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Symplectic groups over noncommutative rings and maximal representations

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Abstract

Maximal representations into Lie groups of Hermitian type have been introduced in [7], and further studied in [2, 6, 26]. All maximal representation are discrete embeddings, and spaces of maximal representations are unions of connected components of the character varieties, hence they provide examples of so-called higher Teichmüller spaces. Connected components of spaces of maximal representations have complicated topology which is not well understood.

In this thesis, we study classical Hermitian Lie groups of tube type and give a parametrization of spaces of decorated (maximal) representations of the fundamental group of a punctured surface into a Hermitian Lie group of tube type. Using this parametrization, we describe the topology and the structure of the spaces of maximal representations.

In the first chapter, we introduce coordinates on the space of Lagrangian decorated representations of the fundamental group of a surface with punctures into the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. These coordinates provide a noncommutative generalization of the parametrization of the space of representations into $\mathrm{SL}(2, \mathbb{R})$ given by V. Fock and A. Goncharov. The locus of positive coordinates maps to the space of decorated maximal representations. We use this to determine the homotopy type and the homeomorphism type of the space of decorated maximal representations, and when $n = 2$, to describe its finer structure as a smooth locus and kind of singularities.

In the second chapter, we study Hermitian Lie groups of tube type and their complexifications uniformly as $\mathrm{Sp}_2(A)$ over some special real algebra A. We use this approach to describe the flag variety of such groups corresponding to a maximal parabolic subgroup, a maximal compact subgroup and different models of the symmetric space. For complexified groups this construction is new. Further, we introduce in these terms coordinates on the space of decorated maximal representations of the fundamental group of a punctured surface into a Hermitian Lie group of tube type and use them to determine the homotopy type and the homeomorphism type of the space of decorated maximal representations.

Zusammenfassung

Maximale Darstellungen in Hermitesche Lie-Gruppen wurden in [7] eingeführt und danach in [2, 6, 26] untersucht. Alle maximalen Darstellungen sind diskrete Einbettungen, und Räume der maximalen Darstellungen sind Vereinigungen von Zusammenhangskomponenten von der Charaktervarietät. Somit liefern sie Beispiele von den sogenannten höheren Teichmüller Räumen. Zusammenhangskomponenten der Räume der maximalen Darstellungen haben komplizierte Topologie, die noch nicht wohlverstanden ist.

In dieser Doktorarbeit untersuchen wir klassische Hermitesche Lie-Gruppen von Tubentyp und parametrisieren Räume der dekorierten (maximalen) Darstellungen der Fundamentalgruppe einer punktierten Fläche in eine Hermitesche Lie Gruppe vom Tubentyp. Mithilfe von dieser Parametrisierung beschreiben wir die Topologie und die Struktur der Räume der maximalen Darstellungen.

Im ersten Kapitel führen wir Koordinaten auf dem Raum von mit Lagrange Unterräumen dekorierten Darstellungen der Fundamentalgruppe einer punktierten Fläche in die symplektische Gruppe $\mathrm{Sp}(2n, \mathbb{R})$ ein. Diese Koordinaten liefern eine nicht kommutative Verallgemeinerung von den von V. Fock und A. Goncharov eingeführten Parametrisierungen der Räume von Darstellungen in $\mathrm{SL}(2, \mathbb{R})$. Der Unterraum von positiven Koordinaten wird auf den Raum von maximalen Darstellungen abgebildet. Wir verwenden das, um den Homotopietyp und Homeomorphietyp des Raums der dekorierten maximalen Darstellungen zu bestimmen und im Falle $n = 2$ seine feinere Struktur sowie die Glattheitsbereich und Typen der Singularitäten zu beschreiben.

Im zweiten Kapitel untersuchen wir Hermitesche Lie-Gruppen vom Tubentyp und ihre Komplexifizierungen auf einheitliche Weise als $\mathrm{Sp}_2(A)$ für spezielle reelle Algebren A. Wir verwenden diesen Ansatz, um die zu den maximalen parabolischen Untergruppen zugehörigen Fahnenvarietäten, maximale kompakte Untergruppen und verschiedene Modelle der symmetrischen Räume von diesen Gruppen zu beschreiben. In diesen Termen führen wir Koordinaten auf dem Raum der dekorierten maximalen Darstellungen der Fundamentalgruppe einer punktierten Fläche in eine Hemitesche Lie-Gruppe vom Tubentyp ein und verwenden sie, um den Homotopietyp und den Homeomorphietyp des Raums der dekorierten maximalen Darstellungen zu bestimmen.

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Contents

0 Introduction

0.1 Higher Teichmüller theory: from hyperbolic structures to representation varieties

Higher Teichmüller theory was developed as a generalization of classical Teichmüller theory that studies moduli spaces of complex structures on a fixed topological surface S of negative Euler characteristic. This moduli space is called *Teichmüller space* $\mathcal{T}(S)$, and it can also be seen as the moduli space of marked complete hyperbolic structures on the surface S. Teichmüller space $\mathcal{T}(S)$ can be naturally embedded into the representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{PSL}(2,\mathbb{R})$ as a connected component which consists entirely of discrete and faithful representations.

Higher Teichmüller theory generalizes this approach and studies representations of $\pi_1(S)$ into a reductive Lie group G of higher rank. A *higher Teichmüller space* is a subset of $\mathrm{Rep}(\pi_1(S), G) := \mathrm{Hom}(\pi_1(S), G)/G$ which is a union of connected components that consist entirely of discrete and faithful representations. There are two well-known families of Higher Teichmüller spaces: Hitchin components and spaces of maximal representations.

Hitchin components are defined when G is a split real simple Lie group (e.g. $\mathrm{SL}(n,\mathbb{R})$) [11, 20, 21]. The space of maximal representations is defined when G is a noncompact simple Lie group of Hermitian type (e.g. $\mathrm{Sp}(2n,\mathbb{R})$) [6, 7]. They have been discovered from very different points of view and by very different methods. They also have different properties, e.g. Hitchin components are always contractible and homeomorphic to an Euclidean ball. In contrast, connected components of spaces of maximal representations have nontrivial complicated topology. Nevertheless, as described before, they also share many properties [6, 21]. Moreover, in the case when $G = \mathrm{PSL}(2,\mathbb{R})$, the Hitchin component and the space of maximal representations agree and coincide with the Teichmüller space $\mathcal{T}(S)$ [28].

However, higher Teichmüller spaces do not exist for every Lie group G. Discrete and faithful representations form in general only a closed subset of $\mathrm{Rep}(\pi_1(S), G)$ but not connected components. In fact, there are special families of Lie groups for which higher Teichmüller spaces exist.

One example of Lie groups which admit higher Teichmüller spaces is conjectured to be Lie groups with a notion of positivity. The theory of Θ-positivity was developed by O. Guichard and A. Wienhard and generalizes Lusztig's total positivity for split real Lie groups and maximality for Hermitian Lie groups to a larger class of simple Lie groups (e.g. $\mathrm{SO}(p,q)$, $p \neq q$) [16].

0.2 Fock–Goncharov's $\mathcal{X}$-space

In their seminal paper [11], Fock and Goncharov introduced an $\mathcal{X}$-moduli space, which is closely related to the variety of representations of the fundamental group of a surface $S_{g,k}$ of genus g with k punctures into a split real simple Lie group G. They introduced explicit cluster $\mathcal{X}$-coordinates on this space associated to an ideal triangulation of $S_{g,k}$. Changing the triangulation, the coordinates change by positive rational functions. Thus the locus of positive coordinates is independent of the choice of triangulation. When G is $\mathrm{SL}(2,\mathbb{R})$, the positive locus in the $\mathcal{X}$-space is closely related to the Teichmüller space, and the Fock–Goncharov coordinates are extensions of Thurston's shear coordinates. When G is a split real group of higher rank, this moduli space gives higher Teichmüller space, and the positive locus of the $\mathcal{X}$-space is closely related to the Hitchin component in the representation variety.

The set of positive representations of Fock–Goncharov and the Hitchin components account only for one family of higher Teichmüller spaces, another family is given by maximal representations into Lie groups of Hermitian type. The symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$ form essentially the only family of Lie groups that are both split real forms and of Hermitian type.

In the first chapter of this thesis, we generalize the work of Fock and Goncharov in the following way. We introduce a new moduli space, an $\mathcal{X}$-space of representations of the fundamental group of $S_{g,k}$ into the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, and describe non-commutative A_1-type cluster coordinates on them. We show that the positive locus of the $\mathcal{X}$-space corresponds precisely to maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$; we use this to determine the homeomorphism type and the homotopy type of the space of maximal representations, and for $\mathrm{Sp}(4,\mathbb{R})$ also its finer structure as a smooth locus and kind of singularities.

In Fock–Goncharov's work, an important role is played by Lusztig's total positivity, in our work, a similar role is played by positivity related to the Maslov index. As such, our work fits well in the framework of Θ-positivity, recently introduced by O. Guichard and A. Wienhard [14–16, 28], that generalizes Lusztig's total positivity and provides a unifying framework for the different higher Teichmüller spaces.

When the Fock–Goncharov's approach is applied to the group $\mathrm{Sp}(2n,\mathbb{R})$, they define a positive locus in the space of symplectic representations. It is important to remark that the positive locus that our approach gives in the space of symplectic representations is larger than the Fock–Goncharov's one (see Section 1.3.6 in Chapter 1 for more details). This is because the two theories are based on two different Θ-positive structures on $\mathrm{Sp}(2n,\mathbb{R})$: respectively the one for split groups and the one for groups of Hermitian type. The perspective chosen in the present thesis is the one which is suitable for describing the spaces of maximal representations.

We now describe our results in more detail.

0.3 Generalization of $\mathcal{X}$-moduli space

In the first chapter of the thesis, we introduce the space of *decorated symplectic representations* (i.e. a representation $\pi_1(S_{g,k}) \to \mathrm{Sp}(2n, \mathbb{R})$ together with a consistent choice of Lagrangian subspaces, which are fixed by peripheral elements in $\pi_1(S_{g,k})$) which serves as our $\mathcal{X}$-space.

Fixing an ideal triangulation $\mathcal{T}$ of $S_{g,k}$, we introduce systems of $\mathcal{X}$-coordinates, using invariant of triples, 4-tuples, and 5-tuples of Lagrangian subspaces. A system of $\mathcal{X}$-coordinates consists of a triangle invariant for each triangle, which is given by the Maslov index of the three Lagrangians associated to the vertices of the triangle, an edge invariant for every edge of the triangulation, which can be seen as a cross-ratio function of four Lagrangians, and an angle invariant, associated to each corner of a triangle, which comes from an invariant of 5-tuples of Lagrangians. We then describe in detail a map denoted by rep from the set $\mathcal{X}(\mathcal{T})$ of $\mathcal{X}$-coordinates to the space of decorated representations. A special role is played by the set $\mathcal{X}^+(\mathcal{T})$ of positive $\mathcal{X}$-coordinates, those for which the triangle invariants are equal to n, the edge invariants are just n-tuples of positive real numbers, and the angle invariants take values in $\mathrm{O}(n)$.

Theorem 0.3.1. *The map* rep *induces a proper surjection with generically finite fibers from $\mathcal{X}^+(\mathcal{T})$ to the space of decorated maximal representations*

Let us emphasize that the correspondence between positive $\mathcal{X}$-coordinates and decorated maximal representations is not a one-to-one. To every decorated maximal representation corresponds a system of positive $\mathcal{X}$-coordinates, but in general only the edge invariants are uniquely determined, the angle invariants involve some choices. We also explicitly describe the fibers of the map rep (Proposition 1.3.8 and Theorem 1.5.18).

0.4 Topology of the space of maximal representations

We now discuss the applications to the topology of the space of (decorated) maximal representations. Let us point out that contrary to the space of positive representations or the Hitchin component, which are contractible, the space of maximal representations has non-trivial topology. In the case of maximal representations of fundamental groups of closed surfaces, the topology of the space of maximal representations has been studied using the theory of Higgs bundles in [1, 5, 12, 13]. These techniques do not apply easily to the case of maximal representations of fundamental groups of surface with punctures, in particular since we do not fix the holonomy along peripheral curves on the surface.

Here we rely on Theorem 0.3.1 and the positive locus of the $\mathcal{X}$-coordinates to determine the topology of the space of maximal representations. Note that the positive locus of the $\mathcal{X}$-coordinates does not parametrize the space of decorated maximal representations, but maps surjectively to it. The fibers of this surjection are complicated to describe, because they depend on the shape of the edge invariants. Studying this fibration, we can describe precisely the homeomorphism type of the space of decorated maximal representations:

Theorem 0.4.1. *The space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *is homeomorphic to*

$$\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1} / \mathrm{O}(n)$$

where $\mathrm{Sym}^+(n,\mathbb{R})$ *is the space of all symmetric positive definite matrices and* $\mathrm{O}(n)$ *acts by simultaneous conjugation in every factor.*

As consequence of this statement, we derive the homotopy type of the space of decorated maximal representations any connected central extension of $\mathrm{PSp}(2n,\mathbb{R})$, see Theorem 1.6.6.

Theorem 0.4.2. *The space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *admit as a deformation retract the space* $\mathrm{O}(n)^{2g+k-1}/\mathrm{O}(n)$*, where the action of* $\mathrm{O}(n)$ *is by simultaneous conjugation.*

As a corollary, we obtain a different proof of [26, Theorem 7.2.7] on the number of connected components.

Corollary 0.4.3. *The space of maximal representations and the space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *have* 2^{2g+k-1} *connected components. The space of decorated maximal representations into* $\mathrm{PSp}(2n,\mathbb{R})$ *has* 2^{2g+k-1} *connected components when n is even; it is connected if n is odd.*

When $n = 2$, we analyze this space in more detail and show that all connected components except one are orbifolds, one connected component contains a non-orbifold singularity, see Section 1.4.3.

0.5 Hermitian Lie groups of tube type

In the second chapter of this thesis, we study classical Hermitian Lie groups of tube type. We prove that all of them can be seen as a noncommutative analog of the symplectic group $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. More precisely, it is possible to see all classical Hermitian Lie groups of tube type uniformly as $\mathrm{Sp}_2(A,\sigma)$ over some special noncommutative $\mathbb{R}$-algebra with an anti-involution (A,σ) or as $\mathrm{Sp}_2(G,\sigma)$ where G is a Lie group of some special type.

To be exact, sometimes, the entire algebra (A,σ) is to large to construct the group $\mathrm{Sp}_2(A,\sigma)$, and it is reasonable to consider a suitable Lie subgroup G of $A^\times$ that is closed under σ and such that the Lie algebra B of G admits a G-invariant proper convex cone B^{sym}_+ inside the space of σ-symmetric elements $B^{sym} := \mathrm{Fix}_B(\sigma)$. For such G, the group $\mathrm{Sp}_2(G,\sigma)$ can be defined. Moreover, the case of $\mathrm{Sp}_2(A,\sigma)$ can be seen as a special case of $\mathrm{Sp}_2(G,\sigma)$ taking $G = A^\times$.

In fact, the group $\mathrm{Sp}_2(G,\sigma)$ generalizes the case of $\mathrm{Sp}_2(A,\sigma)$ at the cost of additional complications. Therefore, first in Section 2.1, we discuss the easier case, defining the group $\mathrm{Sp}_2(A,\sigma)$ and studying its properties, and only later in Section 2.6, we give the most general definition of $\mathrm{Sp}_2(G,\sigma)$.

For example, the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ discussed in the first chapter is isomorphic to $\mathrm{Sp}_2(A,\sigma)$ for $A = \mathrm{Mat}(n,\mathbb{R})$ and the anti-involution σ corresponds to the matrix transposition. The group $\mathrm{U}(n,n)$ can be seen as $\mathrm{Sp}_2(A,\sigma)$ for $A = \mathrm{Mat}(n,\mathbb{C})$ and the anti-involution σ corresponds to the complex conjugation composed with the matrix transposition. The group $\mathrm{SO}^*(4n)$ is isomorphic to $\mathrm{Sp}_2(A,\sigma)$ for $A = \mathrm{Mat}(n,\mathbb{H})$ and the anti-involution σ corresponds to the quaternionic conjugation composed with the matrix transposition. In contrast, the group $\mathrm{Spin}(2,n)$ can only be seen as $\mathrm{Sp}_2(G,\sigma)$, where G is the so-called Clifford group, but it cannot be described as $\mathrm{Sp}_2(A,\sigma)$.

Moreover, using this approach, it becomes possible to describe a wider class of groups. Namely, we can see in this picture groups that are complexifications of Hermitian groups of tube type (e.g. $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{GL}(4n,\mathbb{C})$ and $\mathrm{O}(4n,\mathbb{C})$). For this wider class of groups, we study their maximal compact subgroups and the space of isotropic A- and G-lines as the flag variety corresponding to a maximal parabolic subgroup of $\mathrm{Sp}_2(A,\sigma)$, resp. $\mathrm{Sp}_2(G,\sigma)$. In fact, this flag variety generalizes the real projective space $\mathbb{RP}^2$ which the group $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ is acting on. We discuss properties of the action of $\mathrm{Sp}_2(A,\sigma)$ and $\mathrm{Sp}_2(G,\sigma)$ on this flag variety and find out what are invariants of tuples of isotropic lines under this action. These invariants are closely related to the well-known invariants as the Maslov index and the cross ratio that we discussed in the first chapter for such action of the group $\mathrm{Sp}(2n,\mathbb{R})$ on the Lagrangian Grassmannian.

Further, we discuss the symmetric spaces of $\mathrm{Sp}_2(A,\sigma)$ and $\mathrm{Sp}_2(G,\sigma)$. We are considering two cases: classical Hermitian Lie groups of tube type and their complexifications. In both cases, we can describe their symmetric spaces with models. More precisely, we construct the upper half space model, the projective model, the precompact model (that is usually called bounded model in the literature) and the complex structure model (for real groups) and the quaternionic structure model (for complexified group). We also discuss the natural compactification of these symmetric spaces and an analog of the Shilov boundary for complexified groups. These models are well-known for Hermitian Lie groups [6], but in the case of the complexified groups these results are new.

At the end of the second chapter, we discuss decorated maximal representations of the fundamental group of a punctured surface $S_{g,k}$ into classical Hermitian Lie groups that we see as $\mathrm{Sp}_2(G,\sigma)$. We define noncommutative positive $\mathcal{X}$-coordinates in terms of the group G that generalize positive $\mathcal{X}$-coordinates associated to an ideal triangulation of the surface that we defined in the first chapter. As before, we associate to every edge of the ideal triangulation an n-tuple of positive real numbers where n is the rank of G. The angle invariants take value in the group

$$U(G,\sigma) = \{g \in G \mid \sigma(g)g = 1\}.$$

As in the first chapter, we obtain the map rep that maps surjectively the space of positive $\mathcal{X}$-coordinated onto the space of decorated maximal representations.

Analysing this map rep, we obtain the generalization of the Theorem 0.4.1 describing the homeomorphism type of the space of decorated maximal representations into $\mathrm{Sp}_2(G,\sigma)$:

Theorem 0.5.1. *The space of decorated maximal representation into* $\mathrm{Sp}_2(G,\sigma)$ *is homeomorphic to*

$$(B_+^{sym})^{6g+3k-6} \times U(G,\sigma)^{2g+k-1}/U(G,\sigma)$$

where $U(G,\sigma)$ *acts by simultaneous conjugation in every factor.*

As corollary from this Theorem, we derive the homotopy type of the space of decorated maximal representations into $\mathrm{Sp}_2(G,\sigma)$:

Theorem 0.5.2. *The space of decorated maximal representations admits as a deformation retract the space* $U(G,\sigma)^{2g+k-1}/U(G,\sigma)$*. The quotient is taken by the action of* $U(G,\sigma)$ *on* $U(G,\sigma)^{2g+k-1}$ *by simultaneous conjugation.*

0.6 Structure of the thesis

The present thesis contains the Introduction and two Chapters. The first Chapter is dedicated to the study of the decorated representations into the group $\mathrm{Sp}(2n,\mathbb{R})$. In Section 1.1, we introduce the invariants of Lagrangians which are used to define coordinates. In Section 1.2, we introduce the spaces of decorated representations, recall the definition and key properties of maximal representations. In Section 1.3, we introduce positive $\mathcal{X}$-coordinates, and construct the map to decorated maximal representations. The applications for the topology of the space of maximal representations are proven in Section 1.4. The general $\mathcal{X}$-coordinates are introduced in Section 1.5, and in Section 1.6 we generalize them to representations into central extensions of $\mathrm{PSp}(2n,\mathbb{R})$. The first Chapter is part of the joint work with Daniele Alessandrini, Olivier Guichard and Anna Wienhard and is published on the arXiv as [2]. Main contributions of the author in this project are the definition of general $\mathcal{X}$-coordinates, the standard form of a pair of bilinear forms and the description of the topology and homotopy type of the space of maximal representations.

The second Chapter is dedicated to the study of Hermitian groups of tube type in terms of the symplectic group Sp_2 over noncommutative algebras. In Sections 2.1, we introduce Hermitian algebras with anti-involution (A,σ), their complexifications and the group $\mathrm{Sp}_2(A,\sigma)$, discuss their properties and give examples. In Section 2.2, we construct the space of isotropic lines and discuss the action of $\mathrm{Sp}_2(A,\sigma)$ on it and find invariants of tuples of isotropic lines. In Sections 2.3 and 2.4, we construct different models of symmetric space of $\mathrm{Sp}_2(A,\sigma)$, discuss its compactification and Shilov boundary. In Section 2.5, we implement these models for examples of classical Hermitian Lie groups and their complexifications. In Section 2.6, we define Hermitian Lie algebras with an anti-involution (B,σ) and Lie groups (G,σ) corresponding to such Lie algebras. In Section 2.7, we define the group $\mathrm{Sp}_2(G,\sigma)$ and discuss its properties. In Section 2.8, we construct the space of isotropic G-lines and discuss invariants of tuples of isotropic G-lines. In Sections 2.9, we introduce different models of symmetric space of $\mathrm{Sp}_2(G,\sigma)$, discuss its compactification and Shilov boundary. In Section 2.10, we describe the group $\mathrm{Spin}(2,n)$ as $\mathrm{Sp}_2(G,\sigma)$. In Section 2.11, decorated maximal representations into $\mathrm{Sp}_2(G,\sigma)$ are discussed. The second Chapter will appear in a joint work with Daniele Alessandrini, Arkady Berenstein, Vladimir Retakh and Anna Wienhard. Main contributions of the author in this project are the development of the general theory of $\mathrm{Sp}_2(G,\sigma)$, describing the right conditions for G such that the group $\mathrm{Sp}_2(G,\sigma)$ is well-defined, including the group $\mathrm{Spin}(2,n)$ into this context and the description of models of the symmetric spaces for complexified groups.

The Appendix contains a description of the invariants of pairs of non-degenerate symmetric bilinear forms that are used in Section 1.5 and explicit constructions of isomorphisms between matrix algebras that are used in Sections 2.5 to construct examples of symmetric spaces.

1 Noncommutative coordinates for symplectic representations

1.1 Invariants of Lagrangian subspaces

1.1.1 Lagrangian Grassmannian

We consider the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ where ω is the standard symplectic form on $\mathbb{R}^{2n}$, i.e.

$$\omega(x, y) = \sum_{i=1}^{n} x_i y_{n+i} - \sum_{i=1}^{n} x_{n+i} y_i,$$

for $x = \sum_{i=1}^{2n} x_i e_i$, $y = \sum_{i=1}^{2n} y_i e_i$ where $(e_1, \dots, e_{2n})$ is the standard basis of $\mathbb{R}^{2n}$. With respect to the standard basis, ω can be written as

$$\omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \tag{1.1.1}$$

Every basis of $\mathbb{R}^{2n}$ such that ω, expressed in that basis, has the form (1.1.1) is called a *symplectic basis*. We will usually write a symplectic basis as $(\mathbf{e}, \mathbf{f})$, where $\mathbf{e} = (e_1, \dots, e_n), \mathbf{f} = (f_1, \dots, f_n)$, and $\omega(e_i, f_j) = \delta_{ij}$.

We denote by $\mathrm{Sp}(2n, \mathbb{R})$ the *symplectic group*,

$$\mathrm{Sp}(2n, \mathbb{R}) = \{g \in \mathrm{GL}(2n, \mathbb{R}) \mid g^T \omega g = \omega\},$$

and by $\mathrm{PSp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/\{\pm \mathrm{Id}\}$ the *projective symplectic group*.

Definition 1.1.1. A subspace L of $\mathbb{R}^{2n}$ is called *Lagrangian* if $\dim(L) = n$ and $\omega(u, v) = 0$ for all $u, v \in L$. The set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$ is called *Lagrangian Grassmannian*, we denote this set by $\mathrm{Lag}(2n, \mathbb{R})$.

Definition 1.1.2. A *framed Lagrangian* is a pair $(L, \mathbf{v})$, where $L \in \mathrm{Lag}(2n, \mathbb{R})$ and $\mathbf{v}$ is a basis of L. The set of all framed Lagrangians of $(\mathbb{R}^{2n}, \omega)$ is called *framed Lagrangian Grassmannian*, we denote this set by $\mathrm{Lag}^{fr}(2n, \mathbb{R})$. The natural projection to $\mathrm{Lag}(2n, \mathbb{R})$ turns this space into a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts naturally on $\mathrm{Lag}(2n, \mathbb{R})$ and $\mathrm{Lag}^{fr}(2n, \mathbb{R})$:

$$g(L) := \{g(x) \mid x \in L\},$$
$$g(L, (v_1, \dots, v_n)) := (g(L), (g(v_1), \dots, g(v_n))).$$

These actions are transitive, hence the spaces $\mathrm{Lag}(2n, \mathbb{R})$ and $\mathrm{Lag}^{fr}(2n, \mathbb{R})$ are homogeneous spaces over the symplectic group. To better see this structure, consider the stabilizers of a point:

$$P = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(L), \tag{1.1.2}$$
$$U = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}((L, v)). \tag{1.1.3}$$

The group P is a parabolic subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, and $U \subset P$ is its unipotent subgroup. As homogeneous spaces, we have

$$\mathrm{Lag}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/P,$$
$$\mathrm{Lag}^{fr}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/U.$$

Anyway, the action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathrm{Lag}(2n, \mathbb{R})$ is not effective, it has kernel $\{\pm \mathrm{Id}\}$. The actual group of symmetries of $\mathrm{Lag}(2n, \mathbb{R})$ is the projective symplectic group $\mathrm{PSp}(2n, \mathbb{R})$.

Definition 1.1.3. Two Lagrangians $L_1, L_2 \in \mathrm{Lag}(2n, \mathbb{R})$ are called *transverse* if $L_1 \oplus L_2 = \mathbb{R}^{2n}$.

We now describe charts for $\mathrm{Lag}(2n, \mathbb{R})$. Since we will work in these charts regularly, we describe them and the coordinate changes in detail. Given a Lagrangian L_∞, we denote by $\mathcal{U}_{L_\infty}$ the subset of $\mathrm{Lag}(2n, \mathbb{R})$ consisting of all the Lagrangians transverse to L. This is an open dense subset of $\mathrm{Lag}(2n, \mathbb{R})$. Fixing a Lagrangian $L_0 \in \mathcal{U}_{L_\infty}$ any other Lagrangian $L \in \mathcal{U}_{L_\infty}$ is the graph of a linear map $L_{L_0 \to L_\infty} : L_0 \to L_\infty$, i.e. for each $v \in L_0$, $L_{L_0 \to L_\infty}(v)$ is the unique element in L_∞ such that $v + L_{L_0 \to L_\infty}(v) \in L$. If L is also transverse to L_0, this map, which we denote just by L if there is no danger of confusion, is a linear isomorphism.

We will often use an explicit matrix expression for this linear map. If we choose $\mathbf{e} = (e_1, \ldots, e_n)$ a basis of L_0, there exists a unique basis $\mathbf{f} = (f_1, \ldots, f_n)$ of L_∞ such that $(\mathbf{e}, \mathbf{f})$ is a symplectic basis. Given a symplectic basis $(\mathbf{e}, \mathbf{f})$, we will more generally write then

$$L_{\mathbf{e}} := \mathrm{Span}(\mathbf{e}),$$
$$L_{\mathbf{f}} := \mathrm{Span}(\mathbf{f}).$$

We write $[L_{L_{\mathbf{e}} \to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}}$ for the matrix of the map $L_{L_{\mathbf{e}} \to L_{\mathbf{f}}}$ with respect to the bases $\mathbf{e}, \mathbf{f}$. It is easy to check that this matrix is symmetric. The linear map L and its matrix $[L_{L_{\mathbf{e}} \to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}}$ will be used often in this thesis.

We thus have a map

$$\Psi_{(\mathbf{e},\mathbf{f})} : \mathcal{U}_{L_{\mathbf{f}}} \ni L \to [L_{L_{\mathbf{e}} \to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}} \in \mathrm{Sym}(n, \mathbb{R})$$

This map is a homeomorphism to the vector space of symmetric matrices. To see that it is invertible, the inverse map is given by the formula

$$L_{\mathbf{e},\mathbf{f}}(A) := L = \mathrm{Span}(\mathbf{e} + \mathbf{f}A)$$

The set
$$\{(\mathcal{U}_{L_{\mathbf{f}}}, \Psi_{(\mathbf{e},\mathbf{f})}) \mid (\mathbf{e},\mathbf{f}) \text{ symplectic basis } \}$$
is a manifold atlas for the space $\mathrm{Lag}(2n,\mathbb{R})$.

Remark 1.1.4. We can write the transition functions of this atlas. Assume $(\mathbf{e},\mathbf{f})$ and $(\mathbf{e}',\mathbf{f}')$ are two symplectic bases. There is a unique symplectic matrix $B \in \mathrm{Sp}(2n,\mathbb{R})$ such that $(\mathbf{e}',\mathbf{f}') := (\mathbf{e},\mathbf{f})B^{-1}$. Write B as
$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}),$$
where the B_{ij} are $n \times n$ matrices. For every $L \in \mathcal{U}_{L_f} \cap \mathcal{U}_{L_{f'}}$, denote by
$$\begin{aligned} A &:= \Psi_{(\mathbf{e},\mathbf{f})}(L) \\ A' &:= \Psi_{(\mathbf{e}',\mathbf{f}')}(L) \end{aligned}$$
Then
$$A' = (B_{11} + B_{12}A)^{-1}(B_{21} + B_{22}A) \in \mathrm{Sym}(n,\mathbb{R}). \tag{1.1.4}$$
Remark 1.1.5. Formula (1.1.4) also represents the action of the matrix B on $\mathrm{Lag}(2n,\mathbb{R})$, when restricted to a coordinate chart $\mathcal{U}_{L_f}$: for a Lagrangian L such that both $L, B(L) \in \mathcal{U}_{L_f}$,
$$\begin{aligned} A &:= \Psi_{(\mathbf{e},\mathbf{f})}(L) \\ A^B &:= \Psi_{(\mathbf{e},\mathbf{f})}(B(L)) \end{aligned}$$
we have
$$A^B = (B_{11} + B_{12}A)^{-1}(B_{21} + B_{22}A) \in \mathrm{Sym}(n,\mathbb{R}).$$
In fact the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is formally similar to the action by Möbius transformations of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{CP}^1$ (which is the case $n = 1$).

The action of $\mathrm{Sp}(2n,\mathbb{R})$ on pairs of transverse Lagrangians is transitive, but the action of $\mathrm{Sp}(2n,\mathbb{R})$ on triples, quadruples and 5-tuples of pairwise transverse Lagrangians is not transitive any more. We will now describe invariants of such tuples of Lagrangians, which will lie the foundation for the rest of the thesis.

Similarly, the action of $\mathrm{Sp}(2n,\mathbb{R})$ on pairs $((L,v),L')$, where $(L,v) \in \mathrm{Lag}^{fr}(2n,\mathbb{R})$, $L' \in \mathrm{Lag}(2n,\mathbb{R})$ and L, L' are transverse, is transitive and free. But when we consider pairs $(L,v),(L',v') \in \mathrm{Lag}^{fr}(2n,\mathbb{R})$, the action is not transitive any more, and we describe invariants of such pairs.

1.1.2 Maslov index

In this section we review properties of the Maslov index of three pairwise transverse Lagrangians, for a more general discussion we refer the reader to [22].

Let L_1, L_2, L_3 be three pairwise transverse Lagrangians. As in the previous section, we consider the linear map $L_{3L_1 \to L_2}$. When this does not cause confusion, we will denote the linear map just by L_3.

Using the symplectic form ω, we can define a bilinear form β_3 on L_1 in the following way: for $v_1, v_2 \in L_1$

$$\beta_3(v_1, v_2) := \omega(v_1, L_3(v_2)).$$

We also denote the bilinear form β_3 by $[L_1, L_3, L_2]$.

Proposition 1.1.6. *The bilinear form $\beta_3 = [L_1, L_3, L_2]$ is non degenerate and symmetric.*

Proof. Since $L_3(v) + v \in L_3$ for all $v \in V_1$,

$$0 = \omega(L_3 v + v, L_3 w + w) = \omega(L_3 v, w) + \omega(v, L_3 w).$$

Therefore,

$$\beta_3(v, w) = \omega(v, L_3 w) = -\omega(L_3 v, w) = \omega(w, L_3 v) = \beta_3(w, v)$$

The form β_3 is non-degenerate because L_3 is a linear isomorphism between two transverse Lagrangians L_1 and L_2, i.e. $\omega|_{L_1 \times L_2}$ is non-degenerate. $\square$

We will denote the signature of β_3 by

$$\mathrm{sgn}(\beta_3) = (p, q),$$

where p is the dimension of a maximal subspace of L_1 on which β_3 is positive definite and q is the dimension of a maximal subspace of L_1 on which β_3 is negative definite. They satisfy $p + q = n$. We will also sometimes express the signature as

$$\mathrm{dsgn}(\beta_3) = p - q \in \{-n, -n+2, \dots, n-2, n\}.$$

Definition 1.1.7. The *Maslov index* of the triple of Lagrangians (L_1, L_3, L_2) is the signature $\mathrm{dsgn}([L_1, L_3, L_2])$ and denoted by $\mu(L_1, L_3, L_2)$.

For $n = 1$, the three Lagrangians (L_1, L_3, L_2) correspond to distinct points in the circle $\mathbb{RP}^1$. The Maslov index is 1 if the three points are cyclically ordered, and it is -1 if they are in the reverse cyclic order.

Proposition 1.1.8 (Properties of Maslov index)**.** *The Maslov index*

- *is invariant under the action of* $\mathrm{Sp}(2n, \mathbb{R})$ *on* $\mathrm{Lag}(2n, \mathbb{R})$*;*
- *is anti-symmetric when two of its variables are exchanged;*
- *satisfies the cocycle relation, i.e. for all pairwise transverse* $L_1, L_2, L_3, L_4 \in \mathrm{Lag}(2n, \mathbb{R})$

$$\mu(L_1, L_2, L_3) - \mu(L_1, L_2, L_4) + \mu(L_1, L_3, L_4) - \mu(L_2, L_3, L_4) = 0$$

- *the group* $\mathrm{Sp}(2n, \mathbb{R})$ *acts transitively on the set of triples of pairwise transverse Lagrangians with the same Maslov index, i.e.* $\mathrm{Sp}(2n, \mathbb{R})$*-orbits of pairwise transverse triples of Lagrangians are in 1-1 correspondence with the Maslov indices.*

1.1.3 Cross ratio

Let L_1, L_2, L_3, L_4 be four Lagrangians such that L_3 and L_4 are transverse to L_1 and L_2. We use the linear isomorphisms $L_3\colon L_1 \to L_2$ and $L_4\colon L_2 \to L_1$ to introduce the map

$$[L_1, L_3, L_2, L_4] := L_4 \circ L_3\colon L_1 \to L_1$$

which is a linear automorphism of L_1.

Definition 1.1.9. The map

$$[L_1, L_3, L_2, L_4]\colon L_1 \to L_1$$

is called the *cross ratio* of the 4-tuple of Lagrangians (L_1, L_3, L_2, L_4).

For related invariants of 4 Lagrangians, see [3, 4, 18, 23–25, 29]. For $n = 1$, the cross ratio is a linear map from a line to itself. This is just the multiplication by a scalar, which is exactly the cross ratio of four lines in $\mathbb{R}^2$ in the classical sense.

Proposition 1.1.10 (Properties of cross ratio)**.**

- *The cross ratio is equivariant under the action of* $\mathrm{Sp}(2n, \mathbb{R})$ *on* $\mathrm{Lag}(2n, \mathbb{R})$.
- $[L_1, L_3, L_2, L_4] = [L_1, L_4, L_2, L_3]^{-1}$*;*
  $[L_1, L_3, L_2, L_4] = L_3^{-1} \circ [L_2, L_4, L_1, L_3] \circ L_3$.
- *The group* $\mathrm{Sp}(2n, \mathbb{R})$ *acts transitively on quadruples of pairwise transverse Lagrangians having conjugate cross ratios, i.e. the* $\mathrm{Sp}(2n, \mathbb{R})$*-orbits of pairwise transverse quadruples of Lagrangians are in 1-1 correspondence with the conjugacy classes of cross ratios.*

Proposition 1.1.11. *The cross ratio* $B := [L_1, L_3, L_2, L_4]$ *is a symmetric linear map with respect to the bilinear forms* $[L_1, L_3, L_2]$ *and* $[L_1, L_4, L_2]$.

Proof. Let $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$ be a symmetric bilinear form on L_2. Let $v, w \in L_1$. Then:

$$\beta_3(Bv, w) = \omega(L_4 L_3 v, L_3 w) = -\omega(L_3 w, L_4 L_3 v) =$$
$$= -\beta_4(L_3 w, L_3 v) = -\beta_4(L_3 v, L_3 w) = -\omega(L_3 v, L_4 L_3 w) =$$
$$= \omega(L_4 L_3 w, L_3 v) = \beta_3(Bw, v) = \beta_3(v, Bw).$$ □

Corollary 1.1.12. *If* $[L_1, L_3, L_2]$ *and* $[L_2, L_4, L_1]$ *are positive definite, then* $-[L_1, L_3, L_2, L_4]$ *is diagonalizable with positive eigenvalues.*

Proof. We set as before $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$. Let $\mathbf{e}$ be a basis of L_1 such that $[\beta_3]_{\mathbf{e}} = \mathrm{Id}$ and $[B]_{\mathbf{e}} = -\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. We take the unique basis $\mathbf{f}$ of L_2 such that $\omega(\mathbf{e}, \mathbf{f}) = \mathrm{Id}$. Then $L_3(\mathbf{e}) = \mathbf{f}$ and $[L_3]_{\mathbf{e},\mathbf{f}} = \mathrm{Id}$.

In the basis $\mathbf{f}$ the bilinear form β_4 is diagonal because for every two basis vectors f_i, f_j

$$\beta_4(f_i, f_j) = \omega(f_i, L_4(f_j)) = \omega(L_3 L_3^{-1}(f_i), L_4 L_3 L_3^{-1}(f_j)) =$$
$$= \omega(L_3 e_i, B e_j) = -\omega(B e_j, L_3 e_i) = -\beta_3(B e_j, e_i) = \lambda_i \delta_{ij}$$

Since β_4 is positive definite, we have $\lambda_i > 0$ for all i. □

1.1.4 Angles

We will also make use of invariants of five Lagrangians, here we describe it in the simplest case, when all the Maslov indices are maximal. For the general version of this invariant, see Section 1.5.1. Let $L_1, \dots, L_5$ be pairwise-transverse Lagrangians, which we will think as the vertices of a pentagon, as in Figure 1.1.1. Assume that

$$\mu(L_1, L_3, L_2) = \mu(L_2, L_4, L_1) = \mu(L_1, L_5, L_3) = n.$$

The bilinear forms $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$ are positive definite, therefore, by Corollary 1.1.12, there exists a basis $\mathbf{e_1}$ of L_1 such that $[\beta_3]_{\mathbf{e_1}} = \mathrm{Id}$ and $[L_1, L_3, L_2, L_4]_{\mathbf{e_1}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n > 0$.

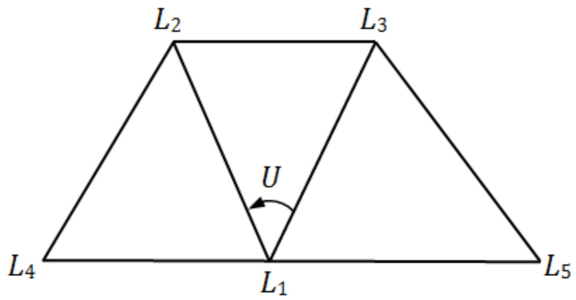

Figure 1.1.1:

We can do the same for the quadruple (L_3, L_2, L_1, L_5) and find a basis $\mathbf{g}$ of L_3 such that the bilinear forms $[L_3, L_2, L_1]_g = \mathrm{Id}$ and $-[L_3, L_2, L_1, L_5]_{\mathbf{g}} = \operatorname{diag}(\mu_1, \dots, \mu_n)$ with $\mu_1 \geq \dots \geq \mu_n > 0$.

We take the unique basis $\mathbf{e_2}$ on L_1 such that $\omega(\mathbf{g}, \mathbf{e_2}) = \mathrm{Id}$. In the basis $\mathbf{e_2}$ of L_1 we have

$$[\beta_3]_{\mathbf{e_2}} = [L_1, L_2, L_3]_{\mathbf{e_2}} = [L_3, L_2, L_1]_{\mathbf{g}} = \mathrm{Id}\,.$$

Let $U \in \mathrm{O}(n)$ be the change-of-basis matrix from the basis $\mathbf{e_2}$ to the basis $\mathbf{e_1}$. We will call this matrix an inner *angle* in the pentagon of Lagrangians $(L_1, L_4, L_2, L_3, L_5)$ (see Figure 1.1.1).

The matrix U is not uniquely defined because the bases $\mathbf{e_1}$ and $\mathbf{g}$ are not unique. In general, U is only well defined as an element of the double coset space $\mathrm{Stab}_1 \setminus \mathrm{O}(n) / \mathrm{Stab}_2$, where

$$\mathrm{Stab}_1 := \{A \in \mathrm{O}(n) \mid \ A \operatorname{diag}(\lambda_1, \dots, \lambda_n) A^T = \operatorname{diag}(\lambda_1, \dots, \lambda_n)\},$$

$$\mathrm{Stab}_2 := \{A \in \mathrm{O}(n) \mid \ A \operatorname{diag}(\mu_1, \dots, \mu_n) A^T = \operatorname{diag}(\mu_1, \dots, \mu_n)\}.$$

We denote by $[L_1, L_5, L_3, L_2, L_4]$ the class of U in $\mathrm{Stab}_1 \setminus \mathrm{O}(n) / \mathrm{Stab}_2$. If bases $\mathbf{e_1}$ and $\mathbf{e_2}$ are chosen as above, we will write

$$U =: [L_1, L_5, L_3, L_2, L_4]_{e_1, e_2}.$$

1.2 Representation varieties

One goal of this thesis is to give a parametrization of spaces of representations of the fundamental group of a punctured surface into $\mathrm{Sp}(2n, \mathbb{R})$, which can be viewed as a non-commutative generalization of the parametrization of representations into $\mathrm{SL}(2, \mathbb{R})$ by Thurston and Penner coordinates. We will in fact not directly parameterize the representation variety, but an extension of it, which we call decorated or framed representations.

1.2.1 Representation spaces

Let S be a punctured surface of genus g with $k > 0$ punctures. We assume that the Euler characteristic $\chi(S)$ of S is negative. In this case the fundamental group $\pi_1(S)$ of S is free with $2g + k - 1 = |\chi(S)| + 1 \geq 2$ generators.

Definition 1.2.1. An element $g \in \pi_1(S)$ is called *peripheral* if g is freely homotopic to a loop contained in an arbitrarily small neighborhood of a puncture. We denote by $\pi_1^{per}(S)$ the subset of $\pi_1(S)$ containing all peripheral elements. Since we consider only punctured surfaces, $\pi_1^{per}(S) \neq \varnothing$.

By $\mathrm{Hom}(\pi_1(S), G)$ we denote the set of all representations of the fundamental group $\pi_1(S)$ of the surface S into some Lie group G. The group G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation.

Definition 1.2.2. The quotient space

$$\mathrm{Rep}(\pi_1(S), G) := \mathrm{Hom}(\pi_1(S), G)/G$$

is called the *moduli space of representations*. We denote by $[\rho]$ the class in $\mathrm{Rep}(\pi_1(S), G)$ of the representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$.

Remark 1.2.3. The action of G on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation is not proper, hence the quotient is, in general, not Hausdorff. The action is proper on the subset of reductive representations, which has an Hausdorff quotient, usually called the character variety. In this thesis, it is more natural to consider the quotient of all representations, and to deal with a quotient space which is not Hausdorff.

Definition 1.2.4. A representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ will be called *peripherally parabolic* if for every $g \in \pi_1^{per}(S)$, the matrix $\rho(g)$ lies in a subgroup conjugate to P (see Formula (1.1.2)).

In other words, a representation is parabolic if and only if every peripheral element leaves invariant a Lagrangian in $(\mathbb{R}^{2n}, \omega)$. We will denote by $\mathrm{Hom}^P(\pi_1(S), G)$ the subset of $\mathrm{Hom}(\pi_1(S), G)$ consisting of peripherally parabolic representations.

Definition 1.2.5. The quotient space

$$\mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) := \mathrm{Hom}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))/\,\mathrm{Sp}(2n, \mathbb{R})$$

is called the *moduli space of peripherally parabolic representations*.

Remark 1.2.6. The space $\operatorname{Rep}(\pi_1(S), G)$ does not depend very much on the surface S, because it depends only on $\pi_1(S)$, and there are several surfaces with the same fundamental group. For this reason, it is not easy to study this space using topological decompositions of S. In the space $\operatorname{Rep}^P(\pi_1(S), G)$ however we put conditions on the peripheral elements in $\pi_1(S)$, and thus it depends on and is more closely related to the topology of S.

Definition 1.2.7. A representation $\rho \in \operatorname{Hom}(\pi_1(S), \operatorname{Sp}(2n, \mathbb{R}))$ will be called *peripherally unipotent* if for every $g \in \pi_1^{per}(S)$, the matrix $\rho(g)$ lies in a subgroup conjugate to U (see Formula (1.1.3)).

In other words, a representation is peripherally unipotent if and only if every peripheral element leaves invariant a framed Lagrangian in $(\mathbb{R}^{2n}, \omega)$. We will denote by $\operatorname{Hom}^U(\pi_1(S), G)$ the subset of $\operatorname{Hom}(\pi_1(S), G)$ consisting of peripherally unipotent representations.

Definition 1.2.8. The quotient space

$$\operatorname{Rep}^U(\pi_1(S), \operatorname{Sp}(2n, \mathbb{R})) := \operatorname{Hom}^U(\pi_1(S), \operatorname{Sp}(2n, \mathbb{R}))/\operatorname{Sp}(2n, \mathbb{R})$$

is called the *moduli space of peripherally unipotent representations.*

1.2.2 Decorated representations

For a peripherally parabolic representation there might be many ways to choose the invariant Lagrangians. A decoration is a special way to make this choice.

Definition 1.2.9. A *decoration* of ρ is a map

$$D\colon \pi_1^{per}(S) \to \operatorname{Lag}(2n, \mathbb{R})$$

satisfying the following properties:

(a) $D(g)$ is invariant under $\rho(g)$ for all $g \in \pi_1^{per}(S)$.

(b) If $g_1, g_2 \in \pi_1^{per}(S)$, $h \in \pi_1(S)$ such that $hg_1h^{-1} = g_2$, then

$$\rho(h)(D(g_1)) = D(g_2).$$

(c) For every $k \in \mathbb{Z} \setminus \{0\}$ and for every $g \in \pi_1^{per}(S)$,

$$D(g) = D(g^k).$$

A *decorated representation* is a pair (ρ, D), where ρ is a representation and D a decoration of ρ.

Remark 1.2.10. By properties a), b), c) of decorations, for every puncture, one has to choose a Lagrangian for only one peripheral element going around the puncture. Then the Lagrangians associated to the other peripheral elements going around the same puncture are determined.

We denote by $\mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ the set of all decorated representations. The action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ and on $\mathrm{Lag}(2n,\mathbb{R})$ induces an action on $\mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. We will study the quotient:

Definition 1.2.11. The quotient space

$$\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$

is called the *moduli space of decorated representations.* We denote by $[\rho, D]$ the class of (ρ, D) in the moduli space of decorated representation.

