing order of Φ we see that $n = \Lambda_y - D_y$ at $y \in \mathsf{Fix}(\sigma)$. Moreover, u defines a polynomial germ $[u] \in H^0(N, \mathcal{O}_Y)$, such that $\sigma^*[u] = -[u]$.

If we apply i) to $\alpha = [us_+ + s_-] \in H^0(N, (E_L, \hat{\sigma}))^*$ the Hecke transformation $(\hat{E}_L^{(N,\alpha)}, \hat{\Phi}_L^{(N,\alpha)})$ is isomorphic to (E, Φ). There is a trivial isomorphism on $Y \setminus \mathsf{Fix}(\sigma)$, where the two Higgs bundles are isomorphic to $L \oplus \sigma^* L$. Respective the fixed frame s_+, s_a on (E, Φ) and the induced frame on $(\hat{E}_L^{(N,\alpha)}, \hat{\Phi}_L^{(N,\alpha)})$ this extends to an isomorphism of holomorphic vector bundles. Local descriptions of the Higgs field respective these frames were computed in 7, 8 and agree. Hence, the map extends to an isomorphism of Higgs bundles. In conclusion, the map $F_D \to \mathcal{S}_D$ defined in i) is a fiberwise isomorphism.

$\square$

COROLLARY 2.4.9. *Consider $\mathcal{M}^\sigma(Y, p^*L, \lambda)$, such that $Z(\lambda) = \mathsf{Fix}(\sigma)$ and λ has only simple zeroes. Then*

$$\mathcal{M}^\sigma(Y, p^*L, \lambda) = \mathsf{Prym}_\Lambda.$$

PROOF. In this case, the only σ-Higgs divisor is $D = 0$. So by Theorem 2.4.11 and Theorem 2.4.8, we have $\mathcal{M}^\sigma(Y, p^*L, \lambda) \cong \mathcal{S}_0 \cong F_0 \cong \mathsf{Prym}_\Lambda$. $\square$

To describe the extra data for zeroes of even order, it is more convenient to use extension classes. We will give an interpretation in terms of Hecke parameters in Proposition 2.7.1.

PROPOSITION 2.4.10. *Consider $\mathcal{M}^\sigma(Y, p^*M, \lambda)$, such that $Z(\lambda) = \mathsf{Fix}(\sigma) \sqcup \{y, \sigma^* y\}$, the zeroes y and $\sigma^* y$ are of order $m \geq 1$, and all other zeroes are simple. Fix a σ-Higgs divisor D and $L \in \mathsf{Prym}_{\Lambda - D}(Y)$. Then*

$$\mathsf{Eig}_D^{-1}(L) \cong \{[c] \in H^0(\Lambda_y y, L^2 p^* M) \mid \mathsf{ord}_y[c] = D_y\} \cong \mathbb{C}^* \times \mathbb{C}^{\Lambda_y - D_y - 1}.$$

The last isomorphism is determined by the choice of a local coordinate (U, z) centred at y and a local frame $s \in H^0(U, L^2 p^* M)$.

PROOF. By assumption, we can trivialize the covering in a neighbourhood U of $p(y) \in X$. Let $p^{-1}(U) = U_y \sqcup U_{\sigma y}$, such that $y \in U_y$. Let $(E, \Phi) \in M^\sigma_\lambda$. By the σ-invariance $(E,\Phi)|_{p^{-1}(U)}$ is uniquely determined by $(E,\Phi)|_{U_y}$. So we need to parametrise the possible $(E,\Phi)|_{U_y}$ with eigenvalues $\pm\lambda|_{U_y}$. Regarding $(E,\Phi)|_{U_y}$ as a $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle on U_y, we will use the description of $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers with reducible spectral curve developed in [GO13] Section 7. Write (E,Φ) as an extension

$$0 \to (L, \lambda) \to (E, \Phi) \to (L^*, -\lambda) \to 0.$$

These extensions are parametrised by the hypercohomology group $\mathbb{H}^1(L^2, 2\lambda)$ of the complex of locally free sheaves

$$\mathcal{O}_Y(L^2) \xrightarrow{2\lambda} O_Y(L^2 p^* M).$$

(see Section 6.1 for an introduction to hypercohomology). The 5-term exact sequence of (one of) the associated spectral sequences reveals that

$$\mathbb{H}^1(L^2, 2\lambda) \cong H^0(\Lambda, L^2 p^* M) := \bigoplus_{y \in \ \mathsf{supp}(\Lambda)} \mathcal{O}(L^2 p^* M)_y / \sim,$$

where for $v, v' \in \bigoplus_{y \in \ \mathsf{supp}(\Lambda)} \mathcal{O}(L^2 p^* M)_y$

$$v \sim v' \quad \Leftrightarrow \quad v = v' + f\lambda \qquad \text{with} \quad f \in \bigoplus_{y \in \mathsf{supp}(\Lambda)} \mathcal{O}_y.$$

By Theorem 2.4.8, the extension data at the simple zeroes in $\mathsf{Fix}(\sigma)$ is uniquely determined by σ-invariance. Hence, the fibers of Eig_D are parametrised by

$$H^0(\Lambda_y y, L^2 p^* M),$$

where we consider $\Lambda_y y$ as a divisor supported at the point y. Furthermore, one can explicitly construct a Higgs bundle

$$\left(E = L \oplus_{C^\infty} L^{-1}, \bar\partial_E = \begin{pmatrix} \bar\partial_L & b \\ 0 & \bar\partial_{L^{-1}} \end{pmatrix}, \Phi = \begin{pmatrix} \lambda & c \\ 0 & -\lambda \end{pmatrix}\right)$$

from the extension data $[c] \in H^0(\Lambda, L^2 p^* M)$ by extending $[c]$ to a smooth section $c \in \mathcal{A}^0(Y, L^2 p^* M)$ and solving the equation

$$\bar\partial c = 2b\lambda.$$

for $b \in \mathcal{A}^{(0,1)}(Y, L^2)$. In this way, we see that $\mathsf{div}(\Phi)_y = D_y$ if and only if $D_y = \mathsf{ord}_y c$. So $(E,\Phi) \in \mathcal{S}_D$ are parametrized by the polynomial germs $[c] \in H^0(\Lambda_y y, L^2 p^* M)$ with $\mathsf{ord}_y([c]) = D_y$. $\square$

THEOREM 2.4.11. *Fix a moduli space $\mathcal{M}^\sigma(Y, p^*M, \lambda)$ and a compatible σ-Higgs divisor D. Then the stratum $\mathcal{S}_D$ is a holomorphic fiber bundle*

$$(\mathbb{C}^*)^r \times \mathbb{C}^s \to \mathcal{S}_D \to \mathsf{Prym}_{\Lambda - D}$$

with

$$r = \tfrac{1}{2}\left(\#Z(\lambda) - \#\mathsf{Fix}(\sigma)\right)$$
$$s = \tfrac{1}{2}\left(\deg(\Lambda) - \deg(D) - \#Z(\lambda)\right).$$

In particular, the dimension of the stratum is given by

$$\deg(M) - \frac{1}{2}\deg(D) + g - 1.$$

Proof. The extra data depends only on the structure of the Higgs bundle at $Z(\lambda)$. If we have more than one higher order zero in λ, the fiber of Eig_D is a Cartesian product of the Hecke parameters described in Theorem 2.4.8 and the extension data of Proposition 2.4.10. The coordinates in Proposition 2.4.10 depend holomorphically on the choice of a local frame $s \in H^0(U, L^2\pi^*M)$. Hence, the argument given in the proof of Proposition 2.4.7 establishes the structure of a holomorphic fiber bundle on $\mathcal{S}_D$. □

2.5. Stratification of singular fibers of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin system

In this section, we specify the stratification result to singular fibers of the K-twisted $\mathsf{SL}(2,\mathbb{C})$-Hitchin system.

Definition 2.5.1. Let $q_2 \in H^0(X, K^2)$. A Higgs divisor $D \in \mathsf{Div}(X)$ is a divisor, such that for all $p \in Z(q_2)$

$$0 \leq D_p \leq \lfloor \frac{1}{2}\mathsf{ord}_p(q_2) \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

For $q_2 \in H^0(K^2)$ let

$$n_{\text{even}} = \#\{p \in Z(q_2) \mid p \text{ zero of even order}\}$$
$$n_{\text{odd}} = \#\{p \in Z(q_2) \mid p \text{ zero of odd order.}\}$$

Theorem 2.5.2. *Let $q_2 \in H^0(K^2)$ be a quadratic differential on X, such that $n_{\text{odd}} \geq 1$. Then there is a stratification*

$$\mathsf{Hit}^{-1}(q_2) = \bigcup_D \mathcal{S}_D$$

by locally closed subsets $\mathcal{S}_D$ indicated by Higgs divisors D. Each stratum has the structure of a holomorphic fiber bundle

$$(\mathbb{C}^\times)^r \times \mathbb{C}^s \to \mathcal{S}_D \to \mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma}),$$

where

$$r = n_{\text{even}}, \quad \text{and} \quad s = 2g - 2 - \deg(D) - n_{\text{even}} - \frac{1}{2}n_{\text{odd}}.$$

In general, the dimension of a stratum $\mathcal{S}_D$ is given by

$$3g - 3 - \deg(D).$$

Proof. The stratification by Higgs divisors is obtained in the same way as in Theorem 2.4.3. We analysed the map $p^* : \mathsf{Hit}^{-1}(q_2) \to \mathcal{M}^\sigma(\tilde{\Sigma}, \tilde{\pi}^*K, \lambda)$ in Section 2.2. It is bijective, as $n_{\text{odd}} > 0$. We showed above how $\mathcal{M}^\sigma_\lambda$ is stratified by σ-Higgs divisors (Theorem 2.4.3). The σ-Higgs divisors on $(\tilde{\Sigma}, \sigma, \lambda)$ correspond to the pullbacks $\tilde{\pi}^*D$ of Higgs divisors $D \in \mathsf{Div}(X)$ associated to q_2. The fiber bundle structure of the strata was described in Theorem 2.4.11. We have $\#\mathsf{Fix}(\sigma) = n_{\text{odd}}$, $\#Z(\lambda) = n_{\text{odd}} + 2n_{\text{even}}$, and

$$g(\tilde{\Sigma}) = 2g - 1 + \tfrac{n_{\text{odd}}}{2}.$$

Hence, the dimension of the stratum $\mathcal{S}_D$ is given by $3g - 3 - \deg(D)$. $\square$

REMARK 2.5.3. We will show in Theorem 5.10 using analytic techniques that the fiber bundle structures of the strata are smoothly trivial.

2.5.1. Algebraic and Geometric Interpretations.
The semi-abelian spectral data developed above shows that the torus structure is not completely lost, once we degenerate to the singular locus. Starting with a torus of codimension 1 in the first degeneration, the deeper we go into the singular locus the lower the dimension of the subtori. All singular $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers with irreducible and reduced spectral curve have a subtorus of at least dimension $g-1$. Furthermore, these subtori are abelian torsors over the Prym variety of the normalised spectral cover.
The highest dimensional stratum $\mathcal{S}_0$ corresponds to Higgs bundles $(E,\Phi) \in \mathsf{Hit}^{-1}(q_2)$ with non-vanishing Higgs field. One can show, that this stratum corresponds to the locally free sheaves on the singular spectral curve by the Beauville-Narasimhan-Ramanan correspondence [BNR89]. The lowest dimensional stratum $\mathcal{S}_{D_{\max}}$ contains the Higgs bundles $(E,\Phi) \in \mathsf{Hit}^{-1}(q_2)$ with maximal vanishing order. At a zero of odd order $2m+1$, they can be locally written as

$$\Phi(z) = z^m \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z$$

and at a zero of order $2m$, they are diagonalizable with local eigensections $\pm z^m \, \mathrm{d}z$. If $B \in \mathsf{Div}(X)$ is the branch divisor of the covering by the normalised spectral curve, we have

$$D_{\max} = \frac{1}{2}(\mathsf{div}(q_2) - B).$$

The stratum $\mathcal{S}_{D_{\max}}$ has no Hecke parameters or extension data. Let $R = \frac{1}{2}\tilde{\pi}^*B$ the ramification divisor. Then $\mathcal{S}_{D_{\max}}$ is obtained as the algebraic pushforward along $\tilde{\pi} : \tilde{\Sigma} \to X$ of the line bundles $L(R)$ with $L \in \mathsf{Prym}_R(\tilde{\Sigma})$ (cf. Lemma 2.3.10). Restricted to certain subsets of the singular locus one recovers the subintegrable systems described by Hitchin [Hit19].

2.6. Singular fibers with locally irreducible spectral curve

In this section, we start analysing how the strata fit together to form the singular Hitchin fiber. We will consider the case, where the spectral curve is locally irreducible, i. e. where $a_2 \in H^0(X, M^2)$ has only zeroes of odd order. To do so, we need to compactify the moduli of Hecke parameters of the highest stratum. We show that these singular Hitchin fibers are themselves holomorphic fiber bundles over twisted Prym varieties with fibers given by the compactified moduli of Hecke parameters. This allows an explicit description of the fibers for the first degenerations.

2.6.1. Hecke transformations - revisited. We will again work in the setting introduced in Section 2.4. Let $p : Y \to X$ a covering of Riemann surfaces and $\sigma : Y \to Y$ the involution changing the sheets. Let M be a line bundle on X with a section $\lambda : Y \to p^*M$, such that $\hat{\sigma}\lambda = -\lambda$. Let $\Lambda = \mathsf{div}(\lambda)$.

In this section, we consider $\mathcal{M}^\sigma_\lambda = \mathcal{M}^\sigma(Y, p^*M, \lambda)$, such that $Z(\lambda) = \mathsf{Fix}(\sigma)$. Under identifying $\mathcal{M}^\sigma_\lambda$ with the Hitchin fiber via pullback, this extra condition is equivalent to the spectral curve being locally irreducible.

Let us shortly recall how we used Hecke transformations in Section 2.4. We saw in Theorem 2.4.8 that for fixed $L \in \mathsf{Prym}_\Lambda(Y)$ the Higgs bundles $(E, \Phi) \in \mathcal{S}_0$, which project to L, are parametrized by

$$\mathsf{Eig}_\Lambda^{-1}(L) = H^0(\Lambda, (E, \hat{\sigma}))^*/G_\Lambda.$$

After choosing frames of L at $Z(\lambda)$, the Hecke parameters are decoded in the polynomial germs u. We reconstructed a σ-invariant Higgs bundle from the spectral data (L, u) as the Hecke transformation of (E_L, Φ_L) at Λ in direction of $\alpha = us_+ + s_-$ introducing the new transition function

$$\hat{\psi}_{01} = \begin{pmatrix} 1 & -uz^{-\Lambda_p} \\ 0 & z^{-\Lambda_p} \end{pmatrix}$$

(see Theorem 2.4.8). To compactify the moduli of Hecke parameters, we need to allow Hecke parameters $\alpha \in H^0(\Lambda, (E, \hat{\sigma}))$, which vanish on $\mathsf{supp}\Lambda$ (cf. Lemma 2.4.5). Fix $L \in \mathsf{Prym}_\Lambda(Y)$ and a frame $s \in H^0(U, L)$ in a neighbourhood U of $Z(\lambda)$.

Definition 2.6.1. Let $\alpha \in H^0(\Lambda, (E, \hat{\sigma})) \setminus \{0\}$. Define $(\hat{E}_L^{(\Lambda,\alpha)}, \hat{\Phi}_L^{(\Lambda,\alpha)})$ by introducing a new transition function

$$\hat{\psi}^y_{01} = \begin{pmatrix} b^{-1} & -az^{-\Lambda_p} \\ 0 & bz^{-\Lambda_p} \end{pmatrix},$$

respective s_+, s_- for all $y \in Z(\lambda)$, where a, b are defined through $\alpha = as_+ + bs_-$ (see Section 2.3 for details on the notation).

Recall that the Hecke transformation is invariant under the group action of

$$G_\Lambda = \{\phi \in H^0(\Lambda, \mathcal{O}^*_Y) \mid \sigma^*\phi = \phi\}$$

on the Hecke parameters $H^0(\Lambda, (E, \hat{\sigma}))^*$. When we allow the Hecke parameters to vanish, there is another equivalence relation.

Lemma 2.6.2. i) *Let $\alpha \in H^0(\Lambda, (E_L, \hat{\sigma}))$ and $\phi \in G_\Lambda$, then $\hat{E}_L^{(\Lambda,\alpha)} \cong \hat{E}_L^{(\Lambda,\phi\alpha)}$. In particular, Definition 2.6.1 and Definition 2.3.5 agree for $\alpha \in H^0(\Lambda, (E_L, \hat{\sigma}))^*$.*

ii) *Let $\alpha, \alpha' \in H^0(\Lambda, (E_L, \hat{\sigma}))$, such that $\mathsf{div}(\alpha) = \mathsf{div}(\alpha') = D$. Then $\hat{E}_L^{(\Lambda,\alpha)} \cong \hat{E}_L^{(\Lambda,\alpha')}$, whenever the projections of α, α' to $H^0(\Lambda - D, (E_L, \hat{\sigma}))$ agree.*

Proof. In i), the new transition function of the Hecke transformation in direction of $\phi\alpha$ is given by

$$\begin{pmatrix} \phi^{-1}b^{-1} & -\phi az^{-\Lambda_p} \\ 0 & \phi bz^{-\Lambda_p} \end{pmatrix} = \begin{pmatrix} b^{-1} & -az^{-\Lambda_p} \\ 0 & bz^{-\Lambda_p} \end{pmatrix} \begin{pmatrix} \phi^{-1} & 0 \\ 0 & \phi \end{pmatrix}$$

Hence, the transition functions define isomorphic vector bundles. For ii), let $\phi \in H^0(\Lambda, \mathcal{O}_Y)$, such that $\sigma^*\phi = \phi$, then the Hecke transformations respective $\alpha = as_+ + bs_-$ and $\alpha' = (a + \phi\frac{\lambda}{b})s_+ + bs_-$ are isomorphic. The isomorphism is given by the gauge transformation

$$\begin{pmatrix} b^{-1} & -az^{-\Lambda_p} \\ 0 & bz^{-\Lambda_p} \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & (\frac{\phi}{b}z^{\Lambda_p} - a)z^{-\Lambda_p} \\ 0 & bz^{-\Lambda_p} \end{pmatrix}.$$

This provides the equivalence in the a-coordinate. Using the G_Λ action one obtains equivalence ii). □

2.6.2. Weighted projective spaces. We will obtain a topological model for the compact moduli of Hecke parameters by gluing subsets of weighted projective spaces. Let us recall some basic facts about weighted projective spaces.

A weight vector $(i_0, \dots, i_n) \in \mathbb{N}^n$ defines a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by

$$\mathbb{C}^* \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \quad (\lambda, x_0, \dots, x_n) \mapsto (\lambda^{i_0}x_0, \dots, \lambda^{i_n}x_n).$$

The weighted projective space $\mathbb{P}(i_0, \dots, i_n)$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ by this action. We will denote the equivalence class of $(x_0, \dots, x_n)$ by $(x_0 : \dots : x_n)$. Weighted projective spaces are complex orbifolds. We obtain orbifold charts in the same way one defines affine charts of projective space $\mathbb{P}^n$. For example, for points of the form $(1, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$, the $\mathbb{C}^*$-action restricts to an action of $\mathbb{Z}_{i_0}$ given by

$$(1, x_1, \dots, x_n) \mapsto (1, \xi_{i_0}^{i_1}x_1, \dots, \xi_{i_0}^{i_n}x_n),$$

where ξ_{i_0} is a primitive i_0-th root of unity.
Weighted projective spaces are normal toric complex spaces. In an orbifold chart, the torus action is given by

$$(1 : x_1 : \dots : x_n) \mapsto (1 : \lambda_1^{i_1}x_1 : \dots : \lambda_n^{i_n}x_n)$$

for $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. This extends to an analytic action on $\mathbb{P}(i_0, \dots, i_n)$. We call a analytic subspace $Y \subset \mathbb{P}(i_1, \dots, i_n)$ toric, if it is preserved by the torus action.

2.6.3. Compact Moduli of Hecke parameters. In this section, we want to study the compact moduli of Hecke parameters. To do so we will restrict our attention to the Hecke parameters at a single higher order zero.

Let (U, z) be holomorphic disc centred at $0 \in \mathbb{C}$ and $\sigma : z \mapsto -z$. Let $d \in \mathbb{N}$ a odd number and D the divisor with coefficient d at zero. Define

$$\mathsf{Heck}_d = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in H^0(D, \mathcal{O}_U^2) \;\middle|\; \sigma^*a = -a, \; \sigma^*b = b \right\} \Big/ \sim,$$

where

$$\alpha = (a, b) \sim \alpha' = (a', b') \quad \Leftrightarrow \quad \begin{array}{l} \mathsf{ord}_0(\alpha) = \mathsf{ord}_0(\alpha') =: n \\ \text{and } (a, b) = (a', b') \bmod z^{d-n}. \end{array}$$

These are the equivalence classes of relation ii) in Lemma 2.6.2. For $0 \le n < \frac{d}{2}$ let

$$V_n := \{\alpha \in \mathsf{Heck}_d \mid \mathsf{ord}_0(\alpha) = n\}.$$

We can understand the quotient of Heck_d by G_D by gluing subsets, on which we find explicit invariant polynomials. By Proposition 2.3.9, $G_D = \mathbb{C}^* \times H_D$, where

$$H_D = \{1 + \phi_2 z^2 + \cdots \in H^0(D, \mathcal{O}_U)\}$$

is the maximal unipotent normal subgroup. We will first factor through H_D as orbits of unipotent group actions on affine spaces are closed. The resulting intermediate quotient can be factored through $\mathbb{C}^*$. The subsets V_n will correspond to the strata of the stratification 2.4.3.

LEMMA 2.6.3. *Let $0 \leq n \leq \frac{d-3}{2}$. There is a holomorphic map $u_n : V_n \to \mathbb{P}^{\frac{1}{2}(d-2n-1)}$ invariant under the G_D-action and separating the orbits. Its image is an affine chart of $\mathbb{P}^{\frac{1}{2}(d-2n-1)}$. For $n = \frac{d-1}{2}$, the G_D action identifies V_n to a point.*

PROOF. Lets assume $n \leq \frac{d-3}{2}$ is even. Every $\alpha \in V_n$ has a unique representative of the form

$$\begin{pmatrix} \frac{a_{n+1}z^{n+1} + \cdots + a_{d-2}z^{d-2}}{1 + \frac{b_{n+2}}{b_n}z^2 + \cdots + \frac{b_{d-1}}{b_n}z^{d-1}} \\ b_n z^n \end{pmatrix} \mod z^{d-n}$$

with $b_n \neq 0$ in its H_D-orbit. In particular, b_n and the $(n+1)$-th, $\ldots$, $(d-n-2)$-th derivatives of the fraction in the first coordinate define $\frac{1}{2}(d-2n+1)$ holomorphic functions invariant under the H_D-action. This defines a map

$$V_n \to \mathbb{C}^* \times \mathbb{C}^{\frac{1}{2}(d-2n-1)}.$$

The $\mathbb{C}^*$-action acts with weight 1 on every coordinate. By factoring through $\mathbb{C}^*$, we obtain the desired map to an affine chart of $\mathbb{P}^{\frac{1}{2}(d-2n-1)}$. For n odd, every $\alpha \in V_n$ has a unique representative

$$\begin{pmatrix} a_n z^n \\ \frac{b}{1 + \frac{a_{n-2}}{a_n}z^2 + \cdots + \frac{a_{d-2}}{a_2}z^{d-2}} \end{pmatrix} \mod z^{d-n}.$$

By recording a_n and the $(n+1)$-th, $\ldots$, $(d-n-2)$-th derivative of the second coordinate and again factoring through the $\mathbb{C}^*$-action we obtain invariant map

$$V_n \to \mathbb{P}^{\frac{1}{2}(d-2n+1)}.$$

As $a_n \neq 0$, the image is an affine chart. If $n = \frac{d-1}{2}$ is odd, the only H_D-invariant function on V_n is $a_n \neq 0$. Hence, V_n is identified to a point by the $\mathbb{C}^*$-action. Similarly, for $n = \frac{d-1}{2}$ even. $\square$

It seems impossible to find enough invariant functions to define the global quotient Heck_d/G_D. However, we obtain a topological model by gluing the quotients of subsets, which are easier to understand.

PROPOSITION 2.6.4. *There exist finitely many locally closed connected subsets $N_i \subset \mathsf{Heck}_d$, $i \in I$, such that*

i) *for every $n < l \leq \frac{d-1}{2}$ and $\alpha \in V_l$, there exist $i \in I$, such that $\alpha \in N_i$ and $N_i \cap V_n \neq 0$,*

ii) *there exist algebraic maps* $N_i \to \mathbb{P}(1,1,2,3,\dots,m_i)$ *invariant by the action of* G_D*, which separate the* G_D*-orbits. Their images are toric subspaces and contain no singular points.*

PROOF. For $n \leq l \leq \frac{d-1}{2}$ let

$$N_l^n := H_D \cdot \{a = x_1 z + x_3 z^3 + \dots, b = x_0 z + x_2 z^2 + \dots \mod z^{d-n} \mid \\ x_0 = \dots = x_{n-1} = x_{n+1} = \dots = x_{l-1} = 0, x_l \neq 0\}.$$

Let $\alpha = (a,b) \in N_l^n$. If $x_n \neq 0$, we have $\mathsf{ord}_0(\alpha) = n$, hence $\alpha \in V_n$. If $x_n = 0$, we have $\alpha \in V_l$. So N_l^n describes a locally closed subset of V_n containing V_l in its closure. We first want to find invariant polynomials by the H_D-action and then take the quotient by $\mathbb{C}^*$. Let l be odd and n be even, then

$$a = x_l z^l + \dots + x_{d-n-2} z^{d-n-2},$$
$$b = x_n z^n + x_{l+1} z^{l+1} + \dots + x_{d-n-1} z^{d-n-1}$$

Every orbit in N_l^n has a representative of the form

$$\begin{pmatrix} x_l z^l \\ \frac{x_n z^n + x_{l+1} z^{l+1} + \dots + x_{d-n-1} z^{d-n-1}}{1 + \frac{x_{l+2}}{x_l} z^2 + \dots + \frac{x_{d-n-2}}{x_l} z^{d-n-2-l}} \end{pmatrix} \mod z^{d-n}. \tag{9}$$

The n-th, $n+2$-th, …, $(d-l-2)$-th derivative give (after multiplying with their common divisor)

$$\tfrac{1}{2}(d-l-n)$$

homogeneous polynomials of degree

$$1, 2, \dots, \tfrac{1}{2}(d-l-n).$$

The representative in 9 is not quite unique because if we act by $(1+z^{d-n-l}\phi) \in H_D$ the a-coordinate stays unchanged modulo z^{d-n}. However, as we only record up to the $d-l-2$-th derivative of the b-coordinate these homogeneous polynomials are invariant under the H_D-action. Furthermore, it is easy to see that they are independent elements of the algebra of H_D-invariant polynomials on N_l^n because the $n+2k$-th derivative is the first on to contain x_{l+2k}. By recording x_l in addition, we have $\frac{1}{2}(d-l-n+2)$ independent homogeneous polynomials. This defines a map

$$N_l^n \to \mathbb{C}^{\frac{1}{2}(d-l-n+2)}$$

invariant by the H_D-action. Factoring through the $\mathbb{C}^*$-action we obtain the desired algebraic map

$$N_l^n \to \mathbb{P}(1,1,2,3,\dots,\frac{d-l-n}{2}).$$

To show that it separates orbits we first consider $N_l^n \cap V_l$, i. e. $x_n = 0$. Here every element has a unique representative $\frac{b}{a} \mod z^{d-l}$. Those are determined by the invariant polynomials induced from the derivatives $l+1$ till $d-l-2$. Instead on $N_l^n \cap V_n$ we can uniquely represent each element by a u-coordinate, see Lemma 2.6.3. This u-coordinate can be recovered from the invariant polynomials. The $(n+2)$-th derivative decodes x_{l+2}, the $(n+4)$-th x_{l+4} etc. and the $(d-l-2)$-th

decodes x_{d-n-2}. So the map separates orbits.
As $x_l \neq 0$ we see that the image is contained in

$$\{(y_0 : \cdots : y_{\frac{1}{2}(d-l-n)}) \in \mathbb{P}(1, 1, \ldots, \tfrac{1}{2}(d-l-n)) \mid y_0 \neq 0\}.$$

This subset contains no singularity of the weighted projective space. Furthermore, by the explicit description of the homogeneous polynomials it is easy to verify that the image is closed under the torus action of $(\mathbb{C}^*)^{\frac{1}{2}(d-l-n)}$.
Now lets consider the case of even n and even l. Here we have to take a finer decomposition. Let $k > l$ a odd number then

$$
{}^kN_n^l := H_D \cdot \left\{ \begin{array}{l} a = x_1 z + x_3 z^3 + \cdots + x_{d-n-2} z^{d-n-2} \\ b = x_0 z + x_2 z^2 + \cdots + x_{d-n-1} z^{d-n-1} \end{array} \right| \\
\left. \begin{array}{l} x_0 = \cdots = x_{n-1} = x_{n+1} = \cdots = x_{l-1} = 0, x_l \neq 0, \\ x_{l+1} = x_{l+3} = \cdots = x_{k-2} = 0, x_k \neq 0 \end{array} \right\}.
$$

Clearly

$$\bigcup_{\text{odd } k \geq l} {}^kN_n^l = N_n^l.$$

So with these subsets we still satisfy property i). For fixed k we proceed as before by computing the n-th, ..., $(d-k-2)$-th derivative of

$$\frac{x_n z^n + x_l z^l + \cdots + x_{d-1} z^{d-1}}{1 + \frac{x_{k+2}}{x_k} z^2 + \cdots + \frac{x_{d-2}}{x_k} z^{d-2-l}}.$$

They define a map

$${}^kN_l^n \to \mathbb{P}(1, 1, 2, 3 \ldots, \tfrac{1}{2}(d-k-n)).$$

invariant by the action of G_D. For $x_n \neq 0$ we can recover the u-coordinate of ${}^kN_l^n \cap V_n$ as above. If $x_n = 0$ the u-coordinate of the lower stratum is now given by $\frac{a}{b} \mod z^{d-l}$. We recover a_{k+2} from the $l+2$-th derivative, a_{k+4} from the $l+4$-derivative till a_{d-l-2} from the $d-k-2$-th derivative. These uniquely defines the u-coordinate on V_l. With the same argument as above the image contains no singular points and is closed under the torus action.
When n is odd we can obtain the same results by changing the role of a and b. □

THEOREM 2.6.5. *The quotient of* Heck_d *by the action of* G_D *is a union of toric subspaces of weighted projective spaces glued algebraically along torus orbits.*

PROOF. Most of the work was already done in the previous lemma by introducing the sets $N_i \subset \mathsf{Heck}_d$ and the G_D-invariant, orbit-separating maps

$$N_i \to \mathbb{P}(1, 1, 2, 3, \ldots, m_i).$$

These maps identify the quotients N_i/G_D with toric subspaces of weighted projective spaces. We can build a model for the quotient Heck_d/G_D by gluing together this subsets N_i/G_D along their intersection. We are left to show that this happens algebraically along torus orbits. It is enough to show that for all $i \in I$ and $0 \leq l \leq \frac{d-3}{2}$ the intersection $N_i \cap V_l$ is mapped onto a toric subspace under the two maps to weighted projective spaces and that the coordinate change is polynomial.