Remark 1.2.12. We have natural surjective maps

$$\begin{array}{ccc} \mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Hom}^P(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ (\rho, D) & \mapsto & \rho \end{array}.$$

$$\begin{array}{ccc} \mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ [\rho, D] & \mapsto & [\rho] \end{array}.$$

These maps are generically $2^n k : 1$-map, where k is the number of punctures.

1.2.3 Transverse representations

We now fix an ideal triangulation $\mathcal{T}$ of S.

Definition 1.2.13. We say that $(\rho, D) \in \mathrm{Hom}^d(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ is *transverse* with respect to $\mathcal{T}$ if the following condition holds: for every edge e of $\mathcal{T}$ connecting punctures p_i and p_j, for every point $b' \in Int(e)$ and for every curve γ connecting b and b', we require that the Lagrangians $D(\gamma * \alpha_i * \gamma^{-1})$ and $D(\gamma * \alpha_j * \gamma^{-1})$ are transverse, where the curves α_i and α_j are as in Figure 1.2.1.

We denote by $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ the set of all decorated representations which are transverse with respect to the triangulation $\mathcal{T}$.

Remark 1.2.14. The transversality property required in the previous definition does not depend on the choice of the path γ and the base point b. Moreover, this property is invariant under the action of $\mathrm{Sp}(2n,\mathbb{R})$, hence we can define the quotient:

$$\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$

Remark 1.2.15. For each $\mathcal{T}$, the space $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is an open dense subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.

Let T be a triangle of $\mathcal{T}$ with boundary ∂T. Using the orientation of S, we can orient ∂T so that T is to the left from ∂T. This gives us a cyclic order on the vertices $\{p_1, p_2, p_3\}$ of T. We assume that (p_1, p_2, p_3) are in positive cyclic order.

Definition 1.2.16. Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$, and consider elements $g_1, g_2, g_3 \in \pi_1^{per}(S,b)$ that go around p_1, p_2, p_3 (see Figure 1.2.2). We can consider the Maslov index $\mu^T := \mu(D(g_1), D(g_2), D(g_3))$. Since μ is $\mathrm{Sp}(2n,\mathbb{R})$-invariant, μ^T is a well defined invariant of $[\rho, D]$ for each triangle T of $\mathcal{T}$. We call μ^T the *Maslov index of the positive oriented triangle* T for $[\rho, D]$.

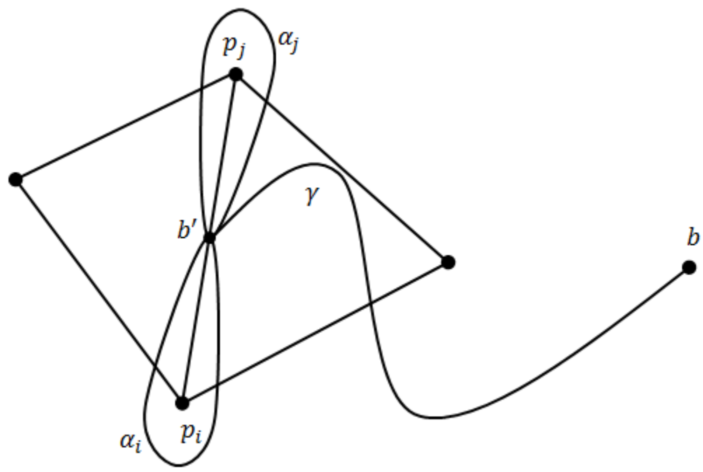

Figure 1.2.1:

1.2.4 Toledo number and maximal representations

An important invariant for representations $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is the Toledo number, here denoted by T_ρ, which was defined in [7] using bounded cohomology. It is a real number which satisfies the Milnor–Wood inequality:

$$-n|\chi(S)| \leq T_\rho \leq n|\chi(S)|.$$

Moreover, for all representations $[\rho] \in \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$, this invariant takes only integer values. The representations where this invariant achieves its maximum have particularly nice geometric properties, see [7].

Definition 1.2.17. A representation $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is called *maximal* if $T_\rho = n|\chi(S)|$.

We denote by $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of $\mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ consisting of all maximal representations. Similarly, we denote by $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ of all decorated maximal representations, and by $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of all decorated maximal representations which are transverse with respect to a chosen triangulation $\mathcal{T}$. The following facts are proven in [7].

Proposition 1.2.18. *[7]*

(a) $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \subset \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. *In particular, the natural projection map*

$$\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \quad \to \quad \mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})).$$

is surjective.

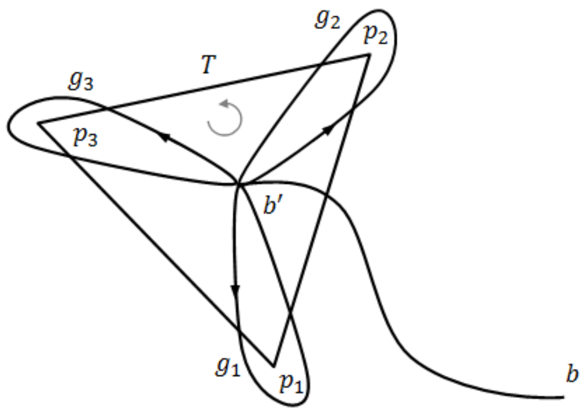

Figure 1.2.2:

(b) Maximal representations are transverse with respect to any ideal triangulation $\mathcal{T}$:

$$\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) = \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})).$$

(c) All maximal representations are reductive, hence the spaces $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ and $M^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ are Hausdorff (cfr. Remark 1.2.3).

Remark 1.2.19. A representation $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is called *almost maximal* if $T_\rho > (n-1)|\chi(S)|$ (see [9]). The Remark 1.2.18 (c) holds also for the subsets of the moduli spaces consisting of all almost maximal representations.

We now show that the Toledo number of a decorated representation can be computed easily using an ideal triangulation. In the special case of a pair of pants the following proposition was proven in [26].

Proposition 1.2.20. *Let $\mathcal{T}$ be an ideal triangulation of S and $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. The Toledo number T_ρ of ρ can be computed from the following formula:*

$$T_\rho = \sum_{T \in \mathcal{T}} \mu^T$$

where μ^T is the Maslov index of the positive oriented triangle T for $[\rho, D]$.

Corollary 1.2.21. *The number $\sum_{T \in \mathcal{T}} \mu^T$ only depends on the representation. In particular it does not depend on the choice of decoration nor on the ideal triangulation.*

The fact that $\sum_{T \in \mathcal{T}} \mu^T$ does not depend on the triangulation can also be seen directly since every two triangulations are connected by a sequence of flips, and for a flip the statement follows from the cocycle relation of the Maslov index (see Remark 1.1.8).

As a corollary of the previous proposition, we can recognize decorated maximal representations using a triangulation:

Corollary 1.2.22. *Given a decorated representation ρ, and an ideal triangulation $\mathcal{T}$ of S, we have that ρ is maximal if and only if the Maslov index of each positively oriented triangle T in $\mathcal{T}$ is n.*

The proof of Proposition 1.2.20 will take the rest of this subsection. It will use, as tools, the Souriau index and the rotation number, whose properties we will briefly discuss.

Let $\tilde{G}$ be the universal covering of $G := \mathrm{Sp}(2n, \mathbb{R})$ and $\widetilde{\mathrm{Lag}}(2n, \mathbb{R})$ be the universal covering of $\mathrm{Lag}(2n, \mathbb{R})$. In [7] it is shown that $\tilde{G}$ acts on $\widetilde{\mathrm{Lag}}(2n, \mathbb{R})$ in a compatible way with respect to the action of G on $\mathrm{Lag}(2n, \mathbb{R})$, i.e. for all $\tilde{g} \in \tilde{G}$ and for all $\tilde{L} \in \widetilde{\mathrm{Lag}}(2n, \mathbb{R})$:

$$p(\tilde{g}.\tilde{L}) = p_G(\tilde{g}).p(\tilde{L})$$

where $p \colon \widetilde{\mathrm{Lag}}(2n, \mathbb{R}) \to \mathrm{Lag}(2n, \mathbb{R})$, $p_G \colon \tilde{G} \to G$ are natural projections of coverings, and by . we denote the actions of corresponding groups.

The Souriau Index is a map

$$m \colon \widetilde{\mathrm{Lag}}(2n, \mathbb{R}) \times \widetilde{\mathrm{Lag}}(2n, \mathbb{R}) \to \mathbb{R}$$

which is $\tilde{G}$-invariant and satisfies the following relation: for each $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3 \in \widetilde{\mathrm{Lag}}(2n, \mathbb{R})$

$$m(\tilde{L}_1, \tilde{L}_2) + m(\tilde{L}_2, \tilde{L}_3) + m(\tilde{L}_3, \tilde{L}_1) = \mu(L_1, L_2, L_3)$$

where $L_i = p(\tilde{L}_i)$ for $i \in \{1, 2, 3\}$. See [8] and [26] for a precise definition.

We also need the rotation number $\widetilde{\mathrm{Rot}} \colon \tilde{G} \to \mathbb{R}$, a conjugation invariant function defined in [7] using the theory of bounded cohomology. We will need the following properties:

Lemma 1.2.23 ([26]). *Let $\tilde{g} \in \tilde{G}$, $\tilde{L} \in \widetilde{\mathrm{Lag}}(2n, \mathbb{R})$ and let $p(\tilde{L}) \in \mathrm{Lag}(2n, \mathbb{R})$ be a fixed point of $p_G(\tilde{g}) \in G$. Then*

$$\begin{array}{rccc} \widetilde{\mathrm{Rot}} \colon & \tilde{G} & \to & \mathbb{R} \\ & \tilde{g} & \mapsto & m(\tilde{g}\tilde{L}, \tilde{L}) \end{array}$$

Lemma 1.2.24 ([7, Thm. 12]). *Let $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ and*

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k \mid c_1 \ldots_k [b_g, a_g] \ldots [b_1, a_1] = 1 \rangle$$

be a presentation of $\pi_1(S)$. Let $\tilde{\rho} \in \mathrm{Hom}(\pi_1(S), \tilde{G})$ be a lift of ρ to the universal covering $\tilde{G}$ of $\mathrm{Sp}(2n, \mathbb{R})$. The Toledo number of ρ can be computed as:

$$T_\rho = -\sum_{i=1}^{k} \widetilde{\mathrm{Rot}}(\tilde{\rho}(c_i))$$

We are finally ready to present the

Proof of Proposition 1.2.20. First we fix a presentation of $\pi_1(S)$:

$$\pi_1(S) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_k \mid c_1 \dots {}_k\, [b_g, a_g] \dots [b_1, a_1] = 1 \rangle$$

where g is the genus of S, k is the number of punctures. We choose a lift $\tilde{\rho}\colon \pi_1(S) \to \tilde{G}$. From $\tilde{\rho}$, we can compute T_ρ using Lemma 1.2.24.

We can assume that $\tilde{\rho}(c_i)$ have a fixed point $z_i,\ i \geq 2$ in $\widetilde{\mathrm{Lag}}(2n, \mathbb{R})$. So $\widetilde{\mathrm{Rot}}(\tilde{\rho}(c_i)) = 0$. This is possible since $\rho(c_2), \dots, \rho(c_k)$ have fixed points in $\mathrm{Lag}(2n, \mathbb{R})$. We also denote by y_0 a lift of a fixed point of $\rho(c_1)$.

We denote for all admissible i:

$$A_i := \rho(a_i),\ \tilde{A}_i := \tilde{\rho}(a_i),$$

$$B_i := \rho(b_i),\ \tilde{B}_i := \tilde{\rho}(b_i),$$

$$C_i := \rho(c_i),\ \tilde{C}_i := \tilde{\rho}(\tilde{c}_i).$$

By induction, we denote

$$y_i := \tilde{B}_i^{-1} \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}$$

for $i \in \{1, \dots, g\}$.

We consider a polygon model of S and the ideal triangulation as in Figure 1.2.3, where the vertices of the triangulation are decorated by lifted fixed points of the corresponding peripheral elements, and the edges are marked by letters and arrows corresponding to the generators of the fundamental group and gluing/cutting directions.

To write the sum of Maslov indices, we use the Souriau index [26, 3.2]:

$$\begin{aligned}\sum_{T \in \mathcal{T}} \mu_\rho^T = \sum_{i=1}^{g} &(m(y_{i-1}, \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}) + m(\tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}, \tilde{B}_i \tilde{A}_i y_{i-1}) + \\ &+ m(\tilde{B}_i \tilde{A}_i y_{i-1}, \tilde{A}_i y_{i-1}) + m(\tilde{A}_i y_{i-1}, \tilde{B}_i^{-1} \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1})) + \\ &+ m(y_g, \tilde{C}_k^{-1} \dots \tilde{C}_3^{-1} z_2) + \\ &+ \sum_{i=2}^{k-1} m(\tilde{C}_k^{-1} \dots \tilde{C}_{i+1}^{-1} z_i, \tilde{C}_k^{-1} \dots \tilde{C}_{i+2}^{-1} z_{i+1}) + \sum_{i=2}^{k-1} m(z_{i+1}, z_i) + m(z_2, y_0).\end{aligned}$$

Using the $\tilde{G}$-invariance of the Souriau index and its anti-symmetry we can see that

$$m(\tilde{B}_i \tilde{A}_i y_{i-1}, \tilde{A}_i y_{i-1}) = m(\tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}, y_{i-1}) = -m(y_{i-1}, \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1})$$

$$\begin{aligned}m(\tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}, \tilde{B}_i \tilde{A}_i y_{i-1}) &= m(\tilde{B}_i^{-1} \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}, \tilde{A}_i y_{i-1}) = \\ &= -m(\tilde{A}_i y_{i-1}, \tilde{B}_i^{-1} \tilde{A}_i^{-1} \tilde{B}_i \tilde{A}_i y_{i-1}).\end{aligned}$$

Therefore, the first sum is equal to zero. Moreover,

$$m(\tilde{C}_k^{-1} \dots \tilde{C}_{i+1}^{-1} z_i, \tilde{C}_k^{-1} \dots \tilde{C}_{i+2}^{-1} z_{i+1}) = m(\tilde{C}_{i+1}^{-1} z_i, z_{i+1}) =$$

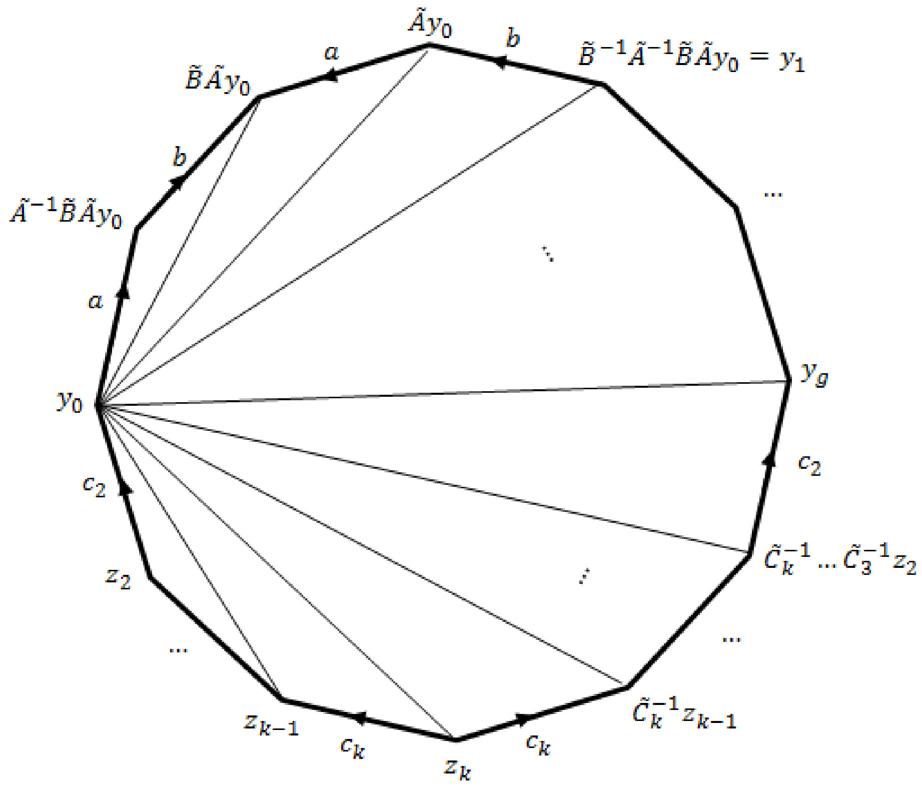

Figure 1.2.3:

$$= m(z_i, \tilde{C}_{i+1} z_{i+1}) = m(z_i, z_{i+1}).$$

Therefore, the second sum is equal to minus the third sum. So we get:

$$\sum_{T \in \mathcal{T}} \mu^T = m(y_g, \tilde{C}_k^{-1} \ldots \tilde{C}_3^{-1} z_2) + m(z_2, y_0) =$$

$$= m(y_g, \tilde{C}_k^{-1} \ldots \tilde{C}_3^{-1} \tilde{C}_2^{-1} z_2) + m(z_2, y_0) =$$

$$= m(\tilde{C}_2 \tilde{C}_3 \ldots \tilde{C}_k y_g, z_2) + m(z_2, y_0) = m(\tilde{C}_1^{-1} y_0, z_2) + m(z_2, y_0) =$$

$$= m(\tilde{C}_1^{-1} y_0, y_0) = \widetilde{\mathrm{Rot}}(\tilde{C}_1^{-1}) = -\widetilde{\mathrm{Rot}}(\tilde{C}_1) = T_\rho. \quad \square$$

1.3 $\mathcal{X}$-coordinates for maximal representations

In this section we introduce positive $\mathcal{X}$-coordinates. They will give a parametrization of the space of maximal representations: we restrict our attention here to this special case because the definition is significantly simpler than in the general case. The definition of general $\mathcal{X}$-coordinates for decorated representations that are not necessarily maximal will be given in Section 1.5.

1.3.1 Ideal triangulations of surfaces

Let $\mathcal{T}$ be an ideal triangulation of a punctured surface $S = \bar{S} \setminus P$ where $P = \{p_1, \dots, p_k\}$ is the set of punctures. We consider $\mathcal{T}$ as a graph $\mathcal{T} = (P, E)$ embedded in $\bar{S}$ so that the complement of $\mathcal{T}$ in $\bar{S}$ is a disjoint union of triangles which we call faces or triangles of the triangulation $\mathcal{T}$. We denote by F the set of all faces of $\mathcal{T}$.

The $\mathcal{X}$-coordinates will in general consist face invariants, edge invariants, and angle invariants. The face coordinates essentially come form the Maslov index, so they take values in a discrete set, and for positive $\mathcal{X}$-coordinates, they are all constant equal to n, so that we can suppress them. For the angle coordinates it is important to introduce the angles of the triangulation, which is what we do know.

For each edge $e \in E$ there are up to homotopy two parametrizations $\vec{e}\colon [0,1] \to e$ and $\vec{e}^{\,-1}\colon [0,1] \to e$, where $\vec{e}^{\,-1}(t) = \vec{e}(1-t)$. The restrictions $\vec{e}, \vec{e}^{\,-1}\colon (0,1) \to e \setminus P$ are bijective. The choice of $\vec{e}$ for $e \in E$ is called an orientation of the edge $e \in E$. We denote by E_{or} the set of all oriented edges of $\mathcal{T}$.

The orientation of S defines maps:

$$r\colon E_{or} \to F$$

$$l\colon E_{or} \to F$$

which associate to an oriented edge $\vec{e}$ the unique face whose closure contains this edge and which lies to the right (resp. to the left) of $\vec{e}$.

Definition 1.3.1. An ideal triangulation $\mathcal{T}$ together with a chosen orientation for every edge is called an *oriented ideal triangulation.*

Definition 1.3.2 (Positive and negative angles)**.** The triple $(\vec{e}_1, \vec{e}_2, f) \in E_{or}^2 \times F$ is called a *positive angle* of the triangulation $\mathcal{T}$ if

- $\vec{e}_1(1) = \vec{e}_2(0) \subseteq P \cap \bar{f}$,
- $l(\vec{e}_1) = l(\vec{e}_2) = f$.

Similarly, the triple $(\vec{e}_1, \vec{e}_2, f) \in E_{or}^2 \times F$ is called a *negative angle* of the triangulation $\mathcal{T}$ if

- $\vec{e}_1(1) = \vec{e}_2(0) \subseteq P \cap \bar{f}$,
- $r(\vec{e}_1) = r(\vec{e}_2) = f$.

We denote by W^+ (resp. W^-) the set of all positive (resp. negative) angles of $\mathcal{T}$, and by W the set of all angles of $\mathcal{T}$, i.e. $W = W^+ \cup W^-$.

For each angle $w = (\vec{e}_1, \vec{e}_2, f)$ the opposite angle is defined as:

$$w^{-1} = (\vec{e}_2^{\,-1}, \vec{e}_1^{\,-1}, f) \in W.$$

Obviously, the opposite angle of a positive angle is negative and vice versa.

Definition 1.3.3 (Positive triple)**.** We call a triple of different positive angles (w_1, w_2, w_3) *positive* if

$$w_1 = (\vec{e}_1, \vec{e}_2, f),\ w_2 = (\vec{e}_2, \vec{e}_3, f),\ w_3 = (\vec{e}_3, \vec{e}_1, f)$$

for $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in E_{or}$.

Obviously, the positivity of a triple of positive angles is invariant under cyclic permutations.

For simplicity we will draw orientation of angles using arrows as on Figure 1.3.1

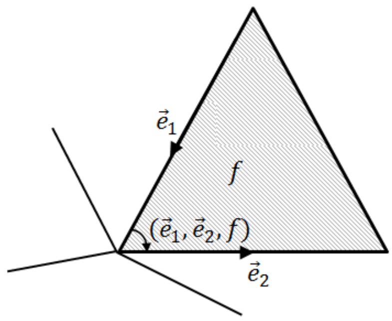

Figure 1.3.1:

1.3.2 Positive $\mathcal{X}$-coordinates

Let S be a surface with an oriented ideal triangulation $\mathcal{T}$. We use the notation introduced in Section 1.3.1.

Definition 1.3.4 (Positive $\mathcal{X}$-coordinates)**.** A *system of positive $\mathcal{X}$-coordinates* of rank n on $(S, \mathcal{T})$ is a map

$$x\colon E \sqcup W^+ \to \mathbb{R}^n_{>0} \sqcup \mathrm{O}(n)$$

such that

- the *edge invariant* $x(e)$ for an edge $e \in E$ is an n-tuple of positive real numbers $x(e) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{>0}$ with $\lambda_i \geq \lambda_{i+1}$;
- the *angle invariant* $x(w)$ for a positive angle $w \in W^+$ is an orthogonal matrix $x(w) \in \mathrm{O}(n)$. The angle coordinates are subject to the following relation: for each positive triple of positive angles (w_1, w_2, w_3) we require

$$x(w_3)x(w_2)x(w_1) = \mathrm{Id}\,.$$

We denote by $\mathcal{X}^+(S, \mathcal{T}, n)$ the set of all positive systems of $\mathcal{X}$-coordinates of rank n on $(S, \mathcal{T})$.

As a convenient notation, if $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ is a system of $\mathcal{X}$-coordinates and $w \in W^-$ is a negative angle, we will write $x(w) = x(w^{-1})^{-1}$.

Given a system of positive $\mathcal{X}$-coordinates, we can construct a decorated transverse homomorphism of the fundamental group $\pi_1(S, b)$ for an appropriately chosen $b \in S$. We describe this procedure in two steps, first constructing the homomorphism and then the decoration.

For this we lift the triangulation $\mathcal{T}$ of S to a triangulation $\tilde{\mathcal{T}} = (\tilde{P}, \tilde{E})$ of the universal covering $\tilde{S}$ of S.

We define a graph Γ on the surface in the following way: in every triangle we choose three points close to the three edges, these points will be the vertices of the graph. The edges of Γ are segments connecting the three points in one triangle and segments connecting the two points in neighboring triangles that are close to the same edge of the triangulation (see Figure 1.3.2).

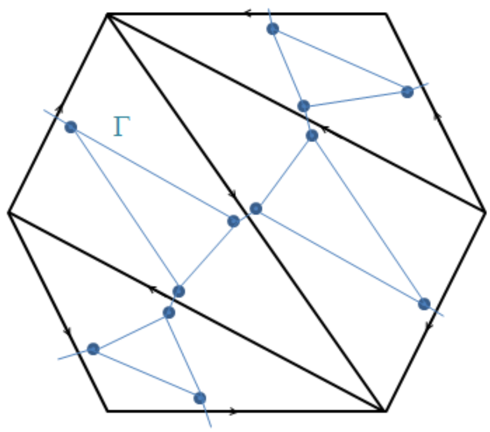

Figure 1.3.2:

We assume that the base point b coincide with one of vertices of Γ. Now, every element $\alpha \in \pi_1(S, b)$ has a representative which is a closed simplicial path in the graph Γ. We can write α as composition of paths

$$\alpha = \alpha_k \circ \cdots \circ \alpha_1,$$

where every α_i is a path along one edge of Γ.

To define the representation $\rho = \mathrm{rep}^+(x)$, we will associate to every α the matrix

$$\rho(\alpha) = A_k \cdots A_1.$$

We introduce the following notation, if $x(r)$ is an edge invariant, i.e. it a an n-tuple of positive real numbers $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{>0}$ with $\lambda_i \geq \lambda_{i+1}$, then $\mathrm{diag}(x(r))$ denotes the diagonal matrix whose ith-entry is λ_i.

Then A_i is defined as follows:

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from *the right to the left* assuming that the edge $\vec{r}$ is oriented upwards, we have

$$A_i := \begin{pmatrix} 0 & -\sqrt{\operatorname{diag}(x(r))} \\ \sqrt{\operatorname{diag}(x(r))}^{-1} & 0 \end{pmatrix}$$

where $\sqrt{\operatorname{diag}(x(r))}$ is a coordinatewise positive square root.

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from *the left to the right* assuming that the edge $\vec{r}$ is oriented upwards, we have

$$A_i := -\begin{pmatrix} 0 & -\sqrt{\operatorname{diag}(x(r))} \\ \sqrt{\operatorname{diag}(x(r))}^{-1} & 0 \end{pmatrix}$$

where $\sqrt{\operatorname{diag}(x(r))}$ is a coordinatewise positive square root.

- If α_i is along an edge of Γ that follows the angle w of the triangulation, consider the matrices

$$\hat{U} := \begin{pmatrix} x(w)^T & 0 \\ 0 & x(w)^T \end{pmatrix},$$

$$T_r = \begin{pmatrix} -\operatorname{Id} & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{pmatrix}, \quad T_l = (T_r)^{-1}.$$

We have $A_i = T_r\hat{U}$ (resp. $A_i = T_l\hat{U}$) if when going from α_{i-1} to α_i we are turning to the right (resp. to the left). Notice that, T_r and $\hat{U}$ commute: $T_r\hat{U} = \hat{U}T_r$, $T_l\hat{U} = \hat{U}T_l$.

All the matrices A_i are symplectic, so $\rho(\alpha) \in \operatorname{Sp}(2n, \mathbb{R})$. It is easy to check that this matrix only depends on the homotopy class of α, and that the map is a group homomorphism. In this way we constructed a representation $\rho \in \operatorname{Hom}(\pi_1(S, b), \operatorname{Sp}(2n, \mathbb{R}))$.

We now construct a decoration D for this representation. First, consider the case of a puncture that is a vertex of an edge of $\mathcal{T}$ which is close to the basepoint b. A simple peripheral element of $\pi_1(S, b)$ around this puncture can be represented by a circle c going around this puncture. Then going around c we always are turning either to the right or to the left. Therefore, either $L_{\mathbf{e}} = \operatorname{Span}(\mathbf{e})$ or $L_{\mathbf{f}} = \operatorname{Span}(\mathbf{f})$ is preserved by $\rho(c)$, where $(\mathbf{e}, \mathbf{f})$ is the standard symplectic basis of $(\mathbb{R}^{2n}, \omega)$ (see Figure 1.3.3).

Now we extend this definition to general punctures. First, we note that if α is any path in the graph Γ, we write $\alpha = \alpha_k \circ \cdots \circ \alpha_1$, where every α_i is a path along one edge of Γ. The definition of the matrix $\rho(\alpha)$ given above can be applied also to this path α, even if it is not closed.

Finally, for each simple peripheral curve γ around some puncture p with start- and endpoint b, we can take a point b' which lies in a triangle adjacent to p. Then we can

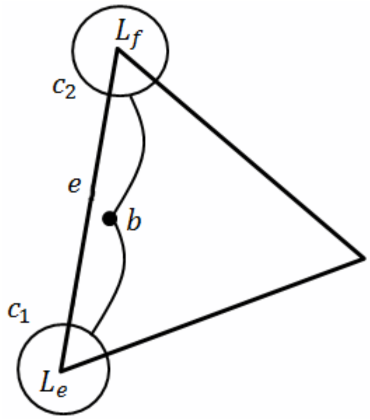

Figure 1.3.3:

decompose γ up to homotopy into a path α from b to b', circle c around p and the inverse path α^{-1} from b' to b. The representation ρ associates to this element the matrix

$$\rho(\gamma) = \rho(\alpha^{-1})\rho(c)\rho(\alpha)$$

We have already seen how to construct a Lagrangian L preserved by the matrix $\rho(c)$, we can then associate to γ the matrix $D(\gamma) := \rho(\alpha^{-1})L$.

For each non-simple peripheral curve which is a power of some simple one, we define a decoration of non-simple peripheral curve to be the decoration of the corresponding simple curve. All other non-simple curves are of the form $\gamma = \beta^{-1}\alpha^n\beta$, where α is simple closed curve, β is some closed curve. So we define $D(\gamma) := \rho(\beta).D(\alpha)$.

In this way, starting from a system of X-coordinates x, we defined an element $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$. We define $\mathrm{rep}^+(x) := (\rho, D)$.

1.3.3 Properties of the map rep^+

We now describe properties of the map

$$\mathrm{rep}^+ : \mathcal{X}^+(S,\mathcal{T},n) \to \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R})).$$

For this we introduce the notion of coordinates that are admissible with respect to a decorated representation $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$. Note that we can lift the decoration D to a map $\tilde{D} \colon \tilde{P} \to \mathrm{Lag}(2n,\mathbb{R})$.

Definition 1.3.5. $x \in \mathcal{X}^+(S,\mathcal{T},n)$ is called *admissible* for a maximal representation $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ if

- for each edge $e \in \tilde{E}$ on the boundary of the triangles $T = (t_1, t_3, t_2)$ and $T' = (t_2, t_4, t_1)$ of $\tilde{\mathcal{T}}$, the cross ratio $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$ is conjugated to $-\mathrm{diag}(x(e))$;

- for each pentagon in $\tilde{\mathcal{T}}$ as in Figure 1.3.4, the orthogonal matrix $x(w)$ belongs to the double coset $[\tilde{D}(t_1), \tilde{D}(t_5), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$.

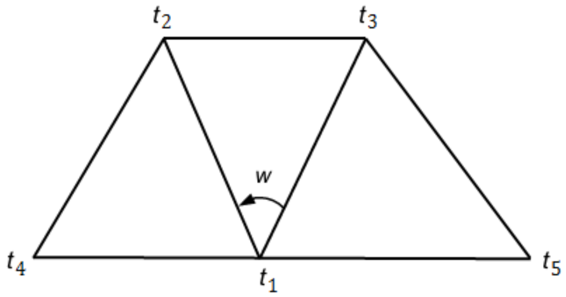

Figure 1.3.4:

Remark 1.3.6. This definition is independent on the choice of $(\rho, D) \in [\rho, D]$ and of the lift $\tilde{D}$ of D.

Proposition 1.3.7. *For every $x \in \mathcal{X}^+(S, \mathcal{T}, n)$, the image* $\mathrm{rep}^+(x)$ *is a decorated maximal representation , and x is admissible for the representation* $\mathrm{rep}^+(x)$*.*

Proof. A direct calculation in one triangle shows that for the decoration constructed above each positive oriented triangle has maximal Maslov index. Similarly, a direct calculations in a quadrilateral and in a pentagon show admissibility of x for $\mathrm{rep}(x)$. $\square$

We denote by $[\mathrm{rep}^+](x)$ the conjugacy class of $\mathrm{rep}^+(x)$. We just constructed a map

$$[\mathrm{rep}^+] \colon \mathcal{X}^+(S, \mathcal{T}, n) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R})).$$

This map is surjective (see Corollary 1.3.13) but it is not injective: sometimes changing the angle coordinates, the image representation stays the same. We describe this ambiguity explicitly.

Proposition 1.3.8. *Let $x \in \mathcal{X}^+(S, \mathcal{T}, n)$. Consider two triangles adjacent by an edge e. Let $x(e) = \Lambda$ and consider the angle coordinates be defined as in Figure 1.3.5. Let us change angle coordinates in the following way:*

$$U_1' = WU_1,\ V_1' = V_1W'^{-1},$$

$$U_2' = U_2W^{-1},\ V_2' = W'V_2.$$

We denote by x' the changed coordinates. Then $[\mathrm{rep}^+](x) = [\mathrm{rep}^+](x')$ *if and only if*

$$W \in \mathrm{O}(n) \cap \mathrm{O}(\mathrm{diag}(\Lambda)),$$

$$W' := D^{-1}W^TD,$$

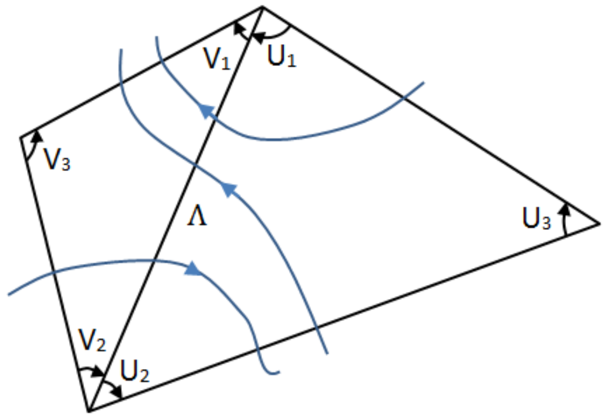

Figure 1.3.5:

$$D := \sqrt{\operatorname{diag}\Lambda}.$$

Moreover, if $[\mathrm{rep}^+](x) = [\mathrm{rep}^+](x')$ *for some* $x, x' \in \mathcal{X}^+(S, \mathcal{T}, n)$ *then* $x(e) = x'(e)$ *for all edges* e *and there exists a finite sequence of changing of angle coordinates defined by formulas above which puts* $x(w)$ *to* $x'(w)$ *for all angles* w.

Remark 1.3.9. The ambiguity in a choice of angle coordinated around an edge e depends on how generic the tuple $x(e) =: \Lambda$ is. Let $\lambda_1 > \cdots > \lambda_k$ are different entries of Λ with multiplicities $l_1, \ldots, l_k$, then $W \in \mathrm{O}(l_1) \times \cdots \times \mathrm{O}(l_k) \leq \mathrm{O}(n)$ (diagonally embedded). In particular, for generic Λ with all entries different, $W \in \mathbb{Z}_2^n$. On the other hand, if $\Lambda = (\lambda, \ldots, \lambda)$ for some $\lambda > 0$, then $W \in \mathrm{O}(n)$.

Proposition 1.3.8 will be proven in Section 1.5 where we treat general $\mathcal{X}$-coordinates.

1.3.4 The set of positive $\mathcal{X}$-coordinates associated to a representation

So far we only constructed a decorated maximal representation given a system of positive $\mathcal{X}$-coordinates. Now we describe how, given an ideal triangulation, we can associate a system of positive $\mathcal{X}$-coordinates to a decorated maximal representation $[(\rho, D)]$ so that $[\mathrm{rep}^+(x)] = [(\rho, D)]$. The basic idea is clear, we want a system of coordinates that is admissible for $[(\rho, D)]$ - so essentially for each edge e of the triangulation there are two adjacent triangles, whose vertices are decorated by four Lagrangian subspaces L_1, L_2, L_3, L_4, and the edge invariant $x(e)$ is the ordered set of eigenvalues of the cross ratio map $[L_1, L_2, L_3, L_4] : L_1 \to L_1$, and for every angle, we have a decoration by five Lagrangians, and the angle coordinate is the angle $[L_1, L_2, L_3, L_4, L_5]$, see Figure 1.1.1. However one has to be a bit careful when making the precise definitions, because we do not only want the the system of coordinates is admissible with respect to $[(\rho, D)]$, but that moreover that $[\mathrm{rep}^+(x)] = [(\rho, D)]$. And in general there are admissible system of $\mathcal{X}$-coordinates $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ for $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ such that $[\mathrm{rep}^+](x) \neq [\rho, D]$.

So we take an ideal triangulation $\mathcal{T}$ of S and choose $b_0 \in S$. Let $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b_0), \mathrm{Sp}(2n, \mathbb{R}))$ be a decorated maximal representation.

We lift the oriented triangulation $\mathcal{T}$ of S to the oriented triangulation $\tilde{\mathcal{T}}$ of the universal covering $\tilde{S}$. We also fix a lift $b \in \tilde{S}$ of $b_0 \in S$. Punctures are lifted to visual boundary points of $\tilde{S}$ (after choice of some Riemannian metric of finite area). Using the decoration D, each boundary point can be decorated by a Lagrangian in a unique way. This decoration is $\pi_1(S, b_0)$-equivariant.

We consider the graph Γ associated to this triangulation as in Section 1.3.2, see Figure 1.3.2. We can assume that Γ is invariant under the action of $\pi_1(S, b_0)$ on $\tilde{S}$. First, we associate a symplectic basis to each vertex of Γ and a tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$ to each edge of lifted triangulation $\mathcal{T}$.

For each vertex b of Γ there is the unique edge r close to which this vertex lies and unique triangle T in which b lies. We take an orientation of the edge $\vec{r}$ such that the vertex b lies to the right from $\vec{r}$. We consider the triangle, which is adjacent to T across the edge r. Thus we have a quadrilateral decorated by Lagrangians (L_1, L_3, L_2, L_4). Since the representation is maximal, the bilinear form $\beta_3 := [L_1, L_2, L_3] \colon L_1 \to L_1^*$ is well defined and positive definite, and the cross ratio map $F := [L_1, L_3, L_2, L_4] \colon L_1 \to L_1$ is well defined and symmetric with respect to β_3 with positive eigenvalues.

We say that the four tuple (L_1, L_2, L_3, L_4) is in standard position with respect to a symplectic basis $(\mathbf{e}, \mathbf{f})$ if $L_1 = L_{\mathbf{e}}$, and $L_2 = L_{\mathbf{f}}$, $[L_3]_{\mathbf{e},\mathbf{f}} = \mathrm{Id}$, and $[L_4]_{\mathbf{e},\mathbf{f}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where $[F]_{\mathbf{e}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

We then define the edge invariant $x(r) = x(\vec{r}) = (\lambda_1, \dots, \lambda_n)$ and associate the symplectic bases $B(b) = (\mathbf{e}, \mathbf{f})$ to the vertex b of Γ.

Because the oriented edge $\vec{r}$ defines the point b uniquely, sometimes we will say that the basis $B(b)$ is associated to the oriented edge $\vec{r}$ and write $B(\vec{r})$.

By construction, the map x for oriented edges is $\pi_1(S, b_0)$-invariant, therefore, x is well-defined for oriented edges of triangulation $\mathcal{T}$ of S. Moreover, the easy calculation shows that $x(\vec{r}) = x(\vec{r}^{\,-1})$, therefore $x(r)$ is well defined and does not depend on the choice of orientation. We have to take care of two other issues:

1. For each oriented edge $\vec{r}$ of triangulation there are two vertices b_1, b_2 of Γ lying close to $\vec{r}$. In general, there are many possibilities to define $B(b_2)$ if $B(b_1)$ is fixed. We fix one of them, which is consistent with the construction of the map rep^+, namely with the matrix associated to the crossing of an edge. Assume $\vec{r}$ is oriented upwards, b_1 lies to the right from $\vec{r}$ and b_2 lies to the left. Let $B(b_1) =: (\mathbf{e}, \mathbf{f})$ then $B(b_2) := (-\mathbf{f}\sqrt{\operatorname{diag}(\lambda_1, \dots, \lambda_n)}, \mathbf{e}\sqrt{\operatorname{diag}(\lambda_1, \dots, \lambda_n)}^{\,-1})$ where $x(r) = (\lambda_1, \dots, \lambda_n)$.

2. The choice of bases B is in general not unique. But it can always be chosen in a ρ-equivariant way with respect to the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases because the lifted decoration by Lagrangians is ρ-equivariant. We will always assume that B is ρ-equivariant.

To define the angle coordinate, we consider a pentagon decorated by Lagrangians as on the Figure 1.3.6. To each oriented diagonal $\vec{r}_0$ and $\vec{r}_1$ of this pentagon are associated bases $B(\vec{r}_0) =: (\mathbf{e_0}, \mathbf{f_0})$ of (L_1, L_2) and $B(\vec{r}_1) =: (\mathbf{e_1}, \mathbf{f_1})$ of (L_3, L_1). So we can define the angle invariant $x(w)$ to be $x(w) := [L_1, L_5, L_3, L_2, L_4]_{\mathbf{e_0},\mathbf{f_1}}$.

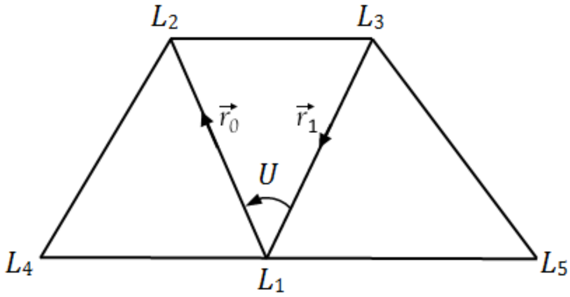

Figure 1.3.6:

Remark 1.3.10. Since the map B is ρ-equivariant, the map x for angles is $\pi_1(S, b_0)$-invariant. Therefore, x is well-defined for all oriented angles of the triangulation $\mathcal{T}$ of S.

Remark 1.3.11. Ordered tuple $x(r) = (\lambda_1, \dots, \lambda_n)$ for each edge r is uniquely defined. In contrast, the matrices U for each angle are in general not uniquely defined by the representation ρ. To define U, we have chosen a map B fixing a symplectic basis for each oriented edge which is not unique in general.

Lemma 1.3.12. *Let* $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$. *Consider* $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ *constructed from* $[\rho, D]$ *as above. Then* $[\mathrm{rep}^+](x) = [\rho, D]$.

Proof. Notice, the bases on vertices of Γ were chosen in compatible way with the construction of the map rep^+, i.e. let b_1, b_2 be vertices of Γ connected by an edge r. To r the matrix E is associated as in the previous section (going along an angle or crossing an edge of triangulation). Then E maps the basis $B(b_1)$ to $B(b_2)$.

Therefore, by induction, for every loop α based in b, $\mathrm{rep}^+(\alpha)(B(b)) = B([\alpha]b)$, where by $[\alpha]b$ we understand the action of $[\alpha] \in \pi_1(S, b)$ on vertices of $\Gamma \subseteq \tilde{S}$. But the choice of B is ρ-equivariant, i.e. $\mathrm{rep}^+(\alpha)(B(b)) = B([\alpha]b) = \rho(\alpha)B(b)$. But the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases is exact, therefore, $\mathrm{rep}^+(\alpha) = \rho(\alpha)$ for all $[\alpha] \in \pi_1(S, b)$, where $\rho(\alpha)$ is written as a matrix with with respect to the basis $B(b)$. □

Corollary 1.3.13. *The map* $[\mathrm{rep}^+]$ *is surjective.*

1.3.5 Change of coordinates

The constructions of positive $\mathcal{X}$-coordinates depends on a choice of ideal triangulation $\mathcal{T}$ of S, however the representation $[\mathrm{rep}^+(x)]$ is independent of the triangulation. If we choose a different ideal triangulation $\mathcal{T}'$ we get a different set of positive $\mathcal{X}$-coordinates.

In the work of Fock and Goncharov, it was essential that the coordinate changes going from one triangulation to another are given by positive rational functions, because these implies that the set of positive representations is independent of the triangulation used to define it. Here we know here a priori that the image of $[\mathrm{rep}^+]$ is independent of the triangulation, because it is the set of maximal representations, which can be defined without reference to any triangulation. It is of interest of interest to understand the coordinate changes.

Since every ideal triangulation $\mathcal{T}'$ can be obtained from any other ideal triangulation $\mathcal{T}$ by a sequence of flips, i.e. changing the triangulation just by taking a quadrilateral and exchanging one diagonal for the other one, the coordinate change of a flip is the central ingredient.

In the case of positive $\mathcal{X}$-coordinates it is quite difficult to write explicit formulas for this coordinate change. In particular the angle coordinates are given rather implicitly. However in the case of "scalar" edge invariants. Let $x(r) = l\,\mathrm{Id}$ then

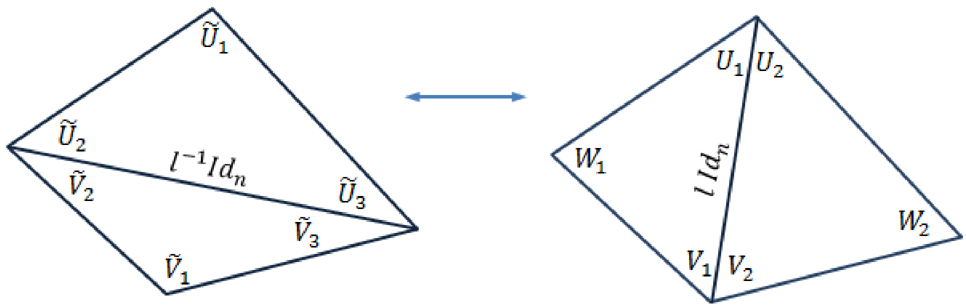

Figure 1.3.7: Flip along “scalar” edge

$$\tilde{U}_1 = U_1U_2,\ \tilde{V}_1 = V_2V_1$$

$$W_1 = \tilde{V}_2\tilde{U}_2,\ W_2 = \tilde{U}_3\tilde{V}_3$$

(triangles and angles are oriented counterclockwise).

1.3.6 Comparison with Fock–Goncharov coordinates

In this section we show that a maximal representation is not always positive in terms of Fock–Goncharov coordinates [11]. To do this, we take a positive 4-tuple of Lagrangians and show that it does not have always positive Fock–Goncharov coordinates.

To do this, first, we fix some symplectic basis $(\mathbf{e},\mathbf{f}) = (e_1,e_2,f_1,f_2)$ on $(\mathbb{R}^4,\omega)$ and consider the following four Lagrangians: $L_1 := L_{\mathbf{e}}$, $L_2 := L_{\mathbf{f}}$, $L_3 := L_{\mathbf{e},\mathbf{f}}(\mathrm{Id})$, $L_4 := L_{\mathbf{e},\mathbf{f}}(-\mathrm{Id})$. This 4-tuple has as $\mathcal{X}$-coordinate $(1,\ldots,1)$.

Since the Fock–Goncharov coordinates are defined for decorations by full flags, we have to choose a line in each Lagrangian. We choose:

$$l_1 = \langle e_1 + \theta e_2 \rangle \leq L_1$$

$$l_2 = \langle f_1 + \lambda f_2 \rangle \leq L_2$$

$$l_3 = \langle e_1 + f_1 + \mu(e_2 + f_2) \rangle \leq L_3$$

$$l_4 = \langle e_1 - f_1 + \nu(e_2 - f_2) \rangle \leq L_4$$

where $\theta, \lambda, \mu, \nu \in \mathbb{R}$ some constants. Then the corresponding full flag for each $i \in \{1, 2, 3, 4\}$ is $(l_i, L_i, l_i^{\perp})$, where $l_i^{\perp} = \{v \in \mathbb{R}^4 \mid \omega(l_i, v) = 0\}$.

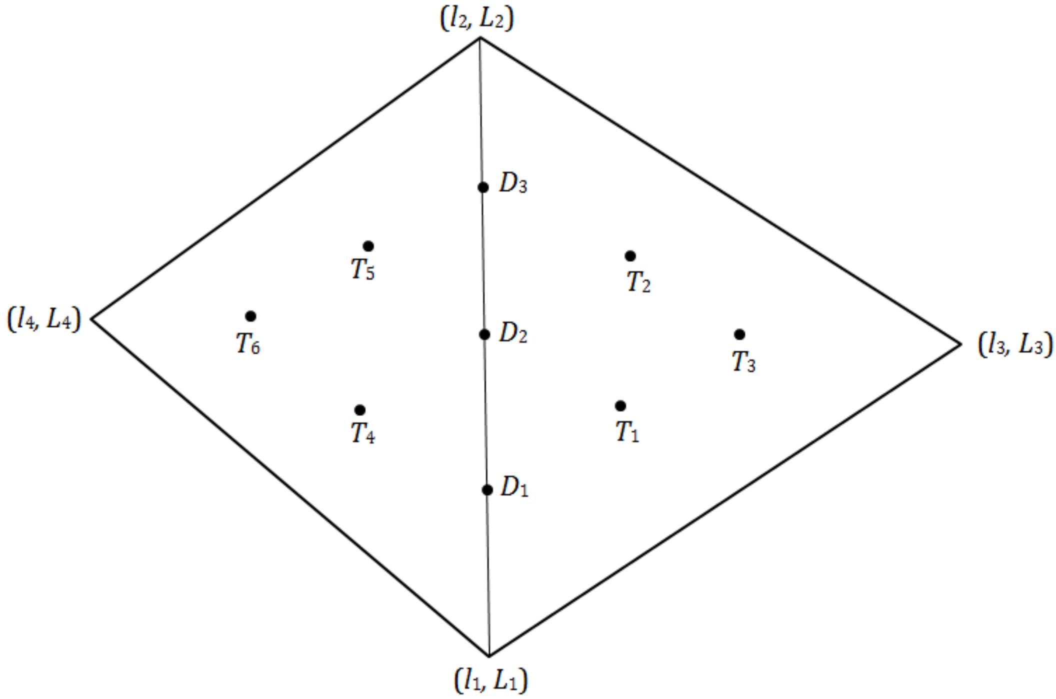

Figure 1.3.8:

So we get the following coordinates:

$$D_1 = -\frac{(\mu\theta + 1)(\lambda - \nu)}{(\nu\theta + 1)(\lambda - \mu)} \quad D_2 = \frac{(\lambda - \mu)(\theta - \nu)}{(\theta - \mu)(\lambda - \nu)} \quad D_3 = -\frac{(\nu\lambda + 1)(\theta - \mu)}{(\mu\lambda + 1)(\theta - \nu)}$$

$$T_1 = -\frac{(\mu\theta + 1)(\theta - \lambda)}{(\lambda\theta + 1)(\theta - \mu)} \quad T_2 = -\frac{(\lambda\theta + 1)(\lambda - \mu)}{(\lambda\mu + 1)(\lambda - \theta)} \quad T_3 = -\frac{(\lambda\mu + 1)(\mu - \theta)}{(\theta\mu + 1)(\mu - \lambda)}$$

$$T_4 = -\frac{(\lambda\theta + 1)(\theta - \nu)}{(\nu\theta + 1)(\theta - \lambda)} \quad T_5 = -\frac{(\nu\lambda + 1)(\lambda - \theta)}{(\theta\lambda + 1)(\lambda - \nu)} \quad T_6 = -\frac{(\theta\nu + 1)(\nu - \lambda)}{(\lambda\nu + 1)(\nu - \theta)}$$

We are going to show that all these coordinates can not be all positive for fixed $\theta, \lambda, \mu, \nu \in \mathbb{R}$. Assume first:

$$\frac{\theta - \lambda}{\lambda\theta + 1} > 0$$

Since $T_2 > 0$, we get

$$\frac{\lambda - \mu}{\lambda\mu + 1} > 0$$

Since $T_5 > 0$, we get

$$\frac{\lambda - \nu}{\nu\lambda + 1} > 0$$

Therefore,

$$D_2 D_3 = -\frac{(\lambda - \mu)(\theta - \nu)}{(\theta - \mu)(\lambda - \nu)} \frac{(\nu\lambda + 1)(\theta - \mu)}{(\mu\lambda + 1)(\theta - \nu)} = -\frac{(\nu\lambda + 1)(\lambda - \mu)}{(\mu\lambda + 1)(\lambda - \nu)} < 0$$

and D_2 and D_3 cannot be positive at the same time.

If we assume

$$\frac{\theta - \lambda}{\lambda\theta + 1} < 0$$

then, since $T_2 > 0$, we get

$$\frac{\lambda - \mu}{\lambda\mu + 1} < 0$$

Since $T_5 > 0$, we get

$$\frac{\lambda - \nu}{\nu\lambda + 1} < 0$$

Therefore,

$$D_2 D_3 = -\frac{(\lambda - \mu)(\theta - \nu)}{(\theta - \mu)(\lambda - \nu)} \frac{(\nu\lambda + 1)(\theta - \mu)}{(\mu\lambda + 1)(\theta - \nu)} = -\frac{(\nu\lambda + 1)(\lambda - \mu)}{(\mu\lambda + 1)(\lambda - \nu)} < 0$$

and D_2 and D_3 cannot be positive at the same time.

This shows that the 4-tuple (L_1, L_2, L_3, L_4) is not positive in the sense of Fock–Goncharov for each choice of lines $l_i \in L_i$, $i \in \{1, 2, 3, 4\}$.

1.4 Topology of the space of maximal representations

We now use positive $\mathcal{X}$-coordinates to understand the topology of the space of (decorated) maximal representations, focussing first on the homotopy type and then on the homeomorphism type. Note that our results are for surfaces with punctures; in the case of a closed surface, topological information about the space of maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ can be obtained using Higgs bundles [1, 5, 12, 13].

1.4.1 Homeomorphism type of the space of maximal representations

In this section we go further to determine not only the homotopy type, but actually the homeomorphism type of the space of decorated maximal representations.

We recall from the description of positive $\mathcal{X}$-coordinates, that if $\mathcal{T}$ is an idea triangulation of the oriented surface S of genus g with k punctures, the three angle coordinates associated to the three corners of one triangle satisfy the relation that their product is equal to the identity. We therefore choose in every triangle two independent angles, the third one is then uniquely defined. We denote the set of chosen independent angles by W'.

The space of positive $\mathcal{X}$-coordinates

$$\mathcal{X}^+(S,\mathcal{T},n) \cong (\mathbb{R}_{>0} \times \mathbb{R}^{n-1}_{\geq 0})^E \times \mathrm{O}(n)^{W'}$$

can be seen as a trivial bundle

$$\theta\colon \mathcal{X}^+(S,\mathcal{T},n) \to (\mathbb{R}_{>0} \times \mathbb{R}^{n-1}_{\geq 0})^E =: B$$

with compact fiber $\mathrm{O}(n)^{W'}$.

Let $y \in B$, then $y(e) = (y_1(e), \dots, y_n(e))$. Consider the set $\{y_1(e), \dots, y_n(e)\}$, let k be the cardinality of this set, so $\{y_1(e), \dots, y_n(e)\} = \{\lambda_1, \dots, \lambda_k\}$ for $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq k-1$. We denote by n_i^e the multiplicity of λ_i in the tuple $(y_1(e), \dots, y_n(e))$.

We define the stabilizer of y to be

$$\mathrm{Stab}(y) := \prod_{e\in E} \mathrm{O}(n_1^e) \times \cdots \times \mathrm{O}(n_r^e).$$

By Proposition 1.3.8 the stabilizer of y acts on the fiber $\theta^{-1}(y) \subseteq \mathcal{X}^+(S,\mathcal{T},n)$ over $y \in B$. So we can consider the following singular fibration:

$$\begin{array}{ccc} \theta^{-1}(y)/\operatorname{Stab}(y) & \hookrightarrow & \mathcal{X}^+(S,\mathcal{T},n)/\sim \\ & & \downarrow \\ & & y \in B \end{array}$$

where the equivalence relation $\sim$ is defined fiberwise by action of $\mathrm{Stab}(y)$ on $\theta^{-1}(y) \cong \mathrm{O}(n)^{W'}$.

By proposition 1.3.8, the map

$$[\mathrm{rep}^+]\colon \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$$

is constant on each orbit of $\mathrm{Stab}(y)$ on $\theta^{-1}(y)$. Therefore, the map

$$[\mathrm{rep}^+]' := [\mathrm{rep}^+] \circ q^{-1}\colon \mathcal{X}^+(S,\mathcal{T},n)/\sim \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$$

well-defined and is a homeomorphism, where

$$q\colon \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{X}^+(S,\mathcal{T},n)/\sim$$

is the quotient map.

Since $\theta^{-1}(y) \cong \mathrm{O}(n)^{W'}$, we have the following description of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$:

$$\begin{array}{ccc} \mathrm{O}(n)^{W'}/\,\mathrm{Stab}(y) & \hookrightarrow & \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ & & \downarrow \\ & & y \in (\mathbb{R}_{>0} \times \mathbb{R}^{n-1}_{\geq 0})^E \end{array}$$

Proposition 1.4.1. *The space $\mathcal{X}^+(S,\mathcal{T},n)$ has $\#W' = 2\#T$ connected components that are all diffeomorphic to each other.*

The connected components of $\mathcal{X}^+(S,\mathcal{T},n)$ can be labeled by elements of the set $\{0,1\}^{W'}$.

Moreover, for each $y \in B$

$$\theta^{-1}(y) = \bigsqcup_{p \in \{0,1\}^{W'}} F_p(y)$$

where $F_p(y)$ is the fiber in the connected component C_p over $y \in B$, $p \in \{0,1\}^{W'}$. For all $y \in B$ and for all $p, q \in \{0,1\}^{W'}$ fibers $F_p(y)$ and $F_q(y)$ are diffeomorphic.

Proof. The set of connected components of $\mathcal{X}^+(S,\mathcal{T},n)$ can be identified with the set $\{0,1\}^{W'}$, where to each independent angle w we associate 0 if it is $x(w) \in \mathrm{SO}(n)$ and 1 otherwise ($x \in \mathcal{X}^+(S,\mathcal{T},n)$).

The diffeomorphism between connected components C_p and C_q for $p, q \in \{0,1\}^{W'}$ is given by multiplication of angle coordinates $x(w)$ with a matrix $U^{p(w)q(w)}$ for all $w \in W'$ where $U \in \mathrm{O}(n) \setminus \mathrm{SO}(n)$. This diffeomorphism is given fiberwise, therefore, $F_p(y)$ and $F_q(y)$ are diffeomorphic for all $y \in B$ and for all $p, q \in \{0,1\}^{W'}$ □

Proposition 1.4.2. *Each connected component C_p is mapped by* [rep$^+$] *surjectively onto some connected component of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.*

Proof. First of all, we fix some connected component C_p and consider the restriction of [rep$^+$] to this component. [rep$^+$](C_p) is path connected and, therefore, is contained in some connected component of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ which we denote by $\mathcal{C}_p$.

Since $\theta|_{C_p} \colon C_p \to B$ is surjective, it is enough to show that [rep$^+$] maps each fiber of C_p surjectively to each fiber of $\mathcal{C}_p$ over B. We take some $y \in B$ and consider the fiber $F(y) \subseteq \mathcal{C}_p$, the fiber $F_p(y) \subseteq C_p$ and $F(y)' := \theta^{-1}(y) \cap [\mathrm{rep}^+]^{-1}(\mathcal{C}_p)$.

Since $F(y) = F(y)'/\,\mathrm{Stab}(y)$ is a quotient be an action of a group, the map $[\mathrm{rep}^+]|_{F(y)'} \colon F(y)' \to F(y) = F(y)'/\,\mathrm{Stab}(y)$ is open.

$F(y)' = \sqcup_{q \in Q} F_q(y)$ where Q is some subset in $\{0,1\}^{\#W'}$ and $p \in Q$. So $F(y)'$ is a union of finitely many diffeomorphic connected components, and $F_p(y)$ is one of them. Therefore $F_p(y)$ is open in $F(y)'$.

Moreover, since $F_p(y)$ is compact, $[\mathrm{rep}^+]F_p(y)$ is open and compact in $F(y)$, so it is closed and, therefore, $[\mathrm{rep}^+]F_p(y) = F(y)$. □

Theorem 1.4.3. *The space of decorated maximal representation $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is homeomorphic to*

$$\mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1} / \mathrm{O}(n)$$

where $\mathrm{Sym}^+(n, \mathbb{R})$ *is the space of all symmetric positive definite matrices and* $\mathrm{O}(n)$ *acts by simultaneous conjugation in every factor.*

Proof. To proof this theorem, first, we need the next technical proposition. But before state it, we fix the following notation:

$$\Delta^n := \{\mathrm{diag}(d_1, \ldots, d_n) \mid d_1 \geq \cdots \geq d_n > 0\} \subset \mathrm{Sym}^+(n, \mathbb{R}).$$

$$\mathrm{Stab}(D) := \mathrm{O}(n) \cap \mathrm{O}(D)$$

for $D \in \Delta^n$. Note that Δ^n is diffeomorphic to $\mathbb{R}_{>0} \times \mathbb{R}^{n-1}_{\geq 0}$. We freely identify edge coordinates with elements of Δ^n.

Proposition 1.4.4. *The space of decorated maximal representation $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is homeomorphic to the singular fibration*

$$\begin{array}{ccc} F_D & \hookrightarrow & E \\ & & \downarrow \\ & & D \in \Delta^n \end{array}$$

which is obtained from the trivial bundle

$$\begin{array}{ccc} \mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} & \hookrightarrow & \mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \Delta^n \\ & & \downarrow \\ & & D \in \Delta^n \end{array}$$

by dividing fiberwise by the action by common conjugation of $\mathrm{Stab}(D)$ *on the fiber over $D \in \Delta^n$, i.e.*

$$F_D = \left(\mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}\right) / \mathrm{Stab}(D)$$

where $\mathrm{Stab}(D)$ *acts on* $\mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}$ *by common conjugation.*

Proof. We consider a special ideal triangulation of S, see Figure 1.4.1.

This triangulation divides the surface in blocks of four different types. The blocks of type 1, see Figure 1.4.2, the clock of type 2, see Figure 1.4.4, and the blocks of type 3 and 4, see Figure 1.4.6.

We parametrize each block and then describe, how to glue the different blocks together. Recall that we chose two independent angles in each triangle, the third angles coordinate is then uniquely determined.

Block of type 1: We choose independent angles as indicated in Figure 1.4.2, with coordinates $U_1, ..., U_6$ and denote by D_0, D_1, D_2, D_3 the edge coordinates (considered as diagonal $n \times n$-matrices, where the entries are ordered by size).

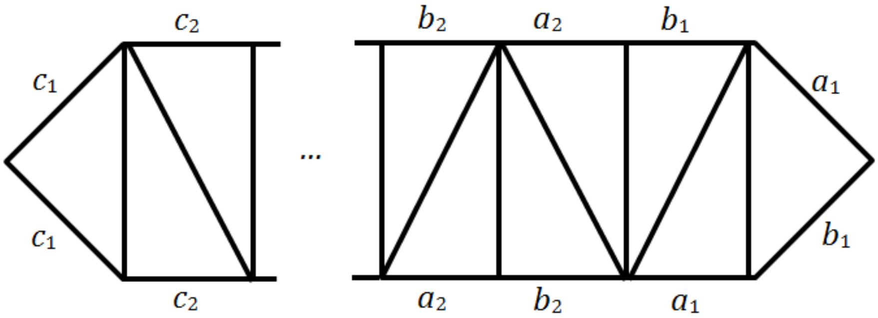

Figure 1.4.1: Triangulation of S. Sides with the same labels are identified

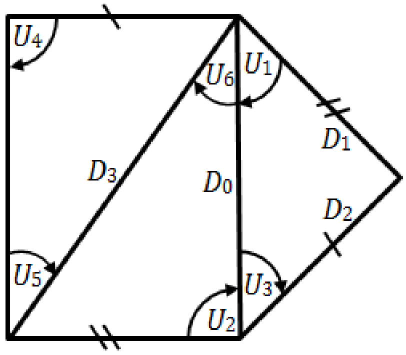

Figure 1.4.2: Block of type 1

We define three maps:

$$f_1(U_1, D_1, U_2) = (U_1 D_1 U_1^{-1}, U_1 U_2^{-1}) =: (S_1, V_1),$$

$$f_2(U_3, D_2, U_4) = (U_3^{-1} D_2 U_3, U_3^{-1} U_4^{-1}) =: (S_2, V_2),$$

$$f_3(U_5, D_3, U_6) = (U_5^{-1} D_3 U_5, U_5^{-1} U_6) =: (S_3, V_3),$$

where S_i are symmetric matrices, and V_i are orthogonal matrices. By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, D_3. We consider (S_i, V_i) as new coordinates on the block of type 1 (see Figure 1.4.3 left).

From the remaining "unused" edge coordinate D_0 we get an additional equivalence relation for the new coordinates $\{(S_i, V_i)\}$ (see Proposition 1.3.8). We could multiply the angle coordinates U_1, U_6, U_2, U_3 by elements of $\mathrm{Stab}(D_0)$. This induces the following equivalence relation:

$$S_1 \sim W S_1 W^{-1}$$

$$S_2 \sim W S_2 W^{-1}$$

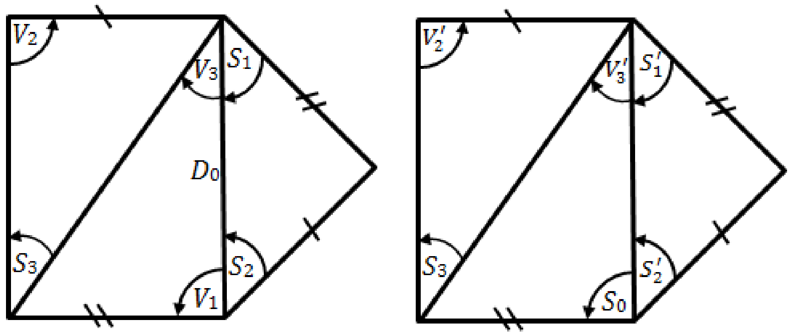

Figure 1.4.3: New coordinates on the block of type 1

$$V_1 \sim V_1 W^{-1}$$
$$V_2 \sim W V_2$$
$$V_3 \sim V_3 W^{-1}$$

for $W \in \mathrm{Stab}(D_0)$. We therefore define the map:

$$f_4(S_1, S_2, V_1, V_2, V_3, D_0) :=$$
$$= (V_1 S_1 V_1^{-1}, V_1 S_2 V_1^{-1}, V_1 D_0 V_1^{-1}, V_1 V_2, V_3 V_1^{-1}) =$$
$$=: (S_1', S_2', S_0, V_2', V_3').$$

By definition, these maps are invariant under changing of angles along the edges with coordinates D_0. We consider $(S_0, S_1', S_2', S_3, V_2', V_3')$ as new coordinates on the block of type 1 (see Figure 1.4.3 right). They define the old edge and angle coordinates exactly up to equivalence relation given by Proposition 1.3.8.

Note, that we have not yet used the left edge. this edge will play a role when gluing the different blocks . Changing of angle coordinates along this edge induces a global conjugation on all new coordinates of the block of type 1.

Block of type 2: We now proceed in a similar way, we choose independent angles as indicated in Figure 1.4.4 with coordinates $U_1, ..., U_8$ and denote by D_0, D_1, D_2, D_3, D_4 the edge coordinates.

We introduce new coordinates (S_i, V_i) on the block of type 2 (see Figure 1.4.5 left) by defining

$$f_1(U_1, D_1, U_2) = (U_2 D_1 U_2^{-1}, U_1^{-1} U_2^{-1}) =: (S_1, V_1),$$
$$f_2(U_3, D_2, U_4) = (U_3^{-1} D_2 U_3, U_3^{-1} U_4^{-1}) =: (S_2, V_2),$$
$$f_3(U_5, D_3, U_6) = (U_5^{-1} D_3 U_5, U_5^{-1} U_6) =: (S_3, V_3),$$
$$f_4(U_7, D_3, U_8) = (U_7 D_4 U_7^{-1}, U_7 U_8^{-1}) =: (S_4, V_4).$$

By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, D_3, D_4.

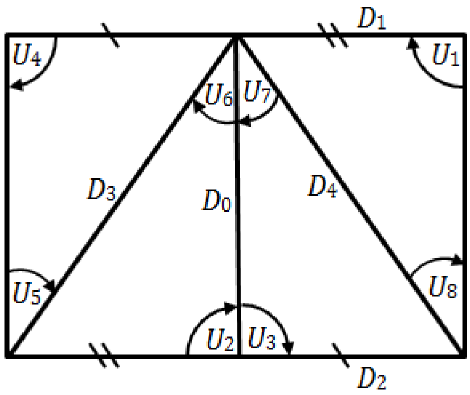

Figure 1.4.4: Block of type 2

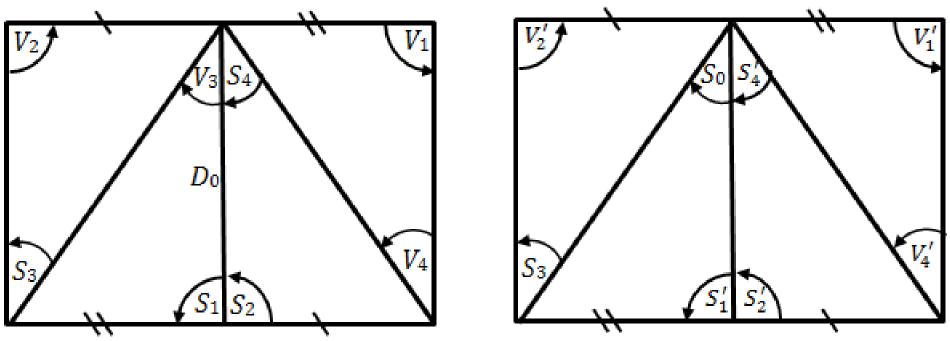

Figure 1.4.5: New coordinates on the block of type 2

The “unused” edge with coordinate D_0 gives us an additional equivalence relation We could multiply U_7, U_6, U_2, U_3 by elements of $\mathrm{Stab}(D_0)$. This induces the following equivalence relation:

$$S_1 \sim WS_1W^{-1}$$
$$S_2 \sim WS_2W^{-1}$$
$$S_4 \sim WS_4W^{-1}$$
$$V_1 \sim V_1W^{-1}$$
$$V_2 \sim WV_2$$
$$V_3 \sim V_3W^{-1}$$
$$V_4 \sim WV_4$$

for $W \in \mathrm{Stab}(D_0)$. Therefore we set :

$$f_4(S_1, S_2, S_4, V_1, V_2, V_3, V_4, D_0) :=$$

$$= (V_3 S_1 V_3^{-1}, V_3 S_2 V_3^{-1}, V_3 S_4 V_3^{-1}, V_3 D_0 V_3^{-1}, V_1 V_3^{-1}, V_3 V_2, V_3 V_4) =$$
$$=: (S_1', S_2', S_4', S_0, V_1', V_2', V_4').$$

and consider $(S_0, S_1', S_2', S_3, S_4', V_1', V_2', V_4')$ as a new coordinates on the block of type 2 (see Figure 1.4.5 right). They define the old edge and angle coordinates exactly up to equivalence relation given by Proposition 1.3.8.

Block of type 3: We choose independent angles as indicated in Figure 1.4.6 left, with coordinates $U_1, ..., U_4$ and denote by D_1, D_2 the edge coordinates. Consider

$$f_1(U_1, D_1, U_2) = (U_2^{-1} D_1 U_2, U_1^{-1} U_2) =: (S_1, V_1),$$

$$f_2(U_3, D_2, U_4) = (U_4 D_2 U_4^{-1}, U_3 U_4^{-1}) =: (S_2, V_2).$$

By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, and we consider (S_i, V_i) as a new coordinates on the block of type 3 (see Figure 1.4.7 left).

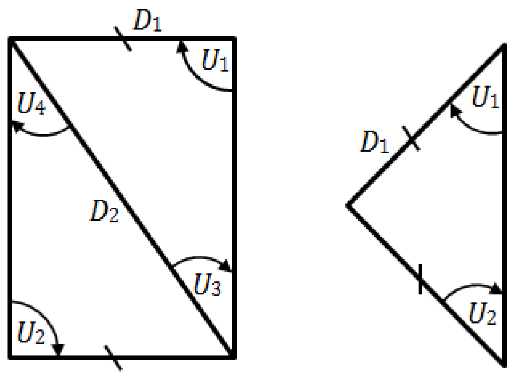

Figure 1.4.6: Block of type 3 (left), block of type 4 (right)

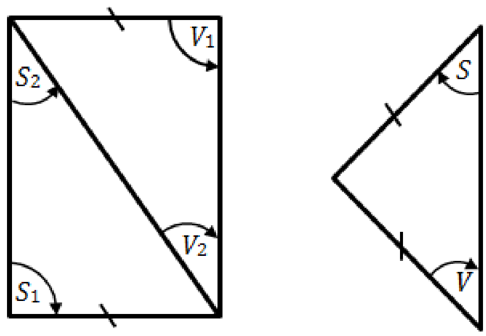

Figure 1.4.7: New coordinates on the block of type 3 (left) and on the block of type 4 (right)

Block of type 4: We choose independent angles as indicated in Figure 1.4.6 right, with coordinates U_1, U_2 and denote by D_1 the edge coordinate. We define

$$f(U_1, D_1, U_2) = (U_1^{-1} D_1 U_1, U_2 U_1) =: (S, V),$$

and consider (S, V) as a new coordinates on the block of type 4 (see Figure 1.4.7 right). Note, the right edge which we have not used yet will play a role in the gluing of blocks. Changing of angle coordinates along this edge induces a global conjugation of the new coordinate.

With this, for every block we have now a parametrization given by several copies of $\mathrm{Sym}^+(n, \mathbb{R})$ and of orthogonal groups $\mathrm{O}(N)$. We now explain how to glue the different blocks.

We will glue blocks from the right to the left as on the Figure 1.4.1 by induction. Assume that the part of the surface laying to left has the parametrization $\mathcal{P}_l = \mathrm{O}(n) \times \mathrm{O}(n) \times \mathcal{P}_r'$, the block lying to the right has parametrization $\mathcal{P}_r = \mathrm{Sym}^+(n, \mathbb{R})^{N_1} \times \mathrm{O}(n)^{N_2}$ for some $N_1, N_2 > 0$ and this is not the last step of gluing so it is not the block of the type 4. We assume as well that changing of angles around the gluing edge by an angle $W \in \mathrm{Stab}(D)$ induces a conjugation of all coordinates in $\mathcal{P}_l$ by W. We can assume that this holds by induction, since in the first step, when gluing a block of type 1 with some other block it holds.

We describe the gluing of two blocks along an edge with coordinate D and coordinates around this edge as in Figure 1.4.8. We denote by K_i the coordinates in $\mathcal{P}_l$.

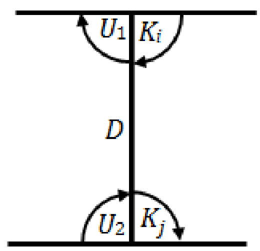

Figure 1.4.8: Gluing, intermediate step

The edge with coordinate D gives us an additional equivalence relation for coordinates $\mathcal{P}_l$ and (U_1, U_2):

$$K_i \sim W K_i W^{-1}$$

$$U_1 \sim U_1 W^{-1}$$

$$U_2 \sim W U_2$$

for $W \in \mathrm{Stab}(D)$ and for all K_i coordinates of $\mathcal{P}_r$. So we can define the map:

$$f_{gl}(U_1, U_2, (K_i), D) := (U_1 U_2, U_1 D U_1^{-1}, (U_1 K_i U_1^{-1})).$$

By definition, this map is invariant under changing of angles along edges with coordinates D. We consider these as the new coordinates on the glued block. They define the old coordinates exactly up to equivalence relation given by Proposition 1.3.8. Note, that there is a right edge which we have not used yet. Changing of angle coordinates along this edge induces conjugation on the new coordinate of glued block.

Now we describe the last step of gluing with a block of type 4. We can write again $\mathcal{P}_r = \mathrm{Sym}^+(n,\mathbb{R})^{N_1} \times \mathrm{O}(n)^{N_2}$ for some $N_1, N_2 > 0$ and $\mathcal{P}_l = \mathrm{Sym}^+(n,\mathbb{R}) \times \mathrm{O}(n)$. Coordinates on the glued edge is D (see Figure 1.4.9).

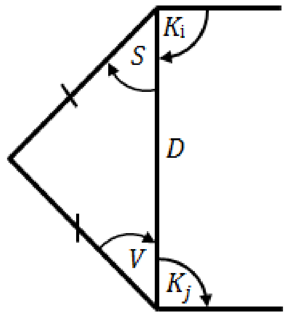

Figure 1.4.9: Gluing, last step

As we have seen, the changing of angles around this edge by some $W \in \mathrm{Stab}(D)$ induces the common conjugation by W of all coordinates in $\mathcal{P}_r$ and $\mathcal{P}_l$. To define the space which is in 1-1 correspondence with the $\mathcal{M}^d_T(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ we have to take a quotient by conjugation depending on D. It can be seen as a singular fibration coming from the projection map:

$$p\colon \mathcal{P} \to \Delta^n$$

of $\mathcal{P} := \mathcal{P}_r \times \mathcal{P}_l \times \Delta^n$ to Δ^n by dividing of the equivalence relation $\sim$ such that for each $K, K' \in \mathcal{P}$ with $p(K) = p(K')$ it is $K \sim K'$ if and only if $(K'_i) = (WK_iW^{-1})$ for some $W \in \mathrm{Stab}(p(K))$, where $K = (K_i), K' = (K'_i)$. $\square$

Now we finish the proof of the theorem:

Notice that $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{Sym}^+(n,\mathbb{R})$ is homeomorphic to $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{O}(n) \times \Delta^n/\sim$ where $\sim$ is the equivalence relation given fibrewise by the action of $\mathrm{Stab}(y) < \mathrm{O}(n)$ for $y \in \Delta^n$ in the following way: $\mathrm{Stab}(y)$ does not act on $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}$, acts by right multiplication on $\mathrm{O}(n)$ and does not act on Δ^n. The homeomorphism

$$\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{O}(n) \times \Delta^n/\sim \longrightarrow \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{Sym}^+(n,\mathbb{R})$$

is given by the diagonalization in the last $\mathrm{Sym}^+(n,\mathbb{R})$-factor

$$(s_1, \ldots, s_{6g+3k-7}, u_1, \ldots, u_{2g+k-1}, v, d) \mapsto (s_1, \ldots, s_{6g+3k-7}, u_1, \ldots, u_{2g+k-1}, vdv^{-1})$$

Now, consider the space $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n) = \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{Sym}^+(n,\mathbb{R})/\,\mathrm{O}(n)$ where $\mathrm{O}(n)$ acts by simultaneous conjugation in every factor. It s homeomorphic to $(\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{O}(n) \times \Delta^n/\sim)/\,\mathrm{O}(n)$ where $\mathrm{O}(n)$ acts by simultaneous conjugation in $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}$ and by left multiplication in $\mathrm{O}(n)$. Consider the following homeomorphism from $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \mathrm{O}(n) \times \Delta^n$ to itself given by the rule:

$$(s_1,\dots,s_{6g+3k-7},u_1,\dots,u_{2g+k-1},v,d) \mapsto (v^{-1}s_1v,\dots,v^{-1}s_{6g+3k-7}v,v^{-1}u_1v,\dots,v^{-1}u_{2g+k-1}v,v,d).$$

After this map, the components $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}$ are invariant under $\mathrm{O}(n)$ action. Therefore,

$$\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7}\times\mathrm{O}(n)^{2g+k-1}\times\mathrm{O}(n)\times\Delta^n/\,\mathrm{O}(n) \cong \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7}\times\mathrm{O}(n)^{2g+k-1}\times\Delta^n.$$

On $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7}\times\mathrm{O}(n)^{2g+k-1}\times\Delta^n$, the equivalence relation $\sim$ acts fibrewise by the simultaneous conjugation of $\mathrm{Stab}(y) < \mathrm{O}(n)$ for $y \in \Delta^n$. By the Proposition 1.4.4, this quotient space is homeomorphic to $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$. □

Remark 1.4.5. From the Theorem 1.4.3 we get:

$$\begin{aligned}\dim \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) &= \dim \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n) = \\ &= (2g+k-2)n(2n+1) = |\chi(S)|\dim(\mathrm{Sp}(2n,\mathbb{R})).\end{aligned}$$

Remark 1.4.6. Consider the subset

$$\Delta^n_{gen} = \{\mathrm{diag}(d_1,\dots,d_n) \mid \forall i \in \{1,\dots,n-1\}(d_i \neq d_{i+1})\} \subset \Delta^n$$

then for all $D \in \Delta^n_{gen}$ it is $\mathrm{Stab}(D) = \mathrm{O}(1)^n$. We can consider the subfibration $E_0 := E|_{\Delta^n_{gen}} \to \Delta^n_{gen}$. Since $\mathrm{Stab}(D) = \mathrm{O}(1)^n$ for all $D \in \Delta^n_{gen}$, we have

$$E_0 = \left(\left(\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1}\right)/\,\mathrm{O}(1)^n\right) \times \Delta^n_{gen}$$

where $\mathrm{O}(1)^n \leq \mathrm{O}(n)$ acts by simultaneous conjugation. This is an orbifold and it is an open dense subset of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$.

Remark 1.4.7. The definition of $E_0 =: E(e_0)$ depends on the edge e_0 along which we were gluing in the last step in the proof of the Proposition 1.4.4. Actually, we can choose any edge to do this last gluing. So for each edge e the constructed as above subspace $E(e)$ is homeomorphic to E_0. Because the property to be an orbifold is a local property, the finite union of all $E(e)$ for all edges e is an orbifold. We denote this subspace by E' and call it *generic part* of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$. It contains all representation with at least one edge coordinate in Δ^n_{gen}. This is an open dense subset of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$.

Corollary 1.4.8. *The space E' for $n = 2$ contains all Zariski dense representations.*

Proof. Let $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) \setminus E'$ and $x \in \mathcal{X}^+(S,\mathcal{T},n)$ such that $[\mathrm{rep}^+](x) = [\rho, D]$. Then for every edge e, $x(e) = (\lambda,\lambda)$ for some $\lambda > 0$. $[\rho, D]$ is a representation into some copy by conjugation of $\mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(2) \leq \mathrm{Sp}(4,\mathbb{R})$, therefore, it is not Zariski dense. □

1.4.2 Homotopy type of the space of maximal representations

Using the topological description of the space of decorated maximal representations form the previous section, we can determine its homotopy type.

Theorem 1.4.9. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ admits as a deformation retract the space $\mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $\mathrm{O}(n)$ on $\mathrm{O}(n)^{2g+k-1}$ by simultaneous conjugation.*

Proof. First, we consider the space $\mathrm{Sym}^+(n,\mathbb{R})$. We consider the following retraction:

$$R\colon \mathrm{Sym}^+(n,\mathbb{R}) \times [0,1] \to \{\mathrm{Id}_n\}$$

where

$$R(A,t) := U^T \operatorname{diag}(\lambda_1(1-t)+t, \dots, \lambda_n(1-t)+t)U$$

such that $A = U^{-1}\operatorname{diag}(\lambda_1,\dots,\lambda_n)U$ for $U \in \mathrm{O}(n)$. The matrix U is not uniquely defined by A, but $R(A,t)$ does not depend on the choice of U. Indeed, if we take another $U' \in \mathrm{O}(n)$ such that $A = (U')^{-1}\operatorname{diag}(\lambda_1,\dots,\lambda_n)U'$, then $U(U')^{-1}$ commutes with $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$. But then it commutes with $\operatorname{diag}(\lambda_1(1-t)+t,\dots,\lambda_n(1-t)+t)$ as well. Therefore, this retraction is well defined.

If we consider the action of $\mathrm{O}(n)$ by conjugation on $\mathrm{Sym}^+(n,\mathbb{R})$, then the retraction R is equivariant with respect to this action. Therefore, using R in every $\mathrm{Sym}^+(n,\mathbb{R})$-factor of $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$, we can retract it to $\mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$. $\square$

As a corollary we also get

Corollary 1.4.10. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$ is homotopically equivalent to $\mathrm{PO}(n)^{2g+k-1}/\,\mathrm{PO}(n)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $\mathrm{PO}(n)$ on $\mathrm{PO}(n)^{2g+k-1}$ by simultaneous conjugation.*

Proof. For representations in $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$ all angle coordinates are in the group $\mathrm{PO}(n)$. So repeating the argument in the proof of Theorem 1.4.9 gives the result. $\square$

As a corollary we obtain the following statement on the number of connected components that had been proven in [26].

Corollary 1.4.11. *[26, Theorem 7.2.7]*

- *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ has 2^{2g+k-1} connected components.*
- *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$ has 2^{2g+k-1} connected components if n is even. If n is odd, it is connected.*

We now turn to determine the number of connected components of the space of maximal representation $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ without any additional decoration. We prove the following theorem:

Theorem 1.4.12. *The number of connected components of $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ agree with the number of connected components of $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. In particular the space of maximal representations has 2^{2g+k-1} connected components.*

First, we need the following lemma:

Lemma 1.4.13. *Let $M \subset \mathrm{Sp}(2n,\mathbb{R})$ be the set of all diagonalizable symplectic matrices with pairwise different eigenvalues. Set $M^d := \{(A, L) \in M \times \mathrm{Lag}(2n,\mathbb{R}) \mid A.L = L\}$. Then the projection map $p\colon M^d \to M$ is a $2^n : 1$-covering map.*

Proof. Observe that since $A \in M$ has pairwise distinct real eigenvalues, it has exactly 2^n invariant Lagrangians, so the map p is a $2^n : 1$-map.

Without lost of generality, consider $A \in M$ a diagonal matrix and L some fixed Lagrangian of A. Since any small variation of A can be written as $B := T(A+\Delta)T^{-1}$ where $T \in \mathrm{Sp}(2n,\mathbb{R})$ close enough to Id and Δ is a small diagonal matrix so that $A + \Delta \in M$, we can take a small neighborhood U of A in M parameterized in this way. Since, $A+\Delta$ has distinct eigenvalues, T is well defined up to right multiplication with a matrix of the following form $\mathrm{diag}(\pm 1, \dots, \pm 1)$. These matrices act trivially on $\mathrm{Lag}(2n,\mathbb{R})$, therefore the invariant Lagrangian for B given by $T.L$ is well defined. For T small enough the rule $B \mapsto T.L$ is a continuous inverse map for $p|_U$. So p is a local homeomorphism.

The map p is a proper local homeomorphism, so it is a covering. $\square$

Remark 1.4.14. Let us make the following observations

- For every $A \in M$ all eigenvalues of A are different from 1. Such elements are *Shilov hyperbolic*, they have a unique attracting Lagrangian and a unique repelling Lagrangian fix point.
- The set M is an open subset of $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R}) \setminus M$ is closed of codimension 2.

For the following discussion, we denote by $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \subset \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ the space of maximal homomorphism and by $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \subset \mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ the space of decorated maximal homomorphisms, without taking conjugacy classes. Note, that the number of connected components of $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is equal to the number of connected components of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. This follows from the fact that the group $\mathrm{Sp}(2n,\mathbb{R})$ is connected. The same holds for $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ and $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. We denote the natural projections:

$$\Psi\colon \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \to \mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})),$$

$$\Psi^d\colon \mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \to \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})).$$

Corollary 1.4.15. *Let $X \subset \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ be the subset containing all maximal representation such that for every $\rho \in X$ all peripheral elements of ρ are Shilov hyperbolic. Let X^d be the preimage of X under the projection $p\colon \mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \to \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. Then the restriction $p|_{X^d}\colon X^d \to X$ is a finite-to-one covering.*

Note that by Remark 1.4.14 X resp. X^d are open subsets in $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ resp. $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})))$, and the complements $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \setminus X$ and $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \setminus X^d$ are closed of codimension at least 2. In particular, X and $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ have the same number of connected components, and in every connected component of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ there is a representation that is contained in X.

Proposition 1.4.16. *The space of maximal homomorphisms $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ has the same number of connected components as the space of decorated maximal homomorphisms. $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$.*

Proof. Let N be the number of connected components of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ and N_d the number of connected components of $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$.

It is immediate that $N_d \geq N$, thus we have to show that $N \geq N_d$. For this we assume that there are two decorated representations (ρ, D_1) and (ρ, D_2), which project to the same (undecorated) representations $\rho \in \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. We show that then (ρ, D_1) and (ρ, D_2) are in the same connected component of $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. Without loss of generality we can assume that $\rho \in X$.

We consider the set of degenerate representations $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. Note that all homomorphisms in

$$D(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) := \Psi^{-1}(\mathcal{D}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})))$$

admit only one decoration. So we can take some representation $\rho \in X$ and connect it by a path $\gamma\colon [0,1] \to \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ to a representation $\rho_0 \in D(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ so that $\gamma([0,1)) \subset X$.

Let (ρ, D_1), (ρ, D_2) be two lifts of ρ in $M^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. We also lift a path γ twice starting from (ρ, D_1) and from (ρ, D_2). Because of compactness of $\mathrm{Lag}(2n, \mathbb{R})$, both of these lifts finish at the same point namely at the unique lift of ρ_0 in $D^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. The concatenation of these two lifted paths gives a path between (ρ, D_1) and (ρ, D_2). This proves that $N_d \leq N$. □

This finishes the proof of the Theorem 1.4.12.

1.4.3 Topology of $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4, \mathbb{R}))$

Theorem 1.4.3 give a description of the homeomorphism type of the space of decorated maximal representations $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ as a singular fibration. When $n = 2$ we can go further to explicitly determine the homeomorphism type of all connected components of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(4, \mathbb{R}))$. In this case the singular fibration is

$$(\mathrm{Sym}^+(2, \mathbb{R})^N \times \mathrm{O}(2)^M \times \Delta^2)/\sim \to \Delta^2$$

where $N = 6g + 3k - 6$, $M = 2g + k - 1$ and the equivalence relation $\sim$ is given by fiberwise action of the group $\mathrm{Stab}(y)$ for $y \in \Delta^2$. Since $n = 2$, there are two possibilities for the stabilizer , we can have $\mathrm{Stab}(y) = \mathrm{O}(1) \times \mathrm{O}(1) < \mathrm{O}(2)$ when $y = (d_1, d_2)$ with $d_1 \neq d_2$, and $\mathrm{Stab}(y) = \mathrm{O}(2)$ for $y = (d, d)$. Since $\mathrm{Stab}(y)$ acts by simultaneous conjugation on all factors, there is a kernel $\{\pm \mathrm{Id}\} \in \mathrm{O}(1) \times \mathrm{O}(1)$ of this action.

We identify $\mathrm{Sym}^+(2, \mathbb{R})$ with $\mathbb{R}_{>0} \times \mathbb{C}$ using the following map:

$$\begin{array}{ccc} \mathbb{R}_{>0} \times \mathbb{C} & \to & \mathrm{Sym}^+(2, \mathbb{R}) \\ (q, r\exp(2i\phi)) & \mapsto & R(\phi)\,\mathrm{diag}(q + r, q)R(-\phi) \end{array}$$

where $R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$. Notice, although $R(\phi)$ is defined only up to sign, the map is well defined and is a homeomorphism.

The action of $R(\phi) \in \mathrm{SO}(2)$, $\phi \in S^1$ is a rotation in $\mathbb{C}$-factor by 2ϕ, for $g = \mathrm{diag}\{1, -1\}$ it is the reflection around the x-axis. Since $\mathrm{O}(2) = \mathbb{Z}_2 \ltimes \mathrm{SO}(2)$ where $\mathbb{Z}_2 = \{\mathrm{Id}, \mathrm{diag}(1, -1)\}$, first we can quotient out the fiberwise action of $\mathrm{Stab}(y) \cap \mathrm{SO}(2)$ and then the global action of $\mathbb{Z}_2$.

We now focus first on analyzing the connected component

$$\mathcal{C}_0 := (\mathrm{Sym}^+(2, \mathbb{R})^N \times \mathrm{SO}(2)^M) / \mathrm{O}(2).$$

Theorem 1.4.17. *The connected component C_0 is homeomorphic to the product $\mathbb{R}^{N+1}_{>0} \times Q$, where $(Q = (S^1)^M \times Q_1)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by the diagonal complex conjugation on each factor. $Q_1 = (\mathbb{C}^N \times \mathbb{R}_{\geq 0})/ \sim_1 \to \mathbb{R}_{\geq 0}$ is a singular fibrations, whose total space is equal to $\mathbb{C}^N \times \mathbb{R}_{>0} \sqcup \mathbb{C}^N / \mathrm{SO}(2) \times \{0\}$. In particular Q_1 is a manifold away from $(0, \ldots, 0) \in \mathbb{C}^N \times \mathbb{R}_{\geq 0}$, and $(0, \ldots, 0)$ is not an orbifold point.*

We subdivide the proof of Theorem 1.4.17 into several Lemmata.

First note that we can write

$$\mathcal{C}_0 = Q_0 / \mathbb{Z}_2$$

$$Q_0 := (\mathrm{Sym}^+(2, \mathbb{R})^N \times \mathrm{SO}(2)^M) / \mathrm{SO}(2)$$

where $\mathrm{SO}(2)$ acts by simultaneous conjugation, and then $\mathbb{Z}_2$ acts by simultaneous conjugation by $\mathrm{diag}(1, -1)$.

Lemma 1.4.18. *Q_0 is homeomorphic to the product of $\mathbb{R}^N_{>0} \times (S^1)^M$ and $Q_1 := \mathbb{C}^N / \mathrm{SO}(2)$. $Q_1' := Q_1 \setminus \{(0, \ldots, 0)\}$ is a manifold diffeomorphic to $\mathbb{R}_{>0} \times \mathbb{C}P^{N-1}$.*

Proof. The homeomorphism of Q_0 with the product of $\mathbb{R}^{N+1}_{>0} \times (S^1)^M$ and the singular fibration $Q_1 = \mathbb{C}^N / \mathrm{SO}(2)$ is given just by the identification above of $\mathrm{Sym}^+(2, \mathbb{R})$ with $\mathbb{R}_{>0} \times \mathbb{C}$, $\mathrm{SO}(2)$ with $U(1) = S^1 \subset \mathbb{C}$.

Moreover $Q_1' = \mathbb{C}^N \setminus \{(0, \ldots, 0)\} / \mathrm{SO}(2) = \mathbb{R}_{>0} \times S^{2N-1} / \mathrm{SO}(2) = \mathbb{R}_{>0} \times \mathbb{C}P^{N-1}$ is a manifold. □

Lemma 1.4.19. *The connected component $\mathcal{C}_0$ is homeomorphic to the product of $\mathbb{R}^N_{>0}$ and the quotient $Q_2 := ((S^1)^M \times Q_1)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by the simultaneous complex conjugation on each factor.*

Let $Q_2' := ((S^1)^M \times Q_1')/\mathbb{Z}_2 \subset Q_2$. Q_2' is a manifold everywhere except for points of the following form: $(s_1, \ldots, s_M, [z_1, \ldots, z_N])$, where all $s_i \in \{\pm 1\}$, $z_i \in \mathbb{R}$.

Proof. We have $(z_1, \ldots, z_N) \sim (z_1 e^{i\phi}, \ldots, z_N e^{i\phi})$ in Q_2 for every $\phi \in \mathbb{R}$, it is

$$(\bar{z}_1, \ldots, \bar{z}_N) \sim (\bar{z}_1 e^{-i\phi}, \ldots, \bar{z}_N e^{-i\phi}) = (\overline{z_1 e^{i\phi}}, \ldots, \overline{z_N e^{i\phi}}),$$

the complex conjugation on Q_1 is well-defined. This gives the homeomorphism given in the statement of the lemma.

Since the action by simultaneous complex conjugation is free and discrete everywhere on $(S^1)^M \times Q_1'$ except for real points, the corresponding quotient is a manifold. $\square$

Lemma 1.4.20. *Q_2' is an orbifold but not a manifold. The real points of Q_2' are orbifold points. Small neighborhoods of these points are homeomorphic to products of Euclidian balls of dimension N and Euclidian balls of dimension $M + N - 1$ modulo the antipodal map.*

Proof. Let $p := (s_1, \ldots, s_M, r, [x_1, \ldots, x_N]) \in (S^1)^M \times Q_1' = (S^1)^M \times \mathbb{C}P^{N-1} \times \mathbb{R}_{>0}$ be some real point. Since at least one $x_i \neq 0$, choose an affine chart of $\mathbb{C}P^{N-1}$ associated to the index i that is homeomorphic to $\mathbb{C}^{N-1}$. Then

$$p \in (S^1)^M \times \mathbb{C}^{N-1} \times \mathbb{R}_{>0}.$$

Note that $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ and since $\mathbb{Z}_2$ acts by complex conjugation on $(S^1)^M \times \mathbb{C}^{N-1} \times \mathbb{R}_{>0}$, we can write:

$$(S^1)^M \times \mathbb{R}_{>0} \times \mathbb{C}^{N-1}/\mathbb{Z}_2 \cong ((S^1)^M \times \mathbb{R}^{N-1})/\mathbb{Z}_2 \times \mathbb{R}^{N-1} \times \mathbb{R}_{>0}$$

where $\mathbb{Z}_2$ acts on $\mathbb{R}$-factors by antipodal map. The fixed points by this $\mathbb{Z}_2$-action are exactly the real points.