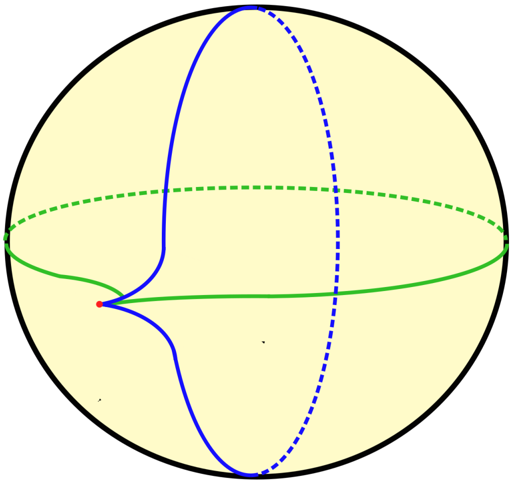

FIGURE 5. The compact moduli of Hecke parameters for $d = 5$: $\mathcal{S}_0$ yellow, $\mathcal{S}_1$ blue, $\mathcal{S}_2$ red, N_2^0 green.

We will show this for $N_l^n \cap V_n$ with $n < l \leq \frac{d-3}{2}$, n even and l odd. For the other cases, it works in the same way. Denote by

$$u_n : V_n \to \mathbb{P}^{\frac{1}{2}(d-2n-1)}, \qquad f_l : N_l^n \to \mathbb{P}(1, 1, 2, \dots, \frac{d-l-n}{2})$$

the G_D-invariant maps defined in Lemma 2.6.3 and Proposition 2.6.4. Let $\alpha \in N_l^n \cap V_n$. We can choose a representative of the form

$$\begin{pmatrix} x_l z^l + x_{l+2} z^{l+2} + \dots + x_{d-n-2} z^{d-n-2} \\ z^n \end{pmatrix}.$$

The image under u_n is given by

$$(1 : 0 : \dots : 0 : x_l : x_{l+2} : x_{d-n-2}) \in \mathbb{P}^{\frac{1}{2}(d-2n-1)}.$$

So $u_n(N_l^n \cap V_n)$ is clearly a union of torus orbits. On the other hand, we can explicitly compute the values of the invariant polynomials defining f_l and obtain

$$\left(x_l \ : \ 1 \ : \ x_{l+2} \ : \ x_{l+4} x_l + x_{l+2}^2 \ : \ \dots \ : \ x_{d-n-2} (x_l)^{\frac{1}{2}(d-l-2-n)} + \dots \right)$$
$$\in \mathbb{P}(1, 1, 2, \dots, \frac{d-l-n}{2}).$$

This is again a union of torus orbits. It is easy to check that the gluing maps are polynomial. $\square$

EXAMPLE 2.6.6. Let us take a closer look at the compact moduli of Hecke parameters at a zero of $a_2 \in H^0(X, M^2)$ of order $d = 5$. This example is sketched in Figure 5.

In this case, we have three strata of Hecke parameters by Lemma 2.6.3 $\mathsf{Heck}_5 = V_0 \cup V_1 \cup V_2$, where $V_0 \cong \mathbb{C}^2, V_1 \cong \mathbb{C}$, and $V_0 \cong \{0\}$. Proposition 2.6.4 defined three subset N_l^n, where $N_1^0 = \mathcal{S}_0 \cup \mathcal{S}_1$ is of dimension 2, $N_2^1 = \mathcal{S}_1 \cup \mathcal{S}_2$ is of dimension 1, and N_2^0 is a one-dimensional subset of $\mathcal{S}_0$ with $\mathcal{S}_2$ in its closure.

Because the glueing maps are algebraic, we can give Heck_5/G_D the structure of a scheme using a pushout-construction (cf. [Sch05]). This is the universal scheme structure $(\mathsf{Heck}_5/G_D, \mathcal{O})$, such that for any other scheme structure $(\mathsf{Heck}_5/G_D, \mathcal{O}')$ the inclusions of the subsets V_i and N_l^n into $(\mathsf{Heck}_5/G_D, \mathcal{O}')$ factor through $(\mathsf{Heck}_5/G_D, \mathcal{O})$. However, because we do not glue along closed subschemes, this scheme structure has some pathologies as explained in [Sch05] Example 3.2. In particular, it is not the scheme structure we will obtain below by embedding the space of Hecke parameters in the singular Hitchin fiber.

2.6.4. Global fibering over twisted Prym varieties. We will show that the singular fibers with locally irreducible spectral curve fiber over twisted Prym varieties with fibers given by the compact moduli of Hecke parameters. As a first step we identify the twisted Prym varieties of the all strata.

DEFINITION 2.6.7. Let $\mathcal{M}_\lambda^\sigma$, such that $\mathsf{Fix}(\sigma) = Z(\lambda)$. Define

$$\mathsf{Eig}_{\mathsf{tw}} : \mathcal{M}_\lambda^\sigma = \bigcup_D \mathcal{S}_D \to \mathsf{Prym}_\Lambda(Y), \quad (E, \Phi) \mapsto \mathsf{Eig}_D(E, \Phi)(-\tfrac{1}{2}D).$$

REMARK 2.6.8. This is well-defined, because D has only even coefficients. If we allow q_2 to have even zeroes, there is no canonical way to identify the twisted Prym varieties of the different strata. See Section 2.7 for more details.

We defined two kinds of u-coordinates: First in Proposition 2.4.6, when parametrising the strata and second in Lemma 2.6.3, when parametrising $V_n \subset \mathsf{Heck}_d$. They are equivalent in the following way.

PROPOSITION 2.6.9. *Let $p \in Y$ and $\Lambda = d \cdot p$. Let $0 \leq n \leq \frac{d-3}{2}$ and $(a, b) \in V_n \subset \mathsf{Heck}_d$. Let $L \in \mathsf{Prym}_\Lambda(Y)$, choose a frame s of L in neighbourhood of p and let $\alpha = as_+ + bs_- \in H^0(\Lambda, (E, \hat{\sigma}))$. Then*

$$(\hat{E}_L^{(\Lambda,\alpha)}, \hat{\Phi}_L^{(\Lambda,\alpha)}) \in \mathcal{S}_{2n\,p},$$

its image under $\mathsf{Eig}_{2n\ p}$ is $L(n\ p)$ and the u-coordinate defined in Proposition 2.4.6 is given by

$$u_n(\alpha) \in \mathbb{C}^{\frac{1}{2}(d-2n-1)} \subset \mathbb{P}^{\frac{1}{2}(d-2n-1)},$$

whit u_n defined in Lemma 2.6.3.

PROOF. The Higgs field of the Hecke transformation at p is given by

$$\hat{\Phi}_L = \hat{\Phi}_L^{(\Lambda,\alpha)} = \hat{\psi}_{01}^{-1} \Phi_L \hat{\psi}_{01} = \begin{pmatrix} \frac{a}{b} z^d & b^2 - a^2 \\ \frac{z^{2d}}{b^2} & -\frac{a}{b} z^d \end{pmatrix} \mathrm{d}z \tag{10}$$

respective the induced frame on $\hat{E}^{(\Lambda,\alpha)}$. A section of the eigen bundle L at p is given by

$$s = \begin{pmatrix} b + a \\ z^d b^{-1} \end{pmatrix}.$$

Let $s = z^{\mathsf{ord}_p(a+b)} \tilde{s}$, then $\tilde{s}$ defines a non-vanishing section of the eigen line bundle $\hat{L} = \mathsf{ker}(\hat{\Phi}_L - \lambda \mathrm{id}_{\hat{E}_L})$. In particular, $\hat{L} = L(np)$ and

$$(\hat{E}_L^{(\Lambda,\alpha)}, \hat{\Phi}_L^{(\Lambda,\alpha)}) \in \mathcal{S}_{2n\,p}.$$

To compute the u-coordinate at p, let us first assume that n is even, i. e. $\mathsf{ord}_p(a+b) = \mathsf{ord}_p(b)$. Then

$$\tilde{s} = \begin{pmatrix} \tilde{b} + \tilde{a} \\ z^{d-2n}\tilde{b}^{-1} \end{pmatrix}$$

with $\tilde{a}$ an odd polynomial of degree $d-n-2$ and $\tilde{b}$ a non-vanishing even polynomial of degree $d-n-1$. The sections $s_\pm$ are given by

$$s_+ = \tilde{s} + \sigma^*\tilde{s} = \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix}, \qquad s_- = \tilde{s} - \sigma^*\tilde{s} = \begin{pmatrix} \tilde{a} \\ z^{d-n}\tilde{b}^{-1} \end{pmatrix}.$$

Hence, the u-coordinate as defined in Proposition 2.4.6 is given by

$$u = \frac{\tilde{a}}{\tilde{b}} \mod z^{d-2n}.$$

Respective the basis $z, z^3, \dots, z^{d-2n-2}$, this exactly gives the coordinates u_n defined in Lemma 2.6.3. When n is odd, a similar consideration gives the result. $\square$

Let $D \in \mathsf{Div}^+(X)$ be a σ-Higgs divisor associated to $\mathcal{M}^\sigma_\lambda$. Define

$$\mathsf{Heck}_D := \bigtimes_{p \in \mathsf{supp}(D)} \mathsf{Heck}_{D_p}.$$

Proposition 2.6.10. *Consider $\mathcal{M}^\sigma_\lambda$, such that $Z(\lambda) = \mathsf{Fix}(\sigma)$. Then the map $\mathsf{Eig}_{\mathrm{tw}} : \mathcal{M}^\sigma_\lambda \to \mathsf{Prym}(Y)$ is a topological fiber bundle with fibers given by the compact moduli of Hecke parameters Heck_D/G_D.*

Proof. By definition, it is clear that $\mathsf{Eig}_{\mathrm{tw}}$ is continuous on each stratum. From Proposition 2.6.9, it is continuous under the degeneration from one stratum to another. Let U a union of neighbourhoods of $Z(\lambda)$ and $V \subset \mathsf{Prym}_\Lambda(Y)$ open such that there exists a local frame of the universal bundle

$$s : U \times V \to \mathcal{L}$$

(cf. proof of Proposition 2.4.7). By applying Hecke transformations we obtain a commuting diagram

$$\begin{tikzcd} \mathsf{Heck}_D/G_D \times V \arrow[r, "\mathsf{Heck}"] \arrow[d, "\mathrm{pr}_2"'] & \mathsf{Eig}_{\mathrm{tw}}^{-1}(V) \arrow[dl, "\mathsf{Eig}_{\mathrm{tw}}"] \\ V & \end{tikzcd}$$

The identification of u-coordinates in the previous proposition shows that this map is bijective. $\square$

Following paragraph 2.49 in [KK83], an analytic subset of a complex space is called reducible, if it is the union of proper analytic subsets. Let X a complex space and $\mathrm{Sing}(X)$ the singular set, then a irreducible component $Z \subset X$ is defined as the closure of a connected component $X \setminus \mathrm{Sing}(X)$. An irreducible component is an irreducible analytic subset.

Corollary 2.6.11. *The space $\mathcal{M}^\sigma_\lambda$ with $Z(\lambda) = \mathsf{Fix}(\sigma)$ is an irreducible complex space. In particular, it is connected.*

PROOF. The space of Hecke parameters of the highest stratum is connected. The twisted Prym variety $\mathsf{Prym}_\Lambda(Y)$ is connected as long as there exists a branch point of p (see Proposition 1.4.8). Furthermore the closure of the highest stratum is the whole singular Hitchin fiber by Theorem 2.6.5 and the previous proposition. In particular, the set of non-singular points is connected and hence $\mathcal{M}_\lambda^\sigma$ is irreducible. □

REMARK 2.6.12. We want to point out that the connectedness was already shown in [GO13].

THEOREM 2.6.13. *The map* $\mathsf{Eig}_{\mathrm{tw}} : \mathcal{M}_\lambda^\sigma \to \mathsf{Prym}_\Lambda(Y)$ *is holomorphic. In particular, the compact moduli of Hecke parameters* $\mathsf{Heck}_\Lambda/G_\Lambda$ *is a complex space.*

PROOF. We will use a version of the Riemann extension theorem for complex spaces to prove the theorem. To do so, we have to reduce the problem to codimension 2. Lets again assume that there is only one higher order zero of λ. We saw in Proposition 2.6.4 that an open neighbourhood N_1^0/G_Λ of the first stratum V_1 in the zeroth stratum V_0 can be identified with an open non-singular toric subspace of a weighted projective space $\mathbb{P}(1,1,2,\dots,n)$. Gluing this open subset to V_0, we obtain a complex manifold $V = V_0 \cup V_1$ of Hecke parameters of the zeroth and first stratum. We can build a holomorphic fibre bundle F_{01}

$$V \to F_{01} \to \mathsf{Prym}_\Lambda(Y)$$

by choosing local frames of $L \in \mathsf{Prym}_\Lambda(Y)$ around $Z(\lambda)$. Through Hecke transformations, we obtain an analytic map to $\mathcal{S}_0 \cup \mathcal{S}_1$, such that the following diagram commutes

$$\begin{tikzcd} F_{01} \arrow[r, "\mathsf{Heck}"] \arrow[d] & \mathcal{S}_0 \cup \mathcal{S}_1 \arrow[dl, "\mathsf{Eig}_{\mathrm{tw}}"] \\ \mathsf{Prym}_\Lambda(Y) & \end{tikzcd}$$

Hence $\mathsf{Eig}_{\mathrm{tw}}$ is holomorphic on $\mathcal{S}_0 \cup \mathcal{S}_1$.

To extend it, we use the Riemann extension theorem (Thm. I.12.13 in [Gra+94]) for reduced locally pure dimensional complex spaces. By Theorem 2.6.11, $\mathcal{M}_\lambda^\sigma$ is an irreducible complex space. Furthermore, the Hitchin map is flat and therefore its fibres are locally pure dimensional (see Thm. II.2.13 in [Gra+94]). Let $p \in \mathcal{M}_\lambda^\sigma \setminus (\mathcal{S}_0 \cup \mathcal{S}_1)$. For a small neighbourhood $U \subset \mathcal{M}_\lambda^\sigma$ of p, we can choose coordinate functions

$$f : V \subset \mathsf{Prym}_\Lambda(Y) \to \mathbb{C}^{\dim \mathsf{Prym}_\Lambda},$$

such that $\mathsf{Eig}_{\mathrm{tw}}(U \cap (\mathcal{S}_0 \cup S_1)) \subset V$. Then $f \circ \mathsf{Eig}_{\mathrm{tw}}$ define holomorphic functions on U away from a analytic subset of codimension 2. By the extensions theorem they extend to U meromorphically. We already showed that Eig_{tw} as defined in Definition 2.6.7 is a continuous extension. Hence $\mathsf{Eig}_{\mathrm{tw}}$ is holomorphic. □

In conclusion, we obtain the following description of singular $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers with locally irreducible spectral curve.

THEOREM 2.6.14. *Let $a_2 \in H^0(X, M^2)$ with only zeroes of odd order. Then* $\mathsf{Hit}^{-1}(a_2)$ *is holomorphic fiber bundle*

$$\mathsf{Heck}_\Lambda/G_\Lambda \to \mathsf{Hit}^{-1}(a_2) \to \mathsf{Prym}_\Lambda(\tilde{\Sigma}).$$

In particular, the singular Hitchin fiber is an irreducible complex space.

2.6.5. The first degenerations.

Zeroes of order 3 Let $a_2 \in H^0(X, M^2)$ with one zero of order 3, such that all other zeroes are simple. In this case, $G_\Lambda \cong \mathbb{C}^*$ is reductive and it is easy to see that the compact moduli of Hecke parameters is given by

$$\mathsf{Heck}_3/\mathbb{C}^* \cong \mathbb{P}^1.$$

So as a direct consequence of Theorem 2.6.13, we obtain:

COROLLARY 2.6.15. *Let $a_2 \in H^0(X, M^2)$ with k zeroes of order 3, such that all other zeroes are simple. Then the singular Hitchin fiber is a holomorphic fiber bundle*

$$(\mathbb{P}^1)^k \to \mathsf{Hit}^{-1}(a_2) \xrightarrow{\mathsf{Eig}_{\mathsf{tw}}} \mathsf{Prym}_\Lambda(\tilde{\Sigma}).$$

In particular, $\mathsf{Hit}^{-1}(a_2)$ *is a toric complex space.*

This example is sketched in Figure 2.

Zeroes of order 5 Let us now consider $a_2 \in H^0(X, M^2)$ with zeroes of order 5.

PROPOSITION 2.6.16. *The compact moduli of Hecke parameters* Heck_5/G_D *is a toric complex space normalised by* $\mathbb{P}(1,1,2)$.

PROOF. In Proposition 2.6.4, we defined an isomorphism from the

$$N_1^0 \to \mathbb{P}(1,1,2) \setminus \{(y_0 : 0 : y_2)\}.$$

Its inverse is given by

$$\mathbb{P}(1,1,2) \setminus \{(y_0 : 0 : y_2)\} \to \mathsf{Heck}_d/G_\Lambda, \quad (y_0 : y_1 : y_2) \mapsto \begin{pmatrix} y_1^2 z \\ y_0 y_1 + y_2 z^2 \end{pmatrix}.$$

This map naturally extends to $(0 : 0 : 1)$ by mapping it onto V_2 consisting of a single point. If $y_0 \neq 0$, the image lies in V_0 and

$$(u_0 \circ \psi)(y_0 : y_1 : y_2) = \frac{y_1}{y_0} z + \frac{y_2}{y_0^2} z^3.$$

Therefore, it extends holomorphically to $y_0 \neq 0, y_1 = 0$. The map is biholomorphic away from the point in the lowest stratum, which is a fixed point of the full-dimensional torus action on $\mathbb{P}(1,1,2)$. Hence, we can pushforward the torus action to the moduli of Hecke parameters. □

COROLLARY 2.6.17. *Let $a_2 \in H^0(X, M^2)$ with k zeroes of order 3 and l zeroes of order 5, such that all other zeroes are simple. Then, up the normalization,* $\mathsf{Hit}^{-1}(a_2)$ *is a holomorphic fiber bundle*

$$(\mathbb{P}^1)^k \times (\mathbb{P}(1,1,2))^l \to \mathsf{Hit}^{-1}(a_2) \xrightarrow{\mathsf{Eig}_{\mathsf{tw}}} \mathsf{Prym}_\Lambda(\tilde{\Sigma}).$$

In particular, $\mathsf{Hit}^{-1}(a_2)$ *is a toric complex space.*

2.7. Singular fibers with irreducible spectral curve

When there exist points of the spectral curve, where it is locally reducible, the singular Hitchin fibers do not fiber over twisted Prym varieties. However, we can still describe the degeneration to lower strata using Hecke transformations. In Section 2.4, it was more convenient to parametrize the extra data at even zeroes with extension data. We will reinterpret these extra data as Hecke parameters now.

Fix $\mathcal{M}_\lambda^\sigma$, such that $\{y, \sigma(y)\} = Z(\lambda) \setminus \mathsf{Fix}(\sigma)$ and all other zeroes of λ are simple. Let D be an associated σ-Higgs divisor. Let $L \in \mathsf{Prym}_{\Lambda - D}$ and $(E, \Phi) \in \mathcal{M}_\lambda^\sigma$ obtained from (E_L, Φ_L) by applying the unique Hecke transformation at all simple zeroes. Choose frames $s_1 \in H^0(U, L)$, $s_2 \in H^0(U, \sigma^* L)$ for a neighbourhood U of y and let

$$s_+ := s_1 \oplus s_2, \quad s_- := s_1 \oplus -s_2.$$

the induced frame of $E|_U = (L \oplus \sigma^* L)|_U$.

Proposition 2.7.1. *Let $l = (\Lambda - D)_y$ and $\alpha = as_+ + bs_- \in H^0(ly, E)^*$, such that $a(0) \neq \pm b(0)$. Then*

$$\left(\hat{E}^{(y+\sigma y, \alpha + \sigma^* \alpha)}, \hat{\Phi}^{(y+\sigma y, \alpha+\sigma^*\alpha)}\right) \in S_D \subset \mathcal{M}_\lambda^\sigma$$

and the extension datum at y introduced in Proposition 2.4.10 is given by

$$\left[\frac{b+a}{b-a}\ \hat{s}_1^2 \ \mathrm{d}z\right] \in H^0(ly, L^2 K).$$

Proof. This is a local computation from the description of the Higgs field given in (10). $\square$

From this description, we see that for the Hecke parameters at even zeroes of the quadratic differential, there are two different ways to degenerate to lower strata:

i) By degenerating to lower strata in the moduli spaces of Hecke parameters, i.e. allowing α to vanish. Here the eigenline bundle of the limit point is twisted by a σ-invariant divisor $D + \sigma^* D$.
ii) By imposing
$$a \equiv b \mod z^l \quad \text{or} \quad a \equiv -b \mod z^l$$
for some $l \leq \mathsf{ord}_y(\lambda)$, while $a(0),\ b(0) \neq 0$. In this case, the eigen line bundle of the limit point is twisted by divisors ly or $l\sigma(y)$, respectively.

Consonant with the previous section, we can find a compactification of the Hecke parameters of the highest stratum by allowing Hecke parameters in $\alpha \in H^0(\Lambda_y y, E)$. Define

$$\mathsf{Heck}_{\Lambda_y} := H^0(\Lambda_y \cdot y, E)/\sim,$$

where $\sim$ denotes the analogue of relation ii) of Lemma 2.6.2. Along the lines of Section 2.6.3, we can study the quotient of $\mathsf{Heck}_{\Lambda_y}$ by the non-reductive group action of $H^0(\Lambda_y\, y, \mathcal{O}_Y^*)$ and obtain a topological model by gluing toric subsets of weighted projective spaces. Following Section 2.6.4 one proves:

THEOREM 2.7.2. *Let $a_2 \in H^0(X, M^2)$ with one zero $x \in X$ of order $2d$, such that all other zeroes are simple. Let $\tilde{\pi}^{-1}(x) = \{y, \sigma y\}$ and $L \in \mathsf{Prym}_\Lambda(\tilde{\Sigma})$. Denote by $(E, \Phi) \in \tilde{\pi}^*\mathsf{Hit}^{-1}(a_2)$ the unique σ-invariant Higgs bundle obtained by applying Hecke transformations to (E_L, Φ_L) at the simple zeroes of a_2. There is a continuous injective map*

$$T_L : \mathsf{Heck}_d / H^0(dy, \mathcal{O}_Y^*) \to \tilde{\pi}^*\mathsf{Hit}^{-1}(q_2),$$

defined by applying Hecke transformations to (E, Φ) at $x \in X$. Its image is the closure of $\mathsf{Eig}_0^{-1}(L)$ in $\mathsf{Hit}^{-1}(a_2)$ and is given by

$$\bigcup_{l_1 + l_2 \leq d} \mathsf{Eig}^{-1}_{D(l_1, l_2)}(L(l_1 y + l_2 \sigma y))$$

with $D(l_1, l_2) = (l_1 + l_2)y + (l_1 + l_2)\sigma y \in \mathsf{Div}^+(Y)$.

PROOF. This theorem is proven by adapting Proposition 2.6.9 to Hecke transformations at the even zeroes of a_2. This allows to compute the eigen line bundles of the limit points determining the image of T_L. □

Define F_{a_2} as the topological fiber bundle over Prym_Λ with fibers given by the moduli of Hecke parameters (cf. Proposition 2.4.7). We can define a continuous map $T : F_{a_2} \to \mathsf{Hit}^{-1}(a_2)$ by applying T_L fiberwise. However, as we will see below, this map is not anymore injective. It has the property that it makes the following diagram commute

$$\begin{tikzcd} T^{-1}(\mathcal{S}_0) \arrow[r, "T"] \arrow[d] & \mathcal{S}_0 \arrow[dl, "\mathsf{Eig}_0"] \\ \mathsf{Prym}_\Lambda & \end{tikzcd}$$

But there is no way to extend the fibering to the whole singular fiber. This was already encountered in [GO13] and [Hit19]. To illustrate why these two properties fail, we describe the case of zeroes of order 2.

EXAMPLE 2.7.3 (Zeros of order 2). Let $a_2 \in H^0(X, M^2)$ having a zero $x \in X$ of order two, such that all other zeroes are simple. Let $\{y, \sigma y\} = \tilde{\pi}^{-1}(x)$. The compact moduli of Hecke parameters at x is given by

$$\left(H^0(y, E_L) \setminus \{0\}\right) / H^0(y, \mathcal{O}_Y^*) = \left(\mathbb{C}^2 \setminus \{0\}\right) / \mathbb{C}^* = \mathbb{P}^1.$$

In this case, the stratification of Hecke parameters by vanishing order is trivial. However, from Theorem 2.5.2 the stratification of $\mathsf{Hit}^{-1}(a_2)$ has two strata. One, where the Higgs field is non-vanishing and one, where it is diagonalizable and vanishes at x of order 1.

Let $L \in \mathsf{Prym}_\Lambda(Y)$ and $\alpha \in F_{q_2}|_L$. Let $U \subset X$ a neighbourhood of x. Choosing frames $s_1 \in H^0(U, L)$ and $s_2 \in H^0(U, \sigma^* L)$, the Hecke parameter α can be written as $\alpha = as_+ + bs_-$. Then the Higgs field of $T(L, \alpha)$ is given by formula (10) in Section 2.6.4. Hence,

$$T(L, \alpha) \in S_0 \quad \Leftrightarrow \quad a_0 \neq \pm b_0.$$

Furthermore, it is easy to check that for $a_0 = b_0$ the eigen line bundle of the Hecke transformation is given by $L(y)$, whereas for $a_0 = -b_0$ it is given by $L(\sigma y)$. We conclude that for given $L \in \mathsf{Prym}_\Lambda(\tilde{\Sigma})$

$$T\left(L, s_+ + s_-\right) = T\left(L(y - \sigma y)), s_+ - s_-\right).$$

In particular, T is not injective and the fibering can not be extended. However, Theorem 2.7.2 defines a holomorphic map from a holomorphic $\mathbb{P}^1$-bundle over Prym_Λ to the Hitchin fiber, which has a holomorphic inverse on the dense stratum $\mathcal{S}_0$. Hence, the normalisation of the Hitchin fiber $\mathsf{Hit}^{-1}(q_2)$ is a $\mathbb{P}^1$-bundle over a Prym variety.

This is the second example discussed in the Introduction and sketched in Figure 3. A degenerating family of abelian varieties with a singular fiber of this kind was constructed in [Mum72] Section 7.

Remark 2.7.4. With our methods, we can not show that the compact moduli of Hecke parameters at locally reducible singularities of the spectral curve is a complex analytic space. If this is the case, the map T defined above is defining a one-sheeted analytic covering in the language of [Gra+94]. The analogue of a birational morphism in the analytic category.

Example 2.7.5 (Zeroes of order 4). Let $a_2 \in H^0(X, M^2)$, such that there is one zero $x \in X$ of order 4 and all other zeroes are simple. Then up to normalisation the singular Hitchin fiber is given by a holomorphic $\mathbb{P}(1,1,2)$-bundle over $\mathsf{Prym}_\Lambda(\tilde{\Sigma})$.
The proof is similar to the proof of Proposition 2.6.16. Choose a local coordinate (U, z) centred at $y \in \tilde{\pi}^{-1}(x)$ and let

$$\mathsf{Heck}_2 = \left\{ \alpha = \begin{pmatrix} a_0 + a_1 z \\ b_0 + b_1 z \end{pmatrix} \in H^0(2 \cdot y, \mathcal{O}_Y^2) \right\} / \sim,$$

where $\alpha \sim \alpha' \Leftrightarrow a_0 = b_0 = 0$. The compact moduli of Hecke parameters at $x \in X$ is the quotient of Heck_2 by $G_2 := H^0(2y, \mathcal{O}_Y^*)$. We defined holomorphic G_2-invariant functions

$$\mathsf{Heck}_2 \setminus \{[0]\} \to \mathbb{P}(1,1,2) \setminus \{(0:0:1)\}, \quad \alpha \mapsto (b_0 : a_0 : b_1 a_0 - b_0 a_1).$$

An inverse is given by

$$\begin{aligned} \Psi : \mathbb{P}(1,1,2) \setminus \{(0:0:1)\} &\to (\mathsf{Heck}_2 \setminus \{[0]\})/G_2, \\ (y_0 : y_1 : y_2) \quad &\mapsto \begin{bmatrix} y_1^2 \\ y_0 y_1 + y_2 z \end{bmatrix}. \end{aligned}$$

Ψ extends to $(0 : 0 : 1)$ by mapping it to $[0] \in \mathsf{Heck}_2/G_2$. The holomorphic transition functions of $\mathcal{S}_0$ define a holomorphic fiber bundle

$$\mathbb{P}(1,1,2) \to F \to \mathsf{Prym}_\Lambda(\tilde{\Sigma})$$

with a holomorphic map $T \circ \Psi : F \to \mathsf{Hit}^{-1}(a_2)$. The $\mathbb{P}(1,1,2)$-bundle over $\mathsf{Prym}_\Lambda(\tilde{\Sigma})$ is normal and $T \circ \Psi$ is biholomorphic on the highest stratum. Hence, this is the normalisation of $\mathsf{Hit}^{-1}(a_2)$.

Corollary 2.7.6. *Let $a_2 \in H^0(X, M^2)$ with at least one zero of odd order, then the $\mathsf{Hit}^{-1}(a_2)$ is irreducible.*

PROOF. Theorems 2.6.10 and 2.7.2 show that $\overline{\mathcal{S}_0} = \mathsf{Hit}^{-1}(a_2)$. Furthermore, $\mathcal{S}_0$ is a $\mathbb{C}^* \times \mathbb{C}^n$-bundle over a twisted Prym variety and hence irreducible by Proposition 1.4.8. In particular, the smooth points of $\mathsf{Hit}^{-1}(a_2)$ are connected and hence $\mathsf{Hit}^{-1}(a_2)$ is irreducible. □

2.7.1. Singular fibers with irreducible, locally reducible spectral curve. We encountered above, that the case of $a_2 \in H^0(X, M^2)$ with only zeroes of even order is very special. This is due the fact that the normalised spectral cover is unbranched. There are two major differences to the cases consider before:

i) The pullback
$$\pi^* : \mathcal{M}_G(X, M) \to \mathcal{M}_G(\tilde{\Sigma}, \pi^* M)$$
is not injective (Proposition 2.2.12).

ii) The twisted Prym variety
$$\mathsf{Prym}_N(\tilde{\Sigma}) = \mathsf{Nm}^{-1}(N^{-1}) \subsetneq \{L \in \mathsf{Pic}(Y) \mid L \otimes \sigma^* L = N^{-1}\}$$
and has two connected components (see Lemma 1.4.12).

The local description of Higgs bundles at the singularities is independent of the global properties of the spectral curve. Hence, the non-abelian spectral data of these Hitchin fibers is given by the moduli of Hecke parameters at even zeroes considered in the previous section.

THEOREM 2.7.7. *Let $a_2 \in H^0(X, M^2)$, such that all zeroes of a_2 have even order. Let I be the unique line bundle on X, such that $\tilde{\pi}^* I = \mathcal{O}_{\tilde{\Sigma}}$. Then there is a stratification*
$$\mathsf{Hit}^{-1}(a_2) = \bigcup_D \mathcal{S}_D$$
by locally closed analytic sets $\mathcal{S}_D$ indicated by Higgs divisors D. Each stratum is a two-sheeted covering of a holomorphic fiber bundle
$$(\mathbb{C}^*)^r \times \mathbb{C}^s \to \mathcal{S}_D \to \mathsf{Prym}_{I(N)}(\tilde{\Sigma}),$$
where
$$r = \#Z(a_2), \quad s = \deg(M) - \deg(D) - \#Z(a_2), \quad \textit{and} \quad N = \tfrac{1}{2}\mathsf{div}(a_2) - D.$$
The dimension of a stratum $\mathcal{S}_D$ is given by
$$\deg(M) + g - 1 - \deg(D).$$

PROOF. Following the receipt of Section 2.4, the pullback defines an eigen line bundle
$$\mathcal{O}(L) = \mathsf{ker}(\tilde{\pi}^*\Phi - \lambda \mathrm{id}_{\tilde{\pi}^* E}).$$
Choosing frames of L at $\tilde{\pi}^* Z(a_2)$, we obtain u-coordinates in $(\mathbb{C}^*)^r \times \mathbb{C}^s$ determining the Hecke transformation
$$0 \to L \oplus \sigma^* L \to \tilde{\pi}^* E \to \mathcal{T}_{\tilde{\Sigma}}(\tilde{\pi}^* N) \to 0.$$
We are left with showing that $L \in \mathsf{Prym}_{I(N)}(\tilde{\Sigma})$. Recall that for $p \in Z(a_2)$ the Hecke transformation of $\tilde{\pi}^* E$ is done σ-invariantly at the preimage $\tilde{\pi}^{-1}(p) = \{p_1, p_2\} \subset Y$. In particular, it define a Hecke transformation $\hat{E}$ of E at $Z(a_2) \subset X$
$$0 \to \hat{E} \to E \to \mathcal{T}_X(N) \to 0,$$

such that $(\hat{E}, \hat{\Phi})$ is everywhere locally diagonalizable on X. It is clear, that

$$\tilde{\pi}^*(\hat{E}, \hat{\Phi}) = (L \oplus \sigma^* L, \operatorname{diag}(\lambda, -\lambda)).$$

Hence, either $\pi_*^\sigma L = \hat{E}$ or $\pi_*^\sigma L = \hat{E} \otimes I$ by Lemma 2.2.8. In both cases, $\det(\pi_*^\sigma L) = \det(\hat{E}) = \mathcal{O}(-N)$. Furthermore, $\det(\pi_*^\sigma \mathcal{O}_{\tilde{\Sigma}}) = I$ by Corollary 2.3.11. So by Lemma 3.2.3

$$\mathcal{O}(-N) = \mathcal{O}(\mathsf{Nm} L) \otimes \det(\pi_*^\sigma \mathcal{O}_{\tilde{\Sigma}}) = \mathcal{O}(\mathsf{Nm} L) \otimes I$$

and hence $L \in \mathsf{Prym}_{I(N)}(\tilde{\Sigma})$.

Reversing this construction, we can construct a σ-invariant Higgs bundle $(E, \Phi, \hat{\sigma})$ for all choices of $L \in \mathsf{Prym}_{I(N)}(\tilde{\Sigma})$ and u-coordinates in $(\mathbb{C}^*)^r \times \mathbb{C}^s$. Hence, $\tilde{\pi}^* \mathsf{Hit}^{-1}(a_2)$ is isomorphic to a holomorphic $(\mathbb{C}^*)^r \times \mathbb{C}^s$-fiber bundle over $\mathsf{Prym}_{I(N)}(\tilde{\Sigma})$. By Proposition 2.2.12, the stratum $\mathcal{S}_D$ is a two-sheeted branched covering over this fiber bundle. □

In contrast, to Corollary 2.7.6, there are two irreducible components.

Theorem 2.7.8. *Let $a_2 \in H^0(X, M^2)$, such that all zeroes of a_2 have even order. Then the singular Hitchin fibers $\mathsf{Hit}^{-1}(a_2)$ is connected and reducible.*

Proof. We showed in Proposition 2.2.12, that $\mathsf{Hit}^{-1}(a_2)$ is a branched two-to-one covering of $\tilde{\pi}^* \mathsf{Hit}^{-1}(a_2)$. Example 2.2.13 shows that on can always find a branch point in the lowest stratum. In particular, we can conclude connectedness of the singular fiber from the connectedness of $\tilde{\pi}^* \mathsf{Hit}^{-1}(a_2)$.

Let $L \in \mathsf{Prym}_{I(\mathsf{div}(a_2))}(\tilde{\Sigma})$ and $p \in \tilde{\pi}^{-1} Z(a_2)$. As we saw in Theorem 2.7.2, we can degenerate from $\mathsf{Eig}_0^{-1}(L) \subset \mathcal{S}_0$ and $\mathsf{Eig}_0^{-1}(L(p - \sigma(p))) \subset \mathcal{S}_0$ to one and the same point in the lower stratum. Moreover, we saw in the proof of Proposition 1.4.9, that we change the connected component of the twisted Prym variety by tensoring with $\mathcal{O}(p - \sigma(p))$ for $p \in \tilde{\Sigma}$. Hence, $\mathsf{Hit}^{-1}(a_2)$ is connected.

However, the fiber is reducible, because $\mathcal{S}_0$ is disconnected. It decomposes by restricting the $\mathbb{C}^* \times \mathbb{C}^n$-bundle to the two connected components of $\mathsf{Prym}_{I(\mathsf{div}(a_2))}(\tilde{\Sigma})$. The closures of these two connected components of $\mathcal{S}_0$ define two irreducible components of $\tilde{\pi}^* \mathsf{Hit}^{-1}(a_2)$. □

2.8. Real points in singular Hitchin fibers

In this section, we are going to study K-twisted $\mathsf{SL}(2, \mathbb{R})$-Higgs bundles with irreducible and reduced spectral curve. We will show that for each stratum they are parametrised by the two-torsion points of the Prym variety and a discrete choice of Hecke parameters at the even zeroes of the quadratic differential. The result for regular Hitchin fibers was considered in [Sch13; Peo13].

A line bundle $L \in \mathsf{Prym}(\tilde{\Sigma})$ is a two-torsion point, if $L^2 \cong O_X$. Under the Prym condition, this is equivalent to $\sigma^* L \cong L$. We call $L \in \mathsf{Prym}_N(\tilde{\Sigma})$ σ-symmetric, if $\sigma^* L \cong L$. Choosing a σ-symmetric base point for the simply transitive action of $\mathsf{Prym}(\tilde{\Sigma})$ on $\mathsf{Prym}_N(\tilde{\Sigma})$ the two-torsion points are bijectively mapped on the σ-symmetric points. Recall that the definition of σ-invariant holomorphic line bundle required the lift $\hat{\sigma}$ to restrict to the identity at all ramification points (cf. Definition 2.2.1).

THEOREM 2.8.1. *Let $q_2 \in H^0(X, K^2)$ a quadratic differential, such that all zeroes have odd order and $D \in \mathsf{Div}(X)$ an associated Higgs divisor. Then the $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles in $\mathcal{S}_D \subset \mathsf{Hit}^{-1}(q_2)$ are parametrised by the σ-symmetric points of $\mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma})$.*

PROOF. Let $N := \Lambda - \tilde{\pi}^* D$ and $L \in \mathsf{Prym}_N(\tilde{\Sigma})$ such that there exists an isomorphism $\phi : \sigma^* L \to L$. ϕ is unique up to $\pm\mathrm{id}$ and restricts to ± 1 at each $p \in \mathsf{Fix}(\sigma) = \tilde{\pi}^{-1} Z(q_2)$. Choose a frame $s \in H^0(U, L)$ at $p \in \mathsf{Fix}(\sigma)$, such that $s = \pm\phi(\sigma^* s)$ for $\phi_p = \pm 1$ respectively. Such frame is uniquely defined up to multiplying by an σ-invariant holomorphic function and therefore defines a unique u-coordinate (cf. Proposition 2.4.6). The induced frame s_+, s_- defines a global splitting

$$(E_L, \Phi_L) = \left(L \oplus L, \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \right).$$

We decompose $N = N_+ + N_-$, such that

$$\mathsf{supp} N_\pm = \{ p \in \mathsf{Fix}(\sigma) \mid \phi|_p = \pm\mathrm{id} \}.$$

The Hecke transformation of (E_L, Φ_L) at N in direction $u = 0$ is given by

$$(\hat{E}_L, \hat{\Phi}_L) = \left(L(N_-) \oplus L(N_+), \begin{pmatrix} 0 & \frac{\lambda\eta_-}{\eta_+} \\ \frac{\lambda\eta_+}{\eta_-} & 0 \end{pmatrix} \right)$$

with $\eta_\pm \in H^0(\tilde{\Sigma}, \mathcal{O}(N_\pm))$ canonical. The induced lift of σ to $L(N_\pm)$ is the identity at all ramification points. Hence, $(\hat{E}_L, \hat{\Phi}_L)$ descends to a $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle on X. If we choose $-\phi$ in the beginning the role of $N_\pm$ are interchanged and we obtain a $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle isomorphic over $\mathsf{SL}(2,\mathbb{C})$.

For the converse, consider a $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle

$$(E, \Phi) = \left(L \oplus L^{-1}, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right) \in \mathcal{S}_D.$$

There are divisors $N_\pm \in \mathsf{Div}^+(\tilde{\Sigma})$, such that

$$\tilde{\pi}^* \alpha = \frac{\lambda\eta_-}{\eta_+}, \quad \tilde{\pi}^* \beta = \frac{\lambda\eta_+}{\eta_-}$$

for $\eta_\pm \in H^0(\tilde{\Sigma}, \mathcal{O}(N_\pm))$ canonical. The eigenline bundles are defined by

$$(\tilde{\pi}^* L)(-N_-) \xrightarrow{(\eta_-, \pm\eta_+)} \tilde{\pi}^* L \oplus \tilde{\pi}^* L^{-1}. \tag{11}$$

and correspond to a σ-symmetric point of $\mathsf{Prym}_N(\tilde{\Sigma})$. Furthermore, the induced isomorphism

$$\phi : \sigma^*((\tilde{\pi}^* L)(-N_-)) \to (\tilde{\pi}^* L)(-N_-)$$

is -1 at $\mathsf{supp} N_-$. So we recover (E, Φ) with the construction in the first part of the proof. $\square$

EXAMPLE 2.8.2. The pullback $\tilde{\pi}^* K^{-\frac{1}{2}} \in \mathsf{Prym}_\Lambda$ is σ-symmetric. The corresponding $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle is the image of the Hitchin section

$$\left(K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}, \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix}\right) \in \mathcal{S}_0.$$

More generally, if $\deg(D) \equiv 0 \mod 2$, there exist line bundles M on X such that $M^2 \cong \mathcal{O}_X(D)$. Then $\tilde{\pi}^*(K^{-\frac{1}{2}} M) \in \mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}$ is σ-symmetric. The corresponding $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle is of the form

$$\left(K^{-\frac{1}{2}} M \oplus K^{\frac{1}{2}} M^{-1}, \begin{pmatrix} 0 & \eta \\ \frac{q_2}{\eta} & 0 \end{pmatrix}\right)$$

with $\eta \in H^0(X, \mathcal{O}(D))$ canonical. These are the only $\mathsf{SL}(2,\mathbb{R})$-Higgs coming from σ-invariant line bundles.

COROLLARY 2.8.3. *Let $q_2 \in H^0(X, K^2)$ be a quadratic differential, such that all zeroes have odd order. Then $\mathsf{Hit}^{-1}(q_2)$ contains*

$$2^{2g-2} \prod_{p \in Z(q_2)} (\mathsf{ord}_p(q_2) + 1)$$

$\mathsf{SL}(2,\mathbb{R})$*-Higgs bundles.*

PROOF. By the previous theorem, every stratum contains 2^{2g-2-n} $\mathsf{SL}(2,\mathbb{R})$-points, where n is the number of zeroes. At a zero $p \in Z(q_2)$, we have $\frac{\mathsf{ord}_p(q_2)+1}{2}$ possible values for D and hence there are

$$\prod_{p \in Z(q_2)} \tfrac{1}{2}(\mathsf{ord}_p(q_2) + 1)$$

different strata. □

EXAMPLE 2.8.4. The regular fibers contain 2^{6g-6} real points. If we have one triple zero and all other zeroes are simple, we have 2^{6g-8} of them. If we have $g-1$ triple and $g-1$ simple zeroes, the number is 2^{5g-5}. For one zero of order $4g-3$ and one simple zero, we have $(2g-1)2^{2g}$ real points. In general, the moduli space of $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles branches over singular locus.

For quadratic differentials with zeroes of even order, there are two Hecke parameters in each stratum leading to $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles. Here, we use the description of the extra data at even zeroes given in Proposition 2.7.1.

THEOREM 2.8.5. *Let $q_2 \in H^0(X, K^2)$ with at least one zero of odd order and $D \in \mathsf{Div}^+(X)$ an associated Higgs divisor. The $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles in the stratum $\mathcal{S}_D \subset \mathsf{Hit}^{-1}(q_2)$ are parametrised by the σ-symmetric points of $\mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma})$ together with a choice of one of two possible Hecke parameters at every even zero p of q_2, where $\frac{1}{2}\mathsf{ord}_p(q_2) \neq D_p$. In particular, each stratum $\mathcal{S}_D$ contains*

$$2^{2g-2+n-n_0}$$

$\mathsf{SL}(2,\mathbb{R})$*-Higgs bundles, where $n = \# Z(q_2)$ and*

$$n_0 = \#\{p \in Z(q_2) \mid \tfrac{1}{2}\mathsf{ord}_p(q_2) = D_p\}.$$

PROOF. Let $L \in \mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma})$, such that there exists an isomorphism $\phi : \sigma^*L \to L$. Fix a choice of $\pm$ at every even zero p, such that $\frac{1}{2}\mathsf{ord}_p(q_2) \neq \tilde{\pi}^*D_p$. Let $Z^e \subset Z(q_2)$ be the set of zeroes of even order and

$$N^{\text{even}} = (\Lambda - \tilde{\pi}^*D)|_{\tilde{\pi}^{-1}Z^e}.$$

Let $N^{\text{even}} = N_+^{\text{even}} + N_-^{\text{even}}$, such that $N_\pm^{\text{even}}$ is supported at the even zeroes assigned a $\pm$ respectively. As seen above, the isomorphism ϕ defines a unique Hecke transformation at $\tilde{\pi}^{-1}(p)$ for all $p \in Z(q_2)$ of odd order. Performing these Hecke transformations, we obtain a σ-invariant Higgs bundle

$$(E, \Phi) = \left(L_1 \oplus L_2, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right)$$

on $\tilde{\Sigma}$ with $L_1 \otimes L_2 \cong \mathcal{O}_{\tilde{\Sigma}}(-N^{\text{even}})$, which is locally diagonalizable over all even zeroes of q_2. If $N^{\text{even}} = 0$, this descends to a $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle on X and we are done. Let $\tilde{p} \in \tilde{\pi}^{-1}Z(q_2)$, such that $N_{\tilde{p}}^{\text{even}} \neq 0$. Choose a frame $s \in H^0(U \cup \sigma(U), L)$ for a neighborhood U of $\tilde{p}$, such that $\phi(\sigma^*s) = s$. Depending on the fixed choice of $\pm$, we define the Hecke parameter $\alpha = [s_\pm]$ respective the induced frame s_+, s_-. By Proposition 2.7.1, this defines a σ-invariant Higgs bundle

$$\left(\hat{E}^{(\tilde{p}+\sigma\tilde{p},\alpha+\sigma^*\alpha)}, \hat{\Phi}^{(\tilde{p}+\sigma\tilde{p},\alpha+\sigma^*\alpha)} \right).$$

Performing Hecke transformations like this over all even zeroes $p \in X$, such that $\frac{1}{2}\mathsf{ord}_p(q_2) \neq \tilde{\pi}^*D_p$, we obtain

$$\left(L_1(N_-^{\text{even}}) \oplus L_2(N_+^{\text{even}}), \begin{pmatrix} 0 & \frac{\alpha\eta_-}{\eta_+} \\ \frac{\beta\eta_+}{\eta_-} & 0 \end{pmatrix} \right).$$

This σ-invariant Higgs bundle descends to a $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle in the desired stratum.

For the converse, let

$$(E, \Phi) = \left(L \oplus L^{-1}, \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \right) \in \mathcal{S}_D.$$

Then

$$\tilde{\pi}^*(E, \Phi) = \left(\tilde{\pi}^*L \oplus \tilde{\pi}^*L^{-1}, \begin{pmatrix} 0 & \frac{\lambda\eta_-}{\eta_+} \\ \frac{\lambda\eta_+}{\eta_-} & 0 \end{pmatrix} \right)$$

for divisors $N_\pm \in \mathsf{Div}^+(\tilde{\Sigma})$ with canonical sections $\eta_\pm$. The eigenline bundles are defined by (11) and define a σ-symmetric element of $\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}$. There is a induced decomposition $N^{\text{even}} = N_+^{\text{even}} + N_-^{\text{even}}$ and hence a choice of $\pm$ for all $p \in Z^e$, where $N_{\tilde{p}}^{\text{even}} \neq 0$, i. e. $\frac{1}{2}\mathsf{ord}_p(q_2) \neq D_p$. □

REMARK 2.8.6. The choice of $\pm$ in the previous theorem actually depends on choosing an isomorphism $\phi : \sigma^*L \to L$. However this isomorphism is unique up to $\pm\mathrm{id}_{\tilde{\Sigma}}$. Choosing $-\phi$ instead of ϕ corresponds to switching all $+$ to $-$ and vice versa. For the $\mathsf{SL}(2,\mathbb{R})$-Higgs bundle, this corresponds to the gauge interchanging the splitting line bundles.

A general formula for the number of real points in a singular Hitchin fiber would be quite complicated. So let us finish by computing this number in some examples.

EXAMPLE 2.8.7. Let $q_2 \in H^0(X, K^2)$ be a quadratic differential with $d < 2g-2$ double zeroes and $4g-4-2d$ simple zeroes. Then the Hitchin fiber contains

$$2^{6g-6-2d} \sum_{k=0}^{d} \binom{d}{k} 2^k$$

real points. Let q_2 be a quadratic differential with one zero of order $2d < 4g-4$ and 4g-4-2d simple zeroes. Then the number is $(4d-3)2^{6g-6-2d}$.

CHAPTER 3

Interlude: Hecke transformations and pushforwards

In the last chapter, we saw that we can recover the pushforward of a line bundle along a two-sheeted covering through the construction of a vector bundle invariant by the involution changing the sheets (see Lemma 2.3.10). For coverings with more than two sheets, this does not work in the same way, because there rarely is a global group action acting transitively on the fibers of the covering. In other words, most coverings are not Galois. However, locally we can still describe the pushforward in this way.

At a ramification point of index $k-1$ there is a local $\mathbb{Z}_k$-action changing the sheets. Choosing a coordinate, such that the covering is given by $z \mapsto z^k$, such local $\mathbb{Z}_k$-action is generated by $\tau : z \mapsto \xi z$, where ξ is a primitive root of unity of order k. Using these local $\mathbb{Z}_k$-actions, we can describe local frames of the pushforward at branch points by constructing τ-invariant frames in the neighbourhoods of the corresponding ramification points via Hecke transformations. This will be very useful for explicit computations. For example to compute the pushforward of a non-degenerate bilinear pairing, which will become important in the following chapter. To this end, we will first introduce Hecke transformations for holomorphic vector bundles of rank > 2. We will use the opportunity to compare this notion to the more general concept of Hecke modifications. Then we will discuss pushforwards describing them in terms of τ-invariant frames. Finally, we will use this result to prove some properties of the pushforward, that can be found in the literature.

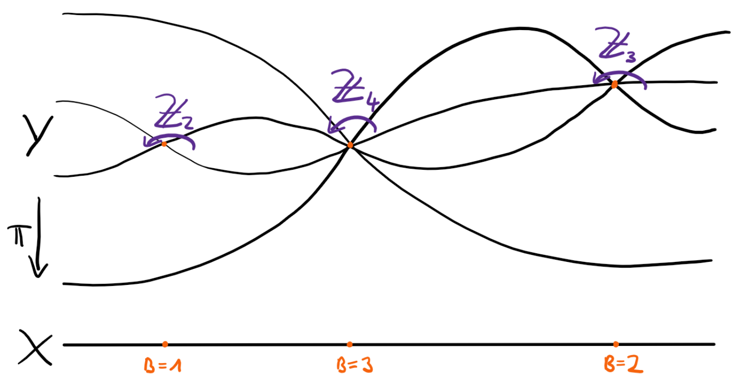

FIGURE 6. Branched covering with local $\mathbb{Z}_k$-actions

3.1. Hecke transformations and Hecke modifications

We introduced Hecke transformations for holomorphic vector bundles of rank 2 in Section 2.3. They were an important tool to study the non-abelian part of the spectral data. Here we shortly introduce this concept for holomorphic vector bundles of higher rank. Similar notions can be found in [HR04; BG10; Hit19].

Let X be a Riemann surface and $x \in X$.

DEFINITION 3.1.1. Let E a holomorphic vector bundle on X. A holomorphic vector bundle $\hat{E}$ is a Hecke transformation of E at x, if there exists an exact sequence

$$0 \to \hat{E} \to E \to \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a torsion sheaf supported at x.

Another way to understand Hecke transformations is by thinking of $\hat{E}$ to be obtain from E by introducing a new transition function ψ_{01} from a neighbourhood U of x to $U \setminus \{x\}$ (using the notation of (1),(2) in Section 2.3). Up to choosing frames on U and $U \setminus \{x\}$, one can always assume ψ_{01} to have Smith normal form

$$\psi_{01} = \begin{pmatrix} z^{l_1} & & \\ & \ddots & \\ & & z^{l_r} \end{pmatrix}, \quad 0 \leq l_1 \leq \cdots \leq l_r,$$

where z is a holomorphic coordinate centred at x. This induces an isomorphism

$$\mathcal{T} = \bigoplus_{i=1}^{r} \mathcal{O}_{X,x} / < z^{l_i} >$$

and hence, $\det(\hat{E}) = \det(E)(-\sum_{i=1}^{r} l_i)$. In the sequel, we will specify a Hecke transformations by giving the transition function ψ_{01} from U to $U \setminus \{x\}$.

REMARK 3.1.2. For $\mathrm{rk}(E) = 2$, we specified a weight and a direction given by a polynomial germ in $E^\vee$ at x in Section 2.3. This uniquely determined a Hecke transformations of a holomorphic vector bundle of rank 2. For higher rank one needs to fix a weighted flag of polynomial germs in $\mathcal{O}(E^\vee)_x$. We will not go further in this direction here. However, one way to obtain more general moduli of Hecke parameters determining the non-abelian part of the spectral data for singular $\mathsf{SL}(n, \mathbb{C})$-Hitchin fibers, $n > 2$, is to study the configuration spaces of such flags.

Pointing in this direction, Furuta and Steer [FS92] used Hecke transformations (without labelling them like this) to obtain a correspondence between isomorphism classes of holomorphic vector V-bundles on a complex 1-dimensional orbifold and parabolic vector bundles on the desingularising Riemann surface.

REMARK 3.1.3. Any extra structure on a holomorphic vector bundle E induces an extra structure on $\hat{E}$. If we fix a Higgs field $\Phi \in H^0(X, \mathrm{End}(E) \otimes M)$, then the Hecke transformations $\hat{E}$ inherits a meromorphic section $\hat{\Phi}$ of $\mathrm{End}(E) \otimes M$. Respective the induced frame on U, it will be given by

$$\hat{\Phi}|_U = \psi_{01}^{-1} \Phi|_U \psi_{01}.$$

In general, it will not be holomorphic. However, as we seen above, if Φ vanishes at $x \in X$, then there are Hecke transformations of E, such that the resulting Higgs field $\hat{\Phi}$ is holomorphic.

Similarly, a hermitian metric h or a bilinear form ω on E induces a hermitian metric $\hat{h}$ respectively a bilinear form $\hat{\omega}$ on $\hat{E}$. In general, they are again singular at $x \in X$. It is straight forward to compute $\hat{h}$ respectively $\hat{\omega}$ at $x \in X$ using the transition function ψ_{01}.

Another generalisation is the notion of Hecke modification (see [Bap10; Won13; HW19]).

Definition 3.1.4. Let E be a holomorphic vector bundle on X. A Hecke modification $(\hat{E}, s)$ of E at $x \in X$ is a holomorphic vector bundle $\hat{E}$ with an isomorphism

$$\hat{E}|_{X\setminus\{x\}} \xrightarrow{s} E|_{X\setminus\{x\}},$$

that extends meromorphically over x with respect to trivialisations.

Clearly Hecke transformations are Hecke modifications, but the converse is false. For a Hecke modification, the Smith normal form

$$s = \begin{pmatrix} z^{l_1} & & \\ & \ddots & \\ & & z^{l_r} \end{pmatrix}, \qquad l_1 \leq \cdots \leq l_r$$

can have exponents $l_i \in \mathbb{Z}$. So in general, neither s nor s^{-1} extend to a sheaf map $\hat{E} \to E$ or $E \to \hat{E}$. Hence, we can not build an exact sequence as above.

Remark 3.1.5. Up to choices of frames of E and $\hat{E}$, the isomorphism s defines a germ at $x \in X$ of the $\mathsf{GL}(n, \mathbb{C})$-valued meromorphic functions with poles in x. In other words, an element of the loop group (see [Won13] Section 1.3). This opens up an alley to the rich theory of loop groups and affine Grassmanians, which seems to be very useful for structuring the study of more complicated degenerations of Hitchin fibers than what we consider in this work.

3.2. The pushforward

In this section, we collect some basic results on the pushforward of a sheaf along an s-sheeted branched covering of Riemann surface $\pi : Y \to X$. Most of the results can be found in the literature. To prove the results, we use an explicit construction of local frames for the pushforward using Hecke transformations. Let $\mathcal{S}$ be a sheaf on Y. Then the pushforward is defined as the sheaf

$$(\pi_*\mathcal{S})(U) := \mathcal{S}(\pi^{-1}(U))$$

for $U \subset X$ open. If $\mathcal{S}$ is analytic, then $\mathcal{S}(\pi^{-1}(U))$ is a $\mathcal{O}_U$-module by

$$\mathcal{O}_U \times \mathcal{S}(\pi^{-1}(U)) \to \mathcal{S}(\pi^{-1}(U)), \quad (\phi, s) \mapsto (\phi \circ \pi)s.$$

Hence, $\pi_*\mathcal{S}$ defines an analytic sheaf on X.

Proposition 3.2.1 ([HSW99] Proposition 4.2, [Gun67] Lemma 10).

i) *Let E be a holomorphic vector bundle of rank r on Y. Then $\pi_*\mathcal{O}(E)$ is locally free sheaf of rank rs.*
ii) *Let E be a holomorphic vector bundle and $\mathcal{S}$ an analytic sheaf on X. Then $\pi_*(\mathcal{S} \otimes \mathcal{O}(\pi^* E)) \cong \pi_*\mathcal{S} \otimes \mathcal{O}(E)$.*

Proof. i) Away from the branch points this is clear. Let $x \in X$ a branch point and $y \in \pi^{-1}(x)$ a ramification point of index $k-1$. Choose coordinate charts (V, z) centred at y and (U, w) centred at x, such that $E|_{\pi^{-1}(U)}$ is trivial and $\pi : V \to U, z \mapsto z^k$. Let $s_1, \ldots, s_r$ be a local frame of $E|_V$. A section $s \in \mathcal{O}(E)_V$ can be written as

$$s = \sum_{i=1}^{r} \phi_i s_i = \sum_{i=1}^{r} \sum_{j=0}^{k-1} z^j \phi_{ij}(z^k) s_i.$$

Hence, there is an isomorphism of $\mathcal{O}_U$-modules $\mathcal{O}(E)_V \cong \mathcal{O}_U^{kr}$. Repeating the consideration for all connected components of $\pi^{-1}(U)$ we obtain the result.

ii) Let $U \subset X$ an open set trivialising E. Then by the definition of pushforward and pullback of analytic sheaves, we have

$$\begin{aligned}
&\pi_*(\mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{O}(\pi^* E))(U)\\
=&\,\mathcal{S}(\pi^{-1}(U)) \otimes_{\mathcal{O}_{\pi^{-1}(U)}} \left(\mathcal{O}_{\pi^{-1}(U)} \otimes_{\pi^{-1}\mathcal{O}_U} (\pi^{-1}\mathcal{O}(E))(\pi^{-1}(U))\right)\\
=&\,\mathcal{S}(\pi^{-1}(U)) \otimes_{\pi^{-1}\mathcal{O}_U} (\pi^{-1}\mathcal{O}(E))(\pi^{-1}(U))\\
=&\,\mathcal{S}(\pi^{-1}(U)) \otimes_{\mathcal{O}_U} \mathcal{O}(E)(U)\\
=&\,(\pi_*\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}(E))(U).
\end{aligned}$$

These isomorphisms clearly commute with restriction maps.

□

Transition functions for the pushforward of a vector bundle. To obtain a better understanding of the pushforward as a vector bundle, we describe it in terms of transition functions.