In a small neighborhood $U_\pm$ of $\pm 1 \in S^1$, the map $U_\pm \ni \pm e^{it} \mapsto t \in V = (-\varepsilon, \varepsilon)$ is a homeomorphism. $\mathbb{Z}_2$-action by conjugation on $U_\pm$ induces the action by antipodal map on V. So the small neighborhood of points $(\pm 1, \ldots, \pm 1, 0, \ldots, 0)$ looks like an Euclidean ball of dimension $M + N - 1$ modulo the antipodal map.

Note that $N + M > 3$. The fact that Q_2' is not a manifold follows from

Proposition 1.4.21. *Let X be a smooth manifold, G be a finite group acting on X by diffeomorphisms. Let X' be the subset of X consisting of points with non-trivial stabilizer in G. Assume, X' is discrete in X.*

If $\dim X \geq 3$, then X/G is not a topological manifold, but $(X \setminus X')/G$ is a smooth manifold.

Proof of Proposition. We prove it by contradiction. Assume, X/G is a manifold.

Note, the quotient map $q\colon X \to X/G$ is open because G acts by diffeomorphism on X. Moreover, $q|_{X\setminus X'}$ is a covering map. Therefore, $(X \setminus X')/G$ is a manifold

Let $x \in X'$ and $y := q(x)$. Since X/G assumed to be a topological manifold, we can take an open neighborhood V of y which is homeomorphic to an Euclidian ball. Then $q^{-1}(V)$ is a union of open sets which are open neighborhoods of points of $q^{-1}(y)$. We can always assume that it is a disjoint union of neighborhoods of points in $q^{-1}(y)$ by taking V small enough.

We take a component U of $q^{-1}(V)$ which is an open connected neighborhood of x. We can take $V' \subset q(U)$ open neighborhood of $y \in X/G$ homeomorphic to an Euclidian ball because q is open. Then $U' := q|_U^{-1}(V')$ is connected, open in X and $q(U') = V' = U'/G_x$, where G_x is the stabilizer of x in G.

The group G_x acts freely and properly discontinuously on $U' \setminus \{x\}$. Therefore, $G \leq \pi_1((U' \setminus \{x\})/G_x) = \pi_1(V' \setminus \{y\}) \neq \{1\}$, but $V' \setminus \{y\}$ is a Euclidian ball without one point of dimension at least 3, so it has a trivial fundamental group. This is a contradiction to the assumption that X/G is a manifold. □

Remark 1.4.22. The condition $\dim X \geq 3$ is essential. To see it, take $X = S^1 \times S^1 \subseteq \mathbb{C}^2$ and $G = \mathbb{Z}_2 = \{1, a\}$ and $a(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Then X/G is homeomorphic to S^2, so it is a manifold.

□

Proposition 1.4.23. *The point* $0 = (0, \ldots, 0) \in Q_1 = \mathbb{C}^N / \operatorname{SO}(2)$ *is not an orbifold singularity.*

Proof. We use the following proposition (see [19, Exercise 3.3.33]):

Proposition 1.4.24. *If M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere; that is $H_i(\partial M; \mathbb{Z}) \cong H_i(S_{n-1}; \mathbb{Z})$ for all i.*

As we have seen, $Q_1 = \mathbb{C}^N / \operatorname{SO}(2)$ is a manifold everywhere except for 0. First of all, we take the following contractible neighborhood of $0 \in Q_1$: $U := B/\operatorname{SO}(2)$ where $B = \{z \in \mathbb{C}^N \mid \|z\| \leq \varepsilon\}$ for some $\varepsilon > 0$. Everywhere except for 0 it is a manifold with boundary $\partial U = S^{2N-1}/\operatorname{SO}(2) \cong \mathbb{C}P^{N-1}$. ∂U is simply connected, so if we assume $0 \in Q_1$ to be an orbifold point, then, by Proposition 1.4.24, ∂U, have to be a finite quotient of homology sphere, but by generalized Poincaré conjecture, ∂U have to be a sphere since it is simply connected. This is a contradiction because $\partial U \cong \mathbb{C}P^{N-1}$. □

Now we consider any of the other connected components

$$\mathcal{C}_q := (\operatorname{Sym}^+(2, \mathbb{R})^N \times \operatorname{SO}(2)^{M-q} \times (J \operatorname{SO}(2))^q)/\operatorname{O}(2)$$

where $J = \operatorname{diag}(1, -1)$, $q \neq 0$. We prove

Theorem 1.4.25. *The connected component $\mathcal{C}_q$ is homeomorphic to*

$$\mathbb{R}^N_{>0} \times \mathbb{R}^N \times ((S^1)^{M-1} \times \mathbb{R}^N)/\mathbb{Z}_2.$$

$((S^1)^{M-1} \times \mathbb{R}^N)/\mathbb{Z}_2$ is a manifold everywhere except for the following points: $(\pm 1, \dots, \pm 1, 0, \dots, 0)$. These points are orbifold points. Small neighborhoods of them are homeomorphic to Euclidian balls modulo the antipodal map.

We can write

$$\mathcal{C}_q = Q_q/\mathbb{Z}_2$$
$$Q_q := (\mathrm{Sym}^+(2,\mathbb{R})^N \times \mathrm{SO}(2)^{M-q} \times (J\,\mathrm{SO}(2))^q)/\,\mathrm{SO}(2)$$

where SO(2) acts by simultaneous conjugation in every factor, and then $\mathbb{Z}_2$ acts by simultaneous conjugation by $\mathrm{diag}(1,-1)$.

Then Theorem 1.4.25 is a direct consequence of the following

Lemma 1.4.26. *Q_q is homeomorphic to*

$$\mathbb{R}^N_{>0} \times (S^1)^{M-1} \times \mathbb{C}^N = \mathbb{R}^N_{>0} \times \mathbb{R}^N \times (S^1)^{M-1} \times \mathbb{R}^N.$$

Proof. As before, we identify $\mathrm{Sym}^+(2,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathbb{C}$, $\mathrm{SO}(2) \cong S^1$. We also identify $J\,\mathrm{SO}(2)$ with $\mathrm{SO}(2) \cong S^1$ by the map $JU \mapsto U$ and write:

$$Q_q \cong \mathbb{R}^N_{>0} \times (S^1)^{M-q} \times (\mathbb{C}^N \times (S^1)^q)/\,\mathrm{SO}(2)$$

where $R(\phi) \in \mathrm{SO}(2)$, $\phi \in S^1$ acts by rotations by 2ϕ on S^1-factors and $\mathbb{C}$-factors around the origin.

Since $q \neq 0$ we can consider the following map:

$$f\colon \mathbb{C}^N \times (S^1)^q \to \mathbb{C}^N \times (S^1)^q,$$

$$f(z_1, \dots, z_N, s_1, \dots, s_q) := (z_1 s_q^{-1}, \dots, z_N s_q^{-1}, s_1 s_q^{-1}, \dots, s_{q-1} s_q^{-1}, s_q).$$

This map is a homeomorphism, and the first $N+q-1$ components are invariant under SO(2)-action. So we can write:

$$Q_q \cong \mathbb{R}^N_{>0} \times (S^1)^{M-1} \times \mathbb{C}^N \times (S^1/\,\mathrm{SO}(2)) = \mathbb{R}^N_{>0} \times (S^1)^{M-1} \times \mathbb{C}^N.$$

Using the identification $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, we obtain

$$Q_q \cong \mathbb{R}^N_{>0} \times \mathbb{R}^N \times (S^1)^{M-1} \times \mathbb{R}^N. \qquad \square$$

Now, we are ready to describe precisely the singular locus of $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$ in terms of representations. First, we consider the following embeddings of groups into $\mathrm{Sp}(4,\mathbb{R})$:

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2) & \hookrightarrow & \mathrm{Sp}(4,\mathbb{R}) \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, U\right) & \mapsto & \begin{pmatrix} aU & bU \\ cU & dU \end{pmatrix} \end{array}$$

and the diagonal embedding of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. These two embeddings induce maps:

$$\mathcal{M}^d(\pi_1(S), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2)) \to \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$$
$$\mathcal{M}^d(\pi_1(S), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) \to \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$$

Notice, these maps are not embeddings.

Theorem 1.4.27.
- *The non-orbifold singular locus of the $\mathcal{C}_0 \subset \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$ agree with the image of $\mathcal{M}^d(\pi_1(S), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2))$ in $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$.*
- *The orbifold singular locus in $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$ agree with the points of the image of $\mathcal{M}^d(\pi_1(S), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ in $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$ that are not in the image of $\mathcal{M}^d(\pi_1(S), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2))$.*

Proof. Follows directly from the Theorems 1.4.17 and 1.4.25. □

1.5 General $\mathcal{X}$-coordinates

In this section we introduce general, not necessarily positive $\mathcal{X}$-coordinates with respect to a chosen ideal triangulation $\mathcal{T}$ of S. General $\mathcal{X}$-coordinates consists of triangle invariants, which are signatures of certain quadratic forms, associated to every triangle of $\mathcal{T}$, edge invariants and angle invariants.

For the edge invariants we had to simultaneously diagonalize pairs of positive definite bilinear forms. Here we would have to simultaneously diagonalize pairs of non-degenerate bilinear forms of varying signature. This is in general impossible. We need to find some analog of this diagonalization process. To do this, we use the following theorem (for the proof and details, see Appendix A.1.4):

Theorem 1.5.1. *Let β_3, β_4 be two symmetric non-degenerate bilinear forms on some vector space L. We consider β_3, β_4 as maps $L \to L^*$ and define the map $\phi := \beta_3^{-1} \circ \beta_4$.*

Then there exists a basis $\mathbf{e}$ of L such that

$$[\phi]_e = X^0(\beta_3, \beta_4) := \begin{pmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ 0 & 0 & \mathcal{K} \end{pmatrix}$$

$$[\beta_3]_e = X^1(\beta_3, \beta_4) := \begin{pmatrix} \mathcal{I}_1^* & 0 & 0 \\ 0 & -\mathcal{I}_2^* & 0 \\ 0 & 0 & \mathcal{I}^{2*} \end{pmatrix}$$

$$[\beta_4]_e = X^2(\beta_3, \beta_4) := X^1(\beta_3, \beta_4) X^0(\beta_3, \beta_4)$$

where for $r = 1, 2$

$$\mathcal{I}_r^* = \begin{pmatrix} I_{1r}^* & 0 & \dots & 0 & 0 \\ 0 & I_{2r}^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_{k_r r}^* \end{pmatrix}, \quad \mathcal{J}_r = \begin{pmatrix} J_{1r} & 0 & \dots & 0 & 0 \\ 0 & J_{2r} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_{k_r r} \end{pmatrix}$$

$$\mathcal{I}^{2*} = \begin{pmatrix} I_1^{2*} & 0 & \dots & 0 & 0 \\ 0 & I_2^{2*} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_s^{2*} \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} K_1 & 0 & \dots & 0 & 0 \\ 0 & K_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & K_s \end{pmatrix}$$

where $n_{ir} := \dim(I_{ir}^*) = \dim(J_{ir})$, $m_j := \dim(I_j^{2*}) = \dim(K_j)$ *and*

$$I_{ir}^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_{ir} \times n_{ir}}$$

$$I_j^{2*} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ & & & & \dots & & \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{m_j \times m_j}$$

J_{ir} are Jordan blocks with eigenvalue $\lambda_{ir} \in \mathbb{R}$, K_j are generalized Jordan blocks with eigenvalue $\mu_j \in \mathbb{C} \setminus \mathbb{R}$ such that $\lambda_{ir} \geq \lambda_{i+1,r}$, $\mu_j \geq \mu_{j+1}$, where for complex numbers the following linear order is used: $x + iy > x' + iy'$ if $x > x'$ or $x = x'$ and $y > y'$.

Remark 1.5.2. The basis $\mathbf{e}$ is in general not unique but the matrices $X^0(\beta_3, \beta_4)$, $X^1(\beta_3, \beta_4)$, $X^2(\beta_3, \beta_4)$ are well defined by β_3, β_4. We denote the edge invariant by

$$X(\beta_3, \beta_4) := (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$$

The triple $X(\beta_3, \beta_4)$ defines $X^0(\beta_3, \beta_4)$, $X^1(\beta_3, \beta_4)$, $X^2(\beta_3, \beta_4)$ uniquely.

Definition 1.5.3. The *signature* of the triple $X(\beta_3, \beta_4) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ is the signature of the bilinear form $X^0(\beta_3, \beta_4)$. We will write $\operatorname{sgn}(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$. This is the triangle invariant.

Definition 1.5.4. We denote by $\mathcal{E}(n)$ the set of all triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}$ are of the form as in the Theorem 1.5.1 with

$$\dim \mathcal{J}_1 + \dim \mathcal{J}_2 + \dim \mathcal{K} = n$$

Definition 1.5.5. If the basis $\mathbf{e}$ of L is chosen so that $[\beta_3]_{\mathbf{e}} = X^1(\beta_3, \beta_4)$, $[\beta_4]_e = X^2(\beta_3, \beta_4)$, we will say that in the basis $\mathbf{e}$ the pair of forms (β_3, β_4) is in the *standard form.*

1.5.1 The angle of five Lagrangians

To define the angel invariant in general $\mathcal{X}$-coordinates we need an invariant of 5 Lagrangians. In the Section 1.1 we already defined this invariant only in the case when all triangles have maximal Maslov index. Now we do it in the general case.

For the 4-tuple (L_1, L_2, L_3, L_4) there exists a basis $\mathbf{e}$ of L_1 such that two bilinear forms $\beta_0 := [L_1, L_3, L_2]$ and $\beta_0' := [L_1, L_4, L_2]$ are in the standard form, i.e. $[\beta_0]_{\mathbf{e}} = X^1(\beta_0, \beta_0')$, $[\beta_0']_{\mathbf{e}} = X^2(\beta_0, \beta_0')$.

For the 4-tuple (L_3, L_2, L_1, L_5) there exists a basis $\mathbf{g}$ of L_3 such that two bilinear forms $\beta_1 := [L_3, L_2, L_1]$ and $\beta_1' := [L_3, L_5, L_1]$ are in the standard form, i.e. $[\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1')$, $[\beta_1']_{\mathbf{g}} = X^2(\beta_1, \beta_1')$. Let $\mathbf{e}'$ be a basis of L_1 such that $\omega(\mathbf{g}, \mathbf{e}') = \mathrm{Id}$.

Notice, $[\beta_0]_{\mathbf{e}'} = [\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1')$ in the basis $\mathbf{e}'$. Therefore, we can take matrices of (p,q)-shape transformations $P_{\beta_0\beta_0'}$ and $P_{\beta_1\beta_1'}$ (for more details see Appendix A.1.6), and define $\mathbf{e_0} := \mathbf{e}P_{\beta_0\beta_0'}$ and $\mathbf{e_1} := \mathbf{e}'P_{\beta_1\beta_1'}$. Then $[\beta_0]_{\mathbf{e_0}} = [\beta_0]_{\mathbf{e_1}} = I_{pq}$ and there exists $U \in \mathrm{O}(p,q)$ such that $\mathbf{e_0} = \mathbf{e_1}U$, where (p,q) is a signature of β_0. We will call this matrix an inner *angle* in the pentagon of Lagrangians $(L_1, L_4, L_2, L_3, L_5)$ (see Figure 1.5.1).

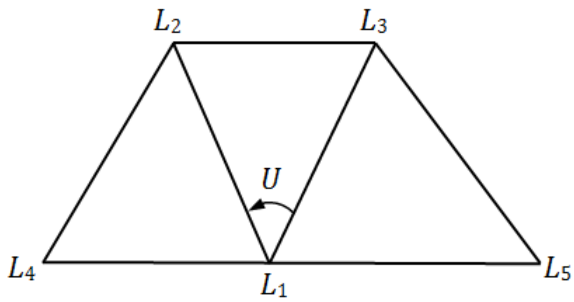

Figure 1.5.1:

Remark 1.5.6. U is well defined only if the bases $\mathbf{e}, \mathbf{e}'$ of L_1 and $\mathbf{g}$ of L_3 are chosen such that

$$\begin{aligned} &[\beta_0]_{\mathbf{e}} = X^1(\beta_0, \beta_0') \quad [\beta_0']_{\mathbf{e}} = X^2(\beta_0, \beta_0') \\ &[\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1') \quad [\beta_1']_{\mathbf{g}} = X^2(\beta_1, \beta_1') \\ &\quad \omega(\mathbf{g}, \mathbf{e}') = \mathrm{Id}\,. \end{aligned} \tag{1.5.1}$$

We denote $[L_1, L_5, L_3, L_2, L_4]_{\mathbf{e},\mathbf{e}'} := U$. We denote by $[L_1, L_5, L_3, L_2, L_4]$ the set of all possible $[L_1, L_5, L_3, L_2, L_4]_{\mathbf{e},\mathbf{e}'}$ when $\mathbf{e}, \mathbf{e}'$ satisfy 1.5.1.

1.5.2 Definition of $\mathcal{X}$-coordinates

Now we can define the general $\mathcal{X}$-coordinates for a triangulated surface $(S, \mathcal{T})$.

Definition 1.5.7. Let S be a surface with an ideal triangulation $\mathcal{T}$. Let E_{or} be the set of oriented edges of $\mathcal{T}$ and W be the set of angles of $\mathcal{T}$, F be the set of all triangles of $\mathcal{T}$.

A system of $\mathcal{X}$*-coordinates* of rank n on $(S, \mathcal{T})$ is a map

$$x\colon F \sqcup E_{or} \sqcup W \to \{(p,q) \mid p,q \in \mathbb{N} \cup \{0\}, p+q=n\} \sqcup \mathcal{E}(n) \sqcup \bigcup_{p+q=n} O(p,q)$$

such that

- $x(T) \in \{(p,q) \mid p,q \in \mathbb{N}\cup\{0\}, p+q=n\}$. We call $x(T)$ *signature* of the triangle T
- $x(\vec{e}) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) \in \mathcal{E}(n)$ for each $\vec{e} \in E_{or}$. $x(\vec{e}^{\,-1}) = \sigma(x(\vec{e}))$, where σ is the *edge reorientation map*:

$$\begin{array}{cccc} \sigma & : & \mathcal{E}(n) & \to & \mathcal{E}(n) \\ & & X(b_1,b_2) & \mapsto & X(b_2^*, b_1^*) \end{array}$$

  where b_1^*, b_2^* are dual bilinear forms. $\operatorname{sgn}(x(\vec{e})) = x(r(\vec{e}))$, i.e. the signature of $x(\vec{e})$ agree with the signature of the triangle $r(\vec{e})$ which lies to the right form $\vec{e}$;
- $x(w) \in \mathrm{O}(p,q)$ for each $w \in W$, where (p,q) is a signature of the triangle defined as above to which this angle corresponds. $x(w)^{-1} = x(w^{-1})$. For each positive triple of positive angles (w_1, w_2, w_3) it is

$$x(w_3)x(w_2)x(w_1) = \mathrm{Id}$$

We denote by $\mathcal{X}(S, \mathcal{T}, n)$ the set of all $\mathcal{X}$-coordinates of rank n on $(S, \mathcal{T})$.

Remark 1.5.8. Since we are going to associate triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ to oriented edges, we will write sometimes $x(\vec{e}) = X_{\vec{e}} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) = X(\beta_1, \beta_2)$ for some pair of forms (β_1, β_2). We will also write $X^i_{\vec{e}}$ for $i \in \{0,1,2\}$ for corresponding $X^i(\beta_1,\beta_2)$ because $X^i(\beta_1, \beta_2)$ is completely determined by the triple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ and the pair (β_1, β_2) is not really important.

Positive $\mathcal{X}$-coordinates are imbedded into the space of general $\mathcal{X}$-coordinates. A coordinate $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ is sent to $x' \in \mathcal{X}(S, \mathcal{T}, n)$ defined by

- $x'(T) = (n, 0)$ for all $T \in F$;
- $x'(e) = (\operatorname{diag} x(e), \varnothing, \varnothing)$ for all $e \in E$;
- $x'(w) = x(w)$ for all $w \in W$.

1.5.3 Construction of a decorated representation using $\mathcal{X}$-coordinates

Let S be a surface with punctures and let $\mathcal{T}$ be an oriented ideal triangulation. Given a decorated representation $[\rho, D] \in \operatorname{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \operatorname{Sp}(2n, \mathbb{R}))$, we can lift the decoration D to a map $\tilde{D}\colon \tilde{P} \to \operatorname{Lag}(2n, \mathbb{R})$.

Definition 1.5.9. A system of $\mathcal{X}$-coordinates $x \in \mathcal{X}(S, \mathcal{T}, n)$ is aid to be *admissible* for a representation $[\rho, D] \in \operatorname{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \operatorname{Sp}(2n, \mathbb{R}))$ if

- for each triangle $T = (t_1, t_3, t_2)$ of $\tilde{\mathcal{T}}$, the signature $x(T)$ agrees with the signature of the bilinear form $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2)]$.

- for each oriented edge $\vec{e} \in \tilde{E}$ on the boundary of the triangles $T = (t_1, t_3, t_2)$ and $T' = (t_2, t_4, t_1)$ of $\tilde{\mathcal{T}}$, the cross ratio $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$ is conjugated to $-X^0([L_1, L_3, L_2], [L_1, L_4, L_2])^{-1}$;

- for each pentagon in $\tilde{\mathcal{T}}$ as in Figure 1.5.2, the orthogonal matrix $x(w)$ belongs to the set $[\tilde{D}(t_1), \tilde{D}(t_5), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$.

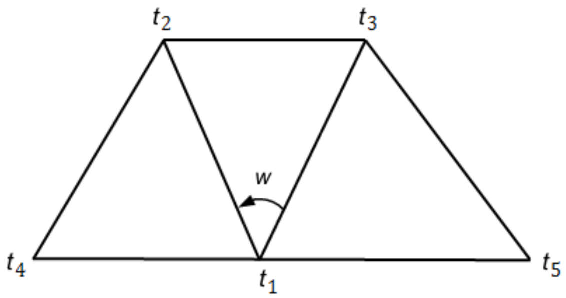

Figure 1.5.2:

We now construct as in Section 1.3.2 a map

$$\mathrm{rep}\colon \mathcal{X}(S, \mathcal{T}, n) \to \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R})),$$

such that, for every $x \in \mathcal{X}(S, \mathcal{T}, n)$, $\mathrm{rep}(x)$ is a decorated representation and x is admissible for the representation $\mathrm{rep}(x)$.

For this we let Γ be the graph on the surface introduced in Section 1.3.2, see Fig. 1.5.3.

To every vertex of Γ we associate an edge coordinate by the rule: let the oriented edge $\vec{r}$ of the triangulation is oriented upwards, then to the point lying to the right from $\vec{r}$ we associate $x(\vec{r})$, to the point lying to the left from $\vec{r}$ we associate $x(\vec{r}^{\,-1})$

We assume that the base point b coincide with one of vertices of Γ. Now, every element $\alpha \in \pi_1(S, b)$ has a representative which is a closed simplicial path in the graph Γ, so

$$\alpha = \alpha_k \circ \cdots \circ \alpha_1,$$

where every α_i is a path along one edge of Γ.

We associate to every α the matrix

$$\rho(\alpha) = A_k \cdots A_1,$$

where A_i is defined as follows:

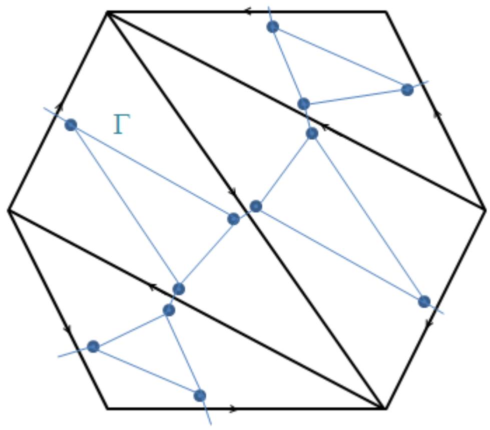

Figure 1.5.3:

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from the right to the left assuming that the edge $\vec{r}$ is oriented upwards, we have
$$E := \begin{pmatrix} 0 & -T^T\Phi \\ T^{-1}\Phi^{-1} & 0 \end{pmatrix}$$
where Φ and T are matrices associated to $x(\vec{r})$ from the definition of the back transformation (see Appendix A.1.5).

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from the left to the right assuming that the edge $\vec{r}$ is oriented upwards, we have
$$E := -\begin{pmatrix} 0 & -T^T\Phi \\ T^{-1}\Phi^{-1} & 0 \end{pmatrix}$$
where Φ and T are matrices associated to $x(\vec{r}^{\,-1})$ from the definition of the back transformation (see Appendix A.1.5).

- If α_i is along an edge of Γ that follows the angle w of the triangulation, consider the matrices
$$\hat{U}(X,Y) := \begin{pmatrix} P_Y^T x(w)^T P_X^{-T} & 0 \\ 0 & P_Y^{-1}x(w)^{-1}P_X \end{pmatrix} \tag{1.5.2}$$
$$T_r(X) = \begin{pmatrix} -\operatorname{Id} & X^1 \\ -X^1 & 0 \end{pmatrix} \quad T_l(X) = (T_r(X))^{-1},$$
where X is the coordinate on the starting vertex of α_i, Y is the coordinate on the ending vertex of α_i, P_X, P_Y are matrices of shape transformations (see Appendix A.1.6) corresponding to X, resp Y. We have $A_i = \hat{U}(X,Y)T_r(X)$

(resp. $A_i = \hat{U}(X,Y)T_l(X)$) if when going from α_{i-1} to α_i we are turning to the right (resp. to the left). Notice that, $\hat{U}(X,Y)T_r(X) = ((\hat{U}^{-1}(Y,X))T_r(Y))^{-1}$.

After multiplication of all these matrices we get a matrix in $\mathrm{Sp}(2n,\mathbb{R})$ for each curve α. So this process gives us a representation $\rho \in \mathrm{Hom}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$.

This representation admits a natural decoration D. To see this, first, we note that the procedure above works also for non-closed curves.

If b lies in the triangle near to the oriented edge $\vec{r}$ which is adjacent to some puncture and the peripheral curve is just a circle c around this puncture. Then going around c we always are turning either to the right or to the left. Therefore, either $L_{\mathbf{est}}$ or $L_{\mathbf{fst}}$ is preserved by $\rho(c)$ (Figure 1.5.4). Finally, for each simple peripheral curve γ around some puncture p with start- and endpoint b, we can take a point b' which lies in the triangle adjacent to p. Then we can decompose γ up to homotopy into a path α from b to b', circle c around p and the inverse path α^{-1} from b' to b. For α we get M_α. The matrix corresponding to c preserves some Lagrangian L. Therefore, $\rho(\gamma)$ preserves $M_\alpha^{-1}.L$, and we define $D(\gamma) := M_\alpha^{-1}.L$

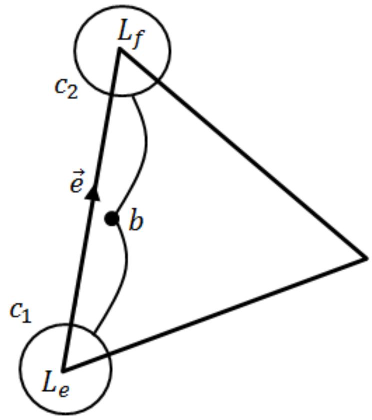

Figure 1.5.4:

For each non-simple peripheral curve which is a power of some simple one, we define a decoration of non-simple peripheral curve to be the decoration of the corresponding simple curve. All other non-simple curves are of the form $\gamma = \beta^{-1}\alpha^n\beta$, where α is simple closed curve, β is some closed curve. So we define $D(\gamma) := \rho(\beta).D(\alpha)$. By construction, this decorated representation is a representative in a standard form of its class. So we define $\mathrm{rep}(x) := (\rho, D)$.

1.5.4 The set of $\mathcal{X}$-coordinates associated to a representation

So far we only constructed a decorated representation given a system of $\mathcal{X}$-coordinates. Now we describe how, given an ideal triangulation, we can associate a system of $\mathcal{X}$-coordinates to a decorated representation $[(\rho, D)]$ so that $[\mathrm{rep}(x)] = [(\rho, D)]$. The procedure described below is very similar to the case of maximal representations. But in this case, one has to be a bit more careful because the cross ratio map is in general not diagonalizable.

We take an ideal triangulation $\mathcal{T}$ of S and choose $b_0 \in S$. Let $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b_0), \mathrm{Sp}(2n, \mathbb{R}))$ be a decorated representation.

We lift the oriented triangulation $\mathcal{T}$ of S to the oriented triangulation $\tilde{\mathcal{T}}$ of the universal covering $\tilde{S}$. We also fix a lift $b \in \tilde{S}$ of $b_0 \in S$. Punctures are lifted to visual boundary points of $\tilde{S}$ (after choice of some Riemannian metric of finite area). Using the decoration D, each boundary point can be decorated by a Lagrangian in a unique way. This decoration is $\pi_1(S, b_0)$-equivariant.

We consider the graph Γ associated to this triangulation as in Section 1.5.3, see Figure 1.5.3. We can assume that Γ is invariant under the action of $\pi_1(S, b_0)$ on $\tilde{S}$. First, we associate a symplectic basis to each vertex of Γ, a pair (p, q) to each triangle and an element from $\mathcal{E}(n)$ to each oriented edge of the lifted triangulation $\mathcal{T}$. For each vertex b of Γ there is the unique edge r close to which this vertex lies and unique triangle T in which b lies. We take an orientation of the edge $\vec{r}$ such that the vertex b lies to the right from $\vec{r}$. We consider the triangle which is adjacent to T across the edge r. Thus we have a quadrilateral decorated by Lagrangians (L_1, L_3, L_2, L_4). The following symmetric non-degenerate bilinear forms on L_1:

$$\beta_3 := [L_1, L_3, L_2],$$

$$\beta_4 := -[L_1, L_4, L_2]$$

are well-defined.

We put the pair (β_3, β_4) to the standard form, i.e. we choose a basis $\mathbf{e} = (e_1, \dots, e_n)$ of L_1 such that

$$([\beta_3]_{\mathbf{e}}, [\beta_4]_{\mathbf{e}}) = (X^1(\beta_3, \beta_4), X^2(\beta_3, \beta_4))$$

Since ω identifies L_2 with L_1^*, we define a basis $\mathbf{f}$ of L_2 to be the dual basis to $\mathbf{e}$. So we get in the notation of the previous section:

$$L_1 = \mathrm{Span}(\mathbf{e}) = L_{\mathbf{e}}, \quad L_2 = \mathrm{Span}(\mathbf{f}) = L_{\mathbf{f}}$$

$$L_3 = \mathrm{Span}(\mathbf{e} + \mathbf{f} X^1(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(X^1(\beta_3, \beta_4))$$

$$L_4 = \mathrm{Span}(\mathbf{e} - \mathbf{f} X^2(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(-X^2(\beta_3, \beta_4))$$

$$\omega(e_i, f_j) = \delta_{ij}$$

In this case, we will say that the four tuple (L_1, L_2, L_3, L_4) is in standard position with respect to a symplectic basis $(\mathbf{e}, \mathbf{f})$.

We define the invariants $x(T) := \mathrm{sgn}(\beta_3)$ for the triangle T, $x(\vec{r}) := X(\beta_3, \beta_4)$ for the oriented edge $\vec{r}$ and associate also the symplectic basis $B(b) := (\mathbf{e}, \mathbf{f})$ to the vertex b of Γ.

Because the oriented edge $\vec{r}$ defines the point b uniquely, sometimes we will say that the basis $B(b)$ is associated to the oriented edge $\vec{r}$ and write $B(\vec{r})$.

To define the angle coordinate, we consider a pentagon decorated by Lagrangians as on the Figure 1.5.5. To each oriented diagonal $\vec{r}_0$ and $\vec{r}_1$ of this pentagon, bases $B(\vec{r}_0) =: (\mathbf{e_0}, \mathbf{f_0})$ of (L_1, L_2) and $B(\vec{r}_1) =: (\mathbf{e_1}, \mathbf{f_1})$ of (L_3, L_1) are associated. So we can define the angle invariant $x(w)$ to be $x(w) := [L_1, L_5, L_3, L_2, L_4]_{\mathbf{e_0},\mathbf{f_1}}$.

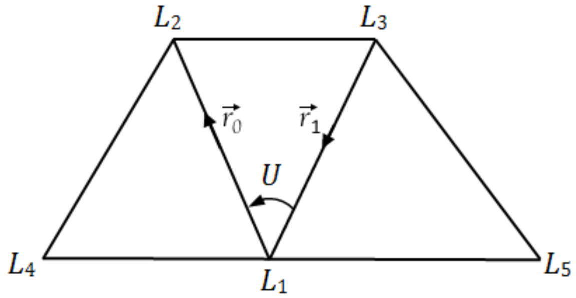

Figure 1.5.5:

Remark 1.5.10. 1. The choice of bases B is in general not unique. But it can be always chosen in a ρ-equivariant way with respect to the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases because the lifted decoration by Lagrangians is ρ-equivariant. We will always assume that B is ρ-equivariant.

2. By construction, the map x is $\pi_1(S, b_0)$-invariant, therefore, x is well-defined for the triangulation $\mathcal{T}$ of S.

3. By construction, $x(\vec{r}^{\,-1}) = X(\beta_4^*, \beta_3^*) = \sigma(X(\beta_3, \beta_4))$. So our definition of the map x for edges is consistent with the definition of $\mathcal{X}$-coordinates.

4. For each oriented edge $\vec{r}$ of triangulation there are two vertices b_1, b_2 of Γ lying close to $\vec{r}$. In general, there is a lot of possibilities to define $B(b_2)$ if $B(b_1)$ is fixed. We need to fix one of them, which is consistent to the definition of the map rep, namely with the matrix associated to the crossing of an edge. We do the following: Assume $\vec{r}$ is oriented upwards, b_1 lies to the right from $\vec{r}$ and b_2 lies to the left. Let $B(b_1) =: (\mathbf{e}, \mathbf{f})$ then $B(b_2) := (-\mathbf{f}\Phi T, \mathbf{e}\Phi^{-1}T^{-T})$ where Φ and T are matrices associated to $x(\vec{r})$ from the definition of the back transformation (see Appendix A.1.5).

5. Coordinate which we associate to an edge are in fact connected with the cross ratio operator in the following way:

$$[L_1, L_3, L_2, L_4]_{\mathbf{e}} = [L_4^{-1}]_{\mathbf{f},\mathbf{e}}[L_3]_{\mathbf{e},\mathbf{f}} = -X^0(\beta_3, \beta_4)^{-1}$$

6. This construction does not depend on the choice of a representative (ρ, D) in the class $[\rho, D]$. The triple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ for each edge is uniquely defined. In contrast, matrices U for each angle are in general not uniquely defined by the representation ρ. To define U, we have chosen a map B which, as we have seen, is in general not unique.

Lemma 1.5.11. *Let* $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$. *Consider* $x \in \mathcal{X}(S, \mathcal{T}, n)$ *constructed from* $[\rho, D]$ *as above. Then* $[\mathrm{rep}](x) = [\rho, D]$.

Proof. Notice, the bases on vertices of Γ were chosen in compatible way with the construction in the previous section, i.e. let b_1, b_2 be vertices of Γ connected by an edge e. To e the matrix E is associated as in the previous section (going along an angle or crossing an edge of triangulation). Then E maps the basis $B(b_1)$ to $B(b_2)$.

Therefore, by induction, for every loop α based in b, $\mathrm{rep}(\alpha)(B(b)) = B([\alpha]b)$, where by $[\alpha]b$ we understand the action of $[\alpha] \in \pi_1(S, b)$ on vertices of $\Gamma \subseteq \tilde{S}$. But the choice of B is ρ-equivariant, i.e. $\mathrm{rep}(\alpha)(B(b)) = B([\alpha]b) = \rho(\alpha)B(b)$. But the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases is exact, therefore, $\mathrm{rep}(\alpha) = \rho(\alpha)$ for all $[\alpha] \in \pi_1(S, b)$, where $\rho(\alpha)$ is written as a matrix with with respect to the basis $B(b)$. $\square$

Corollary 1.5.12. *The map* [rep] *is surjective.*

Definition 1.5.13. Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$, let (ρ, D) be a representative of $[\rho, D]$. Assume, the point b lies in the triangle T_0 near to the upwards oriented edge $\vec{e}$. Assume that peripheral curves α_i (see Figure 1.5.6), $i \in \{1, 2, 3, 4\}$ are decorated by Lagrangians $L_i \in \mathrm{Lag}(2n, \mathbb{R})$.

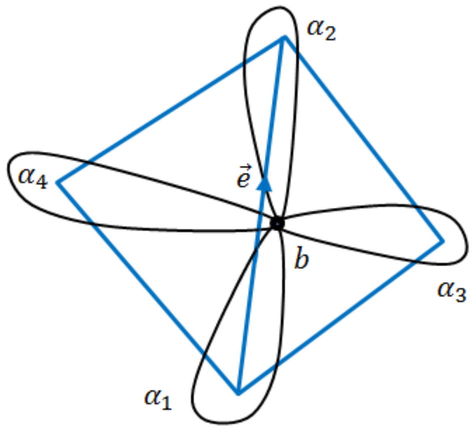

Figure 1.5.6:

We consider bilinear forms β_3, β_4 as above. Then there exists a symplectic basis $(\mathbf{e}, \mathbf{f})$ of $(\mathbb{R}^{2n}, \omega)$ such that

$$L_1 = \mathrm{Span}(\mathbf{e}) = L_{\mathbf{e}}, \quad L_2 = \mathrm{Span}(f) = L_{\mathbf{f}}$$

$$L_3 = \mathrm{Span}(\mathbf{e} + \mathbf{f}X^1(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(X^1(\beta_3, \beta_4))$$

$$L_4 = \mathrm{Span}(\mathbf{e} - \mathbf{f}X^2(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(-X^2(\beta_3, \beta_4))$$

$$\omega(e_i, f_j) = \delta_{ij}$$

The change-of-basis matrix from the standard basis $(\mathbf{e_{st}}, \mathbf{f_{st}})$ to $(\mathbf{e}, \mathbf{f})$ let be T. Then $(\rho', D') := (T^{-1}\rho T, T^{-1}D) \in [\rho, D]$ is called a *representative in standard form* of $[\rho, D]$. It has the following property:

$$D'(\alpha_1) = L_{\mathbf{e_{st}}}, \ D'(\alpha_2) = L_{\mathbf{f_{st}}},$$

$$D'(\alpha_3) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(X^1(\beta_3,\beta_4)),$$
$$D'(\alpha_4) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(-X^2(\beta_3,\beta_4))$$

Corollary 1.5.14. *The map* rep *constructed in the previous section gives us for each $x \in \mathcal{X}(S,\mathcal{T},n)$ a representative in standard form.*

Remark 1.5.15. Let $(S,\mathcal{T})$ be a surface with ideal triangulation. Assume $b \in S$ lies in the triangle T_0 near to the oriented edge $\vec{e}$. We take four peripheral curves α_i, $i \in \{1,2,3,4\}$ as on the Figure 1.5.6.

Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ and $x \in \mathcal{X}(S,\mathcal{T},n)$ is admissible for $[\rho, D]$. Then there exists $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ a representative in standard form such that:

$$D(\alpha_1) = L_{\mathbf{e_{st}}};\ \ D(\alpha_2) = L_{\mathbf{f_{st}}}$$
$$D(\alpha_3) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(X_0^1)$$
$$D(\alpha_4) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(-X_0^2)$$

where $x(\vec{r}) = X_0$. Moreover, (ρ, D) have the same decoration as $\mathrm{rep}(x)$, and ρ and $\mathrm{rep}(x)$ act in the same way on $D(\pi_1^{per}(S,b))$.

Remark 1.5.16. Let $x \in \mathcal{X}(S,\mathcal{T},n)$ be admissible for $[\rho_1, D_1]$ and for $[\rho_2, D_2]$. Then there exist $(\rho_1, D_1) \in [\rho_1, D_1]$ and $(\rho_2, D_2) \in [\rho_2, D_2]$ representatives in a standard form such that $D_1 = D_2$. In particular, the decoration of $\mathrm{rep}(x)$ coincides up to $\mathrm{Sp}(2n,\mathbb{R})$-action with decoration of each decorated representation for which x is admissible.

Remark 1.5.17. If $x \in \mathcal{X}(S,\mathcal{T},n)$ is admissible $\mathcal{X}$-coordinates for $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$, then in general it is wrong that $[\mathrm{rep}(x)] = [\rho, D]$.

As we have seen, angle coordinates are not uniquely defined. Sometimes different collections of angle coordinates define the same representation. Now we are going to find out how the angles can be changed so that the representation stays the same.

We take two adjacent by an edge e triangles. The coordinate on the edge is X_e (oriented as on fig. 1.5.7). The coordinate associated to the opposite orientation of e we denote by $\tilde{X}_e$. Signature of right triangle assume to be $(p,q) = \mathrm{sgn}(X_e^1) = \mathrm{sgn}(\tilde{X}_e^2)$, signature of left triangle assume to be $(p',q') = \mathrm{sgn}(X_e^2) = \mathrm{sgn}(\tilde{X}_e^1)$. We also assume that all angles are oriented counterclockwise with respect to the triangle.

Theorem 1.5.18. *Let $x \in \mathcal{X}(S,\mathcal{T},n)$. Let us change angle coordinates along the edge e in a following way:*

$$U_1' = WU_1,\ V_1' = V_1 W'^{-1},\ U_2' = U_2 W^{-1},\ V_2' = W'V_2 \tag{1.5.3}$$

where

$$W \in \mathrm{O}(p,q) \cap \mathrm{O}(P_{X_e}^{-T} X_e^2 P_{X_e}^{-1})$$
$$W' := D^{-1} W^T D$$
$$D := P_{X_e}^{-T} \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1}$$

This gives us another $x' \in \mathcal{X}(S,\mathcal{T},n)$. Then $[\mathrm{rep}(x)] = [\mathrm{rep}(x')]$.

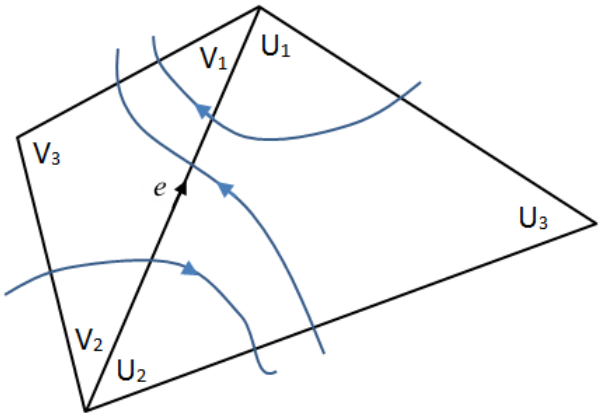

Figure 1.5.7:

Proof. First, we need the following proposition:

Proposition 1.5.19.

$$W' \in O(p', q') \cap O(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1})$$

Proof. First, we note that $D^{-T} I_{p'q'} D^{-1} = P_{X_e} (X_e^2)^{-1} P_{X_e}^T$:

$$\begin{aligned} D^{-T} I_{p'q'} = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} P_{\tilde{X}_e}^T I_{p'q'} = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} \tilde{X}_e^1 P_{\tilde{X}_e}^{-1} = \\ = [(X^2)^{-1} \Phi T = \Phi^{-1} T^{-T} \tilde{X}^1] = P_{X_e} (X_e^2)^{-1} \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1} = \\ = P_{X_e} (X_e^2)^{-1} P_{X_e}^T D \end{aligned}$$

Therefore,

$$W'^T I_{p'q'} W' = D^T W D^{-T} I_{p'q'} D^{-1} W^T D = D^T W P_{X_e} (X_e^2)^{-1} P_{X_e}^T W^T D =$$

$$= [W \in O(P_{X_e}^{-T} X_e^2 P_{X_e}^{-1})] = D^T P_{X_e} (X_e^2)^{-1} P_{X_e}^T D = I_{p'q'}$$

So $W' \in O(p', q')$.

Second, we note that $D^{-T} P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} D^{-1} = I_{pq}$:

$$D^{-T} (P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} P_{\tilde{X}_e}^T P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} =$$

$$= [X^1 \Phi T = \Phi^{-1} T^{-T} \tilde{X}^2] = P_{X_e} X_e^1 \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1} = [P_{X_e}^T I_{pq} P_{X_e} = X_e^1] = I_{pq} D$$

Therefore,

$$\begin{aligned} W'^T (P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) W' = D^T W D^{-T} (P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) D^{-1} W^T D = \\ = D^T W^T I_{pq} W D = [W \in O(p, q)] = D^T I_{pq} D = P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} \end{aligned}$$

So $W' \in O(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1})$.

□

Using the last proposition, it is easy to calculate that:

$$\hat{V}_1 T_r E_{X_e} \hat{U}_1 = \hat{V}_1' T_r E_{X_e} \hat{U}_1'$$

$$\hat{U}_2 T_r E_{X_e} \hat{V}_2 = \hat{U}_2' T_r E_{X_e} \hat{V}_2'$$

$$\hat{V}_1 T_r E_{\tilde{X}_e} \hat{U}_2^{-1} = \hat{V}_1' T_r E_{\tilde{X}_e} \hat{U}_1'$$

So holonomies of all curves are not changed. □

Corollary 1.5.20. *Let $x, x' \in \mathcal{X}(S, \mathcal{T}, n)$ such that* $\mathrm{rep}(x) = \mathrm{rep}(x')$. *Let $w \in W$ be an angle which is adjacent to some oriented edge $\vec{e}$. Then angle coordinates of x along $\vec{e}$ can be changed as above to coordinates $x'' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $x'(w) = x''(w)$ and $x(w') = x''(w')$ for all angles w' which are not adjacent to $\vec{e}$*

Lemma 1.5.21. *The only possible changes of angle coordinates so that the reconstructed representation does not change are given by formulas 1.5.3.*

Proof. (Sketch) We take the surface S of genus g and k punctures and fix the triangulation and the base point as on the picture. For another choice of triangulation the proof is similar.

We take $x, x' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $\mathrm{rep}(x) = \mathrm{rep}(x')$. We assume that x, x' define two different collections of angles $\{U_i\}$ and $\{U_i'\}$. Now we show that by correction of angles $\{U_i\}$ by formulas above we can get the collection $\{U_i'\}$.

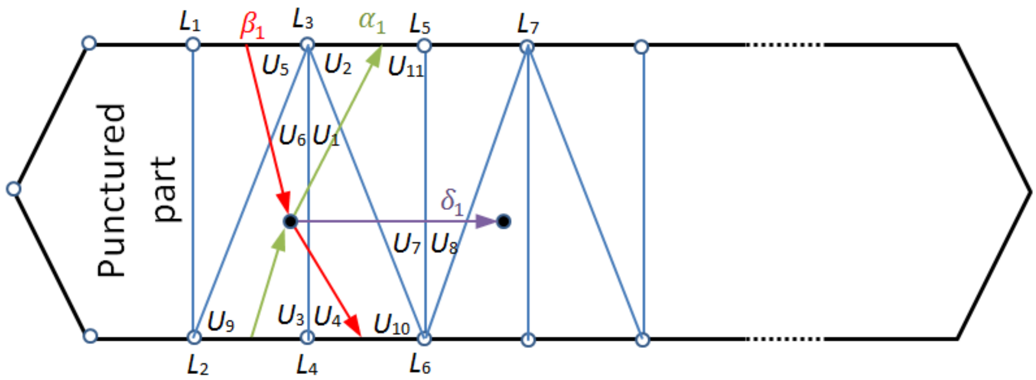

Figure 1.5.8:

Using Corollary 1.5.20 we correct all upper angles ($U_5, U_6, U_1, U_2, U_{11}, \ldots$ see Figure 1.5.8) getting $x'' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $\mathrm{rep}(x) = \mathrm{rep}(x') = \mathrm{rep}(x'')$. Note, that the number of these corrected angles agree with the total number of (non-oriented edges) since we correct each angle along exactly one edge. It makes automatically that some other angles agree ($U_7, U_8, \ldots$) because product of angles in one triangle is always

Id. To see that all other agree, it is enough to look at generators of $\pi_1(S,b)$ $(\alpha_1, \beta_1, \dots$ on Figure 1.5.8). Since their holonomies agree for $\mathrm{rep}(x) = \mathrm{rep}(x') = \mathrm{rep}(x'')$, all other angles $(U_9, U_3, U_4, U_{10}, \dots)$ agree automatically. So we get $x = x''$

$\square$

1.6 $\mathcal{X}$-coordinates for representations into central extensions

We introduced $\mathcal{X}$-coordinates for decorated representations into $\mathrm{Sp}(2n,\mathbb{R})$ using invariants of Lagrangian subspaces. As we remarked before, the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is not effective, but factors through $\mathrm{PSp}(2n,\mathbb{R})$. Therefore, the construction of $\mathcal{X}$-coordinates works as well for decorated representations into $\mathrm{PSp}(2n,\mathbb{R})$. The notions of decoration and transversality are well-defined because the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is just the lift of the action of $\mathrm{PSp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$. We only have to modify the angle invariants, as they now take values in $\mathrm{PO}(p,q)$.