Let E be a holomorphic vector bundle on Y. Choose an open covering $\mathfrak{U}$ of X, such that all branch points lie in a unique open set and $E|_{\pi^{-1}(U)}$ is trivial for all $U \in \mathfrak{U}$. Denote by $\pi^{-1}\mathfrak{U}$ the covering of Y obtained by decomposing each $f^{-1}(U)$ into its connected components. Choose transition functions $\{g_{UV}\} \in \check{H}^1(f^{-1}\mathfrak{U}, \mathcal{GL}(n))$, where $\mathcal{GL}(n)$ denotes the sheaf of $\mathsf{GL}(n, \mathbb{C})$-valued holomorphic functions on X. Let $U, V \in \mathfrak{U}$ contain no branch point, such that $U \cap V \neq \varnothing$. Then $\pi^{-1}(U) = U_1 \sqcup \cdots \sqcup U_s$, $\pi^{-1}(V) = V_1 \sqcup \cdots \sqcup V_s$ and we can numerate them, such that $U_i \cap V_i \neq \varnothing$. Then the transition function of $\pi_* E$ on $U \cap V$ is given by

$$\begin{pmatrix} g_{U_1V_1} & & 0 \\ & \ddots & \\ 0 & & g_{U_sV_s} \end{pmatrix}.$$

Let $x \in X$ a branch point and $U \in \mathfrak{U}$ the unique open set containing it. For simplicity assume that $\pi^{-1}(x) = \{y\}$, i. e. $y \in Y$ is a ramification point of index $s-1$. Choose a coordinate (U, w) centred at x. We enhance the covering $\mathfrak{U}$ by adding an open set $V \subset U$, such that there exists a coordinate $(\pi^{-1}V, z)$ centred at y, such that $\pi|_{\pi^{-1}(V)} : z \mapsto z^s$. Furthermore, we add open sets $W, \tilde{W}$ obtained from U by removing two distinct branch cuts. Now, we can remove U from $\mathfrak{U}$ and it remains a covering.

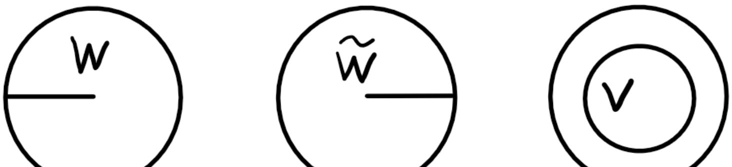

FIGURE 7. Refining the covering

We can identify the sheets $\pi^{-1}W = W_1 \sqcup \cdots \sqcup W_n$ of the covering with the s-th roots of w. Let $s_1, \ldots, s_r$ be a frame of $E|_{\pi^{-1}(U)}$. Then $s_i|_{W_j}$ for $1 \leq i \leq r$, $1 \leq j \leq s$ defines a local frame of $\pi_* E|_W$. Let $s = \sum_{i=1}^{r} \sum_{j=0}^{k-1} z^j \phi_{ij}(z^s) s_i \in \mathcal{O}(\pi_* E)_V$, then

$$s|_{V \cap W} = \sum_{l=1}^{s} s|_{V \cap W_l} = \sum_{l=1}^{s} \sum_{i=1}^{r} \sum_{j=0}^{k-1} (\xi_s^{l-1} z)^j \phi_{ij}(z^s) s_i|_{V \cap W_l},$$

where ξ_s is a primitive s-th root of unity. Let $r = 1$ then the transition function of $\pi_* E$ from V to W is given by

$$g_{VW} = \begin{pmatrix} 1 & z & z^2 & \ldots & z^{s-1} \\ 1 & \xi_s z & \xi_s^2 z^2 & \ldots & \xi_s^{s-1} z^{s-1} \\ 1 & \xi_s^2 z & \xi_s^4 z^2 & \ldots & \xi_s^{s-2} z^{s-1} \\ \vdots & & & & \vdots \\ 1 & \xi_s^{s-1} z & \xi_s^{s-2} z^2 & \ldots & \xi_s z^{s-1} \end{pmatrix}. \tag{12}$$

If $r > 1$, g_{VW} is made up of r blocks like this. In the same way, we obtain the transition function $g_{V\tilde{W}}$. Repeating this procedure at every branch point we obtain transition functions for $\pi_* E$.

PROPOSITION 3.2.2. *Let $x \in X$ be a branch point. There exists a neighbourhood U of x, such that*

$$\pi_* E|_U = \bigoplus_{y \in \pi^{-1}(x)} \pi_*^{\tau_y}(E_y, \hat{\tau}_y),$$

where

i) *τ_y is a generator of the local $\mathbb{Z}_k$-action at a ramification point $y \in \pi^{-1}(x)$ of index $k-1$,*
ii) *$(E_y, \hat{\tau}_y)$ is a holomorphic vector bundle of rank k defined in a neighbourhood of y invariant by the local $\mathbb{Z}_k$-action (cf. Definition 2.2.1), and*
iii) *$\pi_*^{\tau_y}$ is the τ_y-invariant pushforward defined in 2.2.5.*

Proof. Assume for simplicity that $y \in \pi^{-1}(x)$ is ramification point of index $s-1$. Choose local coordinates (V, z) centred at y and (U, w) centred at x, such that $\pi : V \to U, z \mapsto z^s$. Let ξ be a primitive root of unity of order s. We have a local $\mathbb{Z}_s$-action generated by $\tau : V \to V, z \mapsto \xi z$ interchanging the sheets. Let L be a line bundle on Y. We obtain a vector bundle of rank k on V with a natural lift of the $\mathbb{Z}_s$-action by

$$L^\tau := L|_V \oplus \tau^* L|_V \oplus \cdots \oplus (\tau^{s-1})^* L|_V. \tag{13}$$

Here $1 \in \mathbb{Z}_s$ acts by

$$\begin{pmatrix} 0 & & & \tau^* \\ \tau^* & 0 & & \\ & \ddots & \ddots & \\ & & \tau^* & 0 \end{pmatrix}.$$

Let s be local frame of $L|_V$. A diagonalizing frame for the $\mathbb{Z}_s$-action is given by

$$s_i := s + \xi^i \tau^* s + \xi^{2i} \tau^{2*} s + \cdots + \xi^{(s-1)i} \tau^{s-1*} s, \qquad 0 \le i \le s-1.$$

Then $1 \in \mathbb{Z}_s$ acts diagonally by $s_i \mapsto \xi^{s-i} s_i$. We obtain a local τ-invariant holomorphic vector bundle $\hat{E}$ on V by applying a Hecke transformations at y introducing the new transition function

$$\psi_{01} = \begin{pmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z^{s-1} \end{pmatrix}.$$

Then $\pi_*^{\tau_y}(\hat{E}, \hat{\tau}_y)$ defines a rank s holomorphic vector bundle on U. It is easy to check that the transition functions to this specific frame are given by (12). Hence, we constructed a local frame of the pushforward at the branch point x. For a vector bundle E of rank > 1, we obtain the result by applying the construction to a frame $s^1, \dots, s^r$ of $E|_V$. $\square$

Lemma 3.2.3 ([Har83] Exercise IV 2.6). *Let E a rank r holomorphic vector bundle on Y, then*

$$\det(\pi_* E) = \mathsf{Nm}(\det(E)) \otimes \det(\pi_* \mathcal{O}_Y)^r.$$

Furthermore, $\det(\pi_* \mathcal{O}_Y)^2 = \mathcal{O}(-B)$, *where $B \in \mathsf{Div}(X)$ is the branch divisor.*

Proof. Let us first assume $r = 1$ and $E = \det(E) = L$. Let $D \in \mathsf{Div}(Y)$, such that $\mathcal{O}(D) = L$. We can choose transition functions of L, such that for all $y \in \mathsf{supp}(D)$, there exists a coordinate neighbourhood (V, z) and a transition function from V to $W =: V \setminus \{y\}$ given by $g_{VW} = z^{-D_y}$. All other transition functions are trivial. Let $x \in X$ be no branch point, such that $\mathsf{supp}(D) \cap \pi^{-1}(x) \neq \varnothing$. Choose a coordinate neighbourhood (U, w), such that $\pi : V \to U, z \mapsto z$. Then a transition function of $\pi_* L$ from U to $U' = U \setminus \{x\}$ is given by

$$g_{UU'} = \begin{pmatrix} w^{-D_{y_1}} & & \\ & \ddots & \\ & & w^{-D_{y_k}}, \end{pmatrix}$$

where $\pi^{-1}(x) = \{y_1, \dots, y_k\}$ (with possibly $D_{y_i} = 0$ for some i). We have

$$\det(g_{UU'}) = w^{-\sum D_{y_i}} = w^{-\mathsf{Nm}(D)_x}.$$

At a branch point $x \in X$, the pushforward introduces a new transition function as in (12). This extra transition functions do not depend on L and hence their contribution to $\det(\pi_* L)$ is covered by tensoring with $\det(\pi_* \mathcal{O}_Y)$. To see that $\det(\pi_* \mathcal{O}_Y)$ is a square root of the line bundle $\mathcal{O}(-B)$ consider a ramification point $y \in \pi^{-1}(x)$ of index $k-1$. Choose coordinate neighbourhoods (V, z) centred at y and (U, w) centred at x, such that $\pi : z \mapsto z^k = w$. The determinant of the new transition function (12) is $z^{\frac{k(k-1)}{2}}$. Hence, the induced transition function of $\det(\pi_* \mathcal{O}_X)^2$ on $U \setminus \{x\}$ is given by w^{k-1}. Summing up over all ramification points over x, we obtain $\det(\pi_* \mathcal{O}_Y)^2 \cong \mathcal{O}(-B)$.

In higher rank, we introduce a new transitions function as above for each section of a frame $s_1, \dots, s_r$ of E at a ramification point. Hence, the determinant is tensored by $\det(\pi_* \mathcal{O}_Y)^r$.

□

Proposition 3.2.4. *Let E, F holomorphic vector bundles on Y and $\beta : E \otimes F \to \mathbb{C}$ a non-degenerate bilinear pairing. Fix a square root $\mathcal{O}(R)^{\frac{1}{2}}$. Then there is a induced non-degenerate pairing*

$$\pi_*(E \otimes \mathcal{O}(R)^{\frac{1}{2}}) \otimes \pi_*(F \otimes \mathcal{O}(R)^{\frac{1}{2}}) \to \mathbb{C}.$$

Recall that the Ramification divisor R has even degree by the Riemann-Hurwitz formula.

Proof. Let $E' = E \otimes \mathcal{O}(R)^{\frac{1}{2}}, F' = F \otimes \mathcal{O}(R)^{\frac{1}{2}}$. Then β induces a pairing

$$\beta' := (\partial\pi)^{-1}\beta : \quad \mathcal{O}(E') \otimes \mathcal{O}(F') \to \mathfrak{M}_X$$

non-degenerate away from the ramification points of $\pi : Y \to X$. Let $U \subset X$ be trivially covered, then it is clear that β' descends to a non-degenerate bilinear pairing

$$\pi_*\beta' : \pi_* E'|_U \otimes \pi_* F'|_U \to \mathbb{C}.$$

Let $U \subset X$ contain a branch point $x \in X$ and let $y \in Y$ be a corresponding ramification point of index $k-1$. Let (V, z) be a coordinate chart centred at y, such that $\pi|_V : z \mapsto z^k$. There exist frames $s_1, \dots, s_r$ of $E'|_V$ and $t_1, \dots, t_r$ of $F'|_V$, such that $\beta'(s_l, t_m) = \delta_{lm} z^{1-k}$. Define the local $\mathbb{Z}_k$-bundles E^τ, F^τ as in (13) and a bilinear map $\beta^\tau : \mathcal{O}(E^\tau) \otimes \mathcal{O}(F^\tau) \to \mathfrak{M}_X$ by

$$\beta^\tau := \beta' + \tau^*\beta' + \dots + \tau^{(k-1)*}\beta'.$$

For $1 \leq l \leq r$, the sections s_l, t_l induce diagonalizing sections of the $\mathbb{Z}_k$-action on E^τ, F^τ by

$$\begin{aligned} s_{li} &:= s_l + \xi^i \tau^* s_l + \xi^{2i} \tau^{2*} s_l + \dots + \xi^{(k-1)i} \tau^{(k-1)*} s_l, \\ t_{li} &:= t_l + \xi^i \tau^* t_l + \xi^{2i} \tau^{2*} t_l + \dots + \xi^{(k-1)i} \tau^{(k-1)*} t_l. \end{aligned}$$

We have

$$\beta^\tau(s_{li}, t_{lj}) = \begin{cases} z^{1-k} & \text{for } i + j = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Hecke transformation to E^τ, F^τ the bilinear form β^τ is twisted into a non-degenerate $\mathbb{Z}_k$-invariant pairing $\hat{\beta} : \hat{E} \otimes \hat{F} \to \mathbb{C}$. Respective the induced frames $\hat{s}_{lj} = z^j s_{lj}$ of $\hat{E}$ and $\hat{t}_{lj} = z^j t_{lj}$ of $\hat{F}$, we have

$$\hat{\beta}(\hat{s}_{li}, \hat{t}_{mj}) = \begin{cases} 1 & \text{for } l = m \text{ and } i + j = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\hat{\beta}$ descends to a non-degenerate pairing extending $\pi_* \beta'$ over the branch point x. Repeating the consideration at every branch point, we obtain a global non-degenerate pairing as described in the Proposition. $\square$

CHAPTER 4

$\mathfrak{sl}(2)$-type fibers of symplectic and orthogonal Hitchin systems

In this chapter, we define and parametrize so-called $\mathfrak{sl}(2,\mathbb{C})$-type Hitchin fibers. These are singular fibers of symplectic and orthogonal Hitchin systems, which are isomorphic to fibers of an $\mathsf{SL}(2,\mathbb{C})$- respectively $\mathsf{PSL}(2,\mathbb{C})$-Hitchin map. For a $\mathfrak{sl}(2,\mathbb{C})$-type Hitchin fiber, the spectral curve defines a two-sheeted covering of another Riemann surface and these higher rank Hitchin fibers can be identified with fibers of Hitchin maps associated to these two-sheeted coverings. Building on results of Chapter 2, we give a stratification of these singular spaces by fiber bundles over abelian varieties resulting in a global description of the first degenerations. We will compare these semi-abelian spectral data of $\mathfrak{sl}(2)$-type Hitchin fibers for the two Langlands dual groups $\mathsf{Sp}(2n,\mathbb{C})$ and $\mathsf{SO}(2n+1,\mathbb{C})$. We recover a duality on the abelian part of the spectral data. The non-abelian parts of the spectral data for corresponding singular $\mathfrak{sl}(2)$-type Hitchin fibers are isomorphic. This extends the well-known duality of regular Hitchin fibers to $\mathfrak{sl}(2)$-type Hitchin fibers.

We will start by defining $\mathfrak{sl}(2)$-type fibers of the symplectic Hitchin system in Section 4.2.1. Then we will identify these class of singular Hitchin fibers with fibers of an associated $\mathsf{SL}(2,\mathbb{C})$-Hitchin map (Section 4.2.2). Using the results of Chapter 2, we parametrize $\mathfrak{sl}(2)$-type $\mathsf{Sp}(2n,\mathbb{C})$-Hitchin fibers by abelian spectral data in Section 4.2.3.

In the second part, we will repeat these considerations for $\mathsf{SO}(2n+1,\mathbb{C})$. The identification of the $\mathfrak{sl}(2)$-type Hitchin fibers with fibers of an associated $\mathsf{SO}(3,\mathbb{C})$-Hitchin system is proven in Section 4.3.2.

Finally, we will prove the Langlands-Correspondence for $\mathfrak{sl}(2)$-type Hitchin fibers of symplectic and odd orthogonal Hitchin systems in Section 4.4.

4.1. The $\mathsf{Sp}(2n,\mathbb{C})$-Hitchin system

In this section, we take a closer look at the Hitchin system for $\mathsf{Sp}(2n,\mathbb{C})$. Let us first reformulate the definition of a $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundle (see Definition 1.2.1) by using the defining representation on $\mathbb{C}^{2n}$. Let M denote a holomorphic line bundle on X with $\deg(M) > 0$.

LEMMA 4.1.1. *An M-twisted $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundle is a triple (E,Φ,ω) of a*

i) *holomorphic vector bundle E of rank $2n$ with an anti-symmetric bilinear form $\omega \in H^0(X,\bigwedge^2 E^\vee)$, such that $\omega^{\wedge n} \in H^0(X,\det(E^\vee))$ is a non-vanishing, and*
ii) *$\Phi \in H^0(X,\mathrm{End}(E)\otimes M)$, such that $\omega(\Phi\,\cdot,\cdot) = -\omega(\cdot,\Phi\,\cdot)$.*

PROOF. A reduction of structure group of a principal $\mathsf{GL}(2n,\mathbb{C})$-bundle P to a $\mathsf{Sp}(2n,\mathbb{C})$-bundle corresponds to the choice of a section

$$\omega \in H^0\left(X, P \times_{\mathsf{GL}(2n,\mathbb{C})} \mathsf{GL}(2n,\mathbb{C})/\mathsf{Sp}(2n,\mathbb{C})\right).$$

Under the linear representation, this corresponds to the choice of a symplectic form on the fibers. $\square$

THEOREM 4.1.2 (Simplified stability condition [GGR09]). *A $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundle (E,Φ,ω) is stable, if for all isotropic Φ-invariant subbundles $0 \neq F \subsetneq E$*

$$\deg(F) < 0.$$

Let $\mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X,M)$ denote the moduli space of stable M-twisted $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundles on X. For $M = K$, this is a complex symplectic manifold of dimension

$$(2g-2)(2n^2+n).$$

Let $A \in \mathfrak{sp}(2n,\mathbb{C})$, then the the characteristic polynomial of A is of the form

$$T^{2n} + a_2(A)T^{2n-2} + \cdots + a_{2n}(A) \in \mathbb{C}[T].$$

The coefficients $(a_2,\ldots,a_{2n})$ are homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ and the associated Hitchin map is given by

$$\begin{aligned} \mathsf{Hit}_{\mathsf{Sp}(2n,\mathbb{C})}: \quad \mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X,M) &\to B_{2n}(X,M) := \bigoplus_{i=1}^{n} H^0(X, M^{2i}), \\ (E,\Phi) \quad &\mapsto \quad (a_2(\Phi),\ldots,a_{2n}(\Phi)). \end{aligned}$$

For $M = K$, the Hitchin map restricted to a dense subset $B_{2n}^{\mathsf{reg}} \subset B_{2n}$ defines an algebraically completely integrable system.

The characteristic equation of $(E,\Phi) \in \mathsf{Hit}_{\mathsf{Sp}(2n,\mathbb{C})}^{-1}(a_2,\ldots,a_{2n})$ is given by

$$\eta^{2n} + a_2\eta^{2n-2} + \cdots + a_{2n} = 0.$$

Let $p_M : M \to X$ the bundle map, then η can be interpreted as the tautological section $\eta : M \to p_M^* M$. The pointwise eigenvalues of the Higgs field form the spectral curve

$$\Sigma := Z_M(\eta^{2n} + p_M^* a_2 \eta^{2n-2} + \cdots + p_M^* a_{2n}) \subset \mathsf{Tot} M.$$

The projection p_M restricts to a $2n$-sheeted branched covering $\pi : \Sigma \to X$. Recall that in general the spectral curve is singular at the points, where different sheets meet. Due to the specific type of characteristic equation the spectral curve comes with an involutive automorphism $\sigma : \Sigma \to \Sigma$ reflecting in the zero section of M.

For $M = K$, the regular locus B_{2n}^{reg} is the subset of the Hitchin base, where the spectral curve Σ is smooth. The fibers over B_{2n}^{reg} are torsors over the Prym variety

$$\mathsf{Prym}(\Sigma \to \Sigma/\sigma).$$

In contrast to Example 1.3.7, it is not possible to detect the smoothness of symplectic spectral curves with the algebraic discriminant of the characteristic polynomial. If $0 \neq \eta_x \in M_x$ is an eigenvalue of Φ_x of multiplicity 2, then $-\eta_x$ has the same multiplicity. Hence the algebraic discriminant always has double zeroes. The $\mathfrak{sp}(2n,\mathbb{C})$-discriminant $\mathsf{disc}_{\mathfrak{sp}} = \mathsf{disc}_{\mathfrak{sp}(2n,\mathbb{C})}$ introduced in Section 1.3 takes care of this extra symmetry.

LEMMA 4.1.3. *If* $\mathsf{disc}_{\mathfrak{sp}}(a_2,\dots,a_{2n}) \in H^0(X, M^{2n^2})$ *has simple zeroes, then the spectral curve is smooth.*

To proof this lemma, let us take a closer look at the $\mathfrak{sp}(2n,\mathbb{C})$-discriminant. Consider the representation of $\mathfrak{sp}(2n,\mathbb{C})$

$$\{A \in \mathsf{Mat}(n,\mathbb{C}) \mid A^{\mathrm{tr}} J_{2n} + J_{2n} A = 0\}, \text{ where } J_{2n} = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}.$$

A Cartan subalgebra is given by

$$\mathfrak{h} = \{H = \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_n & & \\ & & & & \ddots & \\ & & & & & -h_1 \end{pmatrix} \mid \mathfrak{h}_i \in \mathbb{C}\}.$$

Define $e_i \in \mathfrak{h}^\vee$ by $e_i(H) = h_i$. Then a root system is given by

$$\Delta = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\}.$$

The two types of roots differ by their length. The roots $\pm 2e_i$ have $\sqrt{2}$ times the length of the roots $\pm e_i \pm e_j$ (as depicted in the Dynkin diagram). The Weyl group W preserves the inner product on $\mathfrak{h}$ and hence the set of long/short roots is invariant by the W-action. So we can define invariant polynomials in $\mathbb{C}[\mathfrak{g}]^G$ by the product over the long/short roots. The product over the long roots $\prod_{i=1}^n -4e_i^2$ gives (up to a scalar) the determinant function on $\mathfrak{h}$. We refer to the product over the short roots as the reduced $\mathfrak{sp}(2n,\mathbb{C})$-discriminant

$$\mathsf{disc}_{\mathfrak{sp}}^{\mathsf{red}} := \prod_{i<j} -(e_i \pm e_j)^2.$$

We have

$$\mathsf{disc}_{\mathfrak{sp}} = \det \mathsf{disc}_{\mathfrak{sp}}^{\mathsf{red}}.$$

PROOF OF LEMMA 4.1.3. Let $x \in X$ be a simple zero of

$$\mathsf{disc}_{\mathfrak{sp}}(a_2,\dots,a_{2n}) = a_{2n}\mathsf{disc}_{\mathfrak{sp}}^{\mathsf{red}}(a_2,\dots,a_{2n}) \in H^0(X, M^{2n^2}).$$

If a_{2n} has a simple zero at x and $\mathsf{disc}_{\mathfrak{sp}}^{\mathsf{red}}(a_2,\dots,a_{2n})(x) \neq 0$, then $\pi^{-1}(x) \in \Sigma$ contains a simple ramification point on the zero section. If $\mathsf{disc}_{\mathfrak{sp}}^{\mathsf{red}}(a_2,\dots,a_{2n})$ has a simple zero at x and $a_{2n}(x) \neq 0$, then $\pi^{-1}(x) \in \Sigma$ contains two simple ramification points $0 \neq \lambda, -\lambda \in M_x$. In both cases, the spectral curve is smooth. □

EXAMPLE 4.1.4 ($\mathsf{Sp}(4,\mathbb{C})$). Let $(a_2,a_4) \in B_4(X,M)$. The $\mathfrak{sl}(4,\mathbb{C})$-discriminant is given in terms of the image of the Hitchin map by

$$\mathsf{disc}_{\mathfrak{sl}(4,\mathbb{C})}(a_2,a_4) = a_4(a_2^2 - 4a_4)^2,$$

whereas the $\mathfrak{sp}(4,\mathbb{C})$-discriminant is given by

$$\mathsf{disc}_{\mathfrak{sp}}(a_2,a_4) = a_4(a_2^2 - 4a_4).$$

4.2. $\mathfrak{sl}(2)$-type fibers of symplectic Hitchin systems

4.2.1. $\mathfrak{sl}(2)$-type spectral curves. In this section, we will defined the class of $\mathfrak{sl}(2)$-type fibers of the $\mathsf{Sp}(2n,\mathbb{C})$-Hitchin map. These Hitchin fibers are distinguished by the singularities of the spectral curve, such that for $G = \mathsf{SL}(2,\mathbb{C})$ all Hitchin fibers are of $\mathfrak{sl}(2)$-type.

Let $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$, $\Sigma \subset \mathrm{Tot}(M)$ the spectral curve and σ the involutive biholomorphism reflecting in the zero section of M. This involution defines an algebraic $\mathbb{Z}_2$-action on Σ. We will construct its quotient in the algebraic category (see Remark 1.3.8). A geometric quotient by this action is given by

$$\pi_2 : \Sigma \to \Sigma/\sigma := \mathsf{Spec}(\mathcal{O}_\Sigma^\sigma),$$

where $\mathcal{O}_\Sigma^\sigma$ denotes the sheaf of σ-invariant regular functions on Σ. As π is invariant under the $\mathbb{Z}_2$-action, we obtain the commutative diagram on the right of this paragraph.

$$\begin{tikzcd} & \Sigma \arrow[dl, "\pi_2"'] \arrow[dd, "\pi"] \\ \Sigma/\sigma \arrow[dr, "\pi_n"'] & \\ & X \end{tikzcd}$$

Definition 4.2.1. An element $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$ is called $\mathfrak{sl}(2)$-type, if Σ/σ is smooth. In this case, $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2,\dots,a_n)$ is called $\mathfrak{sl}(2)$-type Hitchin fiber. An $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundle is called $\mathfrak{sl}(2)$-type, if it is contained in an $\mathfrak{sl}(2)$-type Hitchin fiber.

Example 4.2.2.
i) Let $n = 1$. Then $X = \Sigma/\sigma$ is smooth for all $a_2 \in H^0(X,M^2)$ and hence all fibers are $\mathfrak{sl}(2)$-type.
ii) A regular point $(a_2,\dots,a_{2n}) \in B_{2n}^{\mathsf{reg}}(X,M)$ is $\mathfrak{sl}(2)$-type, since in this case Σ is smooth. The fibers are isomorphic to $\mathsf{Prym}(\Sigma \to \Sigma/\sigma)$, which in turn determines a regular Hitchin fiber of the π_n^*K-twisted $\mathsf{SL}(2,\mathbb{C})$-Hitchin system on Σ/σ.
iii) Consider $n = 2$ and $(a_2,a_4) \in B_4(X,M)$, such that Σ is smooth except of one point $p \in \Sigma$ on the zero section. Assume that the spectral curve is locally at p isomorphic to $Z(y^2 - z^2) \subset \mathbb{C}^2$ with $\sigma : \mathbb{C}^2 \to \mathbb{C}^2, (y,z) \mapsto (-y,z)$. The local quotient Σ/σ is isomorphic to the affine curve $\mathsf{Spec}((\mathbb{C}[y,z]/(y^2-z^2))^\sigma)$. There is an isomorphism
$$\left(\mathbb{C}[y,z]/(y^2-z^2)\right)^\sigma \to \mathbb{C}[z], \quad \phi \mapsto \phi(z,z)$$
and hence Σ/σ is smooth at p. In conclusion, $(a_2,a_4) \in B_4 \setminus B_4^{\mathsf{reg}}$ is of $\mathfrak{sl}(2)$-type.

Proposition 4.2.3. *A point $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$ is of $\mathfrak{sl}(2)$-type if and only if all singular points of Σ lie on the zero section of $M \to X$ and only two sheets meet in the singular points. In particular, all singular points of Σ are of type A_k, $k \geq 1$, i. e. nodes or cusps.*

If $\mathsf{disc}_{\mathsf{sp}}^{\mathsf{red}}(a_2,\dots,a_{2n}) \in H^0(X,M^{2n(n-1)})$ has simple zeroes and $Z(a_{2n-2}) \cap Z(a_{2n}) = \varnothing$, then $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$ is of $\mathfrak{sl}(2)$-type. In this case, $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$ is called $\mathfrak{sl}(2,\mathbb{C})$-discriminant type.

Proof. If $(a_2,\dots,a_{2n}) \in B_{2n}(X,M)$ is of $\mathfrak{sl}(2)$-type, there can not be any singular points away from the zero section of M. Otherwise Σ/σ is singular, too. Let $y \in \Sigma$ be a singular point on the zero section. Choose a trivialization $\pi^* M|_U \cong U \times \mathbb{C}$ over a coordinate neighbourhood (U, z) centred at $\pi(y)$ and let (z, λ) be the induced coordinate on M. Then Σ is locally given by the equation

$$q(z, \lambda) := \lambda^{2n} + \lambda^{2n-2} a_2(z) + \cdots + a_{2n}(z) = 0$$

with the involution given by $\sigma : (z, \lambda) \mapsto (z, -\lambda)$. Because $y = (0, 0)$ is a singular point, we have

$$\frac{\partial}{\partial z}|_{(z,\lambda)=(0,0)}\, q = \frac{\partial}{\partial \lambda}|_{(z,\lambda)=(0,0)}\, q = 0.$$

Hence, $\frac{\partial}{\partial z}|_{z=0}\, a_{2n} = 0$, i. e. a_{2n} has a higher order zero at $z = 0$. Now, Σ/σ is locally given by the equation

$$q^\sigma(\eta, z) = \eta^n + \eta^{n-1} a_2 + \cdots + a_{2n} = 0$$

and smooth at $(0, 0)$ by assumption . Therefore,

$$0 \neq \frac{\partial}{\partial \lambda}|_{(z,\lambda)=(0,0)}\, q^\sigma = a_{2n-2}(0).$$

In particular, 0 is a zero of $q(0, \lambda)$ of multiplicity 2 and hence only two sheets meet in the singular point.
Conversely, if a singular point p lies on the zero section and two sheets of the covering π meet there, then Σ is locally given by a polynomial equation of the form $y^2 - z^k = 0$. Let $R = \mathbb{C}[y, z]/(y^2 - z^k)$. The ring of invariant functions R^σ is generated by y^2 and z. In particular,

$$R^\sigma \to \mathbb{C}[z], \quad \phi \mapsto \phi(z^{\frac{k}{2}}, z)$$

defines an isomorphism of coordinate rings. Hence, $\mathsf{Spec}(R^\sigma) \cong \mathbb{C}$ and the quotient is smooth.
The discriminant condition implies that, away from the zero section, the only points, where different sheets meet, are smooth ramification points of ramification index 1. Furthermore, $Z(a_{2n-2}) \cap Z(a_{2n}) = \varnothing$ implies that only two sheets meet at the zero section, in particular at the singular points. Hence, the spectral curve is $\mathfrak{sl}(2)$-type by the first criterion. □

REMARK 4.2.4. Nevertheless there can be smooth ramification points of $\pi : \Sigma \to X$ of higher order on the zero section of M for a $\mathfrak{sl}(2)$-type spectral curve Σ. In local coordinates an example is given by a smooth ramification point of index 3

$$\lambda^4 + z = 0.$$

REMARK 4.2.5. An irreducible algebraic/analytic subset $Z \subset \mathbb{C}^n$ is a C^1-manifold in a neighbourhood of a point p if and if only Z is locally given by analytic/algebraic equations

$$F_1(x_1, \ldots, x_n) = 0, \ldots, F_k(x_1, \ldots, x_n) = 0,$$

such that $D(F_1, \ldots, F_k)$ has maximal rank at p. The backwards implication follows from the implicit function theorem. For the converse see [Mil68, page 13].

PROPOSITION 4.2.6. *Let $p : M^2 \to X$ the bundle map and $\eta : M^2 \to p^*M^2$ the tautological section. Let $(a_2, \dots, a_{2n}) \in B_{2n}(X, M)$ of $\mathfrak{sl}(2)$-type. The reduced spectral curve Σ/σ is the zero divisor of*

$$\eta^n + a_2\eta^{n-1} + \cdots + a_{2n-2}\eta + a_{2n} \in H^0(M^2, p^*M^{2n}).$$

In particular, $K_{\Sigma/\sigma} \cong \pi_n^*\left(M^{2n-2} \otimes K_X\right)$ and $\mathcal{O}(R) \cong \pi_n^* M^{2n-2}$, where $R \in \mathsf{Div}(\Sigma/\sigma)$ is the ramification divisor of $\pi_n : \Sigma/\sigma \to X$.

PROOF. The first assertion is obvious from the consideration in the proof of the precious proposition. By Lemma 1.3.6 and the adjunction formula we have

$$K_{\Sigma/\sigma} = \left(K_{M^2} \otimes p^*M^{2n}\right)|_{\Sigma/\sigma} = \pi_n^*\left(M^{2n-2} \otimes K_X\right).$$

The last assertion follows as $\mathcal{O}(R) = K_{\Sigma/\sigma} \otimes \pi_n^* K_X^{-1}$. □

In the subsequent analysis of Hitchin fibers of $\mathfrak{sl}(2)$-type another version of the spectral curve plays an important role. We can naturally associate a smooth curve to the singular spectral curve Σ by the normalisation $\tilde{\Sigma}$. It can be defined as the unique extension of the covering $\pi|_{\Sigma^\times} : \Sigma^\times \to X^\times$ to a holomorphic covering of Riemann surfaces (here $\cdot^\times$ refers to the complement of ramification/branch points). If Σ/σ is smooth, it can be defined in the same way as the extension of the covering of Riemann surfaces $\pi_2|_{\Sigma^\times} : \Sigma^\times \to (\Sigma/\sigma)^\times$. It also can be defined intrinsically by taking the integral closure of the structure sheaf and defining $\tilde{\Sigma}$ as the associated analytic curve. Hence, all this constructions agree. In total we obtain the commutative diagram in Figure 8.

$$\begin{tikzcd}
 & & \tilde{\Sigma} \arrow[dll, "\tilde{\pi}_2"'] \arrow[dl] \arrow[ddl, "\tilde{\pi}"] \\
\Sigma/\sigma \arrow[dr, "\pi_n"'] & \Sigma \arrow[l, "\pi_2"'] \arrow[d, "\pi"] & \\
 & X &
\end{tikzcd}$$

FIGURE 8. Spectral curves

For $a_{2n} \in H^0(X, M^{2n})$ let

$$n_{\text{odd}} := n_{\text{odd}}(a_{2n}) := \#\{x \in Z(a_{2n}) \mid x \text{ zero of odd order}\}.$$

LEMMA 4.2.7. *Let $(a_2, \dots, a_{2n}) \in B_{2n}(X, M)$ be of $\mathfrak{sl}(2)$-type. Then the genus of Σ/σ is given by*

$$g(\Sigma/\sigma) = n(g - 1 + (n-1)\deg(M)) + 1$$

and the genus of the normalised spectral curve is

$$g(\tilde{\Sigma}) = 2n(g - 1 + (n-1)\deg(M)) + \tfrac{1}{2}n_{\text{odd}} + 1.$$

If $M = K_X$, we have

$$g(\Sigma/\sigma) = n(2n-1)(g-1) + 1$$

and

$$g(\tilde{\Sigma}) = 2n(2n-1)(g-1) + \tfrac{1}{2}n_{\text{odd}} + 1.$$

PROOF. This is immediate from Proposition 4.2.6 and the Riemann-Hurwitz formula. □

4.2.2. $\mathfrak{sl}(2,\mathbb{C})$-type Hitchin fibers are fibers of an $\mathsf{SL}(2,\mathbb{C})$-Hitchin map. In this section, we identify the $\mathfrak{sl}(2)$-type fibers of the symplectic Hitchin system with fibers of a $\mathsf{SL}(2,\mathbb{C})$-Hitchin system on the reduced spectral curve Σ/σ.

PROPOSITION 4.2.8. *Let $p : Y \to X$ be an $s : 1$ covering of Riemann surfaces. Let $R = \mathsf{div}(\partial p) \in \mathsf{Div}(Y)$ the ramification divisor. Let $(E,\Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(Y, p^*M)$, then the pushforward $(p_*(E \otimes \mathcal{O}(R)^{\frac{1}{2}}), p_*\Phi)$ defines a M-twisted $\mathsf{Sp}(2s,\mathbb{C})$-Higgs bundle on X.*

Recall that $\mathcal{O}(R)$ is of even degree by the Riemann-Hurwitz formula.

SHORT PROOF USING PROPOSITION 3.2.4. Let $E' := E \otimes \mathcal{O}(R)^{\frac{1}{2}}$. p_*E' is locally free and

$$p_*\Phi : p_*(E') \to p_*(E' \otimes p^*M) = p_*(E') \otimes M$$

defines a M-twisted Higgs field on p_*E'. By Proposition 3.2.4 the area form on E descends to a symplectic form on p_*E'. Because Φ is trace-free, or equivalently anti-symmetric with respect to the area form, $p_*\Phi$ is anti-symmetric with respect to the induced symplectic form on p_*E. □

For the reader's convenience, we give another proof without using Proposition 3.2.4.

DETAILED PROOF. Let $E' := E \otimes \mathcal{O}(R)^{\frac{1}{2}}$. p_*E' is locally free and

$$p_*\Phi : p_*(E') \to p_*(E' \otimes p^*M) = p_*(E') \otimes M$$

defines a M-twisted Higgs field on p_*E'. The symplectic form $\omega \in H^0(Y, \bigwedge^2 E^\vee)$ induces a degenerate symplectic form $\omega' = \omega(\partial s)^{-1} \in H^0(Y, \bigwedge^2 E^\vee(-R))$ on E'. Let $U \subset X$ be trivially covered, such that $E'|_{p^{-1}(U)}$ is trivial. Hence $p^{-1}(U) = V_1 \sqcup \dots \sqcup V_s$. Let s_{ij} with $i = 1,2; j = 1,\dots,s$ be symplectic frames of $E'|_{V_j}$, i. e.

$$\omega'|_{V_j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

respective s_{1j}, s_{2j}. Then the induced symplectic form on $p_*(E')|_U$ is given by

$$p_*\omega'|_U = \left(\begin{array}{cc|c|cc} 0 & 1 & & & \\ -1 & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & 1 \\ & & & -1 & 0 \end{array}\right)$$

respective the frame s_{ij}. This defines a symplectic form $p_*\omega'$ on $p_*E'|_{Y^\times}$, where $Y^\times = Y \setminus \mathsf{supp}R$. Obviously, $p_*\omega'(p_*\Phi\,\cdot,\cdot) = -p_*\omega'(\cdot, p_*\Phi\,\cdot)$.

To extend the symplectic form to the branch points, we use the description of the algebraic pushforward by local $\mathbb{Z}_k$-invariant bundles at the corresponding ramification point (see Corollary 3.2.2). Let $\omega' := \omega(\partial p)^{-1} \in H^0(\bigwedge^2(E')^\vee)$. Let $y \in Y$ be a ramification point of order k. Choose a coordinate (V,z) centred at y, such that projection map is given by $p : z \mapsto z^k$. Let ξ a primitive root of unity of order k and $\tau : V \to V,\ z \mapsto \xi z$ the local automorphism inducing the $\mathbb{Z}_k$-action. Consider the local holomorphic $\mathbb{Z}_k$-vector bundle

$$F = E'|_V \oplus \tau^* E'|_V \oplus \cdots \oplus (\tau^{k-1})^* E'|_V$$

Let s_1, s_2 be a symplectic frame of $E'|_V$, then

$$s_{ij} := \tfrac{1}{k}(s_i + \xi^j \tau^* s_i + \xi^{2j}(\tau^2)^* s_i + \cdots + \xi^{(k-1)j}(\tau^{k-1})^* s_i)$$

for $i \in \{1,2\}$ and $0 \leq j \leq k-1$ define a frame of F, such that the $\mathbb{Z}_k$-action is given by

$$\mathrm{diag}(1, 1, \xi, \xi, \ldots, \xi^{k-1}, \xi^{k-1}).$$

The induced degenerate symplectic form $\Omega = \omega' + \tau^*\omega' + \cdots + (\tau^{k-1})^*\omega'$ is given by

$$\Omega(s_{1l}, s_{2m}) = \begin{cases} z^{-k+1} & \text{for } l + m = k-1 \\ 0 & \text{otherwise.} \end{cases}$$

To obtain a local $\mathbb{Z}_k$-invariant holomorphic vector bundle $\hat{F}$ descending to $p(V)$ we have to apply a Hecke transformation introducing the new transition function

$$\psi_{01} = \mathrm{diag}(1, 1, z, z, \ldots, z^{k-1}, z^{k-1}).$$

The induced symplectic form is given by

$$\hat{\Omega} = (\psi_{01}^* \Omega)(s_{1l}, s_{2m}) = \begin{cases} 1 & \text{for } l + m = k-1 \\ 0 & \text{otherwise,} \end{cases}$$

where we denote by abuse of notation the induced frame of $\hat{F}$ again by s_{ij}. Hence, $\hat{\Omega}$ descends to a non-degenerate symplectic form on $p_* E'$. Again it is clear that the induced Higgs field $p_*\Phi$ is anti-symmetric with respect to the symplectic form. $\square$

Let $(a_2, \ldots, a_{2n}) \in B_{2n}(X, M)$ of $\mathfrak{sl}(2)$-type. The spectral curve Σ comes with a section $\lambda \in H^0(\Sigma, \pi^* M)$ solving the spectral equation. Hence, $\lambda\sigma^*(\lambda) \in H^0(\Sigma, \pi^* M^2)$ defines a σ-invariant section and descends to $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$.

PROPOSITION 4.2.9. *Let $(a_2, \ldots, a_{2n}) \in B_{2n}(X, M)$ of $\mathfrak{sl}(2)$-type and $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$ the induced section. There is a holomorphic map*

$$\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n}) \to \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(b_2) \subset \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(\Sigma/\sigma, \pi_n^* M).$$

PROOF. Let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n})$. The characteristic polynomial

$$\lambda^{2n} + \pi_n^* a_2 \lambda^{2n-2} + \cdots + \pi_n^* a_{2n}$$

factors through $\lambda^2 + b_2$ and hence defines a generalised eigen bundle E_2 by

$$0 \to E_2 \to \pi_n^* E \xrightarrow{\pi_n^*\Phi^2 + b_2 \mathrm{id}} \pi_n^*(E \otimes M^2) \to E_2 \otimes \pi_n^* M^{2n} \to 0.$$

The dualized exact sequence tensored with $\pi_n^* M^2$ results in

$$0 \to E_2^\vee \otimes \pi_n^* M^{2-2n} \to \pi_n^* E^\vee \xrightarrow{(\pi_n^*\Phi^2 + b_2 \mathrm{id})^\vee} \pi_n^*(E^\vee \otimes M^2) \to E_2^\vee \otimes \pi_n^* M^2 \to 0.$$

The symplectic form ω identifies E with $E^\vee$ and from the anti-symmetry of the Higgs field the bundle map $\pi_n^*\Phi^2 + b_2 \mathrm{id}_{\pi_n^* E}$ is self-dual. Hence, there is an induced isomorphism $E_2 \cong E_2^\vee \otimes \pi_n^* M^{2-2n}$. In particular, ω restricts to a symplectic form

ω_2 on $E_2 \otimes \pi_n^* M^{n-1}$ and the induced Higgs field Φ_2 on E_2 is anti-symmetric with respect to it. Hence, (E_2, Φ_2) is a $\pi_n^* M$-twisted $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle on Σ/σ. $\square$

THEOREM 4.2.10. *Let $(a_2, \dots, a_{2n}) \in B_{2n}(X, M)$ of $\mathfrak{sl}(2)$-type and $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$ the induced section. The holomorphic map*

$$\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \dots, a_{2n}) \to \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(b_2)$$

defined in Proposition 4.2.9 is a biholomorphism. Its inverse is given by Proposition 4.2.8.

PROOF. We need to show that the holomorphic maps defined in Proposition 4.2.8 and 4.2.9 are inverse to each other. Let $(E_2, \Phi_2) \in \mathsf{Hit}^{-1}(b_2)$. By Proposition 4.2.8 $(\pi_{n*}(E_2 \otimes \pi_n^* M^{n-1}), \pi_{n*}\Phi)$ defines a $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundle on X with spectral curve Σ. We have a natural map

$$E_2 \otimes \pi_n^* M^{1-n} \to E_2 \otimes \pi_n^* M^{n-1}$$

by multiplying with the canonical section of $\mathcal{O}(R) \cong \pi_n^* M^{2n-2}$. This induces an inclusion

$$\iota : E_2 \otimes \pi_n^* M^{1-n} \to \pi_n^* \pi_{n*}(E_2 \otimes \pi_n^* M^{n-1}).$$

It is clear by construction that the $\mathsf{im}(\iota) = \mathsf{ker}(\pi_n^*\pi_{n*}\Phi^2 + b_2 \mathrm{id})$.

For the converse, let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \dots, a_{2n})$ and denote by (E_2, Φ_2) the induced $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle on Σ/σ. It is clear that

$$\pi_{n*}(E_2 \otimes \pi_n^* M^{n-1})|_{X^\times} \cong (E, \Phi)|_{X^\times},$$

where $X^\times = X \setminus \pi_n(\mathsf{supp} R)$. We are left with showing that this isomorphism extends over the branch points. Let $x \in X$ be a branch point. For simplicity of notation we assume that it corresponds to a ramification point $y \in Y$ of index $n-1$. Let U be a neighbourhood of x, such that the covering is given by $\pi_n : \pi_n^{-1} U \to U, z \mapsto z^n$. On $V := \pi_n^{-1} U$ we have a local automorphism τ changing the sheets inducing a local $\mathbb{Z}_k$-action on Σ/σ. The pullback $\pi_n^*(E, \Phi)|_V$ is a $\mathbb{Z}_k$-invariant Higgs bundle with trivial determinant. There is a unique way to extend $\pi_n^*\pi_{n*}(E_2 \otimes \pi_n^* M^{n-1}, \Phi_2)|_{\pi_n^{-1} X^\times}$ to a τ-invariant $\mathsf{SL}(2n,\mathbb{C})$-Higgs bundle at $y \in Y$ by Hecke transformations. This is the way the pushforward is constructed in Corollary 3.2.2. Hence, the isomorphism extends over the branch points. $\square$

4.2.3. Semi-abelian spectral data for $\mathfrak{sl}(2)$-type fibers.

In this section, we apply the results of the Chapter 2 to described the $\mathsf{Sp}(2n,\mathbb{C})$-Hitchin fibers of $\mathfrak{sl}(2)$-type.

DEFINITION 4.2.11. Let $a_{2n} \in H^0(X, K^{2n})$. An associated Higgs divisor is a divisor $D \in \mathsf{Div}(X)$, such that $\mathsf{supp}(D) \subset Z(a_{2n})$ and for all $x \in Z(a_{2n})$

$$0 \leq D_x \leq \tfrac{m}{2}.$$

LEMMA 4.2.12. *Let $(a_2, \dots, a_{2n}) \in B_{2n}(X, K)$ of $\mathfrak{sl}(2)$-type. Let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \dots, a_{2n})$ and $x \in Z(a_{2n}) \subset X$ a zero of order m. There exists a* *coordinate neighbourhood (U,z) centred at x and a frame of $E|_U$, such that the Higgs field is given by*

$$\Phi = \left(\begin{array}{cc|ccc} 0 & z^{l_x} & & & \\ z^{m-l_x} & 0 & & & \\ \hline & & * & \cdots & * \\ & & \vdots & \ddots & \vdots \\ & & * & \cdots & * \end{array} \right) \mathrm{d}z$$

for some $0 \leq l_x \leq \frac{m}{2}$. The Higgs divisor of (E,Φ) is the divisor

$$D = \sum_{x \in Z(a_{2n})} l_x.$$

PROOF. By assumption 0 is an eigenvalue of Φ_x of algebraic multiplicity two. Therefore, we can find a neighbourhood U of x, such that $(E,\Phi)|_U = (E_0 \oplus E_1, \Phi_0 \oplus \Phi_1)$, where E_0 is of rank 2 with $\Phi_0(x)$ nilpotent and E_1 is of rank $2n-2$ with Φ_1 having non-zero eigenvalues. Now, we can bring Φ_0 in the desired form by Lemma 2.4.1. □

For $a_{2n} \in H^0(K^{2n})$ let

$$n_{\text{even}} = \#\{x \in Z(a_{2n}) \mid x \text{ zero of even order}\}$$
$$n_{\text{odd}} = \#\{x \in Z(a_{2n}) \mid x \text{ zero of odd order.}\}$$

THEOREM 4.2.13. *Let $(a_2, \ldots, a_{2n}) \in B_{2n}(X,K)$ of $\mathfrak{sl}(2)$-type, such that Σ is irreducible and reduced. There is a stratification*

$$\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n}) = \bigsqcup_D \mathcal{S}_D$$

by locally closed analytic sets $\mathcal{S}_D$ indicated by the Higgs divisors associated to a_{2n}. If a_{2n} has at least on zero of odd order, every stratum S_D is a holomorphic $(\mathbb{C}^*)^r \times (\mathbb{C})^s$-bundle over

$$\mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\pi}_2 : \tilde{\Sigma} \to \Sigma/\sigma)$$

with

$$r = n_{\text{even}}, \quad s = 2n(g-1) - \deg(D) - n_{\text{even}} - \frac{n_{\text{odd}}}{2}.$$

If all zeroes of a_{2n} have even order, each stratum $\mathcal{S}_D$ is a $2:1$-branched covering of a holomorphic fiber bundle described as above. In particular,

$$\dim \mathcal{S}_D = (2n^2+n)(g-1) - \deg(D).$$

PROOF. This is a direct consequence of Theorem 4.2.10 and the stratification result for singular fibers of $\mathsf{SL}(2,\mathbb{C})$-Hitchin systems with irreducible and reduced spectral curve in Theorem 2.4.11. The dimension of the Prym varieties is given by

$$\dim \mathsf{Prym}(\tilde{\pi}_2 : \tilde{\Sigma} \to \Sigma/\sigma) = g(\tilde{\Sigma}) - g(\Sigma/\sigma) = n(2n-1)(g-1) + \frac{n_{\text{odd}}}{2}.$$

□

THEOREM 4.2.14. *Let* $(a_2, \ldots, a_{2n}) \in B_{2n}(X, K)$ *of* $\mathfrak{sl}(2)$*-type, such that* $a_{2n} \in H^0(X, K^{2n})$ *has only zeroes of odd order. Then* $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n})$ *fibers holomorphically over*

$$\mathsf{Prym}_\Lambda(\tilde{\pi}_2 : \tilde{\Sigma} \to \Sigma/\sigma)$$

with fibers given by the compact moduli of Hecke parameters described in Section 2.6.3.

PROOF. This is a direct consequence of Theorem 2.6.14. $\square$

Putting together Corollary 2.6.15, 2.6.17 and Example 2.7.3, 2.7.5 we obtain:

EXAMPLE 4.2.15. Let $(a_2, \ldots, a_{2n}) \in B_{2n}(X, K)$ of $\mathfrak{sl}(2)$-type. Let a_{2n} have k_l zeroes of order l for $l \in \{2, 3, 4, 5\}$ and at least one zero of odd order. Then up to normalisation $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n})$ is given by a holomorphic fiber bundle

$$(\mathbb{P}^1)^{k_2+k_3} \times (\mathbb{P}(1,1,2))^{k_4+k_5} \to \mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n}) \to \mathsf{Prym}_\Lambda(\tilde{\pi}_2 : \tilde{\Sigma} \to \Sigma/\sigma).$$

REMARK 4.2.16. We will show in Theorem 5.10 using analytic techniques that all these fiber bundles are smoothly trivial.

COROLLARY 4.2.17 (Corollary 2.7.6, Theorem 2.7.8). *Let* $(a_2, \ldots, a_{2n}) \in B_{2n}(X, K)$ *of* $\mathfrak{sl}(2)$*-type, such that* $a_{2n} \in H^0(X, K^{2n})$ *has at least one zero of odd order, then* $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n})$ *is an irreducible complex space. If all zeroes of* a_{2n} *have even order, then* $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(a_2, \ldots, a_{2n})$ *is connected and reducible.*

REMARK 4.2.18. Notice that the identification of Hitchin fibers in Theorem 4.2.10 is not restricted to $\mathfrak{sl}(2)$-type Hitchin fibers with irreducible and reduced spectral curve. In particular, the parametrization of singular Hitchin fibers with reducible spectral curve in [GO13] Section 7 describes certain $\mathfrak{sl}(2)$-type Hitchin fibers of the $\mathsf{Sp}(2n, \mathbb{C})$-Hitchin system, for all $n \in \mathbb{N}$.

4.3. $\mathfrak{sl}(2)$-type fibers of odd orthogonal Hitchin systems

4.3.1. The $\mathsf{SO}(2n+1, \mathbb{C})$-Hitchin system. Let $G = \mathsf{SO}(2n+1, \mathbb{C})$ and

$$\mathfrak{so}(2n+1, \mathbb{C}) = \left\{ A \in \mathsf{Mat}(n \times n, \mathbb{C}) \mid A^{\mathrm{tr}} J_{2n+1} + J_{2n+1} A = 0 \right\},$$

where

$$J_{2n+1} = \begin{pmatrix} 0 & \mathrm{id}_n & 0 \\ \mathrm{id}_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then a Cartan subalgebra is given by

$$\mathfrak{h} = \{ H = \begin{pmatrix} h_1 & & & & & & \\ & \ddots & & & & & \\ & & h_n & & & & \\ & & & -h_1 & & & \\ & & & & \ddots & & \\ & & & & & -h_n & \\ & & & & & & 0 \end{pmatrix} \mid h_i \in \mathbb{C} \}.$$

Define by $e_i \in \mathfrak{h}^\vee$ by $e_i(H) = h_i$. Then a root system is given by

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{\pm e_i \mid 1 \leq i \leq n\}.$$

As before the $\mathfrak{so}(2n+1, \mathbb{C})$-discriminant decomposes by the lengths of the roots

$$\mathsf{disc}_{\mathfrak{so}} = \prod_{i=1}^{n} -e_i^2 \mathsf{disc}_{\mathfrak{so}}^{\mathsf{red}}, \quad \text{where} \quad \mathsf{disc}_{\mathfrak{so}}^{\mathsf{red}} = \prod_{i \neq j} -(e_i \pm e_j)^2.$$

The characteristic polynomial of $A \in \mathfrak{so}(2n, \mathbb{C})$ has the form

$$\lambda(\lambda^{2n} + a_2 \lambda^{2n-2} + \cdots + a_{2n}).$$

The coefficients $a_2, \ldots, a_{2n}$ form a basis of the invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$.

Definition 4.3.1. An M-twisted $\mathsf{SO}(m, \mathbb{C})$-Higgs bundle is a triple (E, Φ, ω) of a

i) holomorphic vector bundle E of rank m with $\det(E) \cong \mathcal{O}_X$ together with a holomorphic non-degenerate symmetric bilinear form $\omega \in H^0(X, S^2 E^\vee)$, and

ii) $\Phi \in H^0(X, \mathrm{End}(E) \otimes M)$, such that $\omega(\Phi\,\cdot, \cdot) = -w(\cdot, \Phi\,\cdot)$.

(E, Φ, ω) is called stable, if for all isotropic Φ-invariant subbundles $0 \neq F \subsetneq E$

$$\deg(F) < 0.$$

(see [GGR09] for the simplified stability condition).

Let $\mathcal{M}_{\mathsf{SO}(m,\mathbb{C})}(X, M)$ be the moduli space of stable M-twisted $\mathsf{SO}(m, \mathbb{C})$-Higgs bundles on X. For $m = 2n+1$ the Hitchin map is given by

$$\mathsf{Hit}_{\mathsf{SO}(2n+1,\mathbb{C})} : \mathcal{M}_{\mathsf{SO}(2n+1,\mathbb{C})}(X, M) \to B_{\mathsf{SO}(2n+1,\mathbb{C})}(X, M) := \bigoplus_{i=1}^{n} H^0(X, M^{2i}),$$

$$(E, \Phi, \omega) \mapsto (a_2(\Phi), \ldots, a_{2n}(\Phi)).$$

In particular, we observe that $B_{\mathsf{SO}(2n+1,\mathbb{C})}(X, M) = B_{2n}(X, M)$. Let $(E, \Phi, \omega) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(a_2, \ldots, a_{2n})$, then the characteristic polynomial of Φ is given by

$$\lambda(\lambda^{2n} + a_{2n-2} \lambda^{2n-2} + \cdots + a_{2n}).$$

Hence, the spectral curve decomposes in two irreducible components $\underline{0} \cup \Sigma$, where $\underline{0}$ is the image of the zero section in M and Σ is the $\mathsf{Sp}(2n, \mathbb{C})$-spectral curve associated to $(a_2, \ldots, a_n)$.

From Lemma 4.1.3 we immediately have

Lemma 4.3.2. *If* $\mathsf{disc}_{\mathfrak{so}}(a_2, \ldots, a_{2n}) \in H^0(X, M^{2n^2})$ *has simple zeroes, then* Σ *is smooth.*

Definition 4.3.3. An element of the Hitchin base $\underline{a} \in B_{\mathsf{SO}(2n+1,\mathbb{C})}(X, M)$ is called $\mathfrak{sl}(2)$-type if Σ/σ is smooth. In this case, the corresponding Hitchin fiber $\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a})$ is called $\mathfrak{sl}(2)$-type. A M-twisted $\mathsf{SO}(2n+1, \mathbb{C})$-Higgs bundles is of $\mathfrak{sl}(2)$-type, if it is contained in a $\mathfrak{sl}(2)$-type Hitchin fiber.

Hence, the descriptions and properties of $\mathfrak{sl}(2)$-type spectral curves in Section 4.2.1 carry over to $\mathfrak{sl}(2)$-type Hitchin fibers of the odd orthogonal Hitchin system by adding the irreducible component $\underline{0}$.

4.3.2. Odd orthogonal $\mathfrak{sl}(2,\mathbb{C})$-type fibers as fibers of an $\mathsf{SO}(3,\mathbb{C})$-Hitchin map.

LEMMA 4.3.4. *Let $(E,\Phi,\omega) \in \mathcal{M}_{\mathsf{SO}(2n+1,\mathbb{C})}(X,M)$ of $\mathfrak{sl}(2)$-type. Let $p \in Z(\det(\Phi))$ a zero of order m, then there exists a coordinate neighbourhood (U,z) centred at p and an orthogonal splitting $(E,\Phi)\,|_U = (V_0 \oplus V_1, \Phi_0 \oplus \Phi_1)$, such that V_0 is of rank 3, $\Phi_0(p)$ is nilpotent and V_1 is of rank $2n-2$ containing the eigenspaces to eigenvalues λ with $\lambda(p) \neq 0$. Furthermore, Φ_0 can be locally written with respect to an orthogonal frame as*

$$\Phi_0(z) = z^{l_p} \begin{pmatrix} 0 & 1 - z^{m-2l_p} & 0 \\ z^{m-2l_p} - 1 & 0 & i(z^{m-2l_p}+1) \\ 0 & -i(z^{m-2l_p}+1) & 0 \end{pmatrix} \mathrm{d}z.$$

PROOF. By construction (V_0,Φ_0) is a $\mathsf{O}(3,\mathbb{C})$-Higgs bundle on U. Due to the exceptional isomorphism $\mathsf{SO}(3,\mathbb{C}) \cong \mathsf{PSL}(2,\mathbb{C})$ the Higgs field Φ_0 can be obtained as $\mathsf{ad}(\Psi)$ for a $\mathsf{SL}(2,\mathbb{C})$-Higgs field Ψ (cf. Section 4.4). By Lemma 2.4.1 we can find a local frame, such that

$$\Psi = \begin{pmatrix} 0 & z^{l_p} \\ z^{m-l_p} & 0 \end{pmatrix} \mathrm{d}z.$$

With respect to the induced local frame of V_0 the Higgs field Φ is given by

$$\Phi = \mathsf{ad}(\Psi) = \begin{pmatrix} 0 & -z^{l_p} & 0 \\ -z^{m-l_p} & 0 & z^{l_p} \\ 0 & z^{m-l_p} & 0 \end{pmatrix} \mathrm{d}z$$

and the orthogonal structure induced by the Killing form by

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Choosing an orthogonal frame we obtain the desired form. □

DEFINITION 4.3.5. Let $(E,\Phi,\omega) \in \mathcal{M}_{\mathsf{SO}(2n+1,\mathbb{C})}(X,M)$. The Higgs divisor of (E,Φ,ω) is the divisor

$$D(E,\Phi,\omega) := \sum_{p \in Z(a_{2n})} l_p,$$

where l_p is defined by the previous lemma.

LEMMA 4.3.6. *Let $(E,\Phi,\omega) \in \mathcal{M}_{\mathsf{SO}(2n+1,\mathbb{C})}(X,M)$ of $\mathfrak{sl}(2)$-type and $D = D(E,\Phi)$, then*

i) $\mathsf{ker}(\Phi) \cong M^{-n}(D)$ *and* $\omega\,|_{\mathsf{ker}(\Phi)} = \frac{a_{2n}}{s_D^2} \in H^0(X, M^{2n}(-2D))$, *where s_D denotes the canonical section of $\mathcal{O}(D)$.*

ii) *there is an exact sequence of coherent sheaves*

$$0 \to \mathcal{O}(\mathsf{ker}(\Phi) \oplus \mathsf{ker}(\Phi)^{\perp}) \to \mathcal{O}(E) \to \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a torsion sheaf with $\det(\mathcal{T}) \cong \mathcal{O}(\Lambda - 2D)$.

iii) (E,Φ,ω) *is uniquely determined by D and*

$$\left(\mathsf{ker}(\Phi)^{\perp}, \Phi\,|_{\mathsf{ker}(\Phi)^{\perp}}, \omega\,|_{\mathsf{ker}(\Phi)^{\perp}}\right).$$

PROOF. i) The proof of the first assertion is closely following an argument in Section 4.1/4.2 of [Hit07] using the local form for the Higgs field describe in Lemma 4.3.4. Consider an orthogonal splitting $E = V_0 \oplus V_2 \oplus \cdots \oplus V_n$, such that V_0 is as in the lemma and V_i for $i \geq 2$ is rank 2 containing the eigen spaces to eigenvalues $\pm\lambda_i \neq 0$. Let e_0, e_1, e_2 be an orthogonal frame for V_0, such that Φ_0 has the prescribed form and e_{2i-1}, e_{2i} an orthogonal frame of V_i of eigen sections of Φ. Then the induced alternating bilinear form $\alpha := \omega(\Phi\cdot, \cdot)$ is given by

$$\alpha = iz^l\left(e_2 \wedge (e_3 + ie_1) + z(\cdots)\right) \wedge i\lambda_2(e_3 \wedge e_4) \cdots \wedge i\lambda_n(e_{2n-1} \wedge e_{2n}).$$

Let us assume that with respect to our frame the volume form is given by $\mathrm{vol} = e_0 \wedge \cdots \wedge e_{2n} \in H^0(U, \det(E))$. Then, we can write $\bigwedge^n \alpha \in H^0(U, \bigwedge^{2n} E \otimes M^n)$ as a contraction $i_{v_0}\mathrm{vol}$ with

$$v_0 = -i^{n-1} z^l \lambda_2 \cdots \lambda_n (e_3 + ie_1) + z^{l+1}(\cdots) \in H^0(U, E \otimes M^n).$$

So v_0 defines a non-vanishing section of $H^0(X, E \otimes M^n(-D))$ that spans the kernel of Φ. Hence, $\mathsf{ker} = \mathsf{ker}(\Phi) \cong M^{-n}(D)$.