We can then similarly define a map rep from $\mathcal{X}$-coordinates to the space of transverse decorated representations $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{PSp}(2n,\mathbb{R}))$.

Note that $\mathrm{Sp}(2n,\mathbb{R})$ is a central extension of $\mathrm{PSp}(2n,\mathbb{R})$ by the abelian group $\mathbb{Z}_2$. In this section, we extend the construction of $\mathcal{X}$-coordinates to representations into arbitrary central extensions of $\mathrm{PSp}(2n,\mathbb{R})$. The most interesting cases are the connected coverings of $\mathrm{PSp}(2n,\mathbb{R})$.

Let S be a surface with punctures as above, $\mathcal{T}$ be an ideal triangulation of S. Each representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$ projects to some representation $\rho' \in \mathrm{Hom}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$. Assume ρ' admits a decoration D which is transverse with respect to $\mathcal{T}$. If D is fixed, then $(\rho', D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$.

Definition 1.6.1. The pair (ρ, D) constructed as above is called *decorated representation into the central extension G transverse with respect to $\mathcal{T}$*. The set of all decorated representation into the central extension G transverse with respect to $\mathcal{T}$ is denoted by $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), G)$.

Definition 1.6.2. We denote

$$\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), G) := \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), G)/G$$

Definition 1.6.3. A representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$ is called *maximal* if it projects to a maximal representation $\rho' \in \mathrm{Hom}_{max}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$. The space of all maximal representations into G is denoted by $\mathrm{Hom}_{max}(\pi_1(S), G)$. The space of all maximal decorated representations into G is denoted by $\mathrm{Hom}^d_{max}(\pi_1(S), G)$.

Definition 1.6.4. We denote

$$\mathcal{M}(\pi_1(S), G) := \mathrm{Hom}_{max}(\pi_1(S), G)/G$$

$$\mathcal{M}^d(\pi_1(S), G) := \mathrm{Hom}^d_{max}(\pi_1(S), G)/G$$

Consider the embedding:

$$\begin{array}{rccc} \psi\colon & \mathrm{PO}(p,q) & \hookrightarrow & \mathrm{PSp}(2,\mathbb{R}) \\ & U & \mapsto & \mathrm{diag}(U, U^{-T}) \end{array}$$

and the homomorphism corresponding to the central extension:

$$\pi_G\colon G \to \mathrm{PSp}(2n,\mathbb{R}).$$

Then we define

$$G(p,q) := \pi_G^{-1}(\psi(\mathrm{PO}(p,q)).$$

Before we give the definition of $\mathcal{X}$-coordinates for central extension, we recall that $\mathcal{E}(n)$ is the set of all triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}$ are of the form as in the Theorem 1.5.1 with

$$\dim \mathcal{J}_1 + \dim \mathcal{J}_2 + \dim \mathcal{K} = n.$$

Definition 1.6.5 ($\mathcal{X}$-coordinates for central extension)**.** Let S be a surface with an ideal triangulation $\mathcal{T}$. Let E_{or} be the set of oriented edges of $\mathcal{T}$ and W be the set of angles of $\mathcal{T}$, F be the set of triangles of $\mathcal{T}$.

A system of *$\mathcal{X}$-coordinates* of rank n for the central extension G with respect to $\mathcal{T}$ is a map

$$x\colon F \sqcup E_{or} \sqcup W \to \{(p,q) \mid p,q \in \mathbb{N}\cup\{0\}, p+q=n\} \sqcup \mathcal{E}(n) \sqcup \bigcup_{p+q=n} G(p,q)$$

such that

- the triangle invariant $x(T)$ takes values in $\{(p,q) \mid p,q \in \mathbb{N}\cup\{0\}, p+q=n\}$. We call $x(T)$ also *signature* of the triangle T

- the edge invariant $x(\vec{e})$ is given by $x(\vec{e}) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) \in \mathcal{E}(n)$ for each $\vec{e} \in E_{or}$. $\mathcal{X}(\vec{e}^{-1}) = \sigma(\mathcal{X}(\vec{e}))$, where σ is the *edge reorientation map*:

$$\begin{array}{rccc} \sigma \;:& \mathcal{E}(n) & \to & \mathcal{E}(n) \\ & X(b_1,b_2) & \mapsto & X(b_2^*, b_1^*) \end{array}$$

  where b_1^*, b_2^* are dual bilinear forms to b_1, b_2. $\mathrm{sgn}(x(\vec{e})) = x(r(\vec{e}))$, i.e. the signature of $x(\vec{e})$ agree with the signature of the triangle $r(\vec{e})$ which lies to the right form $\vec{e}$;

- the angle invariant $x(w)$ takes values in $G(p,q)$ for each $w \in W$, where (p,q) is a signature of the triangle defined as above which this angle corresponds to. $U(w^{-1}) = U(w)^{-1}$. For each positive triple of positive angles (w_1, w_2, w_3) is subject to the condition

$$U(w_3)U(w_2)U(w_1) = \mathrm{Id}\,.$$

We denote by $\mathcal{X}_G(S,\mathcal{T},n)$ the set of all $\mathcal{X}$-coordinates of rank n for the central extension G on $(S,\mathcal{T})$.

By the same procedure as for $\mathcal{X}$-coordinates for $\mathrm{Sp}(2n,\mathbb{R})$, see Section 1.5.3, we can construct a map rep_G from the space of $\mathcal{X}$-coordinates to the space of decorated homomorphism $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S),G)$, which induces a surjective map

$$[\mathrm{rep}_G]\colon \mathcal{X}_G(S,\mathcal{T},n)\to \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S),G).$$

Using the map $[\mathrm{rep}_G]$ restricted to the positive locus of $\mathcal{X}_G(S,\mathcal{T},n)$, i.e. the subset of $\mathcal{X}_G(S,\mathcal{T},n)$ such that all triangle invariants are $(n,0)$, as in the Section 1.4, we can study the homotopy type of $\mathcal{M}^d(\pi_1(S),G)$. Namely, we can get the following result:

Theorem 1.6.6. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S),G)$ is homotopically equivalent to $G(n,0)^{2g+k-1}/G(n,0)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $G(n,0)$ on $G(n,0)^{2g+k-1}$ by simultaneous conjugation.*

Theorem 1.6.7. *The space of decorated maximal representation $\mathcal{M}^d(\pi_1(S),G)$ is homeomorphic to*

$$\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6}\times G(n,0)^{2g+k-1}/G(n,0)$$

where $\mathrm{Sym}^+(n,\mathbb{R})$ *is the space of all symmetric positive definite matrices and $G(n,0)$ acts by simultaneous conjugation in every factor.*

2 Hermitian Lie groups and noncommutative algebras

2.1 Symplectic group over algebras with anti-involution

2.1.1 Algebras with an anti-involution

Let A be a unital associative possibly noncommutative finite-dimensional semisimple $\mathbb{R}$-algebra.

Remark 2.1.1. The assumption that A is finite-dimensional over $\mathbb{R}$ implies that A has a well-defined topology.

Definition 2.1.2. An $\mathbb{R}$-linear map $\sigma\colon A \to A$ is called an *anti-involution* if

- $\sigma(ab) = \sigma(b)\sigma(a)$;
- $\sigma^2 = \mathrm{Id}$.

Now we fix a pair (A, σ).

Definition 2.1.3. An element $a \in A$ is called *σ-symmetric* if $\sigma(a) = a$. We denote

$$A^\sigma := A^{sym} := \mathrm{Fix}_A(\sigma) = \{a \in A \mid \sigma(a) = a\}.$$

Remark 2.1.4. We will use the notation A^{sym} when there is only one anti-involution on A. If there are more then one anti-involutions defined on A, then we always use the notation A^σ to emphasize which anti-involution we mean.

Proposition 2.1.5. *Let A be a unital associative $\mathbb{K}$-algebra, of finite dimension n over $\mathbb{K}$ for some field $\mathbb{K}$. Then A is isomorphic to a subalgebra of* $\mathrm{Mat}(n, \mathbb{K})$.

Proof. For every $x \in A$, consider the linear map $L_x : A \to A$ defined by

$$L_x(y) = xy\,.$$

Consider the map

$$A \ni x \to L_x \in \mathrm{Mat}(n, \mathbb{K})\,.$$

This is an injective $\mathbb{K}$-algebra homomorphism (there is no kernel because A is unital). $\square$

Corollary 2.1.6. *Let A be a unital associative $\mathbb{K}$-algebra, which is finite-dimensional over $\mathbb{K}$. The non-invertible elements are all zero-divisors.*

Proof. Using the previous proposition, given $x \in A$, x is invertible if and only if L_x is surjective. If L_x is not surjective, it has a kernel, hence x is a zero-divisor. $\square$

We denote by $A^\times$ the subgroup of invertible elements of A. If $V \subset A$ is a vector subspace, we denote

$$V^\times = A^\times \cap V\,,$$

the set of invertible elements in V.

Corollary 2.1.7. *Let A be a unital associative $\mathbb{R}$-algebra, which is finite-dimensional over $\mathbb{R}$, and let V be a vector subspace of A that contains at least one invertible element. Then $V^\times$ is an open dense subset of V.*

Proof. Using the previous proposition. First notice that $x \in A$ is invertible in A if and only if the linear map L_x is surjective, if and only if L_x is invertible in $\mathrm{Mat}(n, \mathbb{R})$. Using this, since the invertible elements are open in $\mathrm{Mat}(n, \mathbb{R})$, they are open in V.

To see density, consider an invertible element $u \in V$. The subspace $u^{-1} \cdot V$ contains the unit 1. We now prove that density holds for $u^{-1} \cdot V$, and we are done. If $x \in u^{-1} \cdot V$ is not invertible, consider $y_\epsilon = x + \epsilon \cdot 1$. We claim that there are only finitely many values of ϵ such that y_ϵ is not invertible, hence we can approximate x with invertible elements. To prove the claim, notice that the rank of L_{y_ϵ} as a matrix over $\mathbb{R}$ is the same as its rank as a matrix over $\mathbb{C}$. Put L_x in Jordan form, then we can see that L_{y_ϵ} is still in Jordan form, and it is invertible for all values of ϵ different from the eigenvalues of x. $\square$

Corollary 2.1.8. *It follows that $A^\times$ is open and dense in A and $(A^{sym})^\times$ is open and dense in A^{sym}.*

Consider the following map

$$\begin{array}{cccc} \theta\colon & A & \to & A^{sym} \\ & a & \mapsto & \sigma(a)a \end{array}$$

Definition 2.1.9. The subgroup

$$U(A, \sigma) = \{a \in A^\times \mid \theta(a) = 1\}$$

of $A^\times$ is called the *unitary group* of A.

Definition 2.1.10. A subset $C \subset V$ of an $\mathbb{R}$-vector space is a *cone* if it is stable under multiplication by a strictly positive scalar. A cone is *convex* if it is stable by sums of its elements.

Remark 2.1.11. If C is a convex cone, its closure $\overline{C}$ and its interior $\mathring{C}$ are still convex cones. The set of the opposites of the elements of C, denoted by $-C$, is still a convex cone.

Definition 2.1.12. A convex cone C is *proper* if

$$\overline{C} \cap -\overline{C} = \{0\}\,.$$

Definition 2.1.13. Given a subset $D \subset V$, the *convex cone generated* by D, denoted by $C(D)$, is the smallest convex cone containing D, the set of all linear combinations of elements of D with positive coefficients.

Definition 2.1.14. An algebra with an anti-involution (A, σ) is called *Hermitian* if:

1. The convex cone $C(\theta(A^{sym}))$ is proper;

2. A^{sym} does not contain nilpotent elements, i.e. for every $b \in A^{sym}$, $b^2 = 0$ if and only if $b = 0$.

Definition 2.1.15. If (A, σ) is Hermitian, we define

$$A_+^{sym} := C(\theta((A^{sym})^\times)),$$

and $A_{\geq 0}^{sym}$ as the closure of A_+^{sym}. In this case, A_+^{sym} and $A_{\geq 0}^{sym}$ are proper convex cones in A^{sym}.

Remark 2.1.16. We will see later in Corollaries 2.6.20, 2.6.24 and 2.6.42 together with the Corollary 2.7.28 that $A_+^{sym} = \theta(A^\times)$ and it is open subset of A^{sym} and it is contained in $A^\times$. Moreover, from the Corollary 2.6.44, it follows that for every $a \in A$, we have $\sigma(a)a = 0$ if and only if $a = 0$.

Corollary 2.1.17. *A subalgebra of a Hermitian algebra which is closed under σ is also Hermitian.*

Theorem 2.1.18. *The group $U(A, \sigma)$ is compact.*

Proof. The Definition 2.1.14 implies that by Definition 2.6.11, (A, σ) is a weakly Hermitian Lie algebra. We define the following map $\beta \colon A \times A \to \mathbb{R}$:

$$\beta(a_1, a_2) := \operatorname{tr}\left(\frac{\sigma(a_1)a_2 + \sigma(a_2)a_1}{2}\right),$$

where $\operatorname{tr} \colon A^{sym} \to \mathbb{R}$ is the trace map defined in Definition 2.6.28. By the Proposition 2.6.29, this map is an inner product on A^{sym}. It is easy to see, that it is bilinear on A. To see that β is an inner product on A, we have only to check the positive definiteness, i.e. $\beta(a, a) = 0$ if and only if $a = 0$.

For $a \in A$ take its polar decomposition 2.6.40: $a = ua_0$ where $a_0 \in A_+^{sym}$, $u \in U(A, \sigma)$. Then $\beta(a, a) = a_0^2 = 0$ if and only if $a_0 = 0$ if and only if $a = 0$. Therefore, β is an inner product on A. The group $U(A, \sigma)$ acts on A by left multiplication preserving β. Therefore, $U(A, \sigma) \subset \operatorname{Isom}(\beta)$, where $\operatorname{Isom}(\beta)$ is the group of linear transformations of A preserving β which is compact and $U(A, \sigma)$ is a closed subgroup of it. So it is compact as well. □

Remark 2.1.19. In the Corollary 2.6.43, we will see that for a Hermitian algebra (A, σ), the group $U(G, \sigma)$ is a maximal compact subgroup of $A^\times$.

Classical examples

In this subsection, we recall classical algebras with anti-involutions and introduce notation that we will use later.

1. For any field $\mathbb{K}$ such that $\mathbb{R} \subseteq \mathbb{K}$, $A = \mathrm{Mat}(n, \mathbb{K})$, $\sigma(r) := r^T$ is an algebra with an anti-involution. Then $A^\sigma = \mathrm{Sym}(n, \mathbb{K})$ space of all symmetric matrices. If $\mathbb{K} = \mathbb{R}$, (A, σ) is Hermitian with $A^\sigma_+ = \mathrm{Sym}^+(n, \mathbb{R})$ real symmetric positive definite matrices.

2. $A = \mathrm{Mat}(n, \mathbb{C})$, $\bar{\sigma}(r) := \bar{r}^T$ is a Hermitian algebra with $A^{\bar{\sigma}} = \mathrm{Herm}(n, \mathbb{C})$ complex Hermitian matrices and $A^{\bar{\sigma}}_+ = \mathrm{Herm}^+(n, \mathbb{C})$ complex Hermitian positive definite matrices.

3. $A = \mathrm{Mat}(n, \mathbb{H})$, $\sigma_1(r) := \bar{r}^T$, is a Hermitian algebra with $A^{\sigma_1} = \mathrm{Herm}(n, \mathbb{H})$ quaternionic Hermitian matrices and $A^{\sigma_1}_+ = \mathrm{Herm}^+(n, \mathbb{H})$ quaternionic Hermitian positive definite matrices.

4. There is another anti-involution on $A = \mathrm{Mat}(n, \mathbb{H})$, namely $\sigma_0(r) := \sigma(r_1) + \bar{\sigma}(r_2)j$ where $r_1, r_2 \in \mathrm{Mat}(n, \mathbb{C})$. This algebra is not Hermitian.

2.1.2 Sesquilinear forms on A-modules and their groups of symmetries

Let A be a unital associative finite dimensional $\mathbb{R}$-algebra with an anti-involution σ.

Definition 2.1.20. A *σ-sesquilinear form* ω on a right A-module V is a map

$$\omega\colon V \times V \to A$$

such that

$$\omega(x + y, z) = \omega(x, z) + \omega(y, z)$$

$$\omega(x, y + z) = \omega(x, y) + \omega(x, z)$$

$$\omega(x_1 r_1, x_2 r_2) = \sigma(r_1)\omega(x_1, x_2) r_2$$

We denote by

$$\mathrm{Aut}(\omega) := \{f \in \mathrm{Aut}(V) \mid \forall x, y \in V : \omega(f(x), f(y)) = \omega(x, y)\}$$

the group of symmetries of ω. We also define the corresponding Lie algebra:

$$\mathrm{End}(\omega) := \{f \in \mathrm{End}(V) \mid \forall x, y \in V : \omega(f(x), y) + \omega(x, f(y)) = 0\}$$

with the usual Lie bracket $[f, g] = fg - gf$.

Let us take $V = A^2$ (as the set of columns).

Definition 2.1.21.

- A pair (x, y) for $x, y \in A^2$ is called *basis* of A^2 if for every $z \in A^2$ there exist $a, b \in A^2$ such that $z = xa + yb$.
- The element $x \in A^2$ is called *regular* if there exists $y \in A^2$ such that (x, y) is a basis of A^2.
- $l \subseteq A^2$ is called a *line* if $l = xA$ for a regular $x \in A^2$. We denote the space of lines of A^2 by $\mathbb{P}(A^2)$.
- Two regular elements $x, y \in A^2$ are called *linearly independent* if (x, y) is a basis of A^2.
- Two lines l, m are called *transverse* if $l = xA$, $m = yA$ for linearly independent $x, y \in A^2$.
- An element $x \in A^2$ is called *isotropic* with respect to ω if $\omega(x, x) = 0$. The set of all isotropic regular elements of (A^2, ω) is denoted by $\mathrm{Is}(\omega)$.
- A line l is called isotropic if $l = xA$ for an regular isotropic $x \in A^2$. The set of all isotropic lines of (A^2, ω) is denoted by $\mathbb{P}(\mathrm{Is}(\omega))$.

Definition 2.1.22.

- A form ω is called *non-degenerate* if for every regular $x \in A^2$ there exists a $y \in A^2$ such that $\omega(x, y) \in A^\times$.
- A form is called *σ-symmetric* if $\omega(x_2, x_1) = \sigma(\omega(x_1, x_2))$ for all $x_1, x_2 \in A^2$.
- For A Hermitian, a σ-symmetric form is called *σ-inner product* if $\omega(x, x) \in A_+^{sym}$ for all regular $x \in A^2$.
- A form is called *σ-skew-symmetric* if $\omega(x_2, x_1) = -\sigma(\omega(x_1, x_2))$ for all $x_1, x_2 \in A^2$.

Proposition 2.1.23. *For every basis (x, y) of A^2 and for every $z \in A^2$ there exist unique $a, b \in A$ such that $z = xa + yb$. Moreover, for every regular $x \in A^2$, the map*

$$\begin{array}{ccc} A & \to & xA \\ a & \mapsto & xa \end{array}$$

is an isomorphism of right A-modules.

Proof. Take a basis (x, y) of A^2. Consider the following A-homomorphism of right A-modules:

$$\begin{array}{ccc} A^2 & \to & A^2 \\ (a, b) & \mapsto & xa + yb \end{array}.$$

This is also a surjective $\mathbb{R}$-homomorphism of vector spaces of the same dimension. Therefore, it is injective, i.e. (a, b) is uniquely defines by z. The restriction of this homomorphism to $A \times \{0\}$ is an isomorphism $A \to xA$ of right A-modules. $\square$

Definition 2.1.24. We denote $\mathrm{Sp}_2(A,\sigma) := \mathrm{Aut}(\omega)$, $\mathfrak{sp}_2(A,\sigma) := \mathrm{End}(\omega)$ for $\omega(x,y) := \sigma(x)^T\Omega y$ where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark 2.1.25.

$$\mathrm{Sp}_2(A,\sigma) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \sigma(a)c,\, \sigma(b)d \in A^{sym},\ \sigma(a)d - \sigma(c)b = 1 \right\} \subseteq \mathrm{Mat}_2^\times(A)$$

$$\mathfrak{sp}_2(A,\sigma) = \left\{ \begin{pmatrix} x & z \\ y & -\sigma(x) \end{pmatrix} \mid x \in A, y, z \in A^{sym} \right\} \subseteq \mathrm{Mat}_2(A)$$

From now on, we assume $\omega(x,y) := \sigma(x)^T\Omega y$ on A^2.

Proposition 2.1.26. *The form ω is non-degenerate.*

Proof. Let $x = (x_1,x_2)^T \in A^2$ regular. We want to find $y \in A^2$ such that $\omega(y,x) = 1$. Since x is regular, there exists $x' = (x_1', x_2')^T \in A^2$ such that (x,x') is a basis. That means that the matrix $X := \begin{pmatrix} x_1 & x_1' \\ x_2 & x_2' \end{pmatrix}$ is invertible, i.e. there exists the inverse matrix $X^{-1} = \begin{pmatrix} a_1 & a_2 \\ a_1' & a_2' \end{pmatrix}$. Therefore, $a_1x_1 + a_2x_2 = 1$. We take $y := (\sigma(a_2), -\sigma(a_1))^T$, then

$$\omega(y,x) = (a_2, -a_1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_1, a_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1.$$

So ω is non-degenerate. □

Proposition 2.1.27. *An element $x = (x_1,x_2)^T \in A^2$ is isotropic if and only if $\sigma(x_1)x_2 \in A^{sym}$.*

Proof. Direct computation. □

Proposition 2.1.28. *If $x, y \in A^2$ are isotropic and $\omega(x,y) = 1$, then (x,y) is a basis.*

Definition 2.1.29. A basis (x,y) of A^2 is called *symplectic* if x,y are isotropic and $\omega(x,y) = 1$.

Proof. Let $x, y \in A^2$ are isotropic and $\omega(x,y) = 1$. Consider the map

$$\begin{array}{ccc} A^2 & \to & A^2 \\ (a,b) & \mapsto & xa + yb. \end{array}$$

To see that this map is an isomorphism, it is enough to check that it is injective. Assume $xa + yb = 0$ for some $a, b \in A$, then

$$0 = \omega(x, xa + yb) = \omega(x,y)b = b,$$

$$0 = \omega(y, xa + yb) = -\omega(x,y)a = -a.$$

So $a = b = 0$. □

Corollary 2.1.30.

$$\mathrm{Sp}_2(A,\sigma) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right) \text{ is a symplectic basis} \right\}$$

Proposition 2.1.31. *Let $x \in A^2$ regular isotropic, $y \in A^2$ and $\omega(x,y) \in A^\times$. Then (x,y) is a basis of A^2. In particular, y is regular.*

Proof. To see that (x,y) is a basis, it is enough to check that the map

$$\begin{array}{ccc} A^2 & \to & A^2 \\ (a,b) & \mapsto & xa+yb \end{array}$$

is injective. Assume $xa + yb = 0$ for some $a, b \in A$, then

$$0 = \omega(x, xa+yb) = \omega(x,y)b.$$

Since $\omega(x,y) \in A^\times$, $b = 0$.

The element $x \in A^2$ is regular, therefore, by Proposition 2.1.23, if $xa = 0$, then $a = 0$. So, we obtain $z = 0$, i.e. the map above is an isomorphism. □

Proposition 2.1.32. *For every regular isotropic $x \in A^2$, there exists an isotropic $y \in A^2$ such that (x,y) is a symplectic basis.*

Proof. Since ω is non-degenerate, there exists $y' \in A^2$ such that $\omega(x,y') \in A^\times$ and (x,y') is a basis. We take $y'' := y' - \frac{x}{2}\omega(y',x)^{-1}\omega(y,y)$, then

$$\omega(y,y) = \omega(y',y') - \frac{1}{2}\omega(y',x)\omega(y',x)^{-1}\omega(y,y) -$$

$$-\frac{1}{2}\sigma(\omega(y',x)^{-1}\omega(y,y))\omega(x,y') = 0.$$

Since $\omega(x,y') = \omega(x,y'')$, if we take $y := y''\omega(x,y')^{-1}$, we obtain $\omega(x,y) = 1$ and x, y are isotropic, so (x,y) is a symplectic basis. □

Corollary 2.1.33. *The group* $\mathrm{Sp}_2(A,\sigma)$ *acts transitively on regular isotropic elements of* (A^2,ω).

Proof. If $x = (x_1,x_2)^T \in A^2$ is regular isotropic, then there exists $y = (y_1,y_2) \in A^2$ regular isotropic such that (x,y) is a symplectic basis. Then

$$g := \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \mathrm{Sp}_2(A,\sigma)$$

and $g(1,0)^T = x$. □

2.1.3 2x2-matrix algebra over a Hermitian algebra

In this section, we assume (A, σ) to be a Hermitian $\mathbb{R}$-algebra. We consider the following anti-involution on the algebra $\mathrm{Mat}_2(A)$ of 2x2-matrices over A:

$$\begin{array}{rccc} \sigma^T\colon & \mathrm{Mat}_2(A) & \to & \mathrm{Mat}_2(A) \\ & M & \mapsto & \sigma(M)^T \end{array}.$$

We denote

$$\mathrm{Sym}_2(A, \sigma) := \mathrm{Fix}_{\mathrm{Mat}_2(A)}(\sigma^T);$$

$$\mathrm{Sym}_2^{\geq 0}(A, \sigma) := \{M \in \mathrm{Sym}_2(A, \sigma) \mid \sigma(x)^T M x \in A^{sym}_{\geq 0} \text{ for all } x \in A^2\};$$

$$\mathrm{Sym}_2^{+}(A, \sigma) := \{M \in \mathrm{Sym}_2(A, \sigma) \mid \sigma(x)^T M x \in A^{sym}_{+} \text{ for all regular } x \in A^2\};$$

$$\mathrm{U}_2(A, \sigma) := U(\mathrm{Mat}_2^\times(A), \sigma) = \{M \in \mathrm{Mat}_2(A) \mid \sigma(M)^T M = \mathrm{Id}_2\}.$$

Proposition 2.1.34. *For a Hermitian algebra (A, σ), the algebra* $(\mathrm{Mat}_2(A), \sigma^T)$ *is Hermitian.*

Proof. 1. Show that $\mathrm{Sym}_2^{\geq 0}(A, \sigma)$ is a closed proper convex cone. By definition, $\mathrm{Sym}_2^{\geq 0}(A, \sigma)$ is closed in $\mathrm{Sym}_2(A, \sigma)$ because it is defined by a closed condition. It is a cone because for every $\lambda \geq 0$, if $\sigma(x)^T M x \in A^{sym}_{\geq 0}$ for all $x \in A^2$, then $\sigma(x)^T (\lambda M) x = \lambda \sigma(x)^T M x \in A^{sym}_{\geq 0}$ because $A^{sym}_{\geq 0}$ is a cone. It is a convex cone because for $M_1, M_2 \in \mathrm{Sym}_2^{+}(A, \sigma)$,

$$\sigma(x)^T (M_1 + M_2) x = \sigma(x)^T M_1 x + \sigma(x)^T M_2 x \in A^{sym}_{\geq 0}$$

for all $x \in A^2$ because $A^{sym}_{\geq 0}$ is a convex cone. If $M, -M \in \mathrm{Sym}_2^{\geq 0}(A, \sigma)$, then for all $x \in A^2$, $\sigma(x)^T M x, -\sigma(x)^T M x \in A^{sym}_{\geq 0}$. Since the cone $A^{sym}_{\geq 0}$ is proper, $\sigma(x)^T M x = 0$ for all $x \in A^2$. Let $M := \begin{pmatrix} m_{11} & m_{12} \\ \sigma(m_{12}) & m_{22} \end{pmatrix}$ where $m_{11}, m_{22} \in A^{sym}$, $m_{12} \in A$. Take $x = (1, 0)^T$, then $m_{11} = 0$. Take $x = (0, 1)^T$, then $m_{22} = 0$. Take $x = (1, 1)^T$ then

$$\sigma(x)^T M x = \sigma(m_{12}) + m_{12} = 0.$$

i.e. $m_{12} = -\sigma(m_{12})$. Take $x = (1, m_{12})$, then the cone $\mathrm{Sym}_2^{\geq 0}(A, \sigma)$ is proper.

$$\sigma(x)^T M x = 2m_{12}^2 = -2\sigma(m_{12}) m_{12} = 0$$

Because (A, σ) is Hermitian, $m_{12} = 0$. So we obtain $M = 0$.

2. Show that for every $M \in \mathrm{Mat}_2(A)$, $\sigma(M)^T M \in \mathrm{Sym}_2^{\geq 0}(A, \sigma)$. For every $x = (x_1, x_2) \in A^2$,

$$\sigma(x)^T x = \sigma(x_1) x_1 + \sigma(x_2) x_2 \in A^{sym}_{\geq 0}$$

because $A^{sym}_{\geq 0}$ is a convex cone. Therefore,

$$\sigma(x)^T \sigma(M)^T M x = \sigma(M x)^T (M x) \in A^{sym}_{\geq 0}.$$

Therefore, the convex cone $C(\theta(\mathrm{Mat}_2(A)))$ where $\theta(M) := \sigma(M)^T M$ for $M \in \mathrm{Mat}_2(A)$ is contained in the proper convex cone $\mathrm{Sym}_2^{\geq 0}(A,\sigma)$, i.e. it is proper as well.

3. Show that for $M \in \mathrm{Sym}_2(A,\sigma)$, $M^2 = 0$ if and only if $M = 0$. Let $M := \begin{pmatrix} m_{11} & m_{12} \\ \sigma(m_{12}) & m_{22}\end{pmatrix}$ where $m_{11}, m_{22} \in A^{sym}$, $m_{12} \in A$. Assume

$$0 = M^2 = \begin{pmatrix} m_{11}^2 + m_{12}\sigma(m_{12}) & m_{11}m_{12} + m_{12}m_{22} \\ \sigma(m_{12})m_{11} + m_{22}\sigma(m_{12}) & \sigma(m_{12})m_{12} + m_{22}^2 \end{pmatrix}.$$

Since the cone $A^{sym}_{\geq 0}$ is proper,

$$m_{11}^2 + m_{12}\sigma(m_{12}) = 0$$

implies $m_{11} = 0$ and $m_{12} = 0$, and

$$\sigma(m_{12})m_{12} + m_{22}^2 = 0$$

implies $m_{22} = 0$. ☐

Corollary 2.1.35. $\mathrm{U}_2(A,\sigma)$ *is a maximal compact subgroup of* $\mathrm{Mat}_2^\times(A)$

Proof. Follows from the Remark 2.1.19. ☐

Let A be a Hermitian algebra. We consider $A_\mathbb{C} := A \otimes_\mathbb{R} \mathbb{C}$ and extend σ in the complex anti-linear way, i.e. we define $\bar\sigma(x+iy) := \sigma(x) - \sigma(y)i$. In this section, we show that $(A_\mathbb{C}, \bar\sigma)$ is Hermitian.

We embed $A_\mathbb{C}$ into $\mathrm{Mat}_2(A)$ in the following way:

$$\begin{array}{rccc} \Upsilon\colon & A_\mathbb{C} & \to & \mathrm{Mat}_2(A) \\ & x + yi & \mapsto & \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \end{array} \tag{2.1.1}$$

This map is a injective homomorphism of $\mathbb{R}$-algebras. Moreover, the anti-involution $\bar\sigma$ corresponds under this embedding to σ^T. By Corollary 2.1.17, we obtain:

Corollary 2.1.36.
- *The algebra $(A_\mathbb{C}, \bar\sigma)$ is Hermitian.*
- *The group $U(A_\mathbb{C}, \bar\sigma) = \{z \in A_\mathbb{C} \mid \bar\sigma(z)z = 1\}$ is a maximal compact subgroup of $A_\mathbb{C}^\times$.*

Analogously, we can consider the quaternionification $A_\mathbb{H} := A \otimes_\mathbb{R} \mathbb{H}$ of A (for more details about quaternionic extensions of algebras see Section 2.4.1). Then $A_\mathbb{H}$ can be embedded into $\mathrm{Mat}_2(A_\mathbb{C})$ in the following way:

$$\begin{array}{rccc} \Upsilon_\mathbb{H}\colon & A_\mathbb{H} & \to & \mathrm{Mat}_2(A_\mathbb{C}) \\ & x + yj & \mapsto & \begin{pmatrix} x & y \\ -\bar y & \bar x \end{pmatrix}. \end{array} \tag{2.1.2}$$

This map is a injective homomorphism of $\mathbb{C}\{i\}$-algebras. Moreover, the anti-involution σ_1 on $A_{\mathbb{H}}$ defined as follows:

$$\sigma_1(x+yj) = \bar{\sigma}(x) + \sigma(y)j$$

for $x, y \in A_{\mathbb{C}}$, corresponds under this embedding to $\bar{\sigma}^T$. By Corollary 2.1.17, we obtain:

Corollary 2.1.37.

- *The algebra $(A_{\mathbb{H}}, \sigma_1)$ is Hermitian.*
- *The group $U(A_{\mathbb{H}}, \sigma_1) = \{z \in A_{\mathbb{C}} \mid \sigma_1(z)z = 1\}$ is a maximal compact subgroup of $A_{\mathbb{H}}^{\times}$.*

2.1.4 Maximal compact subgroup of $\mathrm{Sp}_2(A, \sigma)$

In this section, we assume (A, σ) to be Hermitian algebra with an anti-involution. Let $(A_{\mathbb{C}}, \sigma_{\mathbb{C}})$ be the complexification of (A, σ), i.e. $A_{\mathbb{C}} := A \otimes_{\mathbb{R}} \mathbb{C}$, $\sigma_{\mathbb{C}}$ is the complex linear extension of σ. We also denote by $\bar{\sigma}_{\mathbb{C}}$ the complex antilinear extension of σ, i.e. for $x, y \in A$

$$\sigma_{\mathbb{C}}(x+yi) = \sigma(x) - \sigma(y)i.$$

We state two theorems that describe maximal compact subgroups of $\mathrm{Sp}_2(A, \sigma)$ and $\mathrm{Sp}_2(A_{\mathbb{C}}, \sigma_{\mathbb{C}})$. The proofs of this theorems will be given in more general case in Sections 2.7.4 and 2.7.5.

Theorem 2.1.38. *The subgroup*

$$\mathrm{KSp}_2(A, \sigma) := \mathrm{Sp}_2(A, \sigma) \cap \mathrm{U}_2(A, \sigma) =$$

$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Mat}_2(A) \,\middle|\, \begin{matrix} \sigma(a)a + \sigma(b)b = 1 \\ \sigma(a)b - \sigma(b)a = 0 \end{matrix} \right\}$$

is a maximal compact subgroup of $\mathrm{Sp}_2(A, \sigma)$.

Corollary 2.1.39. *The embedding Υ from 2.1.1 maps isomorphically $U(A_{\mathbb{C}}, \bar{\sigma})$ to* $\mathrm{KSp}_2(A, \sigma)$.

Theorem 2.1.40. *The subgroup*

$$\mathrm{KSp}_2^c(A_{\mathbb{C}}, \sigma_{\mathbb{C}}) := \mathrm{Sp}_2(A_{\mathbb{C}}, \sigma_{\mathbb{C}}) \cap \mathrm{U}_2(A_{\mathbb{C}}, \bar{\sigma}_{\mathbb{C}}) =$$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Mat}_2(A_{\mathbb{C}}) \,\middle|\, \begin{matrix} \bar{\sigma}_{\mathbb{C}}(a)a + \sigma_{\mathbb{C}}(b)\bar{b} = 1 \\ \bar{\sigma}_{\mathbb{C}}(a)b - \sigma_{\mathbb{C}}(b)\bar{a} = 0 \end{matrix} \right\}$$

is a maximal compact subgroup of $\mathrm{Sp}_2(A_{\mathbb{C}}, \bar{\sigma}_{\mathbb{C}})$.

Corollary 2.1.41. *The embedding $\Upsilon_{\mathbb{H}}$ from 2.1.2 maps isomorphically $U(A_{\mathbb{H}}, \sigma_1)$ to* $\mathrm{KSp}_2^c(A_{\mathbb{C}}, \sigma_{\mathbb{C}})$.

2.1.5 Classical examples

Some classical Lie groups of tube type can be seen as $\mathrm{Sp}_2(A,\sigma)$.

1. For any field $\mathbb{K}$, $A = \mathrm{Mat}(n,\mathbb{K})$, $\sigma(r) = r^T$, we get
$$\mathrm{Sp}_2(A,\sigma) = \mathrm{Sp}(2n,\mathbb{K}).$$
In the case $\mathbb{K} = \mathbb{R}$, (A,σ) is Hermitian.
$$\mathrm{KSp}_2(A,\sigma) = \Upsilon(\mathrm{U}(n)) \cong \mathrm{U}(n).$$
In the case $\mathbb{K} = \mathbb{C}$, (A,σ) is the complexification the Hermitian algebra $\mathrm{Mat}(n,\mathbb{R})$.
$$\mathrm{KSp}_2^c(A,\sigma) \cong \mathrm{Sp}(n).$$

2. For $A = \mathrm{Mat}(n,\mathbb{C})$, $\bar\sigma(r) = \bar r^T$, $\mathrm{Sp}_2(A,\bar\sigma)$ is isomorphic to $\mathrm{U}(n,n)$. To see this, we notice that the standard Hermitian form h of signature (n,n) on $\mathbb{C}^{2n}$ is given by $h(x,y) := i\omega(xT,yT)$ where $T = \mathrm{diag}(\mathrm{Id}_n, -i\,\mathrm{Id}_n))$. In this case $A_\mathbb{C}$ is isomorphic to $\mathrm{Mat}(n,\mathbb{C}) \times \mathrm{Mat}(n,\mathbb{C})$ (see Section A.2.1). Therefore.
$$\mathrm{KSp}_2(A,\sigma) \cong \mathrm{U}(n) \times \mathrm{U}(n).$$

3. For $A = \mathrm{Mat}(n,\mathbb{H})$, $\sigma_1(r) = \bar r^T = \bar\sigma(r_1) - \sigma(r_2)j$ for $r = r_1 + r_2 j$ and $r_1, r_2 \in \mathrm{Mat}(n,\mathbb{C})$. We get in this case $\mathrm{Sp}_2(A,\sigma_1)$ is isomorphic $\mathrm{SO}^*(4n)$ (or some authors use terminology $\mathrm{O}(2n,\mathbb{H})$) considered as the group of isometries of the following quaternionic form β on $\mathbb{H}^{2n}$:
$$\beta(x,y) = \sum_{i=1}^{2n} \bar x_i j y_i = \bar x^T(\mathrm{Id}_{2n} j)y.$$
To see this, we notice that
$$\mathrm{Id}_{2n} j = \sigma_1(T)\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} T$$
for
$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{Id}_n & -\mathrm{Id}_n j \\ -\mathrm{Id}_n j & \mathrm{Id}_n \end{pmatrix}.$$
In this case $A_\mathbb{C}$ is isomorphic to $\mathrm{Mat}(2n,\mathbb{C})$ (see Section A.2.2). Therefore.
$$\mathrm{KSp}_2(A,\sigma) \cong \mathrm{U}(2n).$$

4. For $A = \mathrm{Mat}(n,\mathbb{H})$, $\sigma_0(r) = \sigma(r_1) + \bar\sigma(r_2)j$ for $r = r_1 + r_2 j$ and $r_1, r_2 \in \mathrm{Mat}(n,\mathbb{C})$. We get in this case $\mathrm{Sp}_2(A,\sigma_0)$ is isomorphic $\mathrm{Sp}(n,n)$ considered as the group of isometries of the following quaternionic form ω on $\mathbb{H}^{2n}$:
$$\beta(x,y) = \sum_{i=1}^{2n} \bar x_i y_i = \bar x^T \begin{pmatrix} -\mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix} y =$$

$$= k\left(\sigma_0(x)\begin{pmatrix}\mathrm{Id}_n k & 0\\ 0 & -\mathrm{Id}_n k\end{pmatrix} y\right).$$

To see this, we notice that

$$\begin{pmatrix}\mathrm{Id}_n k & 0\\ 0 & -\mathrm{Id}_n k\end{pmatrix} = \sigma_0(T)\begin{pmatrix}0 & \mathrm{Id}_n\\ -\mathrm{Id}_n & 0\end{pmatrix} T$$

for

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{Id}_n & \mathrm{Id}_n k\\ \mathrm{Id}_n k & \mathrm{Id}_n\end{pmatrix}.$$

The algebra (A, σ_0) is not Hermitian. The maximal compact subgroup of $\mathrm{Sp}(n, n)$ is $\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ (one can see it using the machinery developed in the Section 2.3.2).

2.2 Invariants of isotropic lines

In this section, we assume (A, σ) to be an $\mathbb{R}$- or $\mathbb{C}$-algebra with an anti-involution. We take the group $\mathrm{Sp}_2(A, \sigma)$. It acts on the space of isotropic lines in (A^2, ω):

$$\mathbb{P}(\mathrm{Is}(\omega)) = \{xA \subset A^2 \mid \omega(x, x) = 0,\, x \text{ regular}\}.$$

Similarly to the previous section, here we want to study this action.

2.2.1 Action of $\mathrm{Sp}_2(A, \sigma)$ on isotropic lines

Proposition 2.2.1. $\mathrm{Sp}_2(A, \sigma)$ *acts transitively on* $\mathbb{P}(\mathrm{Is}(\omega))$.

$$\mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}1\\0\end{pmatrix}A\right) := \left\{\begin{pmatrix}x & xy\\ 0 & \sigma(x)^{-1}\end{pmatrix} \mid x \in A^\times, y \in A^{sym}\right\}$$

$$\mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}0\\1\end{pmatrix}A\right) := \left\{\begin{pmatrix}x & 0\\ zx & \sigma(x)^{-1}\end{pmatrix} \mid x \in A^\times, z \in A^{sym}\right\}$$

Proof. $\mathrm{Sp}_2(G, \sigma)$ acts transitively on the space of isotropic lines since it acts transitively on $\mathrm{Is}(\omega)$.

We prove only the statement for the first stabilizer. The second one can be proved analogously.

Since

$$\begin{pmatrix}x & a\\ b & t\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}x\\b\end{pmatrix},$$

$x \in A^\times$ and $b = 0$. Furthermore,

$$\sigma\left(\begin{pmatrix}x & a\\ 0 & t\end{pmatrix}\right)^T \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x & a\\ 0 & t\end{pmatrix} = \begin{pmatrix}0 & \sigma(x)t\\ -\sigma(t)x & -\sigma(t)a + \sigma(a)t\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

we obtain $t = \sigma(x)^{-1}$, $a = xy$ for $y \in A^{sym}$. □

2.2.2 Action of $\mathrm{Sp}_2(A,\sigma)$ on pairs of isotropic lines

Proposition 2.2.2. *Two elements $u, v \in \mathrm{Is}(\omega)$ are linearly independent if and only if, up to action of $\mathrm{Sp}_2(A,\sigma)$, $u = (1,0)^T$, $v = (a,b)^T$ with $b \in A^\times$. Moreover, if $\omega(u,v) = 1$, then $a \in A^{sym}$, $b = 1$.*

Proof. $\mathrm{Sp}_2(A,\sigma)$ acts transitively on $\mathrm{Is}(\omega)$, therefore, up to $\mathrm{Sp}_2(A,\sigma)$-action, we can assume $u = (1,0)^T$. Since u and v are linearly independent, $b \in A^\times$. If, $\omega(u,v) = 1 = b$, then $v = (b,1)^T$ isotropic, i.e.

$$\omega(v,v) = \sigma(b) - b = 0$$

So $b \in A^{sym}$. □

Corollary 2.2.3. *If $x, y \in \mathrm{Is}(\omega)$ linearly independent, then $\omega(x,y) \in A^\times$.*

Proposition 2.2.4. *If (x,y) is a symplectic basis then there exists the unique $g \in \mathrm{Sp}_2(A,\sigma)$ such that $g(1,0)^T = x$, $g(0,1)^T = y$. In particular, $\mathrm{Sp}_2(G,\sigma)$ acts transitively on (G,σ)-symplectic bases.*

Proof. We can assume, $x = (1,0)^T$, $y = (a,1)^T$ and $a \in A^{sym}$. Take $g := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, then $gx = x$, $gy = (0,1)^T$. □

Corollary 2.2.5. *Let xA, yA be two transverse isotropic lines with $x, y \in \mathrm{Is}(\omega)$. Then there exist $M \in \mathrm{Sp}_2(A,\sigma)$ and $y' \in \mathrm{Is}(\omega)$ such that $y'A = yA$ and $Mx = (1,0)^T$, $My' = (0,1)^T$. In particular, $\omega(x,y') = 1$.*

Proposition 2.2.6. $\mathrm{Sp}_2(A,\sigma)$ *acts transitively on pairs of transverse isotropic lines.*

$$\mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}1\\0\end{pmatrix}A, \begin{pmatrix}0\\1\end{pmatrix}A\right) := \left\{\begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1}\end{pmatrix} \mid x \in A\right\} \cong A^\times.$$

Proof. By the Corollary 2.2.5, every pairs of transverse isotropic lines can be mapped to $((1,0)^T A, (0,1)^T A)$ by an element of $\mathrm{Sp}_2(A,\sigma)$. So $\mathrm{Sp}_2(A,\sigma)$ acts transitively on pairs of transverse isotropic lines.

By the Proposition 2.2.1,

$$\mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}1\\0\end{pmatrix}A, \begin{pmatrix}0\\1\end{pmatrix}A\right) =$$

$$= \mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}1\\0\end{pmatrix}A\right) \cap \mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}\left(\begin{pmatrix}0\\1\end{pmatrix}A\right) =$$

$$= \left\{\begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1}\end{pmatrix} \mid x \in A\right\}.$$

□

2.2.3 Action of $\mathrm{Sp}_2(A,\sigma)$ on triples of isotropic lines

Let (x_1A, x_3A, x_2A) be a triple of pairwise transverse isotropic lines where all $x_i \in \mathrm{Is}(\omega)$. Because of transversality of x_1A and x_2A, we can assume $\omega(x_1,x_2)=1$. Up to action of $\mathrm{Sp}_2(A,\sigma)$, we can assume $x_1=(1,0)^T$, $x_2=(0,1)^T$. We can also normalize x_3 so that $\omega(x_1,x_3)=1$. Then $x_3=(b,1)^T$, $b=\omega(x_3,x_2)\in (A^{sym})^\times$.

Proposition 2.2.7. *Orbits of the action of* $\mathrm{Sp}_2(A,\sigma)$ *on triples of pairwise transverse isotropic lines are in 1-1 correspondence with orbits of the following action of* $A^\times$ *on* $(A^{sym})^\times$*:*

$$\begin{array}{rccc}\psi\colon & A^\times \times (A^{sym})^\times & \mapsto & (A^{sym})^\times \\ & (a,b) & \mapsto & ab\sigma(a).\end{array}$$

Proof. Let (l_1,l_3,l_2) is a triple pairwise transverse of isotropic lines. As we have seen, up to $\mathrm{Sp}_2(A,\sigma)$-action, we can assume $l_i=x_iA$ for $x_1=(1,0)^T$, $x_2=(0,1)^T$, $x_3=(1,b)^T$ with $b\in (A^{sym})^\times$. The stabilizer $\mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}((1,0)^TA,(0,1)^TA)\cong A^\times$ acts on x_3 in the following way:

$$\mathrm{diag}(a,\sigma(a)^{-1})x_3=(ab,\sigma(a)^{-1})^T=(ab\sigma(a),1)^Ta^{-1}$$

i.e. $\mathrm{diag}(a,\sigma(a)^{-1})(b,1)^T=(ab\sigma(a),1)^TA$

So we see that in the orbit of $(b,1)^TA$ are exactly all isotropic lines of the form $(b',1)^TA$ where b' is from the orbit of b under ψ. □

Corollary 2.2.8. *If A is the complexification of some Hermitian algebra $(A_\mathbb{R},\sigma_\mathbb{R})$ and σ is the complex linear extension of $\sigma_\mathbb{R}$, then* $\mathrm{Sp}_2(A,\sigma)$ *acts transitively on the set of all triples of pairwise transverse isotropic lines.*

Proof. By the Theorem 2.7.33, the action $\psi_\mathbb{C}$ is transitive. □

Definition 2.2.9. In the case (A,σ) to be Hermitian, the triple (l_1,l_3,l_2) is called *positive* if up to action of $\mathrm{Sp}_2(A,\sigma)$, $l_i=x_i$, $x_1=(1,0)^T$, $x_2=(0,1)^T$, $x_3=(b,1)^T$ with $b\in B^{sym}_+$.

Proposition 2.2.10. *In the case (A,σ) to be Hermitian,* $\mathrm{Sp}_2(A,\sigma)$ *acts transitively on positive triples of isotropic lines.*

The stabilizer of the positive triple

$$\left(\begin{pmatrix}1\\0\end{pmatrix}A,\begin{pmatrix}1\\1\end{pmatrix}A,\begin{pmatrix}0\\1\end{pmatrix}A\right)$$

in $\mathrm{Sp}_2(A,\sigma)$ *coincides with the following subgroup:*

$$\hat{U}:=\left\{\begin{pmatrix}u & 0\\ 0 & u\end{pmatrix}\mid u\in U(A,\sigma)\right\}\cong U(A,\sigma)$$

The stabilizer of every positive triple of isotropic lines is conjugated in $\mathrm{Sp}_2(A,\sigma)$ *to $\hat{U}$.*

Proof. See Proposition 2.8.17. □

2.2.4 Invariants of quadruples of isotropic lines

We consider the following subspace of A:

$$A_0 := \{bb' \mid b, b' \in (A^{sym})^\times\}.$$

$A^\times$ acts on A_0 by conjugation because for $b, b' \in (A^{sym})^\times$, $a \in A^\times$:

$$a(bb')a^{-1} = (ab\sigma(a))(\sigma(a)^{-1}b')a^{-1} \in A_0.$$

Remark 2.2.11. The well-known fact from the linear algebra is that for matrix algebras A over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, it is always $A_0 = A^\times$.

Proposition 2.2.12. *Orbits of the action of* $\mathrm{Sp}_2(A, \sigma)$ *on quadruples of pairwise transverse isotropic lines are in 1-1 correspondence with orbits of the following action of* $A^\times$ *on* A_0*:*

$$\begin{array}{rccc} \eta\colon & A^\times \times A_0 & \mapsto & A_0 \\ & (a, b) & \mapsto & aba^{-1}. \end{array}$$

Proof. Let (l_1, l_3, l_2, l_4) be a quadruple pairwise transverse of isotropic lines. Then up to action of $\mathrm{Sp}_2(A, \sigma)$, we can assume $l_1 = (1, 0)^T A$, $l_2 = (0, 1)^T A$, $l_3 = (b, 1)^T A$, $l_3 = (1, b')^T A$ with $b, b' \in (A^{sym})^\times$. Consider the action of the stabilizer of (l_1, l_2):

$$\mathrm{diag}(a, \sigma(a)^{-1})(b, 1)^T A = (ab\sigma(a), 1)A,$$

$$\mathrm{diag}(a, \sigma(a)^{-1})(1, b')^T A = (1, \sigma(a)^{-1}b'a^{-1})A.$$

We consider the map $(l_1, l_3, l_2, l_4) \mapsto bb' \in A_0$. This map is well-defined, bijective and the action of the stabilizer of (l_1, l_2) (that is isomorphic to $A^\times$) induces the action of $A^\times$ by conjugation on A_0. So we obtain that this two actions are isomorphic. $\square$

Definition 2.2.13. The conjugacy class of A_0 corresponding to the quadruple (l_1, l_3, l_2, l_4) of pairwise transverse isotropic lines is called *cross ratio*.

2.2.5 Examples of matrix algebras

In this section, we construct explicit examples of spaces of isotropic lines for classical matrix algebras. To avoid abusing of notation, we will use the following notation: for complex numbers, we write $\mathbb{C}\{I\}$ to emphasize that the imaginary unit is denoted by I. Similarly, for quaternions, we write $\mathbb{H}\{I, J, K\}$ to emphasize that the imaginary units are denoted by I, J, K. The multiplication rule is then $IJ = K$.

Example 1. Let (A, σ) be $(\mathrm{Mat}(n, \mathbb{R}), \sigma)$, $(\mathrm{Mat}(n, \mathbb{C}), \sigma)$, $(\mathrm{Mat}(n, \mathbb{C}), \bar{\sigma})$ or $(\mathrm{Mat}(n, \mathbb{H}), \bar{\sigma})$ where σ is the transposition, $\bar{\sigma}$ the composition of transposition and complex/quaternionic conjugation.

Every regular element of $x \in A^2$ can be seen as a $2n \times n$-matrix of maximal rank. Columns of this matrix are elements of R^{2n} considered as a right R-module where

R is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If we take the R-span of this columns, we obtain n-dimensional submodule of R^{2n} denoted by $\mathrm{Span}_R(x)$. It is easy to see that the map:

$$\begin{array}{rccc} L\colon & \mathbb{P}(A^2) & \to & \mathrm{Gr}(n, R^{2n}) \\ & xA & \mapsto & \mathrm{Span}_R(x) \end{array}$$

where $\mathrm{Gr}(n, R^{2n})$ is the space of all n-dimensional submodules of R^{2n} is a bijection.

We consider the following form (bilinear or sesquilinear depending on σ) on R^{2n}:

$$\tilde{\omega}(u, v) := \sigma(u) \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} v$$

for $u, v \in R^{2n}$. Then $x \in \mathrm{Is}(\omega)$ if and only if $\mathrm{Span}_R(x)$ is isotropic with respect to $\tilde{\omega}$, that means for all $u, v \in \mathrm{Span}_R(x)$, $\tilde{\omega}(u, v) = 0$. So we obtain that L maps bijectively isotropic lines of A^2 to isotropic n-dimensional submodules of R^{2n}. Such submodules are called *Lagrangian* with respect to $\tilde{\omega}$. The space of all Lagrangian with respect to $\tilde{\omega}$ submodules are denoted by $\mathrm{Lag}(R^{2n}, \tilde{\omega})$.

Example 2. Let $A = \mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes \mathbb{C}\{i\}$ with the anti-involution $\bar{\sigma} \otimes \mathrm{Id}$. We use the map χ form the Section A.2.1 to identify A with $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\}) =: A'$. The induced by $\bar{\sigma} \otimes \mathrm{Id}$ anti-involution

$$\sigma' := \chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (m_2^T, m_1^T).$$

The map χ can be extended componentwise to the map

$$\chi'\colon\ \mathrm{Mat}_2(A) \to \mathrm{Mat}_2(A').$$

Proposition 2.2.14. $\mathrm{Sp}_2(A, \bar{\sigma} \otimes \mathrm{Id})$ *is isomorphic to* $\mathrm{GL}(2n, \mathbb{C})$*.*

Proof. First, we note that

$$A^{\bar{\sigma} \otimes \mathrm{Id}} = \mathrm{Sym}(n, \mathbb{C}\{i\}) + \mathrm{Skew}(n, \mathbb{C}\{i\}) I.$$

It is enough, to identify $\mathfrak{sp}_2(A, \bar{\sigma} \otimes \mathrm{Id})$ and $\mathrm{Mat}_2(A_{\mathbb{R}}) = \mathrm{Mat}(2n, \mathbb{C})$ as Lie algebras. First, we take the map χ' restricted to $\mathfrak{sp}_2(A, \bar{\sigma} \otimes \mathrm{Id})$:

$$\begin{array}{rccc} \chi'\colon & \mathfrak{sp}_2(A, \sigma) & \to & \mathrm{Mat}(2n, \mathbb{C}\{i\}) \times \mathrm{Mat}(2n, \mathbb{C}\{i\}) \\ & \begin{pmatrix} a_1 + a_2 I & b_1 + b_2 I \\ c_1 + c_2 I & -a_1^T + a_2^T I \end{pmatrix} & \mapsto & \left(\begin{pmatrix} a_1 + a_2 i & b_1 + b_2 i \\ c_1 + c_2 i & -a_1^T + a_2^T i \end{pmatrix}, \begin{pmatrix} a_1 - a_2 i & b_1 - b_2 i \\ c_1 - c_2 i & -a_1^T - a_2^T i \end{pmatrix} \right). \end{array}$$

where $a_1, a_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})$, $b_1, c_1 \in \mathrm{Sym}(n, \mathbb{C}\{i\})$, $b_2, c_2 \in \mathrm{Skew}(n, \mathbb{C}\{i\})$. This is an injective homomorphism of $\mathbb{C}\{i\}$-Lie algebras as restriction of injective map. Finally, we take a projection to the first component:

$$\pi_1\colon\ \mathrm{Mat}(2n, \mathbb{C}\{i\}) \times \mathrm{Mat}(2n, \mathbb{C}\{i\}) \to \mathrm{Mat}(2n, \mathbb{C}\{i\}).$$

Easy computation shows that $\pi_1 \circ \chi'$ is an isomorphism. ☐

The set $(A')^2$ can be identified with the space of pairs $(x_1, x_2)^T$ such that $x_1, x_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})$. We define the sesquilinear form:

$$\omega((x_1,x_2)^T,(y_1,y_2)^T) = \sigma'(x_1,x_2)\begin{pmatrix} 0 & (\mathrm{Id}_n,\mathrm{Id}_n) \\ -(\mathrm{Id}_n,\mathrm{Id}_n) & 0 \end{pmatrix}(y_2,y_2)^T =$$

$$= (\sigma(x_2)\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1, \sigma(x_1)\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_2).$$

Therefore,

$$\mathrm{Is}(\omega) = \{(l_1,l_2) \mid l = x_1 \mathrm{Mat}(n,\mathbb{C}\{i\}),\ l_2 = x_2 \mathrm{Mat}(n,\mathbb{C}\{i\}),\ x_1,x_2 \text{ regular},\ \omega(x_1,x_2)=0\}.$$

Since ω is non-degenerate, l_2 is uniquely determined by l_1. Therefore, we can identify:

$$\mathrm{Is}(\omega) \cong \{x \mathrm{Mat}(n,\mathbb{C}\{i\}) \mid x \text{ regular}\}.$$

As in the previous example, we can identify lines in $\mathrm{Mat}(n,\mathbb{C}\{i\})^2$ with Lagrangian subspaces of $(\mathbb{C}^{2n}, \tilde{\omega})$ where:

$$\tilde{\omega}(u,v) = u^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} v.$$

So the space $\mathrm{Is}(\omega)$ can be identified with

$$\mathrm{Is}(\omega) \cong \{(l_1,l_2) \in \mathrm{Gr}(n,\mathbb{C}^{2n})^2 \mid \tilde{\omega}(u,v) = 0 \text{ for all } u \in l_1,\ v \in l_2\}.$$

The form $\tilde{\omega}$ is a non-degenerate. Therefore, for $l \in \mathrm{Gr}(n,\mathbb{C}^{2n})$ there exists exactly one $\tilde{\omega}$-orthogonal complement $l^\perp \in \mathrm{Gr}(n,\mathbb{C}^{2n})$ such that for all $u \in l$, $v \in l^\perp$, $\tilde{\omega}(u,v) = 0$. So we can identify

$$\mathrm{Is}(\omega) \cong \mathrm{Gr}(n,\mathbb{C}^{2n})$$

and $\mathrm{GL}(2n,\mathbb{C})$ acts on $\mathrm{Gr}(n,\mathbb{C}^{2n})$ in the standard way.

Example 3. Let $A = \mathrm{Mat}(n,\mathbb{H}\{i,j,k\}) \otimes \mathbb{C}\{I\}$ with the anti-involution $\bar{\sigma} \otimes \mathrm{Id}$. We use the map ψ form the Section A.2.2 to identify A with $\mathrm{Mat}(2n,\mathbb{C}) =: A'$. The induced by $\bar{\sigma} \otimes \mathrm{Id}$ anti-involution

$$\sigma' := \psi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n,\mathbb{C})$ acts in the following way:

$$m \mapsto -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

We define the following σ'-sesquilinear form on $(A')^2$: for $x, y \in (A')^2$

$$\omega(x,y) = \sigma'(x)^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} y.$$

Proposition 2.2.15. $\mathrm{Sp}_2(A, \bar{\sigma} \otimes \mathrm{Id})$ *is isomorphic to* $\mathrm{O}(4n, \mathbb{C})$.

Proof. $M \in \mathrm{Sp}_2(A', \sigma') \cong \mathrm{Sp}_2(A, \bar{\sigma} \otimes \mathrm{Id})$ if and only if

$$\sigma'(M)^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix},$$

i.e.

$$-\begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \end{pmatrix} M^T \begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \end{pmatrix}.$$

$$\cdot \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix}$$

This is equivalent to:

$$M^T \begin{pmatrix} 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \end{pmatrix}.$$

So the group $\mathrm{Sp}_2(A, \sigma)$ is the group of symmetries of the symmetric bilinear form form

$$\begin{pmatrix} 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \end{pmatrix}$$

on $\mathbb{C}^{4n}$. But all symmetric bilinear forms on $\mathbb{C}^{4n}$ are conjugated. Therefore, $\mathrm{Sp}_2(A, \sigma)$ is isomorphic to $\mathrm{O}(4n, \mathbb{C})$. $\square$

Note that $\mathrm{Is}(\omega) = \mathrm{Is}(\omega')$ for

$$\omega'(x, y) := x^T \begin{pmatrix} 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \end{pmatrix} y.$$

As before, we can identify lines in $(A')^2$ with the space $\mathrm{Gr}(2n, \mathbb{C}^{4n})$ of $2n$-dimensional subspaces of $\mathbb{C}^{4n}$ using the map L (see Example 1). Under this map, the space $\mathrm{Is}(\omega)$ goes to the space $\mathrm{Lag}(\mathbb{C}^{4n}, \tilde{\omega})$ where

$$\tilde{\omega}(u, v) = u^T \begin{pmatrix} 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \\ \begin{matrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \end{pmatrix} v$$

for $x, y \in \mathbb{C}^{4n}$. The group $\mathrm{O}(\tilde{\omega}) \cong \mathrm{O}(4n, \mathbb{C})$ acts on $\mathrm{Lag}(\mathbb{C}^{4n}, \tilde{\omega})$ in the standard way.

2.3 Models for the symmetric space of $\mathrm{Sp}_2(A,\sigma)$ for Hermitian A

The goal of this Chapter is to construct different models of the symmetric space for $\mathrm{Sp}_2(A,\sigma)$ for a Hermitian algebra (A,σ).

2.3.1 Complex structures model

Definition 2.3.1. A *complex structure* on an right A-module V is an A-linear map $J\colon V \to V$ such that $J^2 = -\mathrm{Id}$.

Let $V = A^2$ and ω be the standard symplectic form in A^2. For every complex structure J on A^2, we can define the following σ-sesquilinear form

$$\begin{array}{rccc} h_J\colon & A^2\times A^2 & \to & A \\ & (x,y) & \mapsto & \omega(J(x),y) \end{array}$$

We remind the definition of the σ-inner product:

Definition 2.3.2. A σ-sesquilinear form h on (A^2,ω) is called *σ-inner product* if h is σ-symmetric and for all regular $v\in A^2$, $h(v,v)\in A_+^{sym}$.

We consider the following space:

$$\mathfrak{C} := \left\{J \text{ complex structure on } A^2 \mid h_J \text{ is an } \sigma\text{-inner product}\right\}.$$

Proposition 2.3.3. *Let $J\in\mathfrak{C}$ and $w\in\mathrm{Is}(\omega)$, then $J(w)\in\mathrm{Is}(\omega)$.*

Proof. For $w\in\mathrm{Is}(\omega)$,

$$\omega(J(w),J(w)) = h_J(w,J(w)) = \sigma(h_J(J(w),w)) = \sigma(\omega(w,w)) = 0,$$

therefore, $J(w)\in\mathrm{Is}(\omega)$. □

Definition 2.3.4. The *standard complex* structure on A^2 is the map

$$\begin{array}{rccc} J_0\colon & A^2 & \to & A^2 \\ & (x,y) & \mapsto & (y,-x) \end{array}$$

Theorem 2.3.5. $\mathrm{Sp}_2(A,\sigma)$ *acts on $\mathfrak{C}$ by conjugation. This action is transitive. The stabilizer of the standard complex structure J_0 is* $\mathrm{KSp}_2(A,\sigma)$.

In particular, $\mathfrak{C}$ is a model of the symmetric space of $\mathrm{Sp}_2(A,\sigma)$.

Definition 2.3.6. We call the space $\mathfrak{C}$ the *complex structure model* of the symmetric space of $\mathrm{Sp}_2(A,\sigma)$.

This Theorem will be proved in more general case in the Section 2.9.1.

2.3.2 Symmetric space of $\mathrm{O}(h)$ for an indefinite form h

Definition 2.3.7. The σ-sesquilinear σ-symmetric form h on A^2 such that there exist a basis (e_1, e_2) of A^2 such that $h(e_1, e_1) = -1$, $h(e_2, e_2) = 1$, $h(e_1, e_2) = 0$ is called *indefinite*.

The *standard indefinite form* h_{st} is the σ-sesquilinear σ-symmetric form on A^2 given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the standard basis $((1,0)^T, (0,1)^T)$ of A^2.

We define the group of symmetries of h:

$$\mathrm{O}(h) = \{g \in \mathrm{Aut}(A^2) \mid h(gx, gy) = h(x, y) \text{ for all } x, y \in A^2\}.$$

We define the following spaces:

$$\mathcal{X}_{\mathrm{O}(h)} := \{xA \mid h(x, x) \in A_+^{sym}\}.$$

$$\mathcal{X} := \mathcal{X}_{\mathrm{O}(h_{st})}.$$

Remark 2.3.8. $\mathcal{X}_{\mathrm{O}(h)}$ is well defined because if $xA = yA$, i.e. there exists $a \in A^\times$ such that $y = xa$, then

$$h(y, y) = \sigma(a)h(x, x)a = \sigma(a)\sigma(p)pa = \sigma(pa)pa \in A_+^{sym}$$

where $p \in A^\times$, $\sigma(p)p = h(x, x) \in A_+^{sym}$.

Remark 2.3.9. Since $\mathrm{Aut}(A^2)$ acts transitively on bases of A^2, all $\mathrm{O}(h)$ are isomorphic for indefinite h. Therefore, all $\mathcal{X}_{\mathrm{O}(h)}$ are also isomorphic.

Proposition 2.3.10. $\mathrm{O}(h_{st})$ *acts transitively on* $\mathcal{X}$ *with stabilizer of* $(0,1)^T A$ *equal to* $U(A, \sigma) \times U(A, \sigma)$ *diagonally embedded into* $\mathrm{O}(h_{st})$.

Proof. Since $h_{st}((0,1)^T, (0,1)^T) = 1 \in A_+^{sym}$, the line $(0,1)^T A \in \mathcal{X}$. Let $vA \in \mathcal{X}$ for some $v \in A^2$. Since $h_{st}(v, v) \in A_+^{sym}$, there exists $p \in A$ such that $h_{st}(v, v) = \sigma(p)p$. Let $v' := vp^{-1}$, then $h(v', v') = 1$ and $v'A = vA$.

Consider the following vector $v' := (x, \sigma(v_2)^{-1}\sigma(v_1)x)^T$ where $v = (v_1, v_2)^T$, $x = (1 + v_1\sigma(v_1))^{\frac{1}{2}}$. Then easy calculation shows that $h(v', v') = -1$ and $h(v, v') = 0$. So we can take the following matrix $M := (v', v) \in \mathrm{O}(h_{st})$. Since $M(0,1)^T = v$, we obtain $M(0,1)^T A = vA$, i.e. $\mathrm{O}(h_{st})$ acts transitively on $\mathcal{X}$.

Now, compute the stabilizer of $(0,1)^T A$. Let

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(h_{st})$$

stabilizes $(0,1)^T A$. Then $M(0,1)^T = (b, d)$, i.e. $b = 0$. Moreover

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma(a) & \sigma(c) \\ 0 & \sigma(d) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} -\sigma(a) & \sigma(c) \\ 0 & \sigma(d) \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} =$$

$$= \begin{pmatrix} -\sigma(a)a + \sigma(c)c & \sigma(c)d \\ 0 & \sigma(d)d \end{pmatrix}.$$

Therefore, $\sigma(d)d = 1$, i.e. d is invertible. So we obtain $c = 0$ and $\sigma(a)a = 1$, i.e. $a, d \in U(A, \sigma)$. □

Proposition 2.3.11. *The group $U(A,\sigma) \times U(A,\sigma)$ diagonally embedded into $\mathrm{O}(h_{st})$ is a maximal compact subgroup of $\mathrm{O}(h_{st})$. In particular, $\mathcal{X}$ is a model of the symmetric space of $\mathrm{O}(h_{st})$.*

Proof. First, note that the Lie algebra of $\mathrm{O}(h_{st})$ is:

$$\mathfrak{o}(h_{st}) = \left\{ \begin{pmatrix} a & b \\ \sigma(b) & d \end{pmatrix} \mid \sigma(a) = -a \in A, \sigma(d) = -d \in A, b \in A \right\}.$$

Assume, K is compact subgroup of $\mathrm{O}(h_{st})$ that contains $U(A,\sigma) \times U(A,\sigma)$ as a proper subgroup. Then $\mathrm{Lie}(K)$ contains an element $\begin{pmatrix} a & b \\ \sigma(b) & d \end{pmatrix}$ with $b \neq 0$. Therefore,

$$x := \begin{pmatrix} 0 & b \\ \sigma(b) & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ \sigma(b) & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{Lie}(K)$$

and

$$tx = \begin{pmatrix} 0 & tb \\ t\sigma(b) & 0 \end{pmatrix} \in \mathrm{Lie}(K)$$

for all $t \in \mathbb{R}$. Take a polar decomposition of $b = uy$ where $u \in U(A,\sigma)$, $y \in A^{sym}$. We take the spectral decompositions of y: $y = \sum_{i=1}^k \lambda_i c_i$ where (c_i) is a complete system of orthogonal idempotents, $\lambda_1, \dots, \lambda_k \in \mathbb{R}$.

Further,

$$x^2 = \begin{pmatrix} b\sigma(b) & 0 \\ 0 & \sigma(b)b \end{pmatrix}.$$

Therefore,

$$b\sigma(b) = u \sum_{i=1}^k \lambda_i^2 c_i u^{-1}, \ \sigma(b)b = \sum_{i=1}^k \lambda_i^2 c_i$$

and

$$\exp(tx) = \begin{pmatrix} u \sum_{i=1}^k \cosh(t\lambda_i) c_i u^{-1} & u \sum_{i=1}^k \sinh(t\lambda_i) c_i \\ \sum_{i=1}^k \sinh(t\lambda_i) c_i u^{-1} & \sum_{i=1}^k \cosh(t\lambda_i) c_i \end{pmatrix} \in K.$$

For t goes to infinity, $\exp(xt)$ does not converge even up to subsequence unless all $\lambda_i = 0$. But this means that $b = 0$, so we obtain $K = U(A,\sigma) \times U(A,\sigma)$. This contradicts to our assumption that $U(A,\sigma) \times U(A,\sigma)$ is a proper subgroup of K. $\square$

Proposition 2.3.12. *The following map*

$$\begin{array}{cccl} \Phi\colon & \mathcal{X} & \to & \mathring{D}(A,\sigma) := \{c \in A \mid 1 - \sigma(c)c \in A_+^{sym}\} \\ & (a,b)^T A & \mapsto & ab^{-1} \end{array}$$

is a homeomorphism.

Proof. Let $xA \in \mathcal{X}$ then $x = (a,b)^T$ with $-\sigma(a)a + \sigma(b)b \in A_+^{sym}$, i.e. there exists $p \in A^\times$ such that

$$-\sigma(a)a + \sigma(b)b = \sigma(p)p.$$

Therefore,

$$\sigma(b)b = \sigma(p)p + \sigma(a)a \in A_+^{sym},$$

i.e. $b \in A^\times$. So for $c = ab^{-1}$, $xA = (c,1)^T A$. Moreover, for every line $xA \in \mathcal{X}$, the element $c \in A$ such that $xA = (c,1)^T A$ is well defined and $1 - \sigma(c)c \in A_+^{sym}$.

For every $c \in \mathring{D}(A,\sigma)$, the line $(c,1)^T A \in \mathcal{X}$ because

$$h_{st}((c,1)^T,(c,1)^T) = 1 - \sigma(c)c \in A_+^{sym}.$$

Therefore, Φ is a homeomorphism. $\square$

Corollary 2.3.13. $\mathrm{O}(h_{st})$ *acts on* $\mathring{D}(A,\sigma)$ *via*

$$z \mapsto M.z = (az+b)(cz+d)^{-1}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This transformation is called Möbius transformation.

Remark 2.3.14. Since (A,σ) is Hermitian, by Proposition 2.6.49 the domain $\mathring{D}(A,\sigma)$ is precompact.

2.3.3 Projective model

As usual, we denote by $\sigma_{\mathbb{C}}$ the $\mathbb{C}$-linear extension of σ, i.e.

$$\sigma_{\mathbb{C}}(x+iy) = \sigma(x) + i\sigma(y)$$

for every $x,y \in A$ and by $\bar{\sigma}_{\mathbb{C}}$ the $\mathbb{C}$-antilinear extension of σ, i.e.

$$\bar{\sigma}_{\mathbb{C}}(x+iy) = \sigma(x) - i\sigma(y)$$

for every $x,y \in A$.

As we have seen in the Corollary 2.1.36, $(A_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}})$ is Hermitian.

We extend ω in the following way:

$$\omega_{\mathbb{C}}(x,y) := \sigma(x)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y.$$

The following $\bar{\sigma}$-sesquilinear form is an indefinite form on $A_{\mathbb{C}}^2$:

$$h(x,y) := \bar{\sigma}_{\mathbb{C}}(x)^T \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} y = i\omega_{\mathbb{C}}(\bar{x},y).$$

Indeed,

$$h(y,x) = \bar{\sigma}_{\mathbb{C}}(y)^T \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} x = \bar{\sigma}_{\mathbb{C}}\left(\sigma_1(x)^T \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} y\right) = \bar{\sigma}_{\mathbb{C}}(h(x,y)).$$

Then in the basis $e_1 := \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)^T$, $e_2 := \left(\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right)^T$, the form h is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. h is a $\bar{\sigma}_{\mathbb{C}}$-sesquilinear indefinite form on $A^2_{\mathbb{C}}$.

Note, $\mathrm{Sp}_2(A,\sigma)$ acts on $A^2_{\mathbb{C}}$ preserving $\omega_{\mathbb{C}}$ and h.

Proposition 2.3.15. $\mathrm{Sp}_2(A,\sigma)$ *acts transitively on the space*

$$\mathfrak{P} := \{vA_{\mathbb{C}} \mid v \in \mathrm{Is}(\omega_{\mathbb{C}}),\ h(v,v) \in (A^{\bar{\sigma}}_{\mathbb{C}})_+\} = \mathrm{Is}(\omega_{\mathbb{C}}) \cap \mathcal{X}_{\mathrm{O}(h)}$$

with the stabilizer of $(i,1)^T A_{\mathbb{C}}$ *equal to* $\mathrm{KSp}_2(A,\sigma)$.

In particular, $\mathfrak{P}$ *is a model of the symmetric space of* $\mathrm{Sp}_2(A,\sigma)$.

Definition 2.3.16. We call the space $\mathfrak{P}$ the *projective model* of the symmetric space of $\mathrm{Sp}_2(A,\sigma)$.

This Theorem will be proved in more general case in the Section 2.3.3.

2.3.4 Precompact model

We consider the following $\mathrm{Sp}_2(A_{\mathbb{C}},\sigma_{\mathbb{C}})$-transformation that maps h to the standard indefinite form h_{st}:

$$T := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

i.e. $\bar{\sigma}(T)^T[h]T = \mathrm{diag}(-1,1) = [h_{st}]$. Since $T \in \mathrm{Sp}_2(A_{\mathbb{C}},\sigma_{\mathbb{C}})$, it stabilizes the set $\mathrm{Is}(\omega_{\mathbb{C}})$.

Theorem 2.3.17. *The map*

$$\begin{array}{rccl} \Phi\colon & T^{-1}\mathfrak{P} & \to & \mathring{D}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}}) := \{c \in A^{\sigma_{\mathbb{C}}}_{\mathbb{C}} \mid 1 - \bar{c}c \in (A^{\bar{\sigma}_{\mathbb{C}}}_{\mathbb{C}})_+\} \\ & (a,b)^T A_{\mathbb{C}} & \mapsto & ab^{-1} \end{array}$$

is a homeomorphism. The set $\mathring{D}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}}) \subseteq A^{\sigma_{\mathbb{C}}}_{\mathbb{C}}$ *is precompact.*

This Theorem will be proved in more general case in the Section 2.9.3.

Remark 2.3.18. The group $T^{-1}\mathrm{Sp}_2(A,\sigma)T < \mathrm{Sp}_2(A_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}})$ acts on $\mathring{D}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}})$ by Möbius transformations.

Definition 2.3.19. We call the space $\mathring{D}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}_{\mathbb{C}})$ the *precompact model* of the symmetric space of $\mathrm{Sp}_2(A,\sigma)$.

2.3.5 Compactification and Shilov boundary

We take the topological closure of $\mathring{D}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma})$ in $A^{\sigma_{\mathbb{C}}}_{\mathbb{C}}$:

$$D(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}) := \{c \in A^{\sigma_{\mathbb{C}}}_{\mathbb{C}} \mid 1 - \bar{c}c \in (A^{\bar{\sigma}}_{\mathbb{C}})_{\geq 0}\}$$

The boundary of $D(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma})$ contains the following closed subspace:

$$\check{S}(A^{\sigma_{\mathbb{C}}}_{\mathbb{C}},\bar{\sigma}) := \{c \in A^{\sigma_{\mathbb{C}}}_{\mathbb{C}} \mid 1 - \bar{c}c = 0\}.$$

Definition 2.3.20. We call $\check{S}(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}, \bar{\sigma})$ *Shilov boundary* of the precompact model $\mathring{D}(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}, \bar{\sigma})$.

Note, that
$$\check{S}(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}, \bar{\sigma}) = U(A_{\mathbb{C}}, \bar{\sigma}) \cap A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}$$
and it is compact.

Remark 2.3.21. The map Φ^{-1} extends to the boundary of $D(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}, \bar{\sigma})$ and remains continuous and bijective. Since the boundary is compact, it is a homeomorphism. Therefore, we can see the boundary also in the projective model. In particular, we can see the Shilov boundary there.

The next Proposition describes the Shilov boundary in the projective model.

Proposition 2.3.22. *The preimage of the Shilov boundary $\check{S}(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}, \bar{\sigma}_{\mathbb{C}})$ in $\mathbb{P}(\mathrm{Is}(\omega_{\mathbb{C}}))$ under the map $\Phi \circ T^{-1}$ gives a compact subset of the boundary of the projective model. It consists of all lines of the form $xA_{\mathbb{C}}$ such that $x \in \mathrm{Is}(\omega)$.*

This Proposition will be proved later in the Section 2.9.4.

Corollary 2.3.23. *The space $\mathbb{P}(\mathrm{Is}(\omega))$ of isotropic lines of (A^2, ω) embedded into $\mathbb{P}(\mathrm{Is}(\omega_{\mathbb{C}}))$ as:*
$$xA \mapsto xA_{\mathbb{C}}$$
is a Shilov boundary in the projective model. This is a closed (even compact) orbit of the action of $\mathrm{Sp}_2(A, \sigma)$ on the boundary of the projective model.

2.3.6 Upperhalf space model

We denote as before by $A_{\mathbb{C}}$ the complexification of A, i.e. $A_{\mathbb{C}} := A \otimes_{\mathbb{R}} \mathbb{C}$. We extend σ to $A_{\mathbb{C}}$ complex linearly, i.e. $\sigma_{\mathbb{C}}(x + yi) := \sigma(x) + \sigma(y)i$.

Every element of $z \in A_{\mathbb{C}}^{\sigma_{\mathbb{C}}}$ can be uniquely written as $z = x + yi$ where $x, y \in A^{sym}$. We denote by $\mathrm{Re}(z) := x$, $\mathrm{Im}(z) := y$. We also have a complex conjugation on $A_{\mathbb{C}}$ given by $\bar{z} = x - yi$.

Definition 2.3.24. The *upperhalf space* is
$$\mathfrak{U} := \{z \in A_{\mathbb{C}}^{\sigma_{\mathbb{C}}} \mid \mathrm{Im}(z) \in A_{+}^{sym}\}$$

Theorem 2.3.25. $\mathrm{Sp}_2(A, \sigma)$ *acts transitively on $\mathfrak{U}$ via*
$$z \mapsto M.z = (az + b)(cz + d)^{-1}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
This transformation is called Möbius transformation. *The stabilizer of $1i$ is* $\mathrm{KSp}_2(A, \sigma)$. *In particular, $\mathfrak{U}$ is a model of the symmetric space of* $\mathrm{Sp}_2(A, \sigma)$.

The map:
$$\begin{array}{cccc} F\colon & \mathfrak{P} & \to & \mathfrak{U} \\ & (x_1, x_2)^T A_{\mathbb{C}} & \mapsto & x_1 x_2^{-1} \end{array}$$
defines a $\mathrm{Sp}_2(A, \sigma)$-equivariant homeomorphism between projective model and upperhalf space model.

This Proposition will be proved later in more general case in the Section 2.3.6.

2.3.7 Connection between projective, precompact and upperhalf space models

In this section, construct explicitly an $\mathrm{Sp}_2(A,\sigma)$-equivariant homeomorphism between the projective model and the upperhalf space model of the symmetric space for $\mathrm{Sp}_2(A,\sigma)$.

As we have seen, the map:

$$\begin{array}{cccc} F\colon & \mathfrak{P} & \to & \mathfrak{U} \\ & (x_1,x_2)^T A_{\mathbb{C}} & \mapsto & x_1 x_2^{-1} \end{array}$$

is a homeomorphism.

As we have seen in the Proposition 2.3.17, the map

$$\Phi\circ T^{-1}\colon \mathfrak{P}\to \mathring{D}(A_{\mathbb{C}}^{\sigma_{\mathbb{C}}},\bar{\sigma}_{\mathbb{C}}) := \{c\in A_{\mathbb{C}}^{\sigma_{\mathbb{C}}} \mid 1-\bar{c}c\in (A_{\mathbb{C}}^{\bar{\sigma}_{\mathbb{C}}})_+\}.$$

defines another homeomorphism. These maps F and $\Phi\circ T^{-1}$ can be seen as different coordinate charts for the projective model $\mathfrak{P}$ of the symmetric space for $\mathrm{Sp}_2(A,\sigma)$.

2.4 Models for the symmetric space of $\mathrm{Sp}_2(A,\sigma)$ for complexified A

The goal of this Chapter is to construct different models of the symmetric space for $\mathrm{Sp}_2(A,\sigma)$ where $A=A_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ for some Hermitian algebra $(A_{\mathbb{R}},\sigma_{\mathbb{R}})$.

2.4.1 Quaternionic extensions of algebras

Let $\mathbb{H}$ be the quaternionic skew-field. Sometimes, to make a construction more precise, we will write $\mathbb{H}\{\xi,\eta,\zeta\}$ to emphasize what the imaginary unities of $\mathbb{H}$ are. The multiplication rule is then $\xi\eta=-\eta\xi=\zeta$. Sometimes, we will also write $\mathbb{C}\{\kappa\}$ for $\mathbb{C}$ to emphasize the imaginary unit κ.

If $B_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ is the complexification of some real Lie algebra $B_{\mathbb{R}}$, then it can be embedded into $B_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{H}$ in many different ways. If we write $B:=B_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}\{i\}$, $B_{\mathbb{H}}:=B_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{H}\{i,J,K\}$, it means that B is embedded into $B_{\mathbb{H}}$ by the map induced by the identification $B\ni i\mapsto i\in B_{\mathbb{H}}$.

Let B be a $\mathbb{C}$-algebra, $B_0\subset B$ be $\mathbb{R}$-subalgebra of B, and there is a central element $I\in Z(B)$ such that $I^2=-1$ and $B=B_0\oplus B_0 I$. Then we say that B_0 be a real locus of B with respect to the imaginary unit I. In this case, B is isomorphic to $B_0\otimes_{\mathbb{R}}\mathbb{C}\{I\}$ as $\mathbb{C}\{I\}$-algebras. We take the following $\mathbb{H}$-algebra:

$$\mathbb{H}[B,B_0,I,J] := B_0\otimes_{\mathbb{R}}\mathbb{H}\{I,J,K\}.$$

The algebra B sits in $\mathbb{H}[B,B_0,I,J]$ as described above.

Definition 2.4.1. We call $\mathbb{H}[B,B_0,I,J]$ the *quaternionification* of B with respect to the real locus B_0 and the imaginary unit I.

2.4.2 Quaternionic structures model

Let $(A_{\mathbb{R}}, \sigma_{\mathbb{R}})$ be a Hermitian algebra with anti-involution $\sigma_{\mathbb{R}}$. We consider the complexification $A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. We denote by σ the complex linear extension of $\sigma_{\mathbb{R}}$ and by $\bar{\sigma}$ the complex anti-linear extension of $\sigma_{\mathbb{R}}$. As we have seen, $(A, \bar{\sigma})$ is a Hermitian algebra.

Definition 2.4.2. A *quaternionic structure* on an right A-module V is an additive map $J\colon V \to V$ such that $J^2 = -\operatorname{Id}$ and $J(xa) = J(x)\bar{a}$ for all $x \in V$, $a \in A$.

Let $V = A^2$ and ω be the standard symplectic form in A^2. For every quaternionic structure J on A^2, we can define the form:

$$\begin{array}{cccc} h_J\colon & A^2 \times A^2 & \to & A \\ & (x,y) & \mapsto & \omega(J(x), y) \end{array}$$

that is $\bar{\sigma}$-sesquilinear. Indeed, for $a_1, a_2 \in A$

$$h_J(xa_1, ya_2) = \omega(J(xa_1), ya_2) = \omega(J(x)\bar{a}_1, ya_2) = \bar{\sigma}(a_1) h_J(x,y) a_2.$$

We consider the following space:

$$\mathfrak{C} := \{J \text{ quaternionic structure on } A^2 \mid h_J \text{ is a } \bar{\sigma}\text{-inner product}\}.$$

Definition 2.4.3. The *standard quaternionic structure* on A^2 is the map

$$\begin{array}{cccc} J_0\colon & A^2 & \to & A^2 \\ & (x,y) & \mapsto & (\bar{y}, -\bar{x}) \end{array}$$

Remark 2.4.4. h_{J_0} is the standard $\bar{\sigma}$-inner product on A^2.

Proposition 2.4.5. *Let J be a quaternionic structure on A^2. $J \in \mathfrak{C}$ if and only if there exists a regular isotropic $w \in A^2$ such that $(J(w), w)$ is a symplectic basis.*

Proof. 1. Let $J \in \mathfrak{C}$ and $w \in A^2$ some regular isotropic element. Since $h_J(w,w) \in A_+^{\bar{\sigma}}$, we can normalize w so that $h_J(w,w) = 1$. Then:

$$\omega(J(w), J(w)) = h_J(w, J(w)) = \bar{\sigma}(h_J(J(w), w)) = \bar{\sigma}(\omega(w,w)) = 0,$$

$$\omega(J(w), w) = h_J(w,w) = 1.$$

Therefore, $(J(w), w)$ is a σ-symplectic basis.

2. Let $w \in A^2$ and $(J(w), w)$ is a σ-symplectic basis. Then,

$$h_J(w,w) = \omega(J(w), w) = 1$$

$$h_J(J(w), J(w)) = \omega(J^2(w), J(w)) = \omega(J(w), w) = 1,$$

$$h_J(J(w), w) = \omega(J^2(w), w) = -\omega(w,w) = 0.$$

Therefore, $(w, J(w))$ is an orthonormal basis for h_J, and in this basis, h_J is the standard σ-inner product, so h_J is an $\bar{\sigma}$-inner product. $\square$

Corollary 2.4.6. *For every $J \in \mathfrak{C}$, for every $v \in \mathrm{Is}(\omega)$, $J(v) \in \mathrm{Is}(\omega)$.*

Theorem 2.4.7. $\mathrm{Sp}_2(A,\sigma)$ *acts on* $\mathfrak{C}$ *in the following way:*

$$(g, J) \mapsto g^{-1} \circ J \circ g.$$

This action is transitive. The stabilizer of the standard quaternionic structure is $\mathrm{KSp}_2^c(A,\sigma)$.

In particular, $\mathfrak{C}$ *is a model of the symmetric space of* $\mathrm{Sp}_2(A,\sigma)$.

Proof. 1. First, we prove that $\mathrm{Sp}_2(A,\sigma)$ acts on $\mathfrak{C}$ by conjugation. Let $J \in \mathfrak{C}$, $g \in \mathrm{Sp}_2(A,\sigma)$. Consider $J' := g^{-1} \circ J \circ g$. Then

$$(J')^2 = g^{-1} \circ J \circ g \circ g^{-1} \circ J \circ g = -\,\mathrm{Id}\,.$$

So J' is a quaternionic structure on A^2. For a regular $x \in A^2$,

$$h_{J'}(x,x) = \omega(J'(x),x) = \omega(g^{-1}Jg(x),x) = \omega(Jg(x),g(x)) =$$

$$= h_J(g(x),g(x)) \in A_+^{\bar{\sigma}}.$$

Therefore, $h_{J'}$ is an inner product on A^2, i.e. $J' \in \mathfrak{C}$.

2. Second, we prove that the action is transitive. Let $J \in \mathfrak{C}$, take a symplectic basis $(J(w),w)$ from the Proposition 2.4.5. Since $\mathrm{Sp}_2(A,\sigma)$ acts transitively on symplectic bases, there exists $g \in \mathrm{Sp}_2(A,\sigma)$ which maps the standard symplectic basis to $(J(w),w)$. That means, g maps the standard complex structure J_0 to J. So the action is transitive.

3. Finally, compute the stabilizer of J_0. $g \in \mathrm{Stab}_{\mathrm{Sp}_2(A,\sigma)}(J_0)$ if and only if $g \in \mathrm{Sp}_2(A,\sigma)$ and $g \in \mathrm{O}(h_{J_0}) = \mathrm{U}_2(A_{\mathbb{C}},\bar{\sigma})$, i.e.

$$g \in \mathrm{Sp}_2(A,\sigma) \cap \mathrm{U}_2(A_{\mathbb{C}},\bar{\sigma}) = \mathrm{KSp}_2^c(A,\sigma). \qquad \square$$

Remark 2.4.8. Since any quaternionic structure is a $\mathbb{C}$-antilinear map, if we write the action of $\mathrm{Sp}_2(A,\sigma)$ in the matrix form, we need to add the complex conjugation: i.e. let $[J]$ be the matrix for the quaternionic structure J, then

$$[g^{-1} \circ J \circ g] = g^{-1}[J]\bar{g}.$$

2.4.3 Projective model for $\mathrm{Sp}_2(A,\sigma)$

Now, we consider the following quaternionic extension of A:

$$A_{\mathbb{H}} := \mathbb{H}[A, A_{\mathbb{R}}, i, j] = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}\{i,j,k\}.$$

This space can be embedded into $\mathrm{Mat}_2(A)$ as a subalgebra in the following way:

$$\begin{array}{ccc} A_{\mathbb{H}} & \hookrightarrow & \mathrm{Mat}_2(A) \\ a_1 + a_2 j & \mapsto & \begin{pmatrix} a_1 & a_2 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix}. \end{array}$$

The anti-involution $\bar{\sigma}^T$ on $\mathrm{Mat}_2(A)$ restricts to the following anti-involution on $A_{\mathbb{H}}$:

$$\sigma_1(a_1 + a_2 j) := \bar{\sigma}(a_1) - \sigma(a_2) j,$$

where $a_1, a_2 \in A$. Because $(A, \bar{\sigma})$ is Hermitian, by the Proposition 2.1.34, $(\mathrm{Mat}_2(A), \bar{\sigma}^T)$ is Hermitian and, therefore, $(A_{\mathbb{H}}, \sigma_1)$ is Hermitian as well.

We denote:

$$A_{\mathbb{H}}^{\sigma_1} := \mathrm{Fix}_{A_{\mathbb{H}}}(\sigma_1),\ (A_{\mathbb{H}}^{\sigma_1})_+ := \theta_{\mathbb{H}}(A_{\mathbb{H}}^{\times})$$

where

$$\begin{array}{rccc} \theta_{\mathbb{H}}\colon & A_{\mathbb{H}} & \to & A_{\mathbb{H}}^{\sigma_1} \\ & a & \mapsto & \sigma_1(a)a. \end{array}$$

We also consider the following anti-involution on $A_{\mathbb{H}}$:

$$\sigma_0(a_1 + a_2 j) := \sigma(a_1) + \bar{\sigma}(a_2) j,$$

where $a_1, a_2 \in A$ and extend ω in the following way:

$$\omega_{\mathbb{H}}(x, y) := \sigma_0(x)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y.$$

The following σ_1-sesquilinear form is an indefinite form on $A_{\mathbb{H}}^2$:

$$h(x, y) := \sigma_1(x)^T \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} y.$$

Indeed,

$$h(y, x) = \sigma_1(y)^T \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} x = \sigma_1 \left(\sigma_1(x)^T \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} y \right) = \sigma_1(h(x, y)).$$

Then in the basis $e_1 := \left(\frac{1}{\sqrt{2}}, \frac{j}{\sqrt{2}} \right)^T$, $e_2 := \left(\frac{1}{\sqrt{2}}, -\frac{j}{\sqrt{2}} \right)^T$, the form h is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. h is a σ_1-sesquilinear indefinite form on $A_{\mathbb{H}}^2$.

Proposition 2.4.9. $\mathrm{Sp}_2(A, \sigma)$ *acts on* $A_{\mathbb{H}}^2$ *preserving* h. *So we can see* $\mathrm{Sp}_2(A, \sigma)$ *as a subgroup of* $\mathrm{O}(h)$.

Proof. Let $x, y \in A_{\mathbb{H}}^2$, $M \in \mathrm{Sp}_2(A, \sigma)$, then

$$h(Mx, My) = \sigma_1(Mx)^T \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} My = \sigma_1(x)^T \bar{\sigma}(M)^T j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} My =$$

$$= \sigma_1(x)^T j \sigma(M)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} My = \sigma_1(x)^T \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} y = h(x, y).$$

So M preserves h. $\square$

Every quaternionic structure J on A^2 can be extended additively to a quaternionic structure $J_{\mathbb{H}}$ on $A^2_{\mathbb{H}}$ in the following linear way:

$$J_{\mathbb{H}}(x(a+bj)) := J(x)(\bar{a}+\bar{b}j).$$

where $x \in A^2$, $a,b \in A$.

Proposition 2.4.10. *For every quaternionic structure $J \in \mathcal{S}'$, there exist regular $x,y \in A^2_{\mathbb{H}}$ such that $J_{\mathbb{H}}(x) = xj$, $J_{\mathbb{H}}(y) = -yj$. Elements x,y are uniquely defined up to multiplication by elements of $A_{\mathbb{H}}$.*

Proof. Since $\mathrm{Sp}_2(A,\sigma)$ acts transitively on $\mathcal{S}'$, it is enough to prove the proposition for the standard quaternionic structure J_0.

Since

$$J_0(a_1+a_2j, b_1+b_2j)^T = (\bar{b}_1+\bar{b}_2j, -\bar{a}_1-\bar{a}_2j)^T,$$

we obtain

$$(\bar{b}_1+\bar{b}_2j, -\bar{a}_1-\bar{a}_2j)^T = (a_1+a_2j, b_1+b_2j)^T j = (-a_2+a_1j, -b_2+b_1j)^T$$

if and only if $\bar{a}_1 = b_2$, $\bar{a}_2 = -b_1$, i.e.

$$x = (a_1+a_2j, -\bar{a}_2+\bar{a}_1j)^T = (a_1+a_2j, j(a_1+a_2j))^T = (1, j^T)a,$$

where $a = a_1+a_2j \in A_{\mathbb{H}}$ arbitrary element. Analogously, $y = (j,1)^T a$ where $a = a_1+a_2j \in A_{\mathbb{H}}$ arbitrary element. $\square$

For a quaternionic structure $J \in \mathcal{S}'$, we denote by l_J the $A_{\mathbb{H}}$-line $yA_{\mathbb{H}}$ such that $J_{\mathbb{C}}(y) = -yj$.