Furthermore, using the local form in previous lemma on computes that for $p \in Z(a_{2n})$ we have $\omega|_{\mathsf{ker}} = z^{\mathsf{ord}_p a_{2n} - 2D_p}$. Hence (up to the right choice of s_D)

$$\omega|_{\mathsf{ker}} = \frac{a_{2n}}{s_D^2} \in H^0(X, M^{2n}(-2D)).$$

ii) $\mathsf{ker}^\perp \subset E$ is a Φ-invariant subbundle of rank $2n$, such that

$$E|_U \cong \mathsf{ker} \oplus \mathsf{ker}^\perp|_U$$

for all contractable $U \subset X$, such that $U \cap Z(a_{2n}) = \varnothing$. Hence, the inclusions define an exact sequence of coherent sheaves

$$0 \to \mathcal{O}(\mathsf{ker} \oplus \mathsf{ker}^\perp) \to O(E) \to \mathcal{T} \to 0$$

with $\mathcal{T}$ a torsion sheaf supported on $Z(a_{2n})$. Now, $\det(\mathcal{T})$ can be computed from the local description in Lemma 4.3.4.

iii) Stated differently ii) tells us that E is a Hecke modification of $\mathsf{ker} \oplus \mathsf{ker}^\perp$ (see Definition 3.1.4). We need to show that there is a unique Hecke modification doing the job, i. e. a unique Hecke modification, such that

$$V = \mathsf{ker} \oplus \mathsf{ker}^\perp$$

with its degenerate symmetric bilinear form

$$\beta = \omega|_{\mathsf{ker}} \oplus \omega|_{\mathsf{ker}^\perp}$$

is transformed into a $\mathsf{SO}(2n+1, \mathbb{C})$-bundle $(\hat{V}, \hat{\beta})$. At $p \in Z(a_{2n})$ we have an orthogonal decomposition

$$\left(\mathsf{ker}^\perp, \Phi|_{\mathsf{ker}^\perp}\right)|_U = (V_2 \oplus V_1, \Phi_2 \oplus \Phi_1)$$

by restricting the orthogonal decomposition in Lemma 4.3.4. One the one side V_2 is of rank 2 and $\Phi_2(p)$ is nilpotent, on the other, Φ_1 has non-zero eigenvalues and $\omega|_{V_1}$ is non-degenerate. Thereby, we are left with showing that we can find a unique Hecke modification twisting

$$\left(\mathsf{ker}|_U \oplus V_2, \frac{a_{2n}}{s_D^2} \oplus \omega|_{V_2}\right)$$

into a $\mathsf{SO}(3,\mathbb{C})$-bundle. The existence is clear by b).

Using the local form described in Lemma 4.3.4 one can show that there are local frames e_0 of ker_U and e_1, e_2 of V_2, such that the non-degenerate bilinear form at p is given by

$$\frac{a_{2n}}{s_D^2} \oplus \omega|_{V_2} = \begin{pmatrix} z^{m-2l} & 0 & 0 \\ 0 & z^{m-2l} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m = \mathsf{ord}_p(a_{2n})$ and $l = D_p$. Hence, the Hecke modification can be assumed to take place in $\mathsf{span}\{e_0, e_1\}$.

If there were two Hecke modification,

$$\begin{tikzcd} \mathsf{ker}|_U \oplus V_2 \arrow[r, "s_1"] \arrow[dr, "s_2"'] & F_1 \\ & F_2 \end{tikzcd}$$

such that F_1, F_2 are $\mathsf{SO}(3,\mathbb{C})$-bundles with the induced orthogonal structure, then up to choosing frames $s_1 \circ s_2^{-1}$ reduces to a meromorphic $\mathsf{SO}(2,\mathbb{C})$-gauge (an element of the $\mathsf{SO}(2,\mathbb{C})$-loop group). It is not hard to show, that such a gauge is automatically holomorphic. Hence, the resulting $\mathsf{SO}(3,\mathbb{C})$-bundles F_1, F_2 are isomorphic.

$\square$

PROPOSITION 4.3.7. *Let $\underline{a} \in B_{2n}(X,M)$ with $\mathfrak{sl}(2)$-type spectral curve and $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$ the induced section. The pushforward induces a holomorphic map*

$$\mathcal{M}_{\mathsf{SO}(3,\mathbb{C})}(\Sigma/\sigma, \pi_n^* M) \supset \mathsf{Hit}^{-1}_{\mathsf{SO}(3,\mathbb{C})}(b_2) \to \mathsf{Hit}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a}) \subset \mathcal{M}_{\mathsf{SO}(2n+1,\mathbb{C})}(X,M).$$

PROOF. Let $(E,\Phi,\omega) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(3,\mathbb{C})}(b_2)$. The pushforward

$$\pi_{n*}\left(\mathsf{ker}(\Phi)^\perp \otimes \pi_n^* M^{n-1}, \Phi|_{\mathsf{ker}(\Phi)^\perp}, \partial\pi_n^{-1}\omega|_{\mathsf{ker}(\Phi)^\perp}\right)$$

defines a M-twisted $\mathsf{GL}(2n,\mathbb{C})$-Higgs bundle on X with

$$\det\left(\pi_{n*}\left(\mathsf{ker}(\Phi)^\perp \otimes \pi_n^* M^{n-1}\right)\right) = M^{-n}(\mathsf{Nm} D).$$

and a symmetric bilinear form $\pi_{n*}\left(\partial\pi_n^{-1}\omega|_{\mathsf{ker}(\Phi)^\perp}\right)$, which is non-degenerate away from $Z(a_{2n})$ by Proposition 3.2.4. Furthermore, $\pi_{n*}\Phi$ is anti-symmetric with respect to this symmetric bilinear form. Moreover, we have a induced Higgs divisor given by $\mathsf{Nm}(D)$ that is supported at $Z(a_{2n})$. Now there is a unique way to recover a $\mathsf{SO}(2n+1,\mathbb{C})$-Higgs bundle out of this data by Lemma 4.3.6. $\square$

PROPOSITION 4.3.8. *Let $\underline{a} \in B_{2n}(X, M)$ with $\mathfrak{sl}(2)$-type spectral curve and $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$ the induced section. The pullback along $\pi_n : \Sigma/\sigma \to X$ induces a holomorphic map*

$$\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(a_2, \ldots, a_{2n}) \to \mathsf{Hit}^{-1}_{\mathsf{SO}(3,\mathbb{C})}(b_2) \subset \mathcal{M}_{\mathsf{SO}(3,\mathbb{C})}(\Sigma/\sigma, \pi_n^* M)$$

PROOF. Let $(E, \Phi, \omega) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(a_2, \ldots, a_{2n})$. The pullback of the characteristic polynomial to Σ/σ

$$\lambda \left(\lambda^{2n} + \pi_n^* a_2 \lambda^{2n-2} + \cdots + \pi_n^* a_{2n} \right)$$

factors through $\lambda(\lambda^2 + b_2)$ and hence defines a generalised eigen bundle E_3 on Σ/σ by

$$0 \to E_3 \to \pi_n^* E \xrightarrow{\Psi} \pi_n^*(E \otimes M^3) \to E_3 \otimes \pi_n^* M^{2n+1} \to 0,$$

where

$$\Psi := \pi_n^* \Phi \left(\pi_n^* \Phi^2 + b_2 \mathrm{id}_{\pi_n^* E} \right).$$

The dual exact sequence tensored with $\pi_n^* M^3$ results in

$$0 \to E_3^\vee \otimes \pi_n^* M^{2-2n} \to \pi_n^* E^\vee \xrightarrow{\Psi^\vee} \pi_n^*(E^\vee \otimes M^3) \to E_3^\vee \otimes \pi_n^* M^3 \to 0.$$

The orthogonal bilinear form ω identifies E with $E^\vee$ and from the anti-symmetry of the Higgs field $\Psi^\vee = -\Psi$ under this identification. Hence, ω induces an isomorphism $E_3 \cong E_3^\vee \otimes \pi_n^* M^{2-2n}$. Finally, ω restricts to a symmetric, non-degenerate bilinear form ω_3 on $E_3 \otimes \pi_n^* M^{n-1}$ and the induced Higgs field Φ_3 on E_3 is anti-symmetric with respect to it. Hence, (E_3, Φ_3) is a $\pi_n^* M$-twisted $\mathsf{SO}(3,\mathbb{C})$-Higgs bundle on Σ/σ. □

THEOREM 4.3.9. *Let $\underline{a} \in B_{2n}(X, M)$ with $\mathfrak{sl}(2)$-type spectral curve and let $b_2 \in H^0(\Sigma/\sigma, \pi_n^* M^2)$ the induced section. The holomorphic map between the Hitchin fibers*

$$\mathsf{Hit}^{-1}_{\mathsf{SO}(3,\mathbb{C})}(b_2) \to \mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a})$$

defined in Proposition 4.3.7 is a biholomorphism of complex spaces.

PROOF. Let $(E, \Phi, \omega) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a})$. Then

$$(E_3, \Phi_3) = \mathsf{ker} \left(\pi_n^* \Phi (\pi_n^* \Phi^2 + b_2 \mathrm{id}) \right)$$

decomposes $\pi^*(E, \Phi)$ into rank 3 subbundles. Each sheet of the reduces spectral cover $\pi_n : \Sigma/\sigma \to X$ corresponds to a pair of eigenvalues $\pm\lambda$. The pushforward of $\mathsf{ker}(\Phi_3)^\perp \subset E_3$ reassembles these pieces. The extension of $\pi_{n*}(\mathsf{ker}(\Phi_3)^\perp \otimes \pi_n^* M^{n-1})$ is uniquely determined and actually locally at $p \in Z(a_{2n})$ is a precise copy of $E_3|_U$, where U is neighbourhood of the corresponding zero $p \in Z(b_2)$. Hence, we recover (E, Φ).

The argument for the converse is an adaptation of the argument in the proof of Theorem 4.2.10. We start with $(E_3, \Phi_3) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(3,\mathbb{C})}(b_2)$. Consider the holomorphic map

$$\mathsf{ker}(\Phi_3)^\perp \otimes \pi_n^* M^{1-n} \to \mathsf{ker}(\Phi_3)^\perp \otimes \pi_n^* M^{n-1}$$

by multiplying with $s_R = \partial \pi_n \in H^0(\Sigma/\sigma, \pi_n^* M^{2n-2})$. This induces an inclusion of vector bundles

$$0 \to \mathsf{ker}(\Phi_3)^\perp \otimes \pi_n^* M^{1-n} \to \pi_n^* \pi_{n*} \left(\mathsf{ker}(\Phi_3)^\perp \otimes \pi_n^* M^{n-1} \right).$$

Hence, we recover $\mathsf{ker}^{\perp}$ with the map defined in Proposition 4.3.8. This uniquely determines (E_3, Φ_3) by Lemma 4.3.6 iii). □

Remark 4.3.10 (Alternative approach). Another way to obtain the result for $\mathsf{SO}(2n+1, \mathbb{C})$ is suggested in [Hit07]. Hitchin describes the regular $\mathsf{SO}(2n+1, \mathbb{C})$-Hitchin fibers by relating them to the corresponding $\mathsf{Sp}(2n, \mathbb{C})$-Hitchin fiber on X. Let $(V, \Phi, g) \in \mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a})$ with $\underline{a}$ of $\mathfrak{sl}(2)$-type. Adopting Hitchin's notation, let $V_0 \subset V$ be the kernel line bundle and $\Phi' : V/V_0 \to V/V_0$ the induced Higgs field. It is easy to see that $\omega := g(\Phi' \cdot, \cdot)$ defines a holomorphic anti-symmetric bilinear form on V/V_0 that is non-degenerate, where Φ has distinct eigenvalues. If $\deg(D) \equiv 0 \mod 2n$, where $D = D(V, \Phi)$, we can choose a square root $L^{2n} = K^{-n}(D)$ and define a symplectic Higgs bundle by

$$(E := V/V_0 \otimes L, \phi', \omega).$$

$\bigwedge^n \omega \in H^0(X, \det(E))$ is generically non-zero and $\det(E) = \mathcal{O}_X$ by Lemma 4.3.6 i). Hence, ω is non-degenerate on E. For regular Hitchin fibers, D is always zero and therefore this defines a map

$$\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a}) \to \mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(\underline{a}).$$

Hitchin uses this map to study the regular $\mathsf{SO}(2n+1, \mathbb{C})$-fibers as a covering space of the corresponding symplectic Hitchin fiber and proves the duality in this way (cf. Section 1.4.3). The singular fibers are stratified by the Higgs divisors D. One the highest-dimensional stratum $D = 0$ and we could apply the same argument. But for the lower strata $\deg(D) \mod 2n$ is unconstrained. Hence, this trick does not work.

4.4. Langlands correspondence for $\mathfrak{sl}(2)$-type Hitchin fibers

In this section, we compare the $\mathfrak{sl}(2)$-type Hitchin fibers for the Langlands dual groups $\mathsf{Sp}(2n, \mathbb{C})$ and $\mathsf{SO}(2n+1, \mathbb{C})$ projection to the same point in the Hitchin base. Concerning the abelian part of the spectral data we will recover torsors over dual abelian varieties. This reproves and generalizes the result for regular fibers in [Hit07]. The non-abelian part of the spectral data will not change under the duality. This is a new phenomena. So far we are lacking a conceptual interpretation of this phenomena in terms of mirror symmetry. We will start with the rank 1 case.

For $\mathrm{rk}(\mathfrak{g}) = 1$, we can compare the Hitchin fibers by using the exceptional isomorphisms $\mathsf{Sp}(2, \mathbb{C}) \cong \mathsf{SL}(2, \mathbb{C})$ and $\mathsf{SO}(3, \mathbb{C}) \cong \mathsf{PGL}(2, \mathbb{C})$. The moduli space of $\mathsf{PGL}(2, \mathbb{C})$-Higgs bundles can be constructed as follows (see [Hau13]). First recall that

$$\mathcal{M}_{\mathsf{GL}(1,\mathbb{C})}(X, M) \cong \mathsf{Pic}(X) \times H^0(X, M)$$

is an abelian group with an action on $\mathcal{M}_{\mathsf{GL}(2,\mathbb{C})}(X, M)$. Let

$$(L, \lambda) \in \mathcal{M}_{\mathsf{GL}(1,\mathbb{C})}(X, M) \quad \text{and} \quad (E, \Phi) \in \mathcal{M}_{\mathsf{GL}(2,\mathbb{C})}(X, M),$$

then the action is given by

$$((L, \lambda), (E, \Phi)) \mapsto (E \otimes L, \Phi + \lambda \mathrm{id}_E).$$

Define

$$\mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M) = \mathcal{M}_{\mathsf{GL}(2,\mathbb{C})}(X,M)/\mathcal{M}_{\mathsf{GL}(1,\mathbb{C})}(X,M).$$

Acting with $H^0(X,M)$, we can find a representative for each $\mathsf{PGL}(2,\mathbb{C})$-Higgs bundles with $\mathrm{tr}(\Phi)=0$. Hence,

$$\mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M) \cong \mathsf{Hit}^{-1}_{\mathsf{GL}(2,\mathbb{C})}(B_{\mathsf{SL}(2,\mathbb{C})}(X,M))/\mathsf{Pic}(X),$$

where we think of $B_{\mathsf{SL}(2,\mathbb{C})}(X,M) \subset B_{\mathsf{GL}(2,\mathbb{C})}(X,M)$ by the obvious inclusion. For $N \in \mathsf{Pic}(X)$ define

$$\mathcal{M}^N_{\mathsf{SL}(2,\mathbb{C})}(X,M) = \left\{(E,\Phi) \in M_{\mathsf{GL}(2,\mathbb{C})}(X,M) \mid \det(E) = N, \mathrm{tr}(\Phi) = 0\right\}.$$

The action of $\mathsf{Pic}(X)$ identifies $\mathcal{M}^{N_1}_{\mathsf{SL}(2,\mathbb{C})}(X,M)$ and $\mathcal{M}^{N_2}_{\mathsf{SL}(2,\mathbb{C})}(X,M)$, whenever $\deg(N_1) = \deg(N_2) \mod 2$. Hence, fixing a line bundle $N \in \mathsf{Pic}(X)$ of degree 1, we have

$$\mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M) = \left(\mathcal{M}^{\mathcal{O}_X}_{\mathsf{SL}(2,\mathbb{C})}(X,M) \sqcup \mathcal{M}^N_{\mathsf{SL}(2,\mathbb{C})}(X,M)\right)/\mathsf{Jac}(X)[2], \tag{14}$$

where $\mathsf{Jac}(X)[2] \cong \mathbb{Z}_2^{2g}$ denotes the group of two-torsion points of $\mathsf{Jac}(X)$.

The isomorphism to the moduli space of $\mathsf{SO}(3,\mathbb{C})$-Higgs bundles is defined using the adjoint representation

$$\begin{array}{ccc} \mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M) & \to & \mathcal{M}_{\mathsf{SO}(3,\mathbb{C})}(X,M) \\ (E,\Phi) & \mapsto & \left((E \times_{\mathsf{Ad}} \mathfrak{sl}(2,\mathbb{C})) \otimes \det(E)^{-1}, \mathsf{ad}(\Phi), \omega\right). \end{array}$$

The orthogonal structure ω is induced by the Killing form on $\mathfrak{sl}(2,\mathbb{C})$. Topologically $\mathsf{SO}(3,\mathbb{C})$-Higgs bundles on a Riemann surface are classified by the second Stiefel-Whitney class

$$\mathsf{sw}_2 \in H^2(X,\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

This is the obstruction to lift a $\mathsf{SO}(3,\mathbb{C})$-Higgs bundle to a $\mathsf{Spin}(3,\mathbb{C}) \cong \mathsf{SL}(2,\mathbb{C})$-Higgs bundle. Hence, under the isomorphism $\mathcal{M}^{\mathcal{O}_X}_{\mathsf{SL}(2,\mathbb{C})}(X,M)/\mathsf{Jac}(X)[2]$ is mapped onto the connected component of $\mathsf{SO}(3,\mathbb{C})$-Higgs bundles with $sw_2 = 0$ and $\mathcal{M}^N_{\mathsf{SL}(2,\mathbb{C})}(X,M)/\mathsf{Jac}(X)[2]$ onto the connected component with $sw_2 = 1$.

The Hitchin map

$$\mathsf{Hit}_{\mathsf{PGL}(2,\mathbb{C})} : \mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M) \to H^0(X,M^2)$$

is defined in terms of the decomposition (14) by the $\mathsf{SL}(2,\mathbb{C})$-Hitchin map on each connected component.

For $(E,\Phi) \in \mathcal{M}_{\mathsf{PGL}(2,\mathbb{C})}(X,M)$, there is a well-defined $\mathsf{SL}(2,\mathbb{C})$-Higgs field Φ by (14). In particular, we can define the Higgs divisor $D(E,\Phi)$ as we did in Lemma 4.2.12 (cf. Definition 2.5.1).

THEOREM 4.4.1. *Let $a_2 \in H^0(X,M^2)$, such that the spectral curve is irreducible and reduced, then there is a stratification*

$$\mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2) = \bigsqcup_D \mathcal{S}_D$$

by finitely many locally closed analytic sets $\mathcal{S}_D$ indicated by Higgs divisors D associated to a_2. If there is at least on zero of a_2 of odd order, each stratum is a holomorphic $(\mathbb{C}^*)^r \times \mathbb{C}^s$-bundle over

$$\left(\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma}) \sqcup \mathsf{Prym}_{\tilde{\pi}^*N^{-1}(\Lambda-\tilde{\pi}^*D)}(\tilde{\Sigma})\right)/\mathsf{Jac}(X)[2],$$

where

$$r = n_{\text{even}}, \quad s = 2n(g-1) - \deg(D) - n_{\text{even}} - \frac{n_{\text{odd}}}{2}.$$

If all zeroes of a_2 are of even order, each stratum $\mathcal{S}_D$ is a holomorphic fiber bundle like this over

$$\left(\mathsf{Prym}_{I(\frac{1}{2}\mathsf{div}(q_2)-D)}(\tilde{\Sigma}) \sqcup \mathsf{Prym}_{N^{-1}I(\frac{1}{2}\mathsf{div}(q_2)-D)}(\tilde{\Sigma})\right)/\mathsf{Jac}(X)[2],$$

where I is the unique non-trivial line bundle on X, such that $\tilde{\pi}^*I = \mathcal{O}_X$. A local trivialisation of the fiber bundle $\mathcal{S}_D \subset \mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2)$ induces a local trivialisation of the fiber bundle structure of the corresponding stratum $\mathcal{S}_D \subset \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$ and vice versa.

Proof. Fix a $\mathsf{SL}(2,\mathbb{C})$-representative (E,Φ) of a Higgs bundle in

$$\mathcal{S}_D \subset \mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2) \subset \left(\mathcal{M}^{\mathcal{O}_X}_{\mathsf{SL}(2,\mathbb{C})}(X,M) \sqcup \mathcal{M}^{N}_{\mathsf{SL}(2,\mathbb{C})}(X,M)\right)/\mathsf{Jac}(X)[2]$$

By Theorem 2.4.4, we can associate an eigen line bundle L on the normalised spectral cover $\tilde{\pi} : \tilde{\Sigma} \to X$ to (E,Φ). If $\det(E) = \mathcal{O}_X$, it will lie in $\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma})$ and, if $\det(E) = N$, in $\mathsf{Prym}_{\tilde{\pi}^*N^{-1}(\Lambda-\tilde{\pi}^*D)}(\tilde{\Sigma})$. After choosing frames s of L at $\tilde{\pi}^{-1}Z(a_2)$ the $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle (E,Φ) is uniquely determined by its u-coordinate in $(\mathbb{C}^*)^r \times \mathbb{C}^s$ with r,s as in the Theorem. The action by $\mathsf{Jac}(X)[2]$ lifts to the normalised spectral curve and induces an action

$$\mathsf{Jac}(X)[2] \times \mathsf{Prym}_F(\tilde{\Sigma}) \to \mathsf{Prym}_F(\tilde{\Sigma}), \qquad (J,L) \mapsto \tilde{\pi}^*J \otimes L$$

for $F \in \mathsf{Pic}(X)$. For $F = \mathcal{O}(\Lambda - \tilde{\pi}^*D)$ and $F = \tilde{\pi}^*N^{-1}(\Lambda - \tilde{\pi}^*D)$, this is exactly the action on the eigen line bundle induced from the action of $\mathsf{Jac}(X)[2]$ on (E,Φ).

Recall, that in $\mathsf{SL}(2,\mathbb{C})$-case for $a_2 \in H^0(X,M^2)$ having only zeroes of even order, each stratum was a two-sheeted covering of a fiber bundle over the twisted Prym variety. This was due to the identification of (E,Φ) and $(E\otimes I,\Phi)$ via pullback. However, $I \in \mathsf{Jac}(X)[2]$ and so that for $\mathsf{PSL}(2,\mathbb{C})$-Higgs bundles the pullback is injective.

The non-abelian part of the spectral data decodes the local Hecke parameter at $\tilde{\pi}^{-1}Z(a_2)$ and does not change under the action of $\mathsf{Jac}(X)[2]$ on (E,Φ). Choosing a collection of frames j of J at $Z(a_2)$ we obtain a frame of $\tilde{\pi}^*J \otimes L$ at $\tilde{\pi}^{-1}Z(a_2)$ by $\tilde{\pi}^*j \otimes s$. The u-coordinate does not depend on the choice of j by Proposition 2.4.6. This proves the last assertion. ☐

Theorem 4.4.2. *Let $a_2 \in H^0(X,M^2)$, such that the spectral curve is locally irreducible, then the $\mathsf{PGL}(2,\mathbb{C})$-Hitchin fiber over a_2 is itself a holomorphic fiber bundle over*

$$\left(\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma}) \sqcup \mathsf{Prym}_{\tilde{\pi}^*N^{-1}(\Lambda-\tilde{\pi}^*D)}(\tilde{\Sigma})\right)/\mathsf{Jac}(X)[2]$$

with fibers given by the compact moduli of Hecke parameters.

PROOF. This is a direct consequence of the previous theorem and Theorem 2.6.14. ☐

EXAMPLE 4.4.3. Example 4.2.15 carries over to the $\mathsf{PGL}(2,\mathbb{C})$-case. Let a_{2n} have k_l zeroes of order l for $l \in \{2,3,4,5\}$ and at least one zero of odd order. Then up to normalisation $\mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2)$ is given by a holomorphic

$$(\mathbb{P}^1)^{k_2+k_3} \times (\mathbb{P}(1,1,2))^{k_4+k_5}-$$

bundle over

$$\left(\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma}) \sqcup \mathsf{Prym}_{\tilde{\pi}^*N^{-1}(\Lambda-\tilde{\pi}^*D)}(\tilde{\Sigma})\right)/\mathsf{Jac}(X)[2].$$

COROLLARY 4.4.4. *Let $a_2 \in H^0(X,M^2)$, such that the spectral curve is irreducible and reduced. The Hitchin fibers $\mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2)$ and $\mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$ are related as follows:*

i) *The abelian part of the spectral data are torsors over dual abelian varieties.*
ii) *The complex spaces of Hecke parameters are isomorphic.*

PROOF. This is immediate from the previous theorems and Theorem 1.4.13. In explicit, we showed in Theorem 4.4.1 that a trivialisation of the bundle of Hecke parameters of $\mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$ induces a trivialisation of the bundle of Hecke parameters of $\mathsf{Hit}^{-1}_{\mathsf{PGL}(2,\mathbb{C})}(a_2)$. The identity with respect to corresponding trivialisation induces an isomorphism between the Hecke parameters. ☐

THEOREM 4.4.5. *Let $\underline{a} \in B_{2n}(X,K)$ of $\mathfrak{sl}(2)$-type with irreducible and reduced $\mathsf{Sp}(2n,\mathbb{C})$-spectral curve. All the results from the previous section carry over to the $\mathsf{SO}(2n+1,\mathbb{C})$-case.*

Explicitly, there is a stratification

$$\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a}) = \bigsqcup_D \mathcal{S}_D$$

by fiber bundles over a disjoint union of abelian torsors

as described in Theorem 4.4.1 indicated by Higgs divisors. If a_{2n} has at least one zero of odd order, the abelian torsor is given by

$$\left(\mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma}) \sqcup \mathsf{Prym}_{\tilde{\pi}^*N^{-1}(\Lambda-\tilde{\pi}^*D)}(\tilde{\Sigma})\right)/\mathsf{Jac}(X)[2]$$

When a_{2n} has only zeroes of odd order, or equivalently if Σ is locally irreducible, we obtain a global fibering of the $\mathsf{SO}(2n+1,\mathbb{C})$-Hitchin fiber over the union of abelian torsor as described in Theorem 4.4.2.

Furthermore, Example 4.4.3 describes the first degenerations of singular $\mathfrak{sl}(2)$-type Hitchin fibers for $\mathsf{SO}(2n+1,\mathbb{C})$ up to normalisations.

PROOF. This is immediate from the identification of $\mathfrak{sl}(2)$-type Hitchin fibers for $\mathsf{SO}(2n+1,\mathbb{C})$ with fibers of the π_n^*K-twisted $\mathsf{SO}(3,\mathbb{C})$-Hitchin system on Σ/σ in Theorem 4.3.9. ☐

REMARK 4.4.6. It follows from Theorem 5.10 and the last assertion in Theorem 4.4.1, that all these fiber bundles are smoothly trivial.

In particular, Corollary 4.4.4 generalizes verbatim to higher rank:

Corollary 4.4.7. *Let $\underline{a} \in B_{2n}(X,K)$ of $\mathfrak{sl}(2)$-type, such that the spectral curve is irreducible and reduced. The Hitchin fibers $\mathsf{Hit}^{-1}_{\mathsf{SO}(2n+1,\mathbb{C})}(\underline{a})$ and $\mathsf{Hit}^{-1}_{\mathsf{Sp}(2n,\mathbb{C})}(\underline{a})$ are related as follows:*

i) *The abelian part of the spectral data is a disjoint union of torsors over dual abelian varieties.*
ii) *The complex spaces of Hecke parameters are isomorphic.*

CHAPTER 5

Solution to the decoupled Hitchin equation through semi-abelian spectral data

In this shorter chapter, we will show how to use semi-abelian spectral data for symplectic Higgs bundles of $\mathfrak{sl}(2)$-type to produce solutions to the decoupled Hitchin equation. This generalizes the construction in [Maz+14] and [Fre18b] to singular Hitchin fibers. For singular fibers of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin system it reproves the result obtained in [Moc16] using different methods. In the $\mathsf{SL}(2,\mathbb{C})$-case Mochizuki proved that these solutions to the decoupled Hitchin equation are Limiting Configurations - limits of actual solutions to the Hitchin equation along rays to the ends of the moduli space. We conjecture this to be true for the singular herimitian metrics that we will construct for $\mathsf{Sp}(2n,\mathbb{C})$-Higgs bundles of $\mathfrak{sl}(2)$-type.