We consider the spaces of isotropic elements and isotropic lines of $(A^2_{\mathbb{H}}, \omega_{\mathbb{H}})$:

$$\mathrm{Is}(\omega_{\mathbb{H}}) := \{x \mid x \in A^2_{\mathbb{H}} \text{ regular}, \omega_{\mathbb{H}}(x,x) = 0\},$$

$$\mathbb{P}(\mathrm{Is}(\omega_{\mathbb{H}})) := \{xA \mid x \in \mathrm{Is}(\omega_{\mathbb{H}})\}.$$

We also consider the symmetric space of $\mathrm{O}(h)$:

$$\mathcal{X}_{\mathrm{O}(h)} := \{xA \mid h(x,x) \in (A^{\sigma_1}_{\mathbb{H}})_+\}.$$

Proposition 2.4.11. *The map*

$$\begin{array}{rccl} F\colon & \mathfrak{C} & \to & \mathfrak{P} := \mathcal{X}_{\mathrm{O}(h)} \cap \mathbb{P}(\mathrm{Is}(\omega_{\mathbb{H}})) \\ & J & \mapsto & l_J \end{array}$$

defines is a homeomorphism that is equivariant under the action of $\mathrm{Sp}_2(A,\sigma)$*.*

Definition 2.4.12. We call the space

$$\mathfrak{P} := \mathcal{X}_{\mathrm{O}(h)} \cap \mathbb{P}(\mathrm{Is}(\omega_{\mathbb{H}}))$$

the *projective model* of the symmetric space of $\mathrm{Sp}_2(A,\sigma)$.

Proof. 1. Show that $l_J \in \mathcal{X}_{\mathrm{O}(h)}$. Since $\mathrm{Sp}_2(A,\sigma)$ acts transitively on $\mathfrak{C}$, it is enough to check it the standard quaternionic structure J_0:

$$h((j,1)^T,(j,1)^T) = \sigma_1(j,1)\begin{pmatrix}0 & j\\ -j & 0\end{pmatrix}\begin{pmatrix}j\\1\end{pmatrix} = (-j,1)\begin{pmatrix}j\\1\end{pmatrix} = 2 \in (A_{\mathbb{H}}^{\sigma_1})_+.$$

2. Show that $l_J \in \mathbb{P}(\mathrm{Is}(\omega))$. It is enough to prove it for J_0:

$$\omega((j,1)^T,(j,1)^T) = \sigma_0(j,1)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}j\\1\end{pmatrix} = (-1,j)\begin{pmatrix}j\\1\end{pmatrix} = 0.$$

3. Show that F is surjective. Let $v = u + wj \in A_{\mathbb{H}}$ such that $vA_{\mathbb{H}} \in \mathfrak{P}$. Since $h(v,v) \in (A_{\mathbb{H}}^{\sigma_1})_+$, we can renormalize v so that $h(v,v) = 2$. Since $v \in \mathrm{Is}(\omega)$,

$$0 = \omega_{\mathbb{H}}(v,v) = \omega(u,u) + j\omega(w,w)j + \omega(u,w)j + j\omega(w,u) =$$
$$= \omega(u,u) - \overline{\omega(w,w)} + \left(\omega(u,w) + \overline{\omega(w,u)}\right) j$$

So we have:

$$\omega(u,u) = \overline{\omega(w,w)}$$
$$\omega(u,w) = -\overline{\omega(w,u)}.$$

Moreover,

$$2 = h(v,v) = h(u+wj,u+wj) = h(u,u) - jh(w,w)j + h(u,w)j - jh(w,u).$$

Notice, for $u, w \in A^2$, $h(u,w) = \omega(\bar{u},\bar{w})j = j\omega(u,w)$. Therefore,

$$h(v,v) = \omega(\bar{u},\bar{u})j + \omega(w,w)j - \omega(\bar{u},\bar{w}) + \omega(w,u).$$
$$= 2\omega(w,u) + 2\omega(w,w)j$$

So we have:

$$\omega(w,u) = 1$$
$$\omega(u,u) = \omega(w,w) = 0$$

It means that (w,u) is a symplectic basis of (A^2,ω). We can define the following quaternionic structure: $J(u) = w$, $J(w) = -u$. By the Proposition 2.4.5, $J \in \mathfrak{C}$. Since

$$J_{\mathbb{H}}(v) = J_{\mathbb{H}}(u+wj) = w - uj = -(u+wj)j = -vj,$$

we obtain $F(J) = vA$, i.e. F is surjective.

4. The map F is injective because if $l_J = l_{J'} = yA$ for $J, J' \in \mathcal{S}'$ and some $y = y_1 + y_2 j \in A_{\mathbb{H}}^2$. Then $J(y_1) = J'(y_1) = -y_2$, $J(y_2) = J'(y_2) = y_1$ and (y_1,y_2) is a basis of A^2, i.e. $J = J'$.

5. Now, show the equivariance of F. Let $M \in \mathrm{Sp}_2(A,\sigma)$, $J \in \mathfrak{C}$ and $u, w \in A^2$ such that $w := J(u)$, $J(w) = -u$. Then $MJM^{-1}(Mu) = Mw$, $MJM^{-1}(Mw) = -Mu$. That means that for $v = u + wj$,

$$F(MJM^{-1}) = (Mv)A_{\mathbb{H}} = M(vA_{\mathbb{H}}) = MF(J),$$

i.e. F is equivariant with respect to the $\mathrm{Sp}_2(A,\sigma)$-action. □

Corollary 2.4.13. *The map*

$$\begin{array}{rccc} \tilde{\pi}'' \colon & \mathrm{Sp}_2(A,\sigma)/\operatorname{KSp}_2^c(A,\sigma) & \to & \mathfrak{P} \\ & M \operatorname{KSp}_2^c(A,\sigma) & \mapsto & M(j,1)^T A_{\mathbb{H}} \end{array}$$

is an $\mathrm{Sp}_2(A,\sigma)$*-equivariant homeomorphism.*

2.4.4 Precompact model for $\mathrm{Sp}_2(A,\sigma)$

As we have seen in the Chapter 2.3.2 in the Proposition 2.3.12, the space $\mathcal{X} = \mathcal{X}_{\mathrm{O}(h_{st})}$ for the standard σ_1-indefinite form on $A_{\mathbb{H}}^2$ can be seen as a precompact domain

$$\mathring{D}(A_{\mathbb{H}},\bar{\sigma}) := \{c \in A_{\mathbb{H}} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_+\} \subseteq A_{\mathbb{H}}.$$

To see the symmetric space for $\mathrm{Sp}_2(A,\sigma)$ as a subset of this domain, we need an $A_{\mathbb{H}}$-linear transformation that maps h to the standard indefinite form. We can take the following matrix:

$$T := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}.$$

Then $\sigma_1(T)^T[h]T = \operatorname{diag}(-1,1) = [h_{st}]$ and $T^{-1}\mathfrak{P} \subseteq \mathcal{X}$. Notice, $T \in \mathrm{Sp}_2(A_{\mathbb{H}},\sigma)$, therefore it stabilizes the set of isotropic elements of $(A_{\mathbb{H}}^2,\omega)$.

Proposition 2.4.14. *The image of* $T^{-1}\mathfrak{P}$ *under the homeomorphism* $\Phi \colon \mathcal{X} \to \mathring{D}(A_{\mathbb{H}},\sigma_1)$ *is*

$$\mathring{D}(A_{\mathbb{H}}^{\sigma_0},\sigma_1) := \mathring{D}(A_{\mathbb{H}},\sigma_1) \cap A_{\mathbb{H}}^{\sigma_0} = \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_+\}.$$

Proof. To characterize the image of the symmetric space for $\mathrm{Sp}_2(A,\sigma)$ inside $\mathring{D}(A_{\mathbb{H}},\sigma_1)$, we remind that $(x_1,x_2)^T \in \operatorname{Is}(\omega)$ if and only if $\sigma_0(x_1)x_2 \in A_{\mathbb{H}}^{\sigma_0}$. Therefore, $(c,1)^T$ is isotropic if and only if $\sigma_0(c) \in A_{\mathbb{H}}^{\sigma_0}$, i.e. $c \in A_{\mathbb{H}}^{\sigma_0}$.

$$\Phi(T^{-1}\mathcal{S}'') = \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_+\} \subseteq A_{\mathbb{H}}^{\sigma_0}. \qquad \square$$

Remark 2.4.15. The group $T^{-1}\operatorname{Sp}_2(A,\sigma)T$ acts on $\mathring{D}(A_{\mathbb{H}}^{\sigma_0},\sigma_1)$ by Möbius transformations.

2.4.5 Compactification and Shilov boundary

Let (A,σ) be the complexification of a Hermitian algebra as before. The space

$$\mathring{D}(A_{\mathbb{H}}^{\sigma_0},\sigma_1) = \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_+\}$$

is precompact. Let us take the topological closure of $\mathring{D}(A_{\mathbb{H}}^{\sigma_0},\sigma_1)$ in $A_{\mathbb{H}}^{\sigma_0}$:

$$D(A_{\mathbb{H}}^{\sigma_0},\sigma_1) := \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_{\geq 0}\}.$$

The boundary of $D(A_{\mathbb{H}}^{\sigma_0},\sigma_1)$ contains the following closed subspace:

$$\check{S}(A_{\mathbb{H}}^{\sigma_0},\sigma_1) := \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c = 0\}.$$

Definition 2.4.16. We call $\check{S}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$ *Shilov boundary* of the precompact model $\mathring{D}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$.

Note, that $\check{S}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$ is compact as a closed subspace of a compact.

Remark 2.4.17. The map Φ^{-1} extends to the boundary of $D(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$ and remains continuous and bijective. Since the boundary is compact, it is a homeomorphism. Therefore, we can see the boundary also in the projective model. In particular, we can see the Shilov boundary there.

The next Proposition describes the Shilov boundary in the projective model.

Proposition 2.4.18. *The preimage of the Shilov boundary $\check{S}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$ in* $\mathrm{Is}(\omega_{\mathbb{H}})$ *the map $\Phi \circ T^{-1}$ gives a compact subset of the boundary of the projective model. It consists of all lines of the form $xA_{\mathbb{H}}$ such that $x \in \mathrm{Is}(\omega)$.*

Proof. Note that the line $l \in \mathrm{Is}(\omega_{\mathbb{H}})$ is of the form $xA_{\mathbb{H}}$ for some $x \in \mathrm{Is}(\omega)$ if and only if $\eta(l) = l$ where $\eta\colon A_{\mathbb{H}} \to A_{\mathbb{H}}$ the following involution

$$\eta(c_1 + c_2 j) := c_1 - c_2 j$$

for $c_1, c_2 \in A_{\mathbb{C}\{i\}}$. Notice, η is an involution on $A_{\mathbb{H}}$ and

$$\sigma_1(\sigma_0(c_1 + c_2 j)) = \bar{c}_1 - \bar{c}_2 j = -j\eta(c_1 + c_2 j)j.$$

Assume $c \in \check{S}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$, i.e. $\sigma_1(c)^{-1} = c$, $\sigma_0(c) = c$. Then

$$(\Phi \circ T^{-1})\eta(T \circ \Phi^{-1}(c)) = \Phi\left(\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}\begin{pmatrix} \eta(c) \\ 1 \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} j \\ j\eta(c) \end{pmatrix}\right) =$$

$$= -j\eta(c)^{-1}j = \sigma_1(\sigma_0(c))^{-1} = \sigma_1(c)^{-1} = c$$

i.e. for $l = (c, 1)^T A_{\mathbb{H}}$, $\eta(l) = l$.

If we take a line $xA_{\mathbb{H}}$ for some $x = (x_1, x_2)^T \in \mathrm{Is}(\omega)$, then

$$c := (\Phi \circ T^{-1})(xA) = (x_1 - jx_2)(-jx_1 + x_2)^{-1}.$$

Since $x \in \mathrm{Is}(\omega) \subset \mathrm{Is}(\omega_{\mathbb{H}})$, $c \in A_{\mathbb{H}}^{\sigma_0}$. Further

$$\sigma_1(c)c = \sigma_1(\sigma_0(c))c = (\bar{x}_1 + j\bar{x}_2)(j\bar{x}_1 + \bar{x}_2)^{-1}(x_1 - jx_2)(-jx_1 + x_2)^{-1} =$$

$$= (\bar{x}_1 + j\bar{x}_2)(j\bar{x}_1 + \bar{x}_2)^{-1}(j\bar{x}_1 + \bar{x}_2)(-j)(-jx_1 + x_2)^{-1} =$$

$$= (\bar{x}_1 + j\bar{x}_2)(-j)(-jx_1 + x_2)^{-1} = (-jx_1 + x_2)(-jx_1 + x_2)^{-1} = 1.$$

Therefore, $(\Phi \circ T^{-1})(xA) \in \check{S}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1)$. $\square$

2.4.6 Upperhalf space model for $\mathrm{Sp}_2(A,\sigma)$

Let $A_{\mathbb{R}}$ be an Hermitian $\mathbb{R}$-algebra with an anti-involution $\sigma_{\mathbb{R}}$. We assume $A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}\{I\}$ to be the complexification of $A_{\mathbb{R}}$. We denote here the imaginary unit by I because the algebra A sometimes is already a complex algebra where we just forget about its complex structure, so it may contain i as an element. In order to be more precise, we do not use the letter i in our construction.

We denote by σ the complex linear extension of $\sigma_{\mathbb{R}}$. We denote by $\bar{\sigma}$ the complex antilinear extension of $\sigma_{\mathbb{R}}$.

We denote by $A_{\mathbb{H}}$ the quaternionification of $A_{\mathbb{R}}$, i.e. $A_{\mathbb{H}} := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}\{I,J,K\}$. By our convention form the previous Section 2.4.1, we have $A \subset A_{\mathbb{H}}$.

We extend σ to $A_{\mathbb{H}}$ quaternionic linearly, i.e.

$$\sigma_0 := \sigma(x) + J\sigma(y) = \sigma(x) + \sigma(\bar{y})J = \sigma(x) + \bar{\sigma}(y)J.$$

So $A_{\mathbb{H}}^{\sigma_0} = \mathrm{Fix}_{A_{\mathbb{H}}}(\sigma_0) = A^{\sigma} \oplus A^{\bar{\sigma}}J$ is well defined.

Every element of $z \in A_{\mathbb{H}}^{\sigma_0}$ can be uniquely written as $z = x + yJ$ where $x \in A^{\sigma}$, $y \in A^{\bar{\sigma}}$. We denote by $\mathrm{Re}(z) := x$, $\mathrm{Im}(z) := y$. We also have a quaternionic conjugation on $A_{\mathbb{H}}$ given by $\bar{z} = \bar{x} - J\bar{y} = \bar{x} - yJ$.

Definition 2.4.19. The *upperhalf space* is

$$\mathfrak{U} := \{z \in A_{\mathbb{H}}^{\sigma_0} \mid \mathrm{Im}(z) \in A_{+}^{\bar{\sigma}}\}$$

Proposition 2.4.20. *The following map*

$$\begin{array}{cccc} F\colon & \mathfrak{P} & \to & \mathfrak{U} \\ & (x_1,x_2)^T A_{\mathbb{H}} & \mapsto & x_1 x_2^{-1} \end{array}$$

is a homeomorphism.

Proof. Let $(x_1,x_2)^T A_{\mathbb{H}} \in \mathfrak{P}$. We take such representative $(x_1,x_2)^T$ that $x_2 \in A$. Then

$$0 = \omega_{\mathbb{H}}((x_1,x_2)^T,(x_1,x_2)^T) = \sigma_0(x_1)x_2 - \sigma_0(x_2)x_1,$$

i.e. $\sigma_0(x_1)x_2 = \sigma(x_2)x_1$

$$h((x_1,x_2)^T,(x_1,x_2)^T) = (\sigma_1(x_1),\sigma_1(x_2))\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= \sigma_1(x_1)Jx_2 - \bar{\sigma}(x_2)Jx_1 = \sigma_1(x_1)Jx_2 - J\sigma(x_2)x_1 =$$

$$= \sigma_1(x_1)Jx_2 - J\sigma_0(x_1)x_2 = J(-2\,\mathrm{Im}(\sigma_0(x_1))Jx_2) = 2\overline{\mathrm{Im}(\sigma_0(x_1))}x_2 \in (A_{\mathbb{H}}^{\sigma_1})_{+}.$$

In particular, x_2 ia invertible. If $x_1 = x_{11} + x_{12}j$ then

$$\overline{\mathrm{Im}(\sigma_0(x_1))}x_2 = \sigma(x_{12})x_2 \in A_{+}^{\bar{\sigma}}.$$

Therefore, $x_1 x_2^{-1} \in A_{\mathbb{H}}$ is well-defined. Moreover,

$$\sigma_0(x_1 x_2^{-1}) = \sigma(x_2^{-1})\sigma_0(x_1) = \sigma(x_2^{-1})\sigma(x_2)x_1 x_2^{-1} = x_1 x_2^{-1},$$

i.e. $x_1x_2^{-1} \in A_{\mathbb{H}}^{\sigma_0}$. Furthermore,

$$\operatorname{Im}(x_1x_2^{-1}) = \operatorname{Im}((x_{11} + x_{12}J)x_2^{-1}) = \operatorname{Im}((x_{11}x_2^{-1} + x_{12}\bar{x}_2^{-1}J) = x_{12}\bar{x}_2^{-1} \in A_+^{\bar{\sigma}}$$

if and only if

$$\sigma(x_2)x_{12}\bar{x}_2^{-1}\bar{x}_2 = \sigma(x_2)x_{12} \in A_+^{\bar{\sigma}}$$

if and only if

$$\sigma(x_{12})x_2 \in A_+^{\bar{\sigma}}$$

So we obtain $x_1x_2^{-1} \in \mathcal{S}$. It is easy to check that the map

$$\begin{array}{rccc} F^{-1}\colon & \mathfrak{U} & \to & \mathfrak{P} \\ & z & \mapsto & (z,1)^T A_{\mathbb{H}} \end{array}$$

is inverse to F. Since F and F^{-1} are continuous, F is a homeomorphism. □

Corollary 2.4.21. $\operatorname{Sp}_2(A,\sigma)$ *acts on* $\mathfrak{U}$ *via*

$$z \mapsto M.z = (az+b)(cz+d)^{-1}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Sp}_2(A,\sigma).$$

This transformation is called Möbius transformation. *With respect to this action of* $\operatorname{Sp}_2(A,\sigma)$ *on* $\mathfrak{U}$ *and the natural action on* $\mathfrak{P}$*, the map* F *becomes an* $\operatorname{Sp}_2(A,\sigma)$*-equivariant homeomorphism.*

Proposition 2.4.22. *The map*

$$\begin{array}{rccc} \pi\colon & \operatorname{Sp}_2(A,\sigma) & \to & \mathfrak{U} \\ & M & \to & M.1J \end{array}$$

is continues, proper and surjective, i.e. $\operatorname{Sp}_2(A,\sigma)$ *acts transitively on* $\mathcal{S}$*. The stabilizer of* $1J$ *is* $\operatorname{KSp}_2^c(A,\sigma)$*.*

In particular, $\mathcal{S}$ *is a model for the symmetric space for* $\operatorname{Sp}_2(A,\sigma)$*.*

Proof. Let $z = x + yJ \in \mathcal{S}$ then $y = u^2$ for some $u \in (A^{\bar{\sigma}})^\times$. Then

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & \sigma(u)^{-1} \end{pmatrix}\right) = \pi\left(\begin{pmatrix} u & x\sigma(u)^{-1} \\ 0 & \sigma(u)^{-1} \end{pmatrix}\right) = x + uJ\sigma(u) =$$
$$= x + u\bar{\sigma}(u)J = x + yJ = z$$

An element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stabilizes $1J$ if and only if

$$1J = M.1J = (aJ+b)(cJ+d)^{-1} = (aJ+b)(-\bar{c}+\bar{d}J)^{-1}J.$$

So, $a = \bar{d}$ and $c = -\bar{b}$, i.e. $M \in \operatorname{KSp}_2^c(A,\sigma)$. □

Corollary 2.4.23. *The map* π *induces a homeomorphism*

$$\begin{array}{rccc} \tilde{\pi}\colon & \operatorname{Sp}_2(A,\sigma)/\operatorname{KSp}_2^c(A,\sigma) & \to & \mathfrak{U} \\ & M\operatorname{KSp}_2^c(A,\sigma) & \mapsto & M.1J \end{array}$$

A Möbius transformation $z \mapsto M'.z$ *corresponds under this homeomorphism to the left multiplication* $M\operatorname{KSp}_2^c(A,\sigma) \mapsto M'M\operatorname{KSp}_2^c(A,\sigma)$*.*

2.4.7 Connection between projective, precompact and upperhalf space models

Consider a Hermitian algebra $(A_{\mathbb{R}}, \sigma_{\mathbb{R}})$ and its complexification $A = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

As we have seen, the map:

$$F\colon \begin{array}{ccc} \mathfrak{P} & \to & \mathfrak{U} \\ (x_1, x_2)A_{\mathbb{H}} & \mapsto & x_1 x_2^{-1} \end{array}$$

is a homeomorphism.

As we have seen in the Proposition 2.4.14, the map

$$\Phi \circ T^{-1}\colon \mathfrak{P} \to \mathring{D}(A_{\mathbb{H}}^{\sigma_0}, \sigma_1) = \{c \in A_{\mathbb{H}}^{\sigma_0} \mid 1 - \sigma_1(c)c \in (A_{\mathbb{H}}^{\sigma_1})_+\}.$$

defines another homeomorphism. These maps F and $\Phi \circ T^{-1}$ can be seen as different coordinate charts for the projective model $\mathfrak{P}$ of the symmetric space for $\mathrm{Sp}_2(A, \sigma)$.

2.5 Classical examples

In this Chapter, we construct explicit examples of models of symmetric space for classical Hermitian Lie groups of tube type. We will always denote by $A_{\mathbb{R}}$ a real Hermitian algebra, the complexified algebra will be denoted by $A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The quaternionification of $A_{\mathbb{R}}$ will be denoted by $A_{\mathbb{H}}$.

For the algebras $\mathrm{Mat}(n, \mathbb{R})$ and $\mathrm{Mat}(n, \mathbb{C})$, we denote by σ the transposition. For $\mathrm{Mat}(n, \mathbb{C})$, we denote by $\bar{\sigma}$ the composition of transposition and complex conjugation. For $\mathrm{Mat}(n, \mathbb{H}\{i, j, k\})$, we denote by σ_0 the anti-involution acting in the following way:

$$\sigma_0(a + bj) := a^T + \bar{b}^T j,$$

and by σ_1 the anti-involution acting in the following way:

$$\sigma_1(a + bj) := \bar{a}^T - b^T j$$

for $a, b \in \mathrm{Mat}(n, \mathbb{C}\{i\})$. In particular, we use the same notation in the case $n = 1$, i.e. $\bar{\sigma}$ is the complex conjugation on $\mathbb{C}$.

To denote different models of the symmetric space for a group Γ that can be seen as $\mathrm{Sp}_2(A, \sigma)$ for some reel or complex A and anti-involution σ, we use the following letters: $\mathfrak{U}(\Gamma)$ for the upperhalf space model, $\mathfrak{P}(\Gamma)$ for the projective model, $\mathfrak{B}(\Gamma)$ for the precompact model and $\mathfrak{C}(\Gamma)$ for the complex/quaternionic structure model.

2.5.1 Upperhalf space model

In this section, we construct upperhalf space models for classical Hermitian Lie groups of tube type.

Example 4. Let $A_{\mathbb{R}} := \mathrm{Mat}(n,\mathbb{R})$ with the anti-involution σ, then

$$A = \mathrm{Mat}(n,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Mat}(n,\mathbb{C}),$$

$$\mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{C}),\sigma) = \mathrm{Sp}(2n,\mathbb{C}),\ \mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{R}),\sigma) = \mathrm{Sp}(2n,\mathbb{R}),$$

$$A_{\mathbb{H}} = \mathrm{Mat}(n,\mathbb{H}),\ A^{\sigma} = \mathrm{Sym}(n,\mathbb{C}),\ A^{\bar\sigma}_{+} = \mathrm{Herm}^{+}(n,\mathbb{C}).$$

So we have the following model for the symmetric space for $\mathrm{Sp}(2n,\mathbb{C})$:

$$\mathfrak{U}(\mathrm{Sp}(2n,\mathbb{C})) = \{M_1 + M_2 J \mid M_1 \in \mathrm{Sym}(n,\mathbb{C}),\ M_2 \in \mathrm{Herm}^{+}(n)\} \subset$$

$$\subset \mathrm{Mat}(n,\mathbb{H}).$$

The symmetric space for $\mathrm{Sp}(2n,\mathbb{R})$ is the real locus of this space:

$$\mathfrak{U}(\mathrm{Sp}(2n,\mathbb{R})) = \{M_1 + M_2 J \mid M_1 \in \mathrm{Sym}(n,\mathbb{R}),\ M_2 \in \mathrm{Sym}^{+}(n,\mathbb{R})\} \subset$$

$$\subset \mathfrak{U}(\mathrm{Sp}(2n,\mathbb{C})).$$

Example 5. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n,\mathbb{C}\{i\})$ with the anti-involution $\bar\sigma$. Then

$$A = \mathrm{Mat}(n,\mathbb{C}\{i\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\},$$

$$\mathrm{Sp}_2(A_{\mathbb{R}},\bar\sigma) = \mathrm{U}(n,n),$$

$$\mathrm{Sp}_2(A,\bar\sigma\otimes\mathrm{Id}) = \mathrm{GL}(2n,\mathbb{C}).$$

In the Section A.2.1, we studied the following $\mathbb{C}\{i\}$-algebras isomorphism:

$$\begin{array}{cccc} \chi\colon & \mathrm{Mat}(n,\mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\} & \to & \mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\}) \\ & a + bI & \mapsto & (a+bi, a-bi) \end{array}$$

where $a, b \in \mathrm{Mat}(n,\mathbb{C}\{i\})$.

We have seen,

$$\chi(A_{\mathbb{R}}) = \chi(\mathrm{Mat}(n,\mathbb{C}\{I\})) = \{(m,\bar m) \mid m \in \mathrm{Mat}(n,\mathbb{C}\{i\})\},$$

$$\chi(A^{\bar\sigma\otimes\mathrm{Id}}) = \{(m,m^T) \mid m \in \mathrm{Mat}(n,\mathbb{C}\{I\})\} \cong \mathrm{Mat}(n,\mathbb{C}).$$

$$\chi(A^{\bar\sigma\otimes\bar\sigma}) = \mathrm{Herm}(n,\mathbb{C}\{i\}) \times \mathrm{Herm}(n,\mathbb{C}\{i\}),$$

$$\chi(A^{\bar\sigma\otimes\bar\sigma}_{+}) = \mathrm{Herm}^{+}(n,\mathbb{C}\{i\}) \times \mathrm{Herm}^{+}(n,\mathbb{C}\{i\}).$$

So we have the following model for the symmetric space for $\mathrm{GL}(2n,\mathbb{C})$:

$$\mathfrak{U}(\mathrm{GL}(2n,\mathbb{C})) = \left\{ \begin{pmatrix} m_{11} \\ m_{11}^T \end{pmatrix} + \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} j \;\middle|\; \begin{array}{c} m_{11} \in \mathrm{Mat}(n,\mathbb{C}\{i\}), \\ m_{12}, m_{22} \in \mathrm{Herm}^{+}(n,\mathbb{C}\{i\}) \end{array} \right\} \subset$$

$$\subset \mathbb{H}[\mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\}), \chi(\mathrm{Mat}(n,\mathbb{C}\{I\})), (i,i), j].$$

Since $A_{\mathbb{R}}^{\sigma} = A_{\mathbb{R}} \cap A^{\bar{\sigma}\otimes \mathrm{Id}} = A_{\mathbb{R}} \cap A^{\bar{\sigma}\otimes\bar{\sigma}} = \mathrm{Herm}(n)$, we obtain the symmetric space for $\mathrm{U}(n,n)$ is:

$$\mathfrak{U}(\mathrm{U}(n,n)) \cong \left\{ \begin{pmatrix} m_1 \\ \bar{m}_1 \end{pmatrix} + \begin{pmatrix} m_2 \\ \bar{m}_2 \end{pmatrix} j \;\middle|\; \begin{matrix} m_1 \in \mathrm{Herm}(n, \mathbb{C}\{i\}), \\ m_2 \in \mathrm{Herm}^+(n, \mathbb{C}\{i\}) \end{matrix} \right\} \subset \mathfrak{U}(\mathrm{GL}(2n, \mathbb{C})).$$

To see $\mathfrak{U}(\mathrm{U}(n,n))$ as a subset of $\mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\})$, we have to identify j and $(i,i) = \chi(1 \otimes i)$, so we get

$$\mathfrak{U}(\mathrm{U}(n,n)) =$$
$$= \left\{ (m_1 + m_2 i, \bar{m}_1 + \bar{m}_2 i) \mid m_1 \in \mathrm{Herm}(n,\mathbb{C}\{i\}), m_2 \in \mathrm{Herm}^+(n,\mathbb{C}\{i\}) \right\} \subset$$
$$\subset \mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\}).$$

In a pair $(m_1 + m_2 i, \bar{m}_1 + \bar{m}_2 i)$ for $m_1 \in \mathrm{Herm}(n,\mathbb{C}\{i\}), m_2 \in \mathrm{Herm}^+(n,\mathbb{C}\{i\})$, the second component is completely determined by the first one. It is easy to see, because im_2 is skew-Hermitian and $m_1 + m_2 i$ corresponds to the decomposition of an element from $\mathrm{Mat}(n,\mathbb{C}\{i\})$ in Hermitian and skew-Hermitian part. Therefore, m_1 and m_2 are well-defined by $m_1 + m_2 i$. Therefore, we can identify

$$\mathfrak{U}(\mathrm{U}(n,n)) \cong \{m_1 + m_2 i \mid m_1 \in \mathrm{Herm}(n,\mathbb{C}\{i\}), m_2 \in \mathrm{Herm}^+(n,\mathbb{C}\{i\})\}.$$

Example 6. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n,\mathbb{H}\{i,j,k\})$ with the anti-involution σ_1, then

$$A = \mathrm{Mat}(n,\mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\},$$
$$\mathrm{Sp}_2(A_{\mathbb{R}},\sigma_1) = \mathrm{SO}^*(4n),$$
$$\mathrm{Sp}_2(A,\sigma_1 \otimes \mathrm{Id}) = \mathrm{O}(4n,\mathbb{C}).$$

In the Section A.2.2, we studied the following $\mathbb{C}\{I\}$-$\mathbb{C}\{i\}$-algebras isomorphism:

$$\begin{array}{rccc} \psi\colon & \mathrm{Mat}(n,\mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\} & \to & \mathrm{Mat}(2n,\mathbb{C}\{i\}) \\ & (q_1 + q_2 j) + (p_1 + p_2 j)I & \mapsto & \begin{pmatrix} q_1 + p_1 i & q_2 + p_2 i \\ -\bar{q}_2 - \bar{p}_2 i & \bar{q}_1 + \bar{p}_1 i \end{pmatrix}. \end{array}$$

where $q_1, q_2, p_1, p_2 \in \mathrm{Mat}(n,\mathbb{C}\{i\})$.

We remind, $\psi(\mathrm{Id}_n \otimes I) = \mathrm{Id}_{2n} i$ and

$$\psi(A_{\mathbb{R}}) = \psi(\mathrm{Mat}(n,\mathbb{H}\{i,j,k\})) = \left\{ \begin{pmatrix} q_1 & q_2 \\ -\bar{q}_2 & \bar{q}_1 \end{pmatrix} \mid q_1, q_2 \in \mathrm{Mat}(n,\mathbb{C}) \right\}.$$

Under ψ, the anti-involution $\sigma_1 \otimes \mathrm{Id}$ on $\mathrm{Mat}(n,\mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\}$ indices the following anti-involution

$$\sigma' := \psi \circ (\sigma_1 \otimes \mathrm{Id}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n,\mathbb{C}\{i\}) = \mathrm{Mat}_2(\mathrm{Mat}(n,\mathbb{C}\{i\}))$:

$$\sigma'(m) = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

for $m \in \mathrm{Mat}(2n, \mathbb{C}\{i\})$. So we have:

$$\psi(A^{\sigma_1 \otimes \mathrm{Id}}) = \left\{ m \in \mathrm{Mat}(2n, \mathbb{C}\{i\}) \mid m = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \right\} =$$

$$= \mathfrak{sp}(2n, \mathbb{C}\{i\}).$$

The anti-involution $\sigma_1 \otimes \bar{\sigma}$ on $\mathrm{Mat}(n, \mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\}$ indices the following anti-involution

$$\tilde{\sigma} = \psi \circ (\sigma_1 \otimes \bar{\sigma}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$:

$$\tilde{\sigma}(m) = \bar{m}^T.$$

So, as expected, $(\mathrm{Mat}(2n, \mathbb{C}\{i\}), \tilde{\sigma})$ is a Hermitian algebra and

$$\psi(A_+^{\sigma_1 \otimes \bar{\sigma}}) = \mathrm{Herm}^+(2n).$$

Since $\psi(1 \otimes I) = \mathrm{Id}_{2n} i$, we have to do quaternionification with respect to $\mathrm{Id}\, i$. So the symmetric space is:

$$\mathfrak{U}(\mathrm{O}(4n, \mathbb{C})) = \{M_1 + M_2 J \mid M_1 \in \mathfrak{sp}(2n, \mathbb{C}),\ M_2 \in \mathrm{Herm}^+(2n)\} \subset$$

$$\subset \mathbb{H}[\mathrm{Mat}(2n, \mathbb{C}\{i\}), \psi(\mathrm{Mat}(n, \mathbb{H}\{i,j,k\})), \mathrm{Id}_{2n} i, J].$$

Since $A^{sym}_{\mathbb{R}} = A^{\bar{\sigma} \otimes \mathrm{Id}} \cap A^{\bar{\sigma} \otimes \bar{\sigma}}$, the real locus of this space is the symmetric space of $\mathrm{SO}^*(4n)$:

$$\mathfrak{U}(\mathrm{SO}^*(4n)) \cong$$

$$= \{M_1 + M_2 J \mid M_1 \in \mathfrak{sp}(2n, \mathbb{C}) \cap \mathrm{Herm}(2n),\ M_2 \in \mathfrak{sp}(2n, \mathbb{C}) \cap \mathrm{Herm}^+(2n)\} \subset$$

$$\subset \mathfrak{U}(\mathrm{O}(4n, \mathbb{C})).$$

After identification J and $\mathrm{Id}_{2n} i$, we obtain it as a subset of $\mathrm{Mat}(2n, \mathbb{C}\{i\})$:

$$\mathfrak{U}(\mathrm{SO}^*(4n)) =$$

$$= \{M_1 + M_2 i \mid M_1 \in \mathfrak{sp}(2n, \mathbb{C}) \cap \mathrm{Herm}(2n),\ M_2 \in \mathfrak{sp}(2n, \mathbb{C}) \cap \mathrm{Herm}^+(2n)\} \subset$$

$$\subset \mathrm{Mat}(2n, \mathbb{C}\{i\}).$$

2.5.2 Precompact model

In this section, we construct precompact models for classical Hermitian Lie groups of tube type.

Example 7. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n,\mathbb{R})$ with the anti-involution σ, then

$$\begin{gathered} A = \mathrm{Mat}(n,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Mat}(n,\mathbb{C}) \\ \mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{C}),\sigma) = \mathrm{Sp}(2n,\mathbb{C}), \\ \mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{R}),\sigma) = \mathrm{Sp}(2n,\mathbb{R}), \\ A_{\mathbb{H}} = \mathrm{Mat}(n,\mathbb{H}), \\ \sigma_0(M_1 + M_2 j) = \sigma(M_1) + \bar{\sigma}(M_2) j = M_1^T + \bar{M}_2^T j. \\ \sigma_1(M_1 + M_2 j) = \bar{\sigma}(M_1) - \sigma(M_2) j = \bar{M}_1^T - M_2^T j. \end{gathered}$$

where $M_1, M_2 \in \mathrm{Mat}(n,\mathbb{C})$. Then

$$A_{\mathbb{H}}^{\sigma_0} = \{M_1 + M_2 j \in \mathrm{Mat}(n,\mathbb{H}) \mid M_1 \in \mathrm{Sym}(n,\mathbb{C}),\ M_2 \in \mathrm{Herm}(n,\mathbb{C})\}.$$

So we have the following precompact model for the symmetric space for $\mathrm{Sp}(2n,\mathbb{C})$:

$$\begin{gathered} \mathfrak{B}(\mathrm{Sp}(2n,\mathbb{C})) = \\ = \{M_1 + M_2 j \in A_{\mathbb{H}}^{\sigma_0} \mid \mathrm{Id}_n - (\bar{M}_1 - \bar{M}_2 j)(M_1 + M_2 j) \in \mathrm{Herm}^+(n,\mathbb{H})\} \end{gathered}$$

The symmetric space for $\mathrm{Sp}(2n,\mathbb{R})$ can be seen as the intersection of $\mathfrak{B}(\mathrm{Sp}(2n,\mathbb{C}))$ with $\mathrm{Mat}(n,\mathbb{C}\{j\})$:

$$\begin{gathered} \mathfrak{B}(\mathrm{Sp}(2n,\mathbb{R})) = \\ = \{M_1 + M_2 j \in \mathrm{Sym}(n,\mathbb{C}\{j\}) \mid \mathrm{Id}_n - (M_1 - M_2 j)(M_1 + M_2 j) \in \mathrm{Herm}^+(n,\mathbb{C}\{j\})\} = \\ = \{M \in \mathrm{Sym}(n,\mathbb{C}\{j\}) \mid \mathrm{Id}_n - \bar{M}M \in \mathrm{Herm}^+(n,\mathbb{C}\{j\})\} \subset \mathfrak{B}(\mathrm{Sp}(2n,\mathbb{C})). \end{gathered}$$

Example 8. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n,\mathbb{C}\{I\})$ with the anti-involution $\bar{\sigma}$. Then

$$\begin{gathered} A = \mathrm{Mat}(n,\mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\} = \mathrm{Mat}(n,\mathbb{C}\{I\}) \oplus \mathrm{Mat}(n,\mathbb{C}\{I\}) i, \\ \mathrm{Sp}_2(A, \bar{\sigma} \otimes \mathrm{Id}) = \mathrm{GL}(2n,\mathbb{C}), \\ \mathrm{Sp}_2(A_{\mathbb{R}}, \bar{\sigma}) = \mathrm{U}(n,n), \\ A_{\mathbb{H}} = \mathrm{Mat}(n,\mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{H}\{i,j,k\}. \end{gathered}$$

We use the map ψ from the Section A.2.2 to identify $A_{\mathbb{H}}$ with $\mathrm{Mat}(2n,\mathbb{C})$.

As we have seen, the anti-involution $\bar{\sigma} \otimes \sigma_0$ on $\mathrm{Mat}(n,\mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{H}\{i,j,k\}$ induces the following anti-involution

$$\psi \circ (\bar{\sigma} \otimes \sigma_0) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n,\mathbb{C})$: $m \mapsto \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \bar{m}^T \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$. Therefore,

$$\psi(A_{\mathbb{H}}^{\bar{\sigma}\otimes\sigma_0}) = \left\{ m \in \mathrm{Mat}(2n,\mathbb{C}\{i\}) \mid m = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \bar{m}^T \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \right\}.$$

Similarly, the anti-involution $\bar\sigma\otimes\sigma_1$ on $\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_\mathbb{R}\mathbb{H}\{i,j,k\}$ induces the following anti-involution

$$\psi\circ(\bar\sigma\otimes\sigma_1)\circ\psi^{-1}$$

on $\mathrm{Mat}(2n,\mathbb{C})$: $M\mapsto \bar M^T$ and so $\psi(A_\mathbb{H}^{\bar\sigma\otimes\sigma_1})=\mathrm{Herm}(2n,\mathbb{C})$. So we obtain the following precompact model for the symmetric space of $\mathrm{GL}(2n,\mathbb{C})$:

$$\mathfrak{B}(\mathrm{GL}(2n,\mathbb{C}))=\{M\in\psi(A_\mathbb{H}^{\bar\sigma\otimes\sigma_0})\mid \mathrm{Id}_{2n}-\bar M^TM\in\mathrm{Herm}^+(2n,\mathbb{C})\}.$$

To see the precompact for $\mathrm{U}(n,n)$ as a subspace of $\mathfrak{B}(\mathrm{GL}(2n,\mathbb{C}))$, we have to intersect of with $\psi(\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_\mathbb{R}\mathbb{C}\{j\})$. We remind from the Section A.3.2

$$\psi(\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_\mathbb{R}\mathbb{C}\{j\})=$$

$$=\left\{m\in\mathrm{Mat}(2n,\mathbb{C}\{i\})\mid m=-\begin{pmatrix}0&\mathrm{Id}\\-\mathrm{Id}&0\end{pmatrix}m\begin{pmatrix}0&\mathrm{Id}\\-\mathrm{Id}&0\end{pmatrix}\right\}.$$

Since

$$\psi(\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_\mathbb{R}\mathbb{C}\{j\})\cap\psi(A_\mathbb{H}^{\bar\sigma\otimes\sigma_0})=\left\{\begin{pmatrix}a&b\\-b&a\end{pmatrix}\mid a,b\in\mathrm{Herm}(n,\mathbb{C})\right\},$$

we obtain:

$$\mathfrak{B}(\mathrm{U}(n,n))\cong\left\{\begin{pmatrix}a&b\\-b&a\end{pmatrix}\mid\begin{pmatrix}\mathrm{Id}_n-a^2-b^2&ba-ab\\ab-ba&\mathrm{Id}_n-a^2-b^2\end{pmatrix}\in\mathrm{Herm}^+(2n,\mathbb{C})\right\}\subset$$

$$\subset\mathfrak{B}(\mathrm{GL}(2n,\mathbb{C})).$$

Under the map χ from the Section A.2.1, A can be identified with $\mathrm{Mat}(n,\mathbb{C})\times\mathrm{Mat}(n,\mathbb{C})$, so we obtain the following precompact model for $\mathrm{U}(n,n)$:

$$\mathfrak{B}(\mathrm{U}(n,n))=$$

$$=\{(M,M^T)\mid M\in\mathrm{Mat}(n,\mathbb{C}),\ \mathrm{Id}_n-\bar M^TM\in\mathrm{Herm}^+(n,\mathbb{C}),\mathrm{Id}_n-\bar MM^T\in\mathrm{Herm}^+(n,\mathbb{C})\}.$$

The second component if the pair (M,M^T) is determined by the first one. Moreover, if $\mathrm{Id}_n-\bar M^TM\in\mathrm{Herm}^+(n,\mathbb{C})$ then $\mathrm{Id}_n-\bar MM^T\in\mathrm{Herm}^+(n,\mathbb{C})$. Therefore, we can identify:

$$\mathfrak{B}(\mathrm{U}(n,n))=\{M\in\mathrm{Mat}(n,\mathbb{C})\mid\mathrm{Id}_n-\bar M^TM\in\mathrm{Herm}^+(n,\mathbb{C})\}.$$

Remark 2.5.1. The description for the precompact model of the symmetric space of $\mathrm{U}(n,n)$ seen as $\mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{C}),\bar\sigma)$ agree with the description for the projective model of the symmetric space of $\mathrm{U}(n,n)$ seen as $\mathrm{O}(h_{st})$ for h_{st} the standard indefinite form (see Section 2.3.2).

Example 9. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{H}\{I,J,K\})$. Then

$$A = \mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\}$$

where i is a central element of A such that $i^2 = -1$. Further,

$$A_{\mathbb{H}} = \mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_{\mathbb{R}} \mathbb{H}\{i,j,k\}.$$

We use the map ϕ from the Section A.2.3 to identify $A_{\mathbb{H}}$ with $\mathrm{Mat}(4n, \mathbb{R})$.

As we have seen, the anti-involution $\sigma_1 \otimes \sigma_0$ corresponds under ϕ to the following anti-involution on $\mathrm{Mat}(4n, \mathbb{R})$: $M \mapsto -\Xi M^T \Xi$ where

$$\Xi := \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix}.$$

The anti-involution $\sigma_1 \otimes \sigma_1$ corresponds under ϕ to the transposition on $\mathrm{Mat}(4n, \mathbb{R})$. So we obtain the following precompact model of the symmetric space of $\mathrm{O}(4n, \mathbb{C})$:

$$\mathfrak{B}(\mathrm{O}(4n, \mathbb{C})) = \{M \in \phi(A_{\mathbb{H}}^{\sigma_1 \otimes \sigma_0}) \mid 1 - M^T M \in \mathrm{Sym}^+(4n, \mathbb{R})\}$$

where

$$\phi(A_{\mathbb{H}}^{\sigma_1 \otimes \sigma_0}) = \{M \in \mathrm{Mat}(4n, \mathbb{R}) \mid M = -\Xi M^T \Xi\} \cong \mathfrak{sp}(4n, \mathbb{R}).$$

To see the precompact model $\mathfrak{B}(\mathrm{SO}^*(4n))$ for the symmetric space of $\mathrm{SO}^*(4n)$ as a subspace of $\mathfrak{B}(\mathrm{O}(4n, \mathbb{C}))$, we have to intersect $\mathfrak{B}(\mathrm{O}(4n, \mathbb{C}))$ with $\phi(\mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_{\mathbb{R}} \mathbb{C}\{j\})$. We remind from the Section A.3.2:

$$\phi(\mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_{\mathbb{R}} \mathbb{C}\{j\}) = \{m \in \mathrm{Mat}(4n, \mathbb{R}) \mid M = -\phi(\mathrm{Id}_n \otimes j) M \phi(\mathrm{Id}_n \otimes j)\}.$$

Therefore, we obtain:

$$\mathfrak{B}(\mathrm{SO}^*(4n)) \cong \{M \in \mathfrak{B}(\mathrm{O}(4n, \mathbb{C})) \mid M = -\phi(\mathrm{Id}_n \otimes j) M \phi(\mathrm{Id}_n \otimes j)\}.$$

Under the map ψ from the Section A.2.2, we can identify A with $\mathrm{Mat}(2n, \mathbb{C})$. The anti-involution $\sigma_1 \otimes \mathrm{Id}$ corresponds to the following anti-involution on $\mathrm{Mat}(2n, \mathbb{C})$:

$$m \mapsto -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Therefore, $\psi(A^{\sigma_1 \otimes \mathrm{Id}}) = \mathfrak{sp}(2n, \mathbb{C})$.

The anti-involution $\sigma_1 \otimes \bar{\sigma}$ corresponds to the following anti-involution on $\mathrm{Mat}(2n, \mathbb{C})$: $M \mapsto \bar{M}^T$. Therefore, we obtain the precompact model for $\mathrm{SO}^*(4n)$:

$$\mathfrak{B}(\mathrm{SO}^*(4n)) = \{M \in \mathfrak{sp}(2n, \mathbb{C}) \mid 1 - \bar{M}^T M \in \mathrm{Herm}^+(2n, \mathbb{C})\}.$$

2.5.3 Projective model

In this section, we construct projective models for classical Hermitian Lie groups of tube type. We will see that projective models can be seen in two equivalent ways: in terms of matrix algebras and in terms of subspaces of some modules or vector spaces. Projective models in terms of matrix algebras are denoted by $\mathfrak{P}$, projective models in terms of modules/vector spaces are denoted by $\mathfrak{P}'$.

We will use the following notation: Let R be some division ring, V be a right R-module of dimension $2n$ for some $n \in \mathbb{N}$, b be a (bilinear or sesquilinear) form on V. We denote by $\mathrm{Gr}(k, V)$ the space of all k-dimensional R-submodules of V. We denote by $\mathrm{Lag}(V, b)$ the space of all n-dimensional b-isotropic R-submodules of V, i.e.

$$\mathrm{Lag}(V, b) := \{l \in \mathrm{Gr}(n, V) \mid \forall v \in l,\ b(v, v) = 0\}.$$

The elements of $\mathrm{Lag}(V, b)$ are called b-Lagrangians of V.