Let $(E,\Phi,\omega) \in \mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X,M)$. A hermitian metric h on E compatible with the symplectic structure is a solution to the decoupled Hitchin equation, if the hermitian metric is flat and the Higgs field Φ is normal respective h. For $M=K$ this can be written as

$$F_h = 0, \quad [\Phi \wedge \Phi^{*h}] = 0.$$

Notice that the first formulation stands to reason for M-twisted Higgs bundles. In most cases, there are no smooth solutions to this equation. For $\mathsf{SL}(2,\mathbb{C})$ it is easy to check by a local computations similar to [Maz+14] Section 3.2, that h is singular at all zeroes of $\det(\Phi)$ of odd order (cf. Remark 5.3). Global solutions to the decoupled Hitchin equation can be constructed through the pushforward of a Hermitian-Einstein metric on the eigen line bundle $L \in \mathsf{Prym}_{\Lambda-\tilde{\pi}^*D}(\tilde{\Sigma})$

THEOREM 5.1 ([Moc16] Section 4.3). Let $(E,\Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X,M)$ with irreducible and reduced spectral curve. Let $a_2 = \det(\Phi)$, D its Higgs divisor and for $x \in Z(a_2)$ let $n_x := \mathsf{ord}_x a_2 - 2D_x \in \mathbb{N}_0$. Then there exists a hermitian metric $h_{dc} = h_{dc}(E,\Phi)$ on $E\,|_{X\setminus Z(a_2)}$ solving the decoupled Hitchin equation and inducing a non-singular hermitian metric on $\det(E)$. For $x \in Z(a_2)$ there exists a coordinate (U,w) centred at x and a local frame of $E|_U$, such that the hermitian metric is given by

$$h_{dc} = \begin{pmatrix} |w|^{\frac{1}{2}n_x} & v(w)\frac{|w|^{\frac{1}{2}n_x}}{w^{\frac{1}{2}(n_x-1)}} \\ \bar{v}(w)\frac{|w|^{\frac{1}{2}n_x}}{\bar{w}^{\frac{1}{2}(n_x-1)}} & |w|^{-\frac{1}{2}n_x} \end{pmatrix},$$

if $\mathsf{ord}_x(a_2) \equiv 1 \mod 2$ and by

$$h_{dc} = \begin{pmatrix} |w|^{\frac{1}{2}n_x} & v(w)\frac{|w|^{\frac{1}{2}n_x}}{w^{\frac{1}{2}n_x}} \\ \bar{v}(w)\frac{|w|^{\frac{1}{2}n_x}}{\bar{w}^{\frac{1}{2}n_x}} & |w|^{-\frac{1}{2}n_x} \end{pmatrix},$$

if $\mathsf{ord}_x(a_2) \equiv 0 \mod 2$. The holomorphic functions $v \in \mathcal{O}_U$ are determined through the u-coordinate of (E, Φ) at x.

Proof. Let $(E, \Phi) \in \mathcal{S}_D \subset \mathsf{Hit}^{-1}_{\mathsf{SL}(2,\mathbb{C})}(a_2)$. By Theorem 2.4.4 (E, Φ) defines an eigen line bundle $L := (\mathsf{Eig}_D \circ \tilde{\pi}^*)(E, \Phi) \in \mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma})$. Fix an auxiliary parabolic structure on L by introducing weights $\alpha_p := \frac{1}{2}(\Lambda - \tilde{\pi}^* D)_p$ for all $p \in Z(\tilde{\pi}^* a_2)$. Then the parabolic degree $\mathrm{pdeg}(L, \alpha) = 0$. Hence, there exists a hermitian metric h_L adapted to the parabolic structure that satisfies the Hermitian-Einstein equation

$$F_{h_L} = 0$$

unique up to rescaling by a constant (see [Biq96; Sim90]). This induces a flat hermitian metric $h + \sigma^* h$ on $E_L = L \oplus \sigma^* L$, such that the Higgs field $\Phi_L = \mathrm{diag}(\lambda, -\lambda)$ is normal. The pullback $\tilde{\pi}^*(E, \Phi)$ is obtained from (E_L, Φ_L) by Hecke transformations on $Z(\tilde{\pi}^* a_2)$. Hence, it is clear that the induced hermitian metric h on $E|_{X \setminus Z(a_2)}$ solves the decoupled Hitchin equation.

To show that it induces a non-degenerate hermitian metric on $\det(E) = O_X$ we compute its local shape at $Z(a_2)$. Let $x \in Z(a_2)$ be a zero of odd order and $p \in \tilde{\Sigma}$ its preimage. By [Fre18b] Proposition 3.5, we can choose a frame s of L around p, such that $h_L = |z|^{2\alpha_p}$. Such frame is unique up to multiplying with $c \in \mathsf{U}(1)$ and therefore defines a unique u-coordinate for (E, Φ) at p (cf. Proposition 2.4.6). Applying the Hecke transformation parametrised by u we obtain

$$\hat{h}_L = \begin{pmatrix} |z|^{2\alpha_p} & u\left(\frac{|z|}{z}\right)^{2\alpha_p} \\ \bar{u}\left(\frac{|z|}{\bar{z}}\right)^{2\alpha_p} & |z|^{-2\alpha_p} \end{pmatrix}.$$

This hermitian metric is locally σ-invariant and descends to the singular hermitian metric h_{dc} on $E|_U$ described in the lemma with $v(z^2)z = u(z)$.

Using the description of the Hecke parameters at even zeroes in terms of u-coordinates depicted in Proposition 2.7.1, one can adapt this argument to the zeroes of a_2 of even order. □

Remark 5.2. For the regular fibers of $\mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X, K)$, this resembles the construction of Limiting Configurations in [Fre18b]. In difference to Fredrickson we work with positive weights instead of negatives. This is due to the fact that Fredrickson's construction uses the line bundle L' with the property $\pi_* L' = E$ to reconstruct the Higgs bundle. In terms of $L \in \mathsf{Prym}_\Lambda(\Sigma)$ it is given by $L' =: L \otimes \pi^* K$. To every solution h_L of the Hermitian-Einstein equation on (L, α), as defined in the previous theorem, one obtains a solution of the Hermitian-Einstein equation on $(L', -\alpha)$ in a canonical way by $h' := h_L |\lambda|^{-2}$.

Corollary 5.3. Let $(E, \Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X, K)$, such that $0 \neq \det(\Phi) \in H^0(X, K^2)$ has no global square root and Φ is everywhere locally diagonalizable.

Then the hermitian metric h_{dc} defined in Theorem 5.1 is a smooth solution to the Hitchin equation on (E, Φ).

REMARK 5.4. Let (E, Φ) as in the previous Corollary. (E, Φ) is stable by the irreducibility of the spectral curve. Hence, the rescaled Hitchin equation

$$F_h + t^2[\Phi \wedge \Phi^* h] = 0, \quad t \in \mathbb{C}^*,$$

decouples and the solutions is independently of t given by the hermitian metric h_{dc}. Hence, this hermitian metric is the limit of a constant sequence of solutions to the Hitchin equation along a ray to the ends of the moduli space.

THEOREM 5.5 ([Moc16] Corollary 5.4). Let $(E, \Phi) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(X, K)$ with irreducible and reduced spectral curve, then the solution to the decoupled Hitchin equation h_{dc} is a Limiting Configuration. In explicit, let h_t be the solution to the rescaled Hitchin equation

$$F_{h_t} + t^2[\Phi \wedge \Phi^{*h_t}] = 0, \quad t \in \mathbb{R}_+,$$

then h_t converges to h_∞ in C^∞ on any compact subset of $X \setminus Z(\det(\Phi))$ for $t \to \infty$.

PROOF. For $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers with irreducible and reduced spectral curve the auxiliary parabolic structure is uniquely determined by the condition that the singular hermitian metric h_∞ induces a non-singular hermitian metric on $\det(E)$. Hence, h_∞ coincides with the limiting hermitian metric constructed by Mochizuki and the approximation result follows from his work. □

THEOREM 5.6. Let $(E, \Phi, \omega) \in \mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X, K)$ with irreducible spectral curve of $\mathfrak{sl}(2)$-type. The pushforward of the solution to the decoupled Hitchin equation on the associated $\mathsf{SL}(2,\mathbb{C})$-Higgs bundle $(E_2, \Phi_2) \in \mathcal{M}_{\mathsf{SL}(2,\mathbb{C})}(\Sigma/\sigma, \pi_n^* K)$ defines a solution to the decoupled Hitchin equation $h_{dc} = h_{dc}(E, \Phi)$ on (E, Φ, ω).

PROOF. Let h_2 be the solution to the decoupled Hitchin equation on (E_2, Φ_2) defined in Theorem 5.1. Then $h' := h_2|\partial\pi_n|^{-1}$ defines a degenerate hermitian metric on $E_2 \otimes \pi_n^* K^{n-1}|_{\Sigma/\sigma^\times}$, where $\Sigma/\sigma^\times := (\Sigma/\sigma) \setminus \mathsf{supp} R$. Recall from Theorem 4.2.10 that $\pi_{n*}(E_2 \otimes \pi_n^* K^{n-1}) = E$. Hence, $\pi_{n*}h'$ defines a flat smooth hermitian metric on $E|_{\pi_n(\Sigma/\sigma^\times)}$ compatible with the symplectic form, such that

$$[\pi_{n*}\Phi \wedge \pi_{n*}\Phi^{\pi_{n*}h'}] = 0.$$

We are left with checking that $\pi_{n*}h'$ induces a non-degenerate hermitian metric on $\det(E)$ at the branch points of $\pi_n : \Sigma/\sigma \to X$. Let $p \in \Sigma/\sigma$ be a ramification point of π_n of index $k-1$. If $b_2(p) \neq 0$ we can choose a holomorphic frame, such that $\Phi = \mathrm{diag}(\lambda, -\lambda)$ and $h_2 = \mathrm{diag}(1,1)$. We can choose a local coordinate neighbourhood (U, z) centred at p and (W, w) centred at $\pi_n(p)$, such that $\pi_n : U \to W, z \mapsto z^k$. Then $h' = |z|^{-k+1}h_2$. The induced locally τ-invariant hermitian metric on E_2^τ is given by

$$h^\tau = h' \oplus \tau^* h' \oplus \cdots \oplus \tau^{(k-1)*}h'.$$

Here τ is a generator of the local $\mathbb{Z}_k$-action changing the sheets (cf. Corollary 3.2.2). Applying the Hecke transformation we obtain

$$\begin{aligned}\hat{h} &= \psi_{01}^{*} h^{\tau} \psi_{01} \\ &= \operatorname{diag}(1, \bar{z}, \ldots, \bar{z}^{k-1}, 1, \bar{z}, \ldots, \bar{z}^{k-1}) |z|^{-k+1} \operatorname{diag}(1, z, \ldots, z^{k-1}, 1, z, \ldots, z^{k-1}) \\ &= \operatorname{diag}(|z|^{-k+1}, |z|^{-k+3}, \ldots, |z|^{k-1}, |z|^{-k+1}, |z|^{-k+3}, \ldots, |z|^{k-1})\end{aligned}$$

$\hat{h}$ descends to the degenerate hermitian metric $\pi_{n*}h'$ given at $\pi_n(p)$ by

$$h = \operatorname{diag}(|w|^{\frac{-k+1}{k}}, |w|^{\frac{-k+3}{k}}, \ldots, |w|^{\frac{k-1}{k}}, |w|^{\frac{-k+1}{k}}, |w|^{\frac{-k+3}{k}}, \ldots, |w|^{\frac{k-1}{k}}).$$

If $b_2(p) = 0$ a similar computation gives the result. □

Remark 5.7. At a branch point of $\pi_n : \Sigma/\sigma \to X$ we recover the local form described in [Fre18b] Proposition 3.5.

When considering $\mathfrak{sl}(2)$-type singular fibers for $\mathsf{Sp}(2n, \mathbb{C})$, we see that the possible local forms of the Higgs field at the branch points of the spectral cover (see Proposition 4.2.3), where already covered in the works [Moc16; Fre18b]. Nonzero eigenvalues of the Higgs field of higher multiplicity correspond to smooth ramification points and hence the local situation can appear in the same way for Higgs bundles in a regular $\mathsf{SL}(2n, \mathbb{C})$-Hitchin fiber. Hence, the local approximation problem is covered by [Fre18b] Section 4.1. The singular points lie on the zero section of K and the spectral curve is locally given at a singular point by an equation of the form

$$\lambda^2 - z^k = 0.$$

These are exactly the singularities for singular fibers of $\mathsf{SL}(2, \mathbb{C})$. Furthermore, the Higgs bundle is locally described at a singular point by Lemma 4.2.12. The local approximation result for such local forms was proven in [Moc16] Section 3. This leads to the following conjecture.

Conjecture 5.8. Let $(E, \Phi) \in \mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X, K)$ with irreducible spectral curve of $\mathfrak{sl}(2)$-type. Then the solution $h_{dc}(E, \Phi)$ to the decoupled Hitchin equation is a Limiting Configuration, i. e. let h_t be the solution to the rescaled Hitchin equation

$$F_{h_t} + t^2 [\Phi \wedge \Phi^{*h_t}] = 0, \quad t \in \mathbb{R}_+,$$

then h_t converges to h_∞ in C^∞ on any compact subset of $X \setminus \mathsf{supp}(B)$ for $t \to \infty$.

We can give a proof strategy for the following special case.

Conjecture 5.9. Let $(E, \Phi) \in \mathcal{M}_{\mathsf{Sp}(2n,\mathbb{C})}(X, K)$ of $\mathfrak{sl}(2)$-type, such that (E, Φ) is locally diagonalizable at every higher order zero of $\det(\Phi)$. Then Conjecture 5.8 holds.

Proof Idea. Here one can follow the receipt outlined in [Fre18b] Section 1.2.

i) We already have a preferred solution $h_\infty = h_{dc}(E, \Phi)$ of the decoupled Hitchin equation.

ii) We can build an approximate solution h_t^{app} by gluing the model solutions of [Fre18b] section 4.1 at the branch points $\mathsf{supp}(B) \setminus Z(\det(\Phi))$ and at the simple zeroes of $\det(\Phi)$. As we mentioned above, the local situations considered by Fredrickson cover the possible local forms of Φ at $\mathsf{supp}(B) \setminus Z(\det(\Phi))$ and simple zeroes of $\det(\Phi)$.
iii) To prove that h_t^{app} solves the Hitchin equation up to a small error one can apply [Fre18b] Proposition 4.10. h_t^{app} solves the Hitchin equation away from little discs around the branch points and recall from Corollary 5.3 that under the assumptions of the conjecture this is also true at the higher order zeroes of $\det(\Phi)$. At $\mathsf{supp}B$ and the simple zeroes of $\det(\Phi)$ the Higgs field Φ looks like a $\mathsf{SL}(2n,\mathbb{C})$-Higgs field with smooth spectral curve and is hence estimated by [Fre18b] Proposition 4.10.
iv) In the last step, one needs to show that h_t^{app} is close to an actual solution h_t of the Hitchin equation. This is the part of the proof, where it is not clear to the author, if one can generalize Fredrickson's argument. As pointed out in [Fre18b] Remark 5.3 this requires a global argument.

□

In chapter 2 and 4, we stratified the $\mathfrak{sl}(2)$-type Hitchin fibers by fiber bundles over abelian torsors. Using the solutions to the Hermitian-Einstein equation discussed above, we can prove that all these fiber bundles are smoothly trivial.

Theorem 5.10. All the fiber bundles appearing in the theorems 4.2.13, 4.2.14, 4.4.1, 4.4.2 and 4.4.5 are smoothly trivial.

Proof. In the prove of Theorem 5.1 we saw that a solution to the Hermitian-Einstein equation h_L on the eigen line bundle $L \in \mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma})$ with respect to some auxiliary parabolic structure induces local frames s at $p \in \tilde{\pi}^{-1} Z(a_2)$, such that $h_L = |z|^{2\alpha_p}$. These frames are unique up to multiplying by a constant and therefore define unique u-coordinates at all $p \in Z(\tilde{\pi}^* a_2)$ (cf. Proposition 2.4.6). h_L depends smoothly on $L \in \mathsf{Prym}_{\Lambda - \tilde{\pi}^* D}(\tilde{\Sigma})$ (see [Maz+19] Proposition 3.3). Furthermore, the choice of s depends smoothly on h_L by the explicit argument in [Fre18b] Proposition 3.5. Hence, this defines a smooth trivialisation in the $\mathsf{SL}(2,\mathbb{C})$-case and hence in all other cases. □

CHAPTER 6

Singular fibers with non-reduced spectral curve

In this chapter, we consider spectral data for singular Hitchin fibers with non-reduced spectral curve. These behave quite different than the singular Hitchin fibers with irreducible spectral curve considered above. They typically have plenty of irreducible components. Moreover, the splitting of the spectral data into abelian and non-abelian part does not generalize to this case.

We will describe the Higgs bundles in Hitchin fibers with non-reduced spectral curve as iterative extensions of an associated graded - a direct sum of Higgs bundles, such that their spectral curve is the underlying reduced curve (see Section 6.2). The extensions are parametrized by certain hypercohomology groups and we will start by giving an introduction of this tool in Section 6.1. In general, we encounter three problems with this approach:

i) In general, the summands of the associated graded are not semi-stable and hence there is no classical coarse moduli space of these objects.
ii) If we consider a spectral curve with a non-reduced factor of multiplicity > 2 the iterative extensions depend on each other, which makes it hard to construct a parameter space of spectral data.
iii) The automorphism groups of the associated graded act on the extension data. One has to understand the quotient by these actions.

However, if the underlying reduced spectral curve is irreducible Problem iii) can be understood by classical GIT. Furthermore, Problem i) resolves in several special cases (see Theorem 6.2.5). Moreover, if the graded objects have trivial Higgs field Problem ii) resolves. Summing up, this method works very well for nilpotent cones. In Section 6.3, we will apply it to parametrize certain strata of the nilpotent cone of $\mathsf{SL}(n,\mathbb{C})$.

6.1. Hypercohomology

We start with a very general framework. Let $F : \underline{A} \to \underline{B}$ be a left-exact covariant functor of abelian categories $\underline{A}, \underline{B}$, such that $\underline{A}$ has enough injectives, i. e. there exist injective resolutions. Then the derived functor RF of F is defined as follows: Let $S \in \mathsf{Obj}(\underline{A})$ and choose an injective resolution, i. e. an exact sequence of injective objects

$$0 \to S \to I_1 \to I_2 \to \dots .$$

Then the n-th derived functor $R^nF(S)$ is the n-th homology of the complex

$$0 \to F(I_1) \to F(I_2) \to \dots .$$

Instead of an injective resolution one can also use an acyclic resolution of S for the definition of $R^\bullet F(S)$. $R^\bullet F$ is called the derived functor, because it associates to short exact sequences a long exact sequence. More explicitly, consider a short exact sequence

$$0 \to S_1 \to S_2 \to S_3 \to 0$$

in $\underline{A}$. Then it induces an exact sequence

$$\begin{aligned} 0 \to \quad F(S_1) \quad &\to \quad F(S_2) \quad \to \quad F(S_3) \\ \to R^1 F(S_1) &\to R^1 F(S_2) \to R^1 F(S_3) \\ \to R^2 F(S_1) &\to R^2 F(S_2) \to \dots \end{aligned}$$

EXAMPLE 6.1.1. Let $\underline{\mathsf{Sheaf}}(X)$ the abelian category of coherent sheaves on a complex manifold X and $F : \underline{\mathsf{Sheaf}}(X) \to \underline{\mathsf{Abel}}$ the section functor to the category of abelian groups. By [Wei94] section 2.3, the category $\underline{\mathsf{Sheaf}}(X)$ has enough injectives and the right-derived functor is sheaf cohomology.

To obtain hypercohomology one applies this principal to the category of complexes $\underline{\mathsf{Comp}_{\underline{A}}}$ in some abelian category $\underline{A}$. Lets assume we have a complex

$$S_0 \xrightarrow{d_0} S_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} S_n$$

in $\underline{A}$. If this category has enough injectives, we obtain an injective resolution of the complex. In explicit, a commuting diagram with of injective objects I_{ij}

$$\begin{tikzcd}
S_0 \arrow[r] \arrow[d, "d_0"'] & I_{00} \arrow[r, "\delta_{00}"] \arrow[d, "d_{00}"'] & I_{01} \arrow[r, "\delta_{01}"] \arrow[d, "d_{01}"'] & \cdots \\
S_1 \arrow[r] \arrow[d, "d_1"'] & I_{10} \arrow[r, "\delta_{10}"] \arrow[d, "d_{10}"'] & I_{11} \arrow[r, "\delta_{11}"] \arrow[d, "d_{11}"'] & \cdots \\
\vdots \arrow[d, "d_{n-1}"'] & \vdots \arrow[d, "d_{n-1,0}"'] & \vdots \arrow[d, "d_{n-1,1}"'] & \\
S_n \arrow[r] & I_{n0} \arrow[r, "\delta_{n0}"] & I_{n1} \arrow[r, "\delta_{n1}"] & \cdots
\end{tikzcd}$$

Let again $F : \underline{A} \to \underline{B}$ be a left-exact functor. The n-th hypercohomology group

$$\mathbb{H}^n(S_\bullet)$$

is the n-th homology of the total complex

$$\begin{aligned} &F(I_{00}) \xrightarrow{F(d)_\bullet + F(\delta)_\bullet} F(I_{01}) \oplus F(I_{10}) \xrightarrow{F(d)_\bullet - F(\delta)_\bullet} F(I_{02}) \oplus F(I_{11}) \oplus F(I_{02}) \\ &\dots \quad \xrightarrow{F(d)_\bullet + (-1)^l F(\delta)_\bullet} \bigoplus_{p+q=l} F(I_{pq}) \xrightarrow{F(d)_\bullet + (-1)^{l+1} F(\delta)_\bullet} \dots \end{aligned}$$

There seem to be some subtleties in the construction in this very general context, because the total complex in $\underline{B}$ might not exist, see [Wei94] section 5.7. However, in the concrete situation below the existence will be clear.

So let's get concrete: Let X be Riemann surface and consider $\underline{\mathsf{Compl}}_{\mathsf{Sheaf}(X)}$ the category of complexes of coherent sheaves on X. To every object of this category we can assign a hypercohomology group as we did above. This becomes especially easy for complexes of locally free sheaves. Let

$$(\xi_\bullet, \phi_\bullet) = \left(\xi_0 \xrightarrow{\phi_0} \xi_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{n-1}} \xi_n\right) \in \underline{\mathsf{Compl}}_{\mathsf{Sheaf}(X)}$$

be a complex of locally free sheaves. For all i let E_i be the holomorphic vector bundle, such that $\mathcal{O}_X(E) = \xi_i$. In this setting, an acyclic resolution that is easy to compute is the Dolbeault resolution. It is given by the double complex

$$\begin{tikzcd}
\xi_0 \arrow[r] \arrow[d, "\phi_0"'] & C^\infty(E_0) \arrow[r, "\bar\partial"] \arrow[d, "\phi_0"'] & C^{(0,1)}(E_0) \arrow[r] \arrow[d, "\phi_0"'] & 0 \\
\xi_1 \arrow[r] \arrow[d, "\phi_1"'] & C^\infty(E_1) \arrow[r, "\bar\partial"] \arrow[d, "\phi_1"'] & C^{(0,1)}(E_1) \arrow[r] \arrow[d, "\phi_1"'] & 0 \\
\vdots \arrow[d, "\phi_{n-1}"'] & \vdots \arrow[d, "\phi_{n-1}"'] & \vdots \arrow[d, "\phi_{n-1}"'] & \\
\xi_n \arrow[r] & C^\infty(E_n) \arrow[r, "\bar\partial"] & C^{(0,1)}(E_n) \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{A}(E_i)$ resp. $\mathcal{A}^{(0,1)}(E_i)$ denote the sheaf of smooth sections resp. smooth $(0,1)$-forms of E_i. The total complex is given by

$$\begin{aligned}
\mathcal{A}^0(E_0) &\xrightarrow{\phi+\bar\partial} \mathcal{A}^0(E_1) \oplus \mathcal{A}^{(0,1)}(E_0) \xrightarrow{\phi+\bar\partial} \mathcal{A}^0(E_2) \oplus \mathcal{A}^{(0,1)}(E_1) \xrightarrow{\phi+\bar\partial} \dots \\
\dots &\xrightarrow{\phi+\bar\partial} \mathcal{A}^0(E_n) \oplus \mathcal{A}^{(0,1)}(E_{n-1}) \xrightarrow{\bar\partial} \mathcal{A}^{(0,1)}(E_n) \to 0
\end{aligned}$$

Its homology is the hypercohomology

$$\mathbb{H}^\bullet(\xi_\bullet, \phi_\bullet)$$

of the complex of locally free sheaves $(\xi_\bullet, \phi_\bullet)$.

Spectral sequences. There are two spectral sequences

$${}^{I}E_r^{pq} \quad \text{and} \quad {}^{II}E_r^{pq}$$

converging to $\mathbb{H}^\bullet(\xi_\bullet, \phi_\bullet)$. These are again defined in a very general context. If one has a double complex as above the cohomology of the total complex is approximated by spectral sequences of these types (see [Bla15] Section 1.2).
For us ${}^{I}E_r^{pq}$ will be very useful. To define it we associate to the complex of locally

free sheaves $(\xi_\bullet, \phi_\bullet)$ the cohomology sheaves $\mathcal{H}^i(\xi_\bullet, \phi_\bullet)$, the sheafication of

$$U \mapsto \frac{\mathsf{ker}(\phi_i) : H^0(U, E_i) \to H^0(U, E_{i+1})}{\phi_{i-1}(H^0(U, E_{i-1}))},$$

where $U \subset X$ open. This is the first page of the spectral sequence. The second page is given by the sheaf cohomology

$${}^I E_2^{pq} = H^p(X, \mathcal{H}^q(\xi_\bullet, \phi_\bullet)).$$

If a spectral sequence of a double complex is non-trivial only in the first quadrant, i. e. $E_2^{pq} \neq 0$ only for $p, q \geq 0$, and converges to $\mathbb{H}^q$, one obtains the five-term exact sequence

$$0 \to E_2^{10} \to \mathbb{H}^1 \to E_2^{01} \to E_2^{20} \to \mathbb{H}^2. \tag{15}$$

(see [Wei94] Exercise 5.2.2).

Hypercohomology of Higgs bundles. We are interested in very special of hypercohomology groups. The hypercohomology groups of a complexes of locally free analytic sheaves

$$\mathcal{O}_X(E) \xrightarrow{\Phi} \mathcal{O}_X(E \otimes M),$$

where (E, Φ) is a M-twisted $\mathsf{GL}(n, \mathbb{C})$-Higgs bundle. We will denote it by $\mathbb{H}^i(E, \Phi)$. The total complex of the associated Dolbeault double complex is given by

$$\mathcal{A}^0(E) \xrightarrow{\bar\partial + \Phi} \mathcal{A}^0(E \otimes M) \oplus \mathcal{A}^{(0,1)}(E) \xrightarrow{\bar\partial + \Phi} \mathcal{A}^{(0,1)}(E \otimes M).$$

Explicitly, its first homology group is given by

$$\mathbb{H}^1(E, \Phi) = \frac{\left\{(a, b) \in \mathcal{A}^0(E \otimes M) \oplus \mathcal{A}^{(0,1)}(E) \mid \bar\partial a + \Phi b = 0\right\}}{\left\{(\Phi c, \bar\partial c) \mid c \in \mathcal{A}^0(E)\right\}}.$$

From the five-term exact sequence applied to the spectral sequence ${}^I E_2^{pq}$ we have the following lemma.

LEMMA 6.1.2. *There is a short exact sequence*

$$0 \to H^1(X, \mathsf{ker}(\Phi)) \to \mathbb{H}^1(E, \Phi) \to H^0(X, \mathsf{coker}(\Phi)) \to 0.$$

PROOF. These are the first 3 terms of the five term exact sequence (15) specialized to the complex 6.1. The cohomology sheaves are given by

$$\mathcal{H}^0(E, \Phi) = \mathsf{ker}(\Phi), \quad \mathcal{H}^1(E, \Phi) = \mathsf{coker}(\Phi).$$

This gives the E_2^{10} and E_2^{01} terms. Moreover, on a Riemann surface

$${}^I E_2^{20} = H^2(X, \mathsf{ker}(\Phi)) = 0.$$

Hence, we obtain the short exact sequence described in the lemma. □

PROPOSITION 6.1.3 (Extension classes for Higgs bundles).
Let (E, Φ), (E_1, Φ_1), (E_2, Φ_2) be M-twisted Higgs bundles. Then the extensions

$$0 \to (E_1, \Phi_1) \to (E, \Phi) \to (E_2, \Phi_2) \to 0$$

are parametrized by

$$\mathsf{Ext}[(E_1, \Phi_1), (E_2, \Phi_2)] = \mathbb{H}^1(E_1 \otimes E_2^\vee, \Phi_1 \otimes \Phi_2^\vee).$$

PROOF. Consider an extension as in the proposition. Tensoring by $(E_2^\vee, \Phi_2^\vee)$ we obtain

$$0 \to (E_1 \otimes E_2^\vee, \Phi_1 \otimes \Phi_2^\vee) \to (E \otimes E_2^\vee, \Phi \otimes \Phi_2^\vee) \to (E_2 \otimes E_2^\vee, \Phi_2 \otimes \Phi_2^\vee) \to 0.$$

The induced homomorphism of the associated long exact sequence in hypercohomology is

$$\cdots \to \mathbb{H}^0(E_2 \otimes E_2^\vee, \Phi_2 \otimes \Phi_2^\vee) \xrightarrow{\delta_1} \mathbb{H}^1(E_1 \otimes E_2^\vee, \Phi_1 \otimes \Phi_2^\vee) \ldots$$

It is easy to check that

$$\mathrm{id}_E \in \mathbb{H}^0(E_2 \otimes E_2^\vee, \Phi_2 \otimes \Phi_2^\vee) = \{a \in \mathcal{A}^0(E_2 \otimes E_2^\vee) \mid \bar{\partial} a = 0, \Phi a = 0\}.$$

It becomes clear from chasing the diagram defining δ_1, that the extensions of Higgs bundles as above are parametrised by

$$\delta_1(\mathrm{id}_{E_2}) \in \mathbb{H}^1(E_1 \otimes E_2^\vee, \Phi_1 \otimes \Phi_2^\vee).$$

□

6.2. Non-reduced spectral data

Let X be a Riemann surface. In the following we will consider plenty of sheaf cohomology groups of holomorphic vector bundles F on X. They will all be computed on a fixed Riemann surface X and so we will drop the Riemann surface from the notation, i. e. $H^i(F) = H^i(X, F)$ from now on.

Let $(a_1, \ldots, a_k) \in B_{\mathsf{GL}(k,\mathbb{C})}(X, M)$, such that the associated spectral equation

$$p = \lambda^k + a_1 \lambda^{k-1} + \cdots + a_{k-1}\lambda + a_k.$$

is reduced. Then p^l defines the spectral equation of certain Hitchin fiber of M-twisted $\mathsf{GL}(r, \mathbb{C})$-Higgs bundles with $r = kl$. Denote by $\underline{b} \in B_{\mathsf{GL}(r,\mathbb{C})}(X, M)$ the corresponding point in the $\mathsf{GL}(r, \mathbb{C})$-Hitchin base. Let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(\underline{b})$. Due to the special structure of the spectral curve, there is a filtration of locally free subsheaves

$$0 \subsetneq E_1 \subset E_2 \subset \cdots \subset E_{l-1} \subset E_l = E,$$

where

$$E_i := \mathsf{Ker}\left(p(\Phi)^i : E \to E \otimes M^{ik}\right).$$

Because these subbundles are defined by reducible factors of the characteristic polynomial they are invariant under the Higgs field and we actually have a filtration of Higgs bundles

$$0 \subsetneq (E_1, \Phi_1) \subset (E_2, \Phi_2) \subset \cdots \subset (E_l, \Phi_l) = (E, \Phi_1).$$

We will denote the associated graded by

$$\mathsf{Grad}(E, \Phi) = \bigoplus_{i=1}^{l} (F_i, \Psi_i),$$

where

$$F_i = E_i / E_{i-1} \quad \text{and} \quad \Psi_i : F_i \to F_i \otimes M$$

is the induced Higgs field. The the topological invariant of such filtrations are the ranks $r_i = \mathsf{rank}(F_i)$ and degrees $d_i = \mathsf{deg}(F_i)$.

DEFINITION 6.2.1. Let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(\underline{b})$. The filtration type of (E, Φ) is the pair of l-tuple of integers

$$\underline{r} = (r_1, \dots, r_l), \quad \underline{d} = (d_1, \dots, d_n).$$

$\underline{r}$ will also be referred to as the rank vector.

REMARK 6.2.2. The filtration type is well-defined for polystable Higgs bundles. Not so for semi-stable Higgs bundles. Here points in the closure of the gauge orbit might have different Jordan type. For example, take $E = L \oplus L^{-1}$ with $\deg(L) = 0$ and $\Phi = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ with $0 \neq \alpha \in H^0(L^2K)$. Then the closure of the gauge orbit contains $(E, 0)$.

LEMMA 6.2.3. *Let $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(\underline{b})$ of filtration type $(\underline{r}, \underline{d})$. Then*

i) $r = \sum_{i=1}^{l} r_i$,

ii) *for all $1 \leq i < l$, $r_i \geq r_{i+1}$ and, if $r_i = r_{i+1}$, then*

$$d_{i+1} \leq d_i + r_i k \deg(M),$$

iii) *for all $1 \leq i \leq l$ and Ψ_i-invariant subbundles $V \subset F_i$ of rank r_0,*

$$\frac{\deg(F) + \sum_{j=1}^{i-1} d_j}{\sum_{j=0}^{i-1} r_j} < \mu(E),$$

in particular,

$$\frac{\sum_{j=1}^{i} d_j}{\sum_{j=1}^{i} r_j} < \mu(E).$$

PROOF. For ii) consider the i-th extension

$$0 \to (E_i, \Phi_i) \to (E_{i+1}, \Phi_{i+1}) \to (F_{i+1}, \Psi_{i+1}) \to 0.$$

As we have fixed the filtration type, it is clear that the composition

$$F_{i+1} \xrightarrow{p(\Phi_{i+1})} E_i \otimes M^k \to F_i \otimes M^k$$

has generically rank r_{i+1}. This can only happen, if $r_i \geq r_{i+1}$. If $r_i = r_{i+1}$, there is a induced non-vanishing sheaf homomorphism

$$\bigwedge^{r_i} (p(\Phi_{i+1})|_{F_{i+1}}) : \bigwedge^{r_i} F_{i+1} \to \bigwedge^{r_i} \left(F_i \otimes M^k \right).$$

Hence,

$$d_{i+1} \leq d_i + r_i k \deg(M).$$

To iii): If $V \subset F_i$ is a Ψ_i-invariant subbundle, then $E_{i-1} \oplus V$ is a Φ-invariant subbundle of E. Hence, $\leq$ follows from the poly-stability of E. Assume equality, then by poly-stability we have a splitting

$$(E, \Phi) = (E_{i-1} \oplus V, \Phi_{i-1} \oplus \Psi|_V) \oplus (V', \Psi').$$

But then the map above has everywhere rank $< r_i$. $\square$

THEOREM 6.2.4 ([Lau88]). *Fix a rank vector* $\underline{r} = (r_1, \dots, r_k)$. *Then the subset of all* $(E, \Phi) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(p^l)$ *with filtration type* $(r'_1, \dots, r'_l)$*, such that for all* $1 \leq i \leq l$

$$r_1 + \cdots + r_i \leq r'_1 + \cdots + r'_i$$

is closed.

PROOF. Let $(E_j, \Phi_j) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(p^l), j \in \mathbb{N}$ be a sequence of Higgs bundles with fixed rank vector $\underline{r}$ In the limit the i-th generalised eigenspace E_i can only increase its dimension, as some of the extension data is vanishing in the limit. This is formalised in the work of [Lau88] for the kernel filtration for Higgs bundles. The result generalises to our setting by considering the M^k-twisted Higgs bundle

$$p(\Phi) : E \to E \otimes M^k.$$

$\square$

To obtain spectral data we would like to parametrize the Higgs bundles in $\mathsf{Hit}^{-1}_{\mathsf{GL}(r,\mathbb{C})}(\underline{b})$ with fixed filtration type by the moduli of the associated graded $\mathsf{Grad}(E, \Phi)$ and the extension data, which will be given in terms of hypercohomology groups. However, as we see in the previous lemma the quotients (F_i, Ψ_i) are not stable, but satisfy a twisted stability condition. So in general there is non coarse moduli space of this objects by [Nit91]. To tackle these cases on could try to apply the modern understanding of the instability locus by non-reductive GIT in [Ham].

Let $\mathcal{M}^d_{\mathsf{GL}(r,\mathbb{C})}(X, M)$ denote the moduli space of poly-stable M-twisted Higgs bundles (E, Φ) with $\deg(E) = d$.

THEOREM 6.2.5. *Let* $\underline{b} \in B_{\mathsf{GL}(r,\mathbb{C})}(X, M)$*, such that the spectral equation is given by* p^l *with* p *irreducible of degree* k*. Given* $(E, \Phi) \in \mathsf{Hit}^{-1}(\underline{b})$ *of filtration type*

$$\underline{r} = (mk, k, \dots, k, 0, \dots, 0), \quad \underline{d} = (d_1, \dots, d_l)$$

of length $l - m + 1$ *we can retrieve the following data:*

i) *The associated graded*

$$\mathsf{Grad}(E, \Phi) = \bigoplus_{i=1}^{l-m+1} (F_i, \Psi_i) \in \mathsf{Hit}^{-1}_{\mathsf{GL}(mk,\mathbb{C})}(p^m) \cap \mathcal{M}^{d_1}_{\mathsf{GL}(mk,\mathbb{C})}(X, M) \times \bigtimes_{i=2}^{l-m+1} \mathsf{Hit}^{-1}_{\mathsf{GL}(k,\mathbb{C})}(p) \cap \mathcal{M}^{d_i}_{\mathsf{GL}(k,\mathbb{C})}(X, M),$$

and

ii) *the extension data*

$$\mathsf{Ext}(E, \Phi) \in \bigoplus_{i=2}^{l-m+1} H^1\left(\mathsf{ker}\left(\Phi_{i-1} \otimes \Psi_i^\vee : E_{i-1} \otimes F_i^\vee \to E_{i-1} \otimes F_i^\vee \otimes M\right)\right) \oplus H^0\left(\mathsf{coker}\left(\Phi_{i-1} \otimes \Psi_i^\vee : E_{i-1} \otimes F_i^\vee \to E_{i-1} \otimes F_i^\vee \otimes M\right)\right).$$

The extension data is unique up to an action of

$$\bigtimes_{i=1}^{l-m+1} \mathsf{Aut}(F_i, \Psi_i) = (\mathbb{C}^*)^{l-m+1}.$$

PROOF. The usual definition for stability holds for the first summand of the graded. The other summands have irreducible spectral curve. Hence, they are automatically stable by Lemma 1.3.9. By Proposition 6.1.3 the $i-1$-th extension defines a hypercohomology class in

$$\mathbb{H}^1(E_{i-1} \otimes F_i^\vee, \Phi_{i-1} \otimes \Psi_i^\vee).$$

The hypercohomology group can be computed in terms of kernels and cokernels by Lemma 6.1.2. $\square$

REMARK 6.2.6 (A Problem of Organization). If we want to parametrize a part of the Hitchin fiber with fixed filtration type as in the previous theorem, we need to organize the data. A general approach is to do successive bundles. As base we have the moduli space of graded objects. Then the extension data for the first extension of (F_1, Ψ_1) by (F_2, Ψ_2) defines a bundle over this moduli space. But the extension data of the second extension depends on (E_2, Φ_2) in particular on the extension data of the first extension. Therefore, we need to consider the data of the first extension as a base for the bundle parametrising the second extension. Inductively, we get a bundle structure like

$$\left(\dots \left(\begin{matrix} \left(\begin{matrix} \mathcal{M}_{\mathsf{GL}(r_1,\mathbb{C})\times\cdots\times\mathsf{GL}(r_k,\mathbb{C})} \\ \uparrow \\ \mathsf{Ext}\left[(E_1,\Phi_1),(F_2,\Psi_2)\right] \end{matrix} \right) \\ \uparrow \\ \mathsf{Ext}\left[(E_2,\Phi_2),(F_3,\Psi_3)\right] \end{matrix} \right) \dots \begin{matrix} \\ \\ \vdots \\ \uparrow \\ \mathsf{Ext}\left[(E_{k-1},\Phi_{k-1}),(F_k,\Psi_k)\right] \end{matrix} \right)$$

As a last step one needs to take care of the action of $\bigtimes_{i=1}^k \mathsf{Aut}(E_i)$. In the situation of the previous theorem it is a reductive group. Hence one can use GIT.

REMARK 6.2.7. In this description of spectral data, the abelian part is hidden in the moduli of the associated graded. For example, if one restricts p to have a smooth $\mathsf{GL}(k,\mathbb{C})$-spectral curve Σ, then

$$\mathsf{Hit}^{-1}_{\mathsf{GL}(k,\mathbb{C})}(p) \cap \mathcal{M}^d_{\mathsf{GL}(k,\mathbb{C})}(X, M) \cong \mathsf{Jac}(\Sigma).$$

On the other hand, there is not anymore a splitting of the spectral data in an abelian and non-abelian part as we seen above. Here they depend and determine each. In the following section, we will see that we can completely determine the abelian part of the data by fixing the divisors of some holomorphic section in the extension data. Then there will be non abelian variety visible in the presentation of the spectral data.

6.3. The nilpotent cone for $\mathsf{SL}(n,\mathbb{C})$

In this section, we will apply the description of spectral data for Hitchin fibers with reduced spectral curve developed in the previous section to the nilpotent cone in $\mathsf{SL}(n,\mathbb{C})$. For $\mathsf{SL}(3,\mathbb{C})$ we can give a complete stratification by filtration types. For $\mathsf{SL}(n,\mathbb{C})$ we only describe does with rank vector $(1,\dots,1)$.

6.3.1. $\mathsf{SL}(3,\mathbb{C})$.

THEOREM 6.3.1. *The elements of the nilpotent cone*

$$\mathsf{Hit}^{-1}_{\mathsf{SL}(3,\mathbb{C})}(0) \subset \mathcal{M}_{\mathsf{SL}(3,\mathbb{C})}(X,K)$$

with rank vector $(1,1,1)$ *are stratified by* $a_1,a_2 \in \mathbb{N}$, *such that*

$$2a_1+a_2 \equiv 0 \mod 3, \quad \tfrac{1}{3}(2a_1+a_2) < 2g-2, \quad \tfrac{1}{3}(a_1+2a_2) < 2g-2.$$

The corresponding filtration types are

$$\underline{r} = (1,1,1), \quad \underline{d} = \left(\tfrac{1}{3}(2a_1+a_2)-2g+2, \tfrac{1}{3}(a_2-a_1), 2g-2-\tfrac{1}{3}(a_1+2a_2)\right).$$

For each stratum $\mathcal{S}(a_1,a_2)$, *there exists a holomorphic map to a unbranched* 3^{2g}*-sheeted cover of*

$$\mathsf{Sym}^{a_1}(X) \times \mathsf{Sym}^{a_2}(X)$$

with fiber over divisors A_1, A_2 *given by*

$$H^1(K^{-1}(A_1)) \oplus H^1(K^{-2}(A_1+A_2)) \oplus H^0(A_1, K^{-1}(A_1+A_2)),$$

If $a_1 < 2g-2$, *then this defines a vector bundle of rank*

$$8g-8-a_1-a_2.$$

If $a_1 \geq 2g-2$, *then the dimension of the generic fiber is given by this formula. The dimension of each stratum* $\mathcal{S}(a_1,a_2)$ *at a smooth point is* $8g-8$. *The closure of each stratum defines an irreducible component.*

PROOF. This is a direct application of the methods described above. The moduli space of graded objects is given by

$$\mathcal{M}^{d_1}_{U(1)} \times \mathcal{M}^{d_2}_{U(1)},$$

with $d_1 < 0$, $d_1+d_2 < 0$. Here $\mathcal{M}^d_{U(j)}$ denotes the moduli space of poly-stable holomorphic vector bundles of rank j and degree d. The associated filtration type is

$$\underline{r} = (1,1,1), \quad \underline{d} = (d_1, d_2, -d_1-d_2).$$

Let $(F_1,F_2) \in \mathcal{M}^{d_1}_{U(1)} \times \mathcal{M}^{d_2}_{U(1)}$, then the graded Higgs bundle is the direct sum of line bundles

$$(F_1,0) \oplus (F_2,0) \oplus (F_3,0)$$

with $F_3 = F_1^{-1}F_2^{-1}$. The extension data for the first extension is given by

$$\mathsf{Ext}[(F_1,0),(F_2,0)] = H^1(F_1F_2^{-1}) \oplus H^0(F_1F_2^{-1}K).$$

Fix a choice of first extension $(b_1,\alpha_1) \in \mathsf{Ext}((F_1,0),(F_2,0))$ and denote the resulting rank 2 Higgs bundle by (E_2,Φ_2). For the second extension we compute

$$\Phi_2 \otimes \Psi_3^\vee = \begin{pmatrix} 0 & \alpha_1 \\ 0 & 0 \end{pmatrix} : E_2 \otimes F_3^\vee \to E_2 \otimes F_3^\vee \otimes K$$

using that $\Psi_3 = 0$. So $\mathsf{ker}(\Phi_2 \otimes \Psi_3^\vee) = F_1$ and

$$\mathsf{coker}(\Phi_2 \otimes \Psi_3^\vee) = F_2K \oplus \mathcal{S}_{\alpha_1},$$

where $\mathcal{S}_{\alpha_1}$ is the torsion sheaf defined by

$$0 \to \mathcal{O}_X \xrightarrow{\alpha_1} F_1F_2^{-1}K \to \mathcal{S}_{\alpha_1} \to 0.$$

Hence,

$$\mathsf{Ext}[(E_2, \Phi_2), (F_3, 0)] = H^1(F_1F_2^{-1}) \oplus H^0(F_2F_3^{-1}K) \oplus H^0(\mathsf{div}\alpha_1, F_1F_3^{-1}K).$$

To get the result we organize the data in a different way. First, the divisors of the two holomorphic sections $A_i = \mathsf{div}(\alpha_i)$

$$(\alpha_1, \alpha_2) \in H^0(F_1F_2^{-1}K) \oplus H^0(F_2F_3^{-1}K)$$

determine F_1, F_2, F_3 up to the choice of a third root M of $\mathcal{O}_X(2A_1 + A_2)$. In explicit, $F_1 = MK^{-1}$, $F_2 = M^{-2}(A_1 + A_2)$ and $F_3 = MK(-A_1 - A_2)$. For $a_i := \deg(A_i)$ this defines the holomorphic map to the 3^{2g}-sheeted cover of $\mathsf{Sym}^{a_1}(X) \times \mathsf{Sym}^{a_2}(X)$. In this case, $\mathsf{Aut}_{\mathsf{SL}(3,\mathbb{C})}(F_1 \oplus F_2 \oplus F_3) = (\mathbb{C}^*)^2$. Up the induced $(\mathbb{C}^*)^2$-action the divisor A_1, A_2 determine uniquely α_1, α_2. It is easy to check that the condition for a_1, a_2 are equivalent to stability conditions $d_1 < 0$, $d_1 + d_2 < 0$.

Finally, notice that the extension data of the second extension does not depend on the choice of the extension class b_1. Hence, the fibers of this holomorphic map are given by

$$H^1(K^{-1}(A_1)) \oplus H^1(K^{-2}(A_1 + A_2)) \oplus H^0(A_1, K^{-1}(A_1 + A_2)).$$

The dimension count for $2g - 2 > a_1$ is an application of Riemann-Roch, where we use that $H^0(K^{-1}(A_1)) = 0$. For a_1 above this bound the result holds for generic divisors A_1. This can be seen by a general argument that can be found in [Kas08] page 7. In particular, it holds at a smooth point.

All this strata are disjoint, have the same dimension and their smooth points are connected. Hence, their closures define irreducible components of $\mathsf{Hit}^{-1}_{\mathsf{SL}(3,\mathbb{C})}(0)$ (cf. 2.6.11). $\square$

REMARK 6.3.2. We can explicitly reconstruct a Higgs bundle from the data in the following way. Choose divisors

$$(A_1, A_2) \in \mathsf{Sym}^{a_1}(X) \times \mathsf{Sym}^{a_2}(X).$$

Define the line bundles F_1, F_2, F_3 as above. Choose extension data

$$(b_1, b_2, t) \in H^1(K^{-1}(A_1)) \oplus H^1((K^{-2}(A_1 + A_2)) \oplus H^0(A_1, K^{-1}(A_1 + A_2)).$$

We can extend t to a smooth section $\phi \in \mathcal{A}^0(X, K^{-1}(A_1 + A_2))$ holomorphic in a small neighbourhood U of $Z(\alpha_1)$. We can choose a representative for b_1 vanishing on U. Now we can define a third extension class by the equation

$$\bar{\partial}\phi + b_1\alpha_2 - b_3\alpha_1 = 0.$$

This defines b_3 as $\bar{\partial}\phi + b_1\alpha_2 = 0$ in U. Then the resulting element of the nilpotent cone is given by

$$(E, \bar{\partial}_E, \Phi) = \left(F_1 \oplus F_2 \oplus F_3, \quad \begin{pmatrix} \bar{\partial}_{F_1} & b_1 & b_2 \\ 0 & \bar{\partial}_{F_2} & b_3 \\ 0 & 0 & \bar{\partial}_{F_3} \end{pmatrix}, \quad \begin{pmatrix} 0 & \alpha_1 & \phi \\ 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

THEOREM 6.3.3. *The elements of the nilpotent cone* $\mathsf{Hit}^{-1}_{\mathsf{SL}(3,\mathbb{C})}(0)$ *with rank vector* $(2,1,0)$*, such that* F_1 *is semi-stable, are stratified by an integer* $2g-2 \leq d_1 < 0$ *determining the filtration type*

$$\underline{r} = (2,1,0), \quad \underline{d} = (d_1, -d_1, 0).$$

For each stratum $\mathcal{S}(d_1)$ *there exists a holomorphic map to*

$$\mathcal{M}^{d_1}_{\mathsf{U}(2)}.$$

The fiber over a stable point $F_1 \in \mathcal{M}^{d_1}_{\mathsf{U}(2)}$ *is given by*

$$\mathbb{P}\left(H^1(F_1 \otimes \det(F_1)) \times \left(H^0(F_1 \otimes \det(F_1) \otimes K) \setminus \{0\}\right)\right).$$

of dimension $\geq 4g-5$*. In particular, at smooth points each stratum has dimension* $8g-8$*. The closure of each stratum* $\mathcal{S}(d_1)$ *is an irreducible component.*

PROOF. The stratification is a straightforward application of the receipt given in Theorem 6.2.5. The lower bound for d_1 comes from the existence of a non-zero section $\beta \in H^0(F_1 \otimes \det(F_1) \otimes K)$. Its image is contained in a line bundle $L \subset F_1$. Hence, from the poly-stability of F_1, we have

$$-\deg(F_1) \leq \deg(L) + 2g - 2 \leq 2g - 2.$$

The extension data of the single extension is parametrized by

$$\mathsf{Ext}[F_1 \otimes F_2^\vee, 0] = H^1(F_1 \otimes \det(F_1)) \times \left(H^0(F_1 \otimes \det(F_1) \otimes K) \setminus \{0\}\right)$$

To compute the dimension we use Riemann-Roch. We have

$$\begin{aligned} & \dim_{\mathbb{C}} H^1(F_1 \otimes \det(F_1)) + \dim_{\mathbb{C}} H^0(F_1 \otimes \det(F_1) \otimes K) \\ = \quad & \dim_{\mathbb{C}} H^1(F_1 \otimes \det(F_1) \otimes K) + 4g - 4 \quad \geq \quad 4g-4. \end{aligned}$$

Here $H^0(F_1 \otimes \det(F_1)) = 0$ from the poly-stability of F_1. Hence, total dimension of a stratum is a

$$\begin{aligned} & \dim \mathsf{Ext}[F_1 \otimes F_2^\vee, 0] + \dim \mathcal{M}^{d_1}_{\mathsf{U}(2)} - \dim \mathsf{Aut}(F_1) \\ \geq \quad & 4g-4 \qquad + 4g-3 \qquad -1 \qquad \geq \qquad 8g-8. \end{aligned}$$

The Hitchin map is flat, as it is open (see [Gra+94] Theorem II.2.13) and hence at smooth points the dimension of the fibers is $8g-8$. Hence, all the strata $\mathcal{S}(d_1)$ have full dimension. Furthermore, their smooth points are connected as the smooth points of $\mathcal{M}^{d_1}_{\mathfrak{U}(2)}$, the moduli space of stable holomorphic vector bundles, is connected (see [Tha97]). □

THEOREM 6.3.4. *Together with* $\mathcal{M}_{\mathsf{SU}(3)}$*, the moduli space of holomorphic vector bundles of rank* 3 *with trivial determinant, Theorem 6.3.1 and Theorem 6.3.3 determine a stratification of* $\mathsf{Hit}^{-1}_{\mathsf{SL}(3,\mathbb{C})}(0)$*, such that the irreducible components of* $\mathsf{Hit}^{-1}_{\mathsf{SL}(3,\mathbb{C})}(0)$ *are precisely the closures of those strata.*

PROOF. Every polystable Higgs bundle has a well-defined filtration type and hence is contained in a unique stratum $\mathcal{S}(a_1, a_2), \mathcal{S}(d_1)$ or $\mathcal{M}_{\mathsf{SU}(3)}$. The smooth part of each stratum is connected and hence their closure defines a unique irreducible component. □

REMARK 6.3.5. The intersection of the irreducible components is very complicated. For $\mathsf{SL}(2,\mathbb{C})$ it will be studied in upcoming work [ALS20].

6.3.2. $\mathsf{SL}(n,\mathbb{C})$ and rank vector $(1,\dots,1)$.

THEOREM 6.3.6. *The elements of the nilpotent cone $\mathsf{Hit}^{-1}_{\mathsf{SL}(n,\mathbb{C})}(0)$ with rank vector $(1,\dots,1)$ are stratified by finitely many natural numbers $a_1,\dots,a_{n-1}\in\mathbb{N}$, such that*

$$\sum_{i=1}^{n-1}(n-i)a_i \equiv 0 \mod n$$

and for all $1\leq i\leq n-1$

$$\sum_{j=1}^{i-1} ja_j + i\sum_{j=i}^{n-1} a_j - \tfrac{i}{n}\sum_{j=1}^{n-1} ja_j < i(n-i)(g-1).$$

For each stratum $\mathcal{S}(a_1,\dots,a_{n-1})$ there exists a holomorphic map to a unbranched n^{2g}-sheeted cover of

$$\mathsf{Sym}^{a_1}(X)\times\cdots\times\mathsf{Sym}^{a_{n-1}}(X).$$

For divisors $A_1,A_2,\dots A_{n-1}$ in this product of symmetric products, there exist

$$(F_1,\dots,F_n)\in\mathsf{Pic}(X)^n,$$

such that $\mathcal{O}_X(A_i)=F_iF_{i+1}^{-1}K$, unique up to the choice of the n-th root of a line bundle. The fiber of this holomorphic map is given by

$$\bigoplus_{i=1}^{n-1} H^1(F_1F_{i+1}^{-1})\oplus\bigoplus_{i=1}^{n-2}\bigoplus_{j=i+2}^{n} H^0(A_i,F_iF_j^{-1}K).$$

For generic $A_1,\dots,A_{n-1}$ the dimension of the fiber is given by

$$8g-8-\sum_{i=1}^{n-1}a_i.$$

Hence, the dimension of the strata at smooth points is $8g-8$. The closure of each stratum $\mathcal{S}(a_1,\dots,a_{n-1})$ defines an irreducible component of $\mathsf{Hit}^{-1}_{\mathsf{SL}(n,\mathbb{C})}(0)$.

PROOF. Induction on the proof of theorem 6.3.1. □

REMARK 6.3.7. For $n>3$ not all possible rank vectors are covered by Theorem 6.2.5. And for example for the rank vector $(2,2,0,0)$ for $n=4$, the graded objects are not poly-stable and hence there is no moduli space of graded objects by classical theory. In particular, one does not obtain a complete description of the irreducible components of the nilpotent cone $\mathsf{Hit}^{-1}_{\mathsf{SL}(n,\mathbb{C})}(0)$ for $n>3$ in this way.

CHAPTER 7

Outlook

In this last chapter, we will discuss some open problems and directions for future research.

$\mathfrak{sl}(2)$-type Hitchin fibers for $\mathsf{SO}(2n,\mathbb{C})$. The definition of $\mathfrak{sl}(2)$-type spectral curve is meaningful, whenever the spectral curve has an involutive Deck transformation. This is true for spectral curves of $\mathsf{SO}(2n,\mathbb{C})$-Higgs bundles. In this case, the spectral curve is always singular. This is due to the fact that the determinant of an element of $\mathfrak{so}(2n,\mathbb{C})$ is the square of the Pfaffian. The regular $\mathsf{SO}(2n,\mathbb{C})$-Hitchin fibers are those, where the Pfaffian has simple zeroes. This means that the singularities of the spectral curve are simple nodes lying on the zero section of K. Hence, the regular fibers are of $\mathfrak{sl}(2)$-type (cf. Proposition 4.2.3). The corresponding 2-sheeted covering $\tilde{\Sigma} \to \Sigma/\sigma$ is an unbranched covering of Riemann surfaces and the regular fibers are torsors over the Prym variety associated to this covering. In terms of the semi-abelian spectral data of the associated $\mathsf{SL}(2,\mathbb{C})$-spectral curve, this is the closed stratum.

For $\mathfrak{sl}(2)$-type fibers of the $\mathsf{SO}(2n,\mathbb{C})$-Hitchin system the Pfaffian can have higher order zeroes. A singular Hitchin fiber of this type will be a union of certain strata of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin fiber associated to the covering $\Sigma \to \Sigma/\sigma$. This yields moduli spaces of Hecke parameters different from the ones encountered for $\mathsf{Sp}(2n,\mathbb{C})$ and $\mathsf{SO}(2n+1,\mathbb{C})$. The group $G = \mathsf{SO}(2n,\mathbb{C})$ is self-dual under Langlands duality and it would be interesting to test the abstract formulation of Langlands correspondence for $\mathfrak{sl}(2)$-type Hitchin fibers in Corollary 4.4.7 for the $\mathsf{SO}(2n,\mathbb{C})$-Hitchin system.

Integrable systems on the singular locus and hyperkähler geometry. In his recent paper [Hit19], Hitchin described lower dimensional integrable systems supported on the singular locus of the $\mathsf{SL}(2,\mathbb{C})$-Hitchin map. These subintegrable systems are defined on the subsets $\mathcal{C}_d \subset H^0(X,K^2)$ of quadratic differentials with d double zeroes, such that all other zeroes are simple. However, the fibration by abelian varieties extends.

THEOREM 7.0.1. *There exists a stratification*

$$H^0(X,K^2) = \bigsqcup_{d=0}^{2g-2} \mathcal{Q}_d$$

by locally closed analytic sets $\mathcal{Q}_d$, such that $\mathcal{C}_d \subset \mathcal{Q}_d$ and for each d there exists a fibration by complex tori

$$\mathcal{P}_d \to \mathcal{Q}_d$$

with a smoothly varying polarization.

Here, $\mathcal{Q}_d$ is given by the set of quadratic differentials with $4g-4-2d$ zeroes of odd order. Hitchin proved that $\mathcal{C}_d \subset H^0(X, K^2)$ is a submanifold. We don't expect this to be true for $\mathcal{Q}_d$. However, as the fibration by abelian varieties extends to $\mathcal{Q}_d$, there might also be a way to extend the semi-flat hyperkähler metric defined on the Hitchin subintegrable system.

These semi-flat hyperkähler metrics defined over the singular locus will certainly play an important role in the analysis of the asymptotics of the Hitchin hyperkähler metric along the singular locus. An interesting special case is $\mathcal{C}_{2g-2}$. Here all zeroes of the quadratic differential are double zeroes and therefore the Higgs bundles in the subintegrable system are everywhere locally diagonalizable. We saw in Remark 5.4, that, in this case, the Hitchin equation decouples and hence is invariant under scaling the Higgs field. This suggests that the Hitchin hyperkähler metric restricted to these submanifolds is equal to the semi-flat metric associated to the Hitchin subintegrable system.

Moreover, the direct correspondence of $\mathsf{SL}(2,\mathbb{C})$-Hitchin fibers and $\mathfrak{sl}(2)$-type fibers of symplectic and odd orthogonal Hitchin systems points to the existence of subintegrable systems supported on the singular locus $B_G \setminus B_G^{\mathsf{reg}}$. It seems complicated to generalize Hitchin's method of explicitly computing the sub-algebra generated by the Hamiltonian vector fields of the Hitchin map to higher rank. On the other hand, the fibration by abelian varieties is already known. Possibly, one can define a complex symplectic structure on this abstract torus fibration and show afterwards that the inclusion into $\mathcal{M}_G$ is symplectic.

Singular Hitchin fibers beyond $\mathfrak{sl}(2)$-type. In the present work, we restricted our attention to the class of $\mathfrak{sl}(2)$-type Hitchin fibers. In joint work with Xuesen Na, we are working on a far-reaching generalization of our results. Let $\pi : \Sigma = \Sigma(\underline{a}) \to X$ a $\mathsf{SL}(n,\mathbb{C})$-spectral cover with a decomposition into irreducible components $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_l$, such that all irreducible components are non-reduced. For all i, let $\tilde{\pi}_i : \tilde{\Sigma}_i \to X$ the normalisation of Σ_i and $\lambda_i : \tilde{\Sigma}_i \to \tilde{\pi}_i^* K$ the holomorphic section induced by the inclusion $\Sigma_i \subset \mathsf{Tot}(K)$. We prove, that for all $(E,\Phi) \in \mathsf{Hit}^{-1}_{\mathsf{SL}(n,\mathbb{C})}(\underline{a})$, there exist line bundles $L_i \in \mathsf{Pic}(\Sigma_i)$, such that (E,Φ) is a Hecke modification of

$$\bigoplus_{i=1}^{l} \tilde{\pi}_{i*}(L_i, \lambda_i).$$

on $\pi(\mathsf{Sing}(\Sigma))$. Hence, we obtain semi-abelian spectral data for these class of singular fibers: The abelian part is given by a union of abelian torsors containing the line bundles $(L_1, \ldots, L_l)$ and the non-abelian part by the parameters of Hecke modifications determining the local shape of the Higgs field at the singularities of the spectral curve.

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