Example 10. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{R})$, then

$$A = \mathrm{Mat}(n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Mat}(n, \mathbb{C}),$$

$$\begin{aligned}
\mathrm{Sp}_2(\mathrm{Mat}(n, \mathbb{C}), \sigma) &= \mathrm{Sp}(2n, \mathbb{C}),\\
\mathrm{Sp}_2(\mathrm{Mat}(n, \mathbb{R}), \sigma) &= \mathrm{Sp}(2n, \mathbb{R}),\\
A_{\mathbb{H}} &= \mathrm{Mat}(n, \mathbb{H}),\\
\sigma_0(M_1 + M_2 j) = \sigma(M_1) + \bar{\sigma}(M_2) j &= M_1^T + \bar{M}_2^T j,\\
\sigma_1(M_1 + M_2 j) = \bar{\sigma}(M_1) - \sigma(M_2) j &= \bar{M}_1^T - M_2^T j.
\end{aligned}$$

where $M_1, M_2 \in \mathrm{Mat}(n, \mathbb{C})$. Further, for $x, y \in A^2$

$$\omega(x, y) = \sigma_0(x)^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y,$$

$$h(x, y) = \sigma_1(x)^T \begin{pmatrix} 0 & \mathrm{Id}_n j \\ -\mathrm{Id}_n j & 0 \end{pmatrix} y.$$

We obtain the projective model for $\mathrm{Sp}(2n, \mathbb{C})$:

$$\mathfrak{P}(\mathrm{Sp}(2n, \mathbb{C})) = \{x A_{\mathbb{H}} \mid x \in A_{\mathbb{H}}^2,\ \omega(x, x) = 0,\ h(x, x) \in \mathrm{Herm}^+(n, \mathbb{H})\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{Sp}(2n, \mathbb{C})) \cong \{x A_{\mathbb{H}} \mid x \in A_{\mathbb{H}}^2,\ \omega(x, x) = h(x, x) = 0\} \cong$$

$$\cong \{x A \mid x \in A^2,\ \omega(x, x) = 0\}.$$

The projective model for $\mathrm{Sp}(2n, \mathbb{R})$ can be seen as:

$$\mathfrak{P}(\mathrm{Sp}(2n, \mathbb{R})) = \{x A_{\mathbb{C}\{j\}} \mid x \in A_{\mathbb{C}\{j\}}^2,\ \omega(x, x) = 0,\ h(x, x) \in \mathrm{Herm}^+(n, \mathbb{C}\{j\})\}$$

where $\mathbb{C}\{j\} \subset \mathbb{H}$ and $A_{\mathbb{C}\{j\}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}\{j\} \cong A$. $\mathfrak{P}(\mathrm{Sp}(2n,\mathbb{R}))$ can be embedded into $\mathfrak{P}(\mathrm{Sp}(2n,\mathbb{C}))$ using the following injective map: for $x \in \mathbb{C}\{j\}$, $xA_{\mathbb{C}\{j\}} \mapsto xA_{\mathbb{H}}$. The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{Sp}(2n,\mathbb{R})) \cong \{xA_{\mathbb{C}\{j\}} \mid x \in A_{\mathbb{R}}^2,\ \omega(x,x) = h(x,x) = 0\} \cong$$

$$\cong \{xA_{\mathbb{R}} \mid x \in A_{\mathbb{R}}^2,\ \omega(x,x) = 0\}.$$

We can also construct the projective model in terms of Lagrangians of $\mathbb{H}^{2n}$. Consider $\mathbb{H}^{2n}$ as a right module over $\mathbb{H}$. We can identify a line $xA_{\mathbb{H}}$ for a regular $x \in A_{\mathbb{H}}^2$ with a n-dimensional submodule of $\mathbb{H}^{2n}$ in the following way:

$$L(xA) := \mathrm{Span}_{\mathbb{H}}(xe_1, \dots, xe_n) \subset \mathbb{H}^{2n}$$

where e_i is the i-th basis vector (considered as a column) of the standard basis of $\mathbb{H}^n$. In fact, the map L is well-defined (does not depend on the choice of a regular $x \in xA$) and, moreover, it is a bijection.

We define two forms on $\mathbb{H}^{2n}$: for $u, v \in \mathbb{H}^{2n}$,

$$\tilde{\omega}(u,v) := \sigma_0(u)^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} v,$$

$$\tilde{h}(u,v) := \sigma_1(u)^T \begin{pmatrix} 0 & j\,\mathrm{Id}_n \\ -j\,\mathrm{Id}_n & 0 \end{pmatrix} v.$$

If we take $x \in \mathrm{Is}(\omega)$, then $L(xA) \in \mathrm{Lag}(\mathbb{H}^{2n}, \tilde{\omega})$. Using the map L, we obtain the following projective model for $\mathrm{Sp}(2n,\mathbb{C}) \cong \mathrm{Sp}_2(A,\sigma)$:

$$\mathfrak{P}'(\mathrm{Sp}(2n,\mathbb{C})) = \{l \in \mathrm{Lag}(\mathbb{H}^{2n}, \tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) > 0\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{Sp}(2n,\mathbb{C})) \cong \{l \in \mathrm{Lag}(\mathbb{H}^{2n}, \tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) = 0\} \cong \mathrm{Lag}(\mathbb{C}^{2n}, \tilde{\omega}).$$

The projective model for the symmetric space of $\mathrm{Sp}(2n,\mathbb{R}) \cong \mathrm{Sp}_2(A_{\mathbb{R}}, \sigma_{\mathbb{R}})$ is:

$$\mathfrak{P}'(\mathrm{Sp}(2n,\mathbb{R})) = \{l \in \mathrm{Lag}(\mathbb{C}\{j\}^{2n}, \tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) > 0\}.$$

It can be embedded to the projective model for $\mathrm{Sp}(2n,\mathbb{C})$ using the map:

$$\begin{array}{ccc} \mathrm{Lag}(\mathbb{C}\{j\}^{2n}, \tilde{\omega}) & \to & \mathrm{Lag}(\mathbb{H}^{2n}, \tilde{\omega}) \\ l & \mapsto & \mathrm{Span}_{\mathbb{H}}(l) \end{array}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{Sp}(2n,\mathbb{R})) \cong \{l \in \mathrm{Lag}(\mathbb{C}\{j\}^{2n}, \tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) = 0\} \cong \mathrm{Lag}(\mathbb{R}^{2n}, \tilde{\omega}).$$

Example 11. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{C}\{I\})$ with the anti-involution $\bar{\sigma}$. Then

$$A = \mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\},$$

$$A_{\mathbb{H}} = \mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{H}\{i, j, k\}.$$

We use the map ψ from the Section A.2.2 to identify $A_{\mathbb{H}}$ with $\mathrm{Mat}(2n, \mathbb{C}) =: A'$.

As we already have seen, under ψ, the anti-involution $\bar{\sigma} \otimes \sigma_0$ on $\mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{H}\{i, j, k\}$ induces the following anti-involution

$$\sigma' := \psi \circ (\bar{\sigma} \otimes \sigma_0) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$: $M \mapsto \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \bar{M}^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$. Therefore,

$$\psi(A_{\mathbb{H}}^{\bar{\sigma}\otimes\sigma_0}) = \left\{ M \in \mathrm{Mat}(2n, \mathbb{C}) \mid \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \bar{M}^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \right\}.$$

Similarly, the anti-involution $\bar{\sigma} \otimes \sigma_1$ on $\mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{H}\{i, j, k\}$ induces the following anti-involution

$$\sigma'' := \psi \circ (\bar{\sigma} \otimes \sigma_1) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$: $M \mapsto \bar{M}^T$ and so $\psi(A_{\mathbb{H}}^{\bar{\sigma}\otimes\sigma_1}) = \mathrm{Herm}(2n, \mathbb{C})$.

Further, for $x, y \in (A')^2$

$$\omega(x, y) = \sigma'(x)^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} y =$$

$$= \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \bar{x}^T \begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & 0 & 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \end{pmatrix} y =$$

$$= \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \bar{x}^T \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix} y$$

$$h(x, y) = \sigma''(x)^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} i \\ -\mathrm{Id}_{2n} i & 0 \end{pmatrix} y = \bar{x}^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} i \\ -\mathrm{Id}_{2n} i & 0 \end{pmatrix} y$$

Note, $x \in \mathrm{Is}(\omega)$ if and only if $x \in \mathrm{Is}(\omega')$ where

$$\omega'(x, y) = \bar{x}^T \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix} y$$

We obtain the projective model for $\mathrm{GL}(2n,\mathbb{C})$:

$$\mathfrak{P}(\mathrm{GL}(2n,\mathbb{C})) = \{xA' \mid x \in (A')^2,\ \omega'(x,x) = 0,\ h(x,x) \in \mathrm{Herm}^+(2n,\mathbb{C})\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{GL}(2n,\mathbb{C})) \cong \{xA' \mid x \in (A')^2,\ \omega'(x,x) = h(x,x) = 0\}.$$

The projective model for $\mathrm{U}(n,n)$ can be seen as a subspace of $\mathfrak{P}(\mathrm{GL}(2n,\mathbb{C}))$ in the following way. As we have seen in the Section A.3.1,

$$\psi(A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}\{j\}) = \left\{ \begin{pmatrix} q & p \\ -p & q \end{pmatrix} \mid p,q \in \mathrm{Mat}(n,\mathbb{C}\{i\}) \right\} =$$

$$= \left\{ m \in \mathrm{Mat}(2n,\mathbb{C}\{i\}) \mid m = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} m \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \right\}.$$

Therefore, if we define

$$\delta(x) := -\begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \end{pmatrix} x \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

for $x \in (A')^2_{\mathbb{H}}$. We obtain

$$\mathfrak{P}(\mathrm{U}(n,n)) \cong \{xA' \in \mathfrak{P}(\mathrm{GL}(2n,\mathbb{C})) \mid x \in (A')^2_{\mathbb{H}},\ \delta(x) = x\}$$

We can also see the projective model for $\mathrm{U}(n,n)$ in another way. We consider the isomorphism χ from the Section A.2.1 identifying $\mathrm{Mat}(n,\mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\}$ with $\mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\}) =: A''$. Then the induced by $\bar{\sigma} \otimes \mathrm{Id}$ anti-involution

$$\chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (m_2^T, m_1^T).$$

The induced by $\bar{\sigma} \otimes \bar{\sigma}$ involution

$$\chi \circ (\bar{\sigma} \otimes \bar{\sigma}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n,\mathbb{C}\{i\}) \times \mathrm{Mat}(n,\mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (\bar{m}_1^T, \bar{m}_2^T).$$

Note,

$$(A'')^2 = \mathrm{Mat}(n,\mathbb{C}\{i\})^2 \times \mathrm{Mat}(n,\mathbb{C}\{i\})^2.$$

We take $x_1, x_2, y_1, y_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})^2$, then we can define

$$\omega((x_1,x_2),(y_1,y_2)) :=$$

$$= \chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}(x_1,x_2) \begin{pmatrix} 0 & (\mathrm{Id}_n, \mathrm{Id}_n) \\ -(\mathrm{Id}_n, \mathrm{Id}_n) & 0 \end{pmatrix} (y_1, y_2) =$$

$$= \left(x_2^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1, x_1^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_2 \right),$$

$$h((x_1,x_2),(y_1,y_2)) := \left(\bar{x}_1^T \begin{pmatrix} 0 & \mathrm{Id}_n i \\ -\mathrm{Id}_n i & 0 \end{pmatrix} y_1, \bar{x}_2^T \begin{pmatrix} 0 & \mathrm{Id}_n i \\ -\mathrm{Id}_n i & 0 \end{pmatrix} y_2 \right).$$

We obtain the projective model for $\mathrm{U}(n,n)$:

$$\mathfrak{P}(\mathrm{U}(n,n)) = \left\{ (x_1,x_2)A'' \;\middle|\; \begin{array}{l} x_1, x_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})^2,\ \hat{\omega}(x_1,x_2) = 0, \\ \hat{h}(x_1,x_1), \hat{h}(x_2,x_2) \in \mathrm{Herm}^+(n,\mathbb{C}) \end{array} \right\}$$

where $\hat{\omega}(x_1,x_2) := x_1^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_2$, $\hat{h}(x,y) := \bar{x}_1^T \begin{pmatrix} 0 & \mathrm{Id}_n i \\ -\mathrm{Id}_n i & 0 \end{pmatrix} y_1$. Since $\hat{\omega}$ is non-degenerate, the line $x_2 \mathrm{Mat}(n, \mathbb{C}\{i\}$ is uniquely defined by x_1.

Let us check that for the pair (x_1,x_2) such that $\hat{\omega}(x_1,x_2) = 0$, $\hat{h}(x_1,x_1) \in \mathrm{Herm}^+(n,\mathbb{C})$, we have always $\hat{h}(x_2,x_2) \in \mathrm{Herm}^+(n,\mathbb{C})$. As we have seen in the Section 2.3.7, we can always choose $x_1 = (m_1, 1)^T$, $x_2 = (m_2, 1)^T$. Then

$$\hat{\omega}(x_1,x_2) = m_1^T - m_2 = 0,$$

$$\hat{h}(x_1,x_1) = i(\bar{m}_1^T - m_1) \in \mathrm{Herm}^+(n,\mathbb{C}).$$

These two conditions imply

$$\hat{h}(x_2,x_2) = i(\bar{m}_2^T - m_2) = i(\bar{m}_1 - m_1^T) = i(\bar{m}_1^T - m_1)^T \in \mathrm{Herm}^+(n,\mathbb{C}).$$

Therefore, we can write the following identification:

$$\mathfrak{P}(\mathrm{U}(n,n)) \cong \left\{ x \,\mathrm{Mat}(n, \mathbb{C}\{i\}) \mid \hat{h}(x,x) \in \mathrm{Herm}^+(n,\mathbb{C}) \right\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{U}(n,n)) \cong \left\{ x \,\mathrm{Mat}(n, \mathbb{C}\{i\}) \mid \hat{h}(x,x) = 0 \right\}.$$

To construct the projective model in terms of Lagrangians, similarly to the Example 10, we can identify the space of A'-lines of $(A')^2$ with the space $\mathrm{Gr}(2n, \mathbb{C}^{4n})$ of $2n$-dimensional subspaces of $\mathbb{C}^{4n}$ by the rule:

$$L(xA') := \mathrm{Span}_{\mathbb{H}}(xe_1, \dots, xe_{2n})$$

where e_i is the i-th basis vector (considered as a column) of the standard basis of $\mathbb{C}^{2n}$.

We define two forms on $\mathbb{C}^{4n}$: for $u, v \in \mathbb{C}^{4n}$,

$$\tilde{\omega}(u,v) := \bar{u}^T \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix} v,$$

$$\tilde{h}(u,v) := \bar{u}^T \begin{pmatrix} 0 & i\,\mathrm{Id}_{2n} \\ -i\,\mathrm{Id}_{2n} & 0 \end{pmatrix} v.$$

The projective model for the symmetric space of $\mathrm{GL}(4n,\mathbb{C}) \cong \mathrm{Sp}_2(A,\sigma)$ can be seen as the following space:

$$\mathfrak{P}'(\mathrm{GL}(4n,\mathbb{C})) = \{l \in \mathrm{Lag}(\mathbb{C}^{4n},\tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) > 0\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{GL}(4n,\mathbb{C})) = \{l \in \mathrm{Lag}(\mathbb{C}^{4n},\tilde{\omega}) \mid \forall v \in l \setminus \{0\},\ \tilde{h}(v,v) = 0\}.$$

We can see the the projective model for the symmetric space of $\mathrm{U}(n,n) \cong \mathrm{Sp}_2(A_\mathbb{R},\sigma_\mathbb{R})$ as a subspace of $\mathfrak{P}(\mathrm{GL}(4n,\mathbb{C}))$:

$$\mathfrak{P}'(\mathrm{U}(n,n)) \cong \{l \in \mathfrak{P}(\mathrm{GL}(4n,\mathbb{C})) \mid \delta'(l) = l\}$$

where

$$\begin{array}{cccc} \delta'\colon & \mathbb{C}^{4n} & \to & \mathbb{C}^{4n} \\ & v & \mapsto & \begin{pmatrix} 0 & \mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \end{pmatrix} v. \end{array}$$

We can also see another projective model for the symmetric space of $\mathrm{U}(n,n) \cong \mathrm{Sp}_2(A_\mathbb{R},\bar{\sigma})$ if we identify again $A = \mathrm{Mat}(n,\mathbb{C}) \otimes_\mathbb{R} \mathbb{C}$ with $\mathrm{Mat}(n,\mathbb{C}) \times \mathrm{Mat}(n,\mathbb{C}) =: A'$ by the map χ form the Section A.2.1. As before, we can identify every line $xA' \subset (A')^2$ with pair of n-dimensional subspaces of $\mathbb{C}^{2n}$. We define two forms on $\mathbb{C}^{2n}$: for $u, v \in \mathbb{C}^{2n}$

$$\tilde{\omega}(u,v) := u^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} v,$$

$$\tilde{h}(u,v) := \bar{u}^T \begin{pmatrix} 0 & \mathrm{Id}_n\, i \\ -\mathrm{Id}_n\, i & 0 \end{pmatrix} v.$$

The pair (l_1, l_2) of n-dimensional subspaces of $\mathbb{C}^{2n}$ is called ω-orthogonal if for all $v \in l_1$, $u \in l_2$, $\tilde{\omega}(v,u) = 0$. So we can see the projective model of the symmetric space for $\mathrm{U}(n,n)$:

$$\mathfrak{P}'(\mathrm{U}(n,n)) = \{(l_1,l_2)\ \tilde{\omega}\text{-orthogonal pair} \mid \forall u \in l_1 \cup l_2 \setminus \{0\},\ \tilde{h}(u,u) > 0\}.$$

Since ω is non-degenerate, the space l_2 is completely determined by l_1. And as we have seen for $\mathfrak{P}(\mathrm{U}(n,n))$, if for all $u \in l_1 \setminus \{0\}$, $\tilde{h}(u,u) > 0$, then for all $u \in l_2 \setminus \{0\}$, $\tilde{h}(u,u) > 0$. Therefore, we can identify

$$\mathfrak{P}'(\mathrm{U}(n,n)) \cong \{l \in \mathrm{Gr}(n, \mathbb{C}^{2n}) \mid \forall u \in l \setminus \{0\},\ \tilde{h}(u,u) > 0\}.$$

The Shilov boundary corresponds in this model to the space:

$$\check{S}(\mathrm{U}(n,n)) \cong \{l \in \mathrm{Gr}(n, \mathbb{C}^{2n}) \mid \forall u \in l \setminus \{0\},\ \tilde{h}(u,u) = 0\}.$$

Remark 2.5.2. The description for the projective model of the symmetric space of $\mathrm{U}(n,n)$ seen as $\mathrm{Sp}_2(\mathrm{Mat}(n,\mathbb{C}), \bar{\sigma})$ agree with the description for the projective model of the symmetric space of $\mathrm{U}(n,n)$ seen as $\mathrm{O}(h_{st})$ for h_{st} the standard indefinite form (see Section 2.3.2).

Example 12. Consider the real algebra $A_\mathbb{R} := \mathrm{Mat}(n, \mathbb{H}\{I,J,K\})$ with the anti-involution σ_1. Then

$$A = \mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_\mathbb{R} \mathbb{C}\{i\},$$
$$A_\mathbb{H} = \mathrm{Mat}(n, \mathbb{H}\{I,J,K\}) \otimes_\mathbb{R} \mathbb{H}\{i,j,k\}.$$

As we have seen in the Section A.2.3, the map ϕ defines an $\mathbb{R}$-algebra isomorphism:

$$\phi\colon A_\mathbb{H} \to \mathrm{Mat}(4n, \mathbb{R}) =: A'.$$

Moreover, the anti-involution $\sigma_1 \otimes \sigma_0$ corresponds under ψ to the following anti-involution σ_0' on $\mathrm{Mat}(4n,\mathbb{R})$: $\sigma_0'(M) = -\Xi M^T \Xi$ where

$$\Xi := \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix}.$$

The anti-involution $\sigma_1 \otimes \sigma_1$ corresponds under ϕ to the transposition on $\mathrm{Mat}(4n,\mathbb{R})$.

Further, for $x, y \in (A')^2$

$$\omega(x,y) = \sigma_0'(x)^T \begin{pmatrix} 0 & \mathrm{Id}_{4n} \\ -\mathrm{Id}_{4n} & 0 \end{pmatrix} y =$$

$$= -\Xi x^T \begin{pmatrix} \Xi & 0 \\ 0 & \Xi \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{Id}_{2n} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_{2n} & 0 & 0 \end{pmatrix} y =$$

$$= -\Xi x^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} y$$

$$h(x,y) = x^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} y.$$

By definition of h, we use that

$$\phi(\mathrm{Id}_n \otimes j) = \begin{pmatrix} 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & 0 & 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \end{pmatrix}.$$

Note, $x \in \mathrm{Is}(\omega)$ if and only if $x \in \mathrm{Is}(\omega')$ where

$$\omega'(x,y) = x^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} y.$$

So we obtain the projective model for the symmetric space of $\mathrm{O}(4n,\mathbb{C})$:

$$\mathfrak{P}(\mathrm{O}(4n,\mathbb{C})) = \{xA' \mid x \in (A')^2,\ \omega'(x,x) = 0,\ h(x,x) \in \mathrm{Sym}^+(4n,\mathbb{R})\}.$$

We can see the Shilov boundary in this model as the space:

$$\check{S}(\mathrm{O}(4n,\mathbb{C})) \cong \{xA' \mid x \in (A')^2,\ \omega'(x,x) = h(x,x) = 0\}.$$

The projective model for $\mathrm{SO}^*(4n)$ can be seen as a subspace of $\mathfrak{P}(\mathrm{O}(4n,\mathbb{C}))$ in the following way. As we have seen in the Section A.3.2,

$$\psi(A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}\{j\}) = \{m \in \mathrm{Mat}(4n,\mathbb{R}) \mid m = -\phi(1\otimes j)m\phi(1\otimes j)\}.$$

Therefore,

$$\mathfrak{P}(\mathrm{SO}^*(4n)) \cong \{xA' \in \mathfrak{P}(\mathrm{GL}(2n,\mathbb{C})) \mid x = -\phi(1\otimes j)x\phi(1\otimes j)\}.$$

To see another projective model for the symmetric space of $\mathrm{SO}^*(4n)$, we remind that $A = \mathrm{Mat}(n,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ is to $\mathrm{Mat}(2n,\mathbb{C}) =: A''$ isomorphic under the map ψ from the Section A.2.2. The anti-involution $\sigma_1 \otimes \mathrm{Id}$ corresponds under this map to the anti-involution σ' on $\mathrm{Mat}(2n, \mathbb{C})$ given by:

$$\sigma'(m) = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

The anti-involution $\sigma_1 \otimes \bar{\sigma}$ corresponds under ψ to the complex conjugation composed with transposition on $\mathrm{Mat}(2n, \mathbb{C})$.

We define for $x, y \in (A'')^2$,

$$\omega(x,y) := x^T \begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} y =$$

$$= x^T \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix} y,$$

$$\tilde{h}(x,y) := \bar{x}^T \begin{pmatrix} 0 & \mathrm{Id}_{2n} i \\ -\mathrm{Id}_{2n} i & 0 \end{pmatrix} y.$$

Then the projective model of the symmetric space for $\mathrm{SO}^*(4n)$ can be seen as:

$$\mathfrak{P}(\mathrm{SO}^*(4n)) = \{xA'' \mid x \in (A'')^2, \tilde{h}(x,x) \in \mathrm{Herm}^+(2n, \mathbb{C})\}.$$

We can see the Shilov boundary in this model as the space:

$$\check{S}(\mathrm{SO}^*(4n)) \cong \{xA'' \mid x \in (A'')^2, \tilde{h}(x,x) = 0\}.$$

Now we construct the projective model in terms of Lagrangians. As before, we identify using the map L the space of A'-lines and the space $\mathrm{Gr}(4n, \mathbb{R}^{8n})$ of $4n$-dimensional subspaces of $\mathbb{R}^{8n}$:

$$L(xA') := \mathrm{Span}_{\mathbb{R}}(xe_1, \ldots, xe_{4n})$$

where e_i is the i-th standard basis vector of $\mathbb{R}^{4n}$. We define two forms on $\mathbb{R}^{8n}$: for $u, v \in \mathbb{R}^{8n}$,

$$\tilde{\omega}(u,v) := u^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} v,$$

$$\tilde{h}(u,v) := u^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_n & 0 & 0 & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_n & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} v.$$

The space of $\tilde{\omega}$-isotropic vectors of $\mathbb{R}^{8n}$ is denoted by $\mathrm{Is}(\tilde{\omega})$. Then the projective model of the symmetric space for $\mathrm{O}(4n,\mathbb{C})$ can be seen as:

$$\mathfrak{P}'(\mathrm{O}(4n,\mathbb{C})) = \{l \in \mathrm{Lag}(\mathbb{R}^{8n},\tilde{\omega}) \mid \forall x \in l \setminus \{0\}, \tilde{h}(x,x) > 0\}.$$

We can see the Shilov boundary in this model as the space:

$$\check{S}(\mathrm{O}(4n,\mathbb{C})) \cong \{l \in \mathrm{Lag}(\mathbb{R}^{8n},\tilde{\omega}) \mid \forall x \in l \setminus \{0\}, \tilde{h}(x,x) = 0\}.$$

The projective model for the symmetric space of $\mathrm{SO}^*(4n)$ can be seen as a subspace of $\mathfrak{P}(\mathrm{O}(4n,\mathbb{C}))$:

$$\mathfrak{P}'(\mathrm{SO}^*(4n)) = \{l \in \mathfrak{P}(\mathrm{O}(4n,\mathbb{C})) \mid \delta(l) = l\},$$

where

$$\begin{array}{cccc} \delta\colon & \mathbb{R}^{8n} & \to & \mathbb{R}^{8n} \\ & v & \mapsto & \begin{pmatrix} \phi(\mathrm{Id}_n \otimes j) & 0 \\ 0 & \phi(\mathrm{Id}_n \otimes j) \end{pmatrix} v. \end{array}$$

To see another projective model for the symmetric space of $\mathrm{SO}^*(4n)$, we remind that $A = \mathrm{Mat}(n,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ is to $\mathrm{Mat}(2n,\mathbb{C}) =: A''$ isomorphic under the map ψ from the Section A.2.2. The anti-involution $\sigma_1 \otimes \mathrm{Id}$ corresponds under this map to the anti-involution σ' on $\mathrm{Mat}(2n,\mathbb{C})$ given by:

$$\sigma'(m) = -\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

The anti-involution $\sigma_1 \otimes \bar{\sigma}$ corresponds under ψ to the complex conjugation composed with transposition on $\mathrm{Mat}(2n,\mathbb{C})$.

To construct the projective model in terms of Lagrangians, as before, we identify using the map L the space of A''-lines and the space $\mathrm{Gr}(2n,\mathbb{C}^{4n})$ of $2n$-dimensional subspaces of $\mathbb{C}^{4n}$:

$$L(xA'') := \mathrm{Span}_{\mathbb{R}}(xe_1,\dots,xe_{2n})$$

where e_i is the i-th standard basis vector of $\mathbb{C}^{2n}$. We define two forms on $\mathbb{R}^{8n}$: for $u,v \in \mathbb{R}^{8n}$,

$$\tilde{\omega}(u,v) := u^T \begin{pmatrix} \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} v =$$

$$= u^T \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix} v,$$

$$\tilde{h}(u,v) := \bar{u}^T \begin{pmatrix} 0 & \mathrm{Id}_n i \\ -\mathrm{Id}_n i & 0 \end{pmatrix} v.$$

Then the projective model of the symmetric space for $\mathrm{SO}^*(4n)$ can be seen as:

$$\mathfrak{P}'(\mathrm{SO}^*(4n)) = \{l \in \mathrm{Lag}(\mathbb{C}^{4n}, \tilde{\omega}) \mid \forall x \in l \setminus \{0\}, \tilde{h}(x,x) > 0\}.$$

We can see the Shilov boundary in this model as the space:

$$\check{S}(\mathrm{SO}^*(4n)) \cong \{l \in \mathrm{Lag}(\mathbb{C}^{4n}, \tilde{\omega}) \mid \forall x \in l \setminus \{0\}, \tilde{h}(x,x) = 0\}.$$

2.5.4 Quaternionic structure model

In this section, we construct quaternionic structure model models for classical Hermitian Lie groups of tube type.

Example 13. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{R})$, then

$$A = \mathrm{Mat}(n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Mat}(n, \mathbb{C})$$
$$\mathrm{Sp}_2(A, \sigma) = \mathrm{Sp}(2n, \mathbb{C}),$$
$$\mathrm{Sp}_2(A_{\mathbb{R}}, \sigma) = \mathrm{Sp}(2n, \mathbb{R}).$$

The quaternionic structure on A can be seen as a $2n \times 2n$-matrix J acting on A^2 as $J(x) = J\bar{x}$ for $x \in A^2$. Since $J(J(x)) = J\overline{J\bar{x}} = -x$, $J\bar{J} = -\mathrm{Id}_n$.

The corresponding $\bar{\sigma}$-sesquilinear form is then

$$h_J(x,y) = \omega(J(x), y) = \bar{x}^T J^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y.$$

So we obtain the quaternionic structure model for $\mathrm{Sp}(2n, \mathbb{C})$:

$$\mathfrak{C}(\mathrm{Sp}(2n, \mathbb{C})) := \left\{ J \in \mathrm{Mat}(2n, \mathbb{C}) \mid J^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(n, \mathbb{C}),\ J\bar{J} = -\mathrm{Id} \right\}.$$

The space of complex structures on $A^2_{\mathbb{R}}$ can be seen as a subspace of $\mathfrak{C}(\mathrm{Sp}(2n, \mathbb{C}))$ because every complex structure can be extended in the unique way to the quaternionic structure on A^2 in the following way: for a complex structure J we define:

$$J_{\mathbb{C}}(x + yi) := J(x) - J(y)i$$

where $x, y \in A_{\mathbb{R}}$. So we obtain the inclusion of the complex structure model for $\mathrm{Sp}(2n, \mathbb{R})$ into the quaternionic model for $\mathrm{Sp}(2n, \mathbb{C})$ as subspace of quaternionic structure fixing $A^2_{\mathbb{R}} \subset A^2$:

$$\mathfrak{C}(\mathrm{Sp}(2n, \mathbb{R})) := \left\{ J \in \mathrm{Mat}(2n, \mathbb{R}) \mid J^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in \mathrm{Sym}^+(n, \mathbb{R}),\ J^2 = -\mathrm{Id}_{2n} \right\} =$$
$$= \{J \in \mathfrak{C}(\mathrm{Sp}(2n, \mathbb{C})) \mid J \in \mathrm{Mat}(2n, \mathbb{R})\}$$

Example 14. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{C}\{I\})$. Then

$$A = \mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\},$$

$$\mathrm{Sp}_2(A, \bar{\sigma} \otimes \mathrm{Id}) \cong \mathrm{GL}(2n, \mathbb{C}),$$

$$\mathrm{Sp}_2(A_{\mathbb{R}}, \bar{\sigma}) \cong \mathrm{U}(n, n).$$

We use the map χ from the Section A.2.1, to identify A with $A' := \mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$. The involution $\mathrm{Id} \otimes \bar{\sigma}$ is mapped under χ to the involution

$$(m_1, m_2) \mapsto (\bar{m}_2, \bar{m}_1).$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$.

If we take a quaternionic structure J on A^2 then we define

$$J' := \chi \circ J \circ \chi^{-1}.$$

If we see J' as a pair (J_1, J_2) of $2n \times 2n$ complex matrices then $J_1 \bar{J}_2 = -\mathrm{Id}_{2n}$ because for $(m_1, m_2) \in (A')^2 \cong \mathrm{Mat}(n, \mathbb{C}\{i\})^2 \times \mathrm{Mat}(n, \mathbb{C}\{i\})^2$,

$$J'(m_1, m_2) = (J_1, J_2)(\bar{m}_2, \bar{m}_1) = (J_1 \bar{m}_2, J_2 \bar{m}_1),$$

$$-(m_1, m_2) = (J')^2(m_1, m_2) = (J_1 \overline{J_2 \bar{m}_1}, J_2 \overline{J_1 \bar{m}_2}).$$

The induced by $\bar{\sigma} \otimes \mathrm{Id}$ anti-involution

$$\chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (m_2^T, m_1^T).$$

We take the standard symplectic structure on $(A')^2$: for $x_1, x_2, y_1, y_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})$

$$\omega((x_1, x_2), (y_1, y_2)) = \chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}(x_1, x_2) \begin{pmatrix} 0 & (\mathrm{Id}_n, \mathrm{Id}_n) \\ -(\mathrm{Id}_n, \mathrm{Id}_n) & 0 \end{pmatrix} (y_1, y_2) =$$

$$= \left(x_2^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1, x_2^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1 \right).$$

For a quaternionic structure on $(A')^2$ seen as pair of matrices (J_1, J_2), we define

$$h_{(J_1, J_2)}((x_1, x_2), (y_1, y_2)) :=$$

$$= \left((J_1 \bar{x}_1)^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1, (J_2 \bar{x}_2)^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_2 \right) =$$

$$= \left(\bar{x}_1^T J_1^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_1, \bar{x}_2^T J_2^T \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} y_2 \right).$$

The quaternionic structure model for $\mathrm{GL}(2n,\mathbb{C})$ is then:

$$\mathfrak{C}(\mathrm{GL}(2n,\mathbb{C})) := \left\{ (J_1,J_2) \;\middle|\; \begin{array}{l} J_1, J_2 \in \mathrm{Mat}(2n,\mathbb{C}),\ J_1\bar{J}_2 = -\operatorname{Id}_{2n}, \\ J_1^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix},\ J_2^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(2n,\mathbb{C}) \end{array} \right\}.$$

Since $J_1\bar{J}_2 = -\operatorname{Id}_{2n}$, by given J_1 such that $J_1^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(2n,\mathbb{C})$, we can calculate $J_2 = -\bar{J}_1^{-1}$. Then

$$J_2^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} = -\bar{J}_1^{-T} \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \bar{J}_1^T \right)^{-1} \in \mathrm{Herm}^+(2n,\mathbb{C})$$

if and only if

$$\begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \bar{J}_1^T \in \mathrm{Herm}^+(2n,\mathbb{C})$$

if and only if

$$\bar{J}_1^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(2n,\mathbb{C}).$$

Therefore, we can identify

$$\mathfrak{C}(\mathrm{GL}(2n,\mathbb{C})) \cong \left\{ J \in \mathrm{Mat}(2n,\mathbb{C}) \mid J^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(2n,\mathbb{C}) \right\}.$$

In this presentation of the symmetric space, $\mathrm{GL}(2n,\mathbb{C})$ acts on it in the following way:

$$(g,J) \mapsto -g^{-1} J \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \bar{g}^{-T} \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix}$$

for $g \in \mathrm{GL}(2n,\mathbb{C})$.

Since $\chi(A_\mathbb{R}) = \{(m,\bar{m}) \mid m \in \mathrm{Mat}(n,\mathbb{C}\{i\})\}$, the quaternionic structure model for $\mathrm{U}(n,n) \cong \mathrm{Sp}_2(A_\mathbb{R},\bar{\sigma})$ can be seen as a subset of $\mathfrak{C}(\mathrm{GL}(2n,\mathbb{C}))$ stabilizing $\chi(A_\mathbb{R})$. $(J_1,J_2) \in \mathfrak{C}(\mathrm{GL}(2n,\mathbb{C}))$ stabilizes $\chi(A_\mathbb{R})^2$ if and only if for all $m \in \mathrm{Mat}(n,\mathbb{C}\{i\})^2$,

$$(J_1,J_2)(m,\bar{m}) = (J_1(m), J_2(\bar{m})) = (m', \bar{m}'),$$

for some $m' \in \mathrm{Mat}(n,\mathbb{C})^2$, i.e. $J_1 = J_2$. Therefore,

$$\mathfrak{C}(\mathrm{U}(n,n)) \cong \{(J,J) \in \mathfrak{C}(\mathrm{GL}(2n,\mathbb{C}))\}.$$

We can also see $\mathfrak{C}(\mathrm{U}(n,n))$ directly as the space complex structures on $A_\mathbb{R}^2$:

$$\mathfrak{C}(\mathrm{U}(n,n)) = \left\{ J \in \mathrm{Mat}(2n,\mathbb{C}) \mid \bar{J}^T \begin{pmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{pmatrix} \in \mathrm{Herm}^+(2n,\mathbb{C}), J\bar{J} = -\operatorname{Id}_{2n} \right\}.$$

Example 15. Consider the real algebra $A_{\mathbb{R}} := \mathrm{Mat}(n, \mathbb{H}\{i,j,k\})$. Then $A = \mathrm{Mat}(n, \mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\}$ and

$$\mathrm{Sp}_2(A, \sigma_1 \otimes \mathrm{Id}) \cong \mathrm{O}(4n, \mathbb{C}),$$

$$\mathrm{Sp}_2(A_{\mathbb{R}}, \sigma_1) \cong \mathrm{SO}^*(4n).$$

We use the map ψ from the Section A.2.2, to identify A with $A' := \mathrm{Mat}(2n, \mathbb{C}\{i\})$. The induced by $\mathrm{Id} \otimes \bar{\sigma}$ involution

$$\sigma' := \psi \circ (\mathrm{Id} \otimes \bar{\sigma}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto -\Omega \bar{m} \Omega$$

where $\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in \mathrm{Mat}(2n, \mathbb{C})$. We also denote by $\Omega_0 := \mathrm{diag}(\Omega, \Omega) \in \mathrm{Mat}(4n, \mathbb{C})$.

If we take a quaternionic structure J on A^2 then we define

$$J' := \psi \circ J \circ \psi^{-1}.$$

We can see J' as a complex $4n \times 4n$-matrix acting on $(A')^2$ in the following way: for $x \in (A')^2$,

$$J'(x) := J' \sigma'(x) = -J' \Omega_0 \bar{x} \Omega_0.$$

J' is a quaternionic structure, therefore,

$$-x = (J')^2(x) = J' \Omega_0 \overline{J' \Omega_0 \bar{x} \Omega_0} \Omega_0 = -J' \Omega_0 \bar{J}' \Omega_0 x.$$

So we obtain, J' is a quaternionic structure on A if and only if

$$J' \Omega_0 \bar{J}' \Omega_0 = \mathrm{Id}_{4n}.$$

The induced by $\sigma_1 \otimes \mathrm{Id}$ anti-involution

$$\psi \circ (\sigma_1 \otimes \mathrm{Id}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto -\Omega m^T \Omega.$$

So we define the standard symplectic form ω on $(A')^2$ with respect to this anti-involution: for $x, y \in (A')^2$,

$$\omega(x, y) := -\Omega_0 x^T \Omega_0 \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} y$$

and for a quaternionic structure J', we define

$$h_{J'}(x,y) := \omega(J'(x),y) = \bar{x}^T\Omega_0(J')^T\Omega_0\Omega_0\begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0\end{pmatrix} y =$$

$$= -\bar{x}^T\Omega_0(J')^T\begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0\end{pmatrix} y.$$

The quaternionic structure model for $\mathrm{GL}(2n,\mathbb{C})$ is then:

$$\mathfrak{C}(\mathrm{O}(4n,\mathbb{C})) := \left\{ J' \in \mathrm{Mat}(4n,\mathbb{C}) \;\middle|\; \begin{array}{l} J'\Omega_0\bar{J}'\Omega_0 = \mathrm{Id}_{4n}, \\ -\Omega_0(J')^T\begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0\end{pmatrix} \in \mathrm{Herm}^+(4n,\mathbb{C}) \end{array} \right\}.$$

The model for the symmetric space of $\mathfrak{C}(\mathrm{SO}^*(4n))$ can be seen as a subset of $\mathfrak{C}(\mathrm{O}(4n,\mathbb{C}))$ whose elements commute with σ' i.e. $\sigma'(J'(x)) = J'(\sigma'(x))$. Therefore:

$$\sigma'(J'(x)) = -\Omega\overline{J'(x)}\Omega = \Omega\overline{J'\Omega\bar{x}\Omega}\Omega = -\Omega\bar{J}'\Omega x,$$
$$J'(\sigma'(x)) = -J'\Omega\overline{\sigma'(x)}\Omega = J'\Omega\overline{\Omega\bar{x}\Omega}\Omega = -J'x$$

and we obtain:

$$\mathfrak{C}(\mathrm{SO}^*(4n)) \cong \{J' \in \mathfrak{C}(\mathrm{O}(4n,\mathbb{C})) \mid J' = \Omega\bar{J}'\Omega\}.$$

The space $\mathfrak{C}(\mathrm{SO}^*(4n))$ can be also seen directly as complex structures J on $A^2_{\mathbb{R}}$ such that the form:

$$h_J(x,y) = \sigma_1(x)^T\sigma_1(J)^T\Omega_0 y$$

is positive definite. So we obtain:

$$\mathfrak{C}(\mathrm{SO}^*(4n)) = \left\{ J' \in \mathrm{Mat}(2n,\mathbb{H}\{i,j,k\} \;\middle|\; \begin{array}{l} \sigma_1(J)^T\Omega_0 \in \mathrm{Herm}^+(2n,\mathbb{H}\{i,j,k\}), \\ J^2 = -\mathrm{Id}_{2n} \end{array} \right\}.$$

2.6 Hermitian Lie algebras with anti-involution

2.6.1 Weakly Hermitian Lie algebras

Let (A,σ) be a finite dimensional semisimple $\mathbb{R}$-algebra with an anti-involution. A can be turned into a Lie algebra with the Lie bracket $[x,y] = xy - yx$. Let $B \subseteq A$ be a Lie subalgebra that is closed under σ. We define:

$$B^{sym} := \mathrm{Fix}_B(\sigma) = B \cap A^{sym}.$$

Remark 2.6.1. The theory developed in the Section 2.1 is the special case when $B = A$.

Corollary 2.6.2. *If B is a Lie subalgebra of A containing 1, then $B^\times$ is open and dense in B and $(B^{sym})^\times := B^\times \cap A^{sym}$ is open and dense in $B^{sym} := B \cap A^{sym}$.*

Proof. Follows from the Corollary 2.1.7. □

Definition 2.6.3. A Lie subalgebra B is called *of Jordan type*, if:

1. for every $x, y \in B^{sym}$, $xy \in B$.

Remark 2.6.4. The condition (1) implies that for all $b \in B^{sym}$, $b^2 \in B^{sym}_{\geq 0}$.

Proposition 2.6.5. *Let B be of Jordan type. Then for every $x \in B$ and for every $b \in B^{sym}$, $\sigma(x)b + bx \in B^{sym}$.*

Proof. Take $x^s := \frac{x+\sigma(x)}{2}$, $x^a := \frac{x-\sigma(x)}{2}$, then $x = x^s + x^a$, $x^s \in B^{sym}$ and $\sigma(x^a) = -x^a$. Then we can write

$$\sigma(x)b + bx = (x^s - x^a)b + b(x^s + x^a) = (x^s b + bx^s) + (bx^a - x^a b).$$

Since B is of Jordan type, $x^s b, bx^s \in B$ and so $x^s b + bx^s \in B^{sym}$. Further, $bx^a - x^a b = [b, x^a] \in B$ and $\sigma(bx^a - x^a b) = -x^a b + bx^a$, i.e. $bx^a - x^a b \in B^{sym}$. So we obtain, $\sigma(x)b + bx \in B^{sym}$. □

Definition 2.6.6. We denote

$$B^{anti} := \mathrm{Fix}_B(-\sigma).$$

Remark 2.6.7. The following properties hold:

$$[B^{anti}, B^{anti}] \subseteq B^{anti}, \ \ [B^{anti}, B^{sym}] \subseteq B^{sym}, \ \ [B^{sym}, B^{sym}] \subseteq B^{anti}.$$

In particular, B^{anti} is a sub Lie algebra of B.

Let B be of Jordan type, let G_0 be the unique connected subgroup of $A^\times$ such that $\mathrm{Lie}(G_0) = B$. We assume G_0 to be a Lie subgroup of $A^\times$. We denote:

$$U(G_0, \sigma) = \{u \in G_0 \mid \sigma(u)u = 1\}$$

Remark 2.6.8. $\mathrm{Lie}(U(G, \sigma)) = B^{anti}$.

Proposition 2.6.9. *Let B be of Jordan type with $1 \in B$. For every $g \in G_0$ and for every $b \in B^{sym}$, $\sigma(g)bg \in B^{sym}$.*

Proof. We consider the following map:

$$\begin{array}{rccc} F\colon & U(G, \sigma) \times \exp(B^{sym}) & \to & G \\ & (u, b) & \mapsto & ub \end{array}$$

We notice, that since for all $b \in B^{sym}$, $b^2 \in B^{sym}$, we have $b^n \in B^{sym}$ for all $n \in \mathbb{N}$. Moreover, B^{sym} is closed in A, therefore, $\exp(b) - 1 \in B^{sym}$. Since exp is a diffeomorphism in a small neighborhood of $0 \in B^{sym}$, $T_1 \exp(B^{sym}) = B^{sym}$.

The differential of F at $(1, 1)$ is a bijection. Indeed:

$$D_{(1,1)}F(x, y) = x + y \in B$$

where $x \in B^{anti} = \mathrm{Lie}(U(G,\sigma))$, $y \in B^{sym}$. Therefore, in a small neighborhood V of $(1,1) \in U(G,\sigma) \times \exp(B^{sym})$, F is a homeomorphism. Moreover, G_0 is generated by $F(V)$, therefore, for every $g \in G_0$, there exist $r \geq 0$ and $u_1, \ldots, u_r \in U(G,\sigma)$, $b_1, \ldots, b_r \in \exp(B^{sym})$ such that $g = u_1 b_1 \ldots u_r b_r$.

Since $\sigma(u) = u^{-1}$ for $u \in U(G,\sigma)$, $\sigma(u)bu = u^{-1}bu \in B^{sym}$ for all $b \in B^{sym}$.

Let $b \in B^{sym}$, $b' \in \exp(B^{sym})$, then $b' = 1 + b_0$ for $b_0 \in B^{sym}$.

$$\sigma(b')bb' = b'bb' = (1+b_0)b(b+b_0) = b + b_0 b + b b_0 + b_0 b b_0.$$

By Proposition 2.6.5, $b_0 b + b b_0 = \tilde{b}$ for $\tilde{b} \in B^{sym}$. Therefore, $b_0 b b_0 = b_0 \tilde{b} - (b_0)^2 b \in B$ and, since $b'bb' \in A^{sym}$, we obtain $b'bb' \in B^{sym}$.

Therefore, by induction, we obtain $\sigma(g)bg \in B^{sym}$ for all $g \in G_0$. □

Remark 2.6.10. The group G_0 acts on B^{sym} in the following way:

$$\begin{array}{rccc} \psi\colon & G_0 & \to & \mathrm{Aut}(B^{sym}) \\ & g & \mapsto & [\psi(g)\colon b \mapsto \sigma(g)bg]. \end{array}$$

Definition 2.6.11. A Lie subalgebra B of Jordan type is called *weakly Hermitian*, if:

1. $1 \in B$;
2. The convex cone $C(\theta(B^{sym}))$ is proper;
3. B^{sym} does not contain nilpotent elements, i.e. for every $b \in B^{sym}$, $b^2 = 0$ if and only if $b = 0$.

Definition 2.6.12. If B is weakly Hermitian, we define

$$B^{sym}_{+} := C(\theta((B^{sym})^{\times})),$$

and $B^{sym}_{\geq 0}$ as the closure of B^{sym}_{+}. In this case, B^{sym}_{+} and $B^{sym}_{\geq 0}$ are proper convex cones in B^{sym}.

We recall the definition of Jordan algebra and formally real Jordan algebra.

Definition 2.6.13. Let $(V,\circ)$ be an possibly non-associative algebra over some field $\mathbb{K}$. $(V,\circ)$ said to be a *Jordan algebra* if for all $x, y \in V$

1. $x \circ y = y \circ x$;
2. $(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x))$ (Jordan identity).

A Jordan algebra $(V,\circ)$ is called *formally real* if for all $x, y \in V$, $x^2 + y^2 = 0$ implies $x, y = 0$.

Proposition 2.6.14.

- *For B of Jordan type, the algebra $(B^{sym}, \circ)$ is a Jordan algebra where*

$$x \circ y = \frac{xy + yx}{2}.$$

- *For weakly Hermitian B, the Jordan algebra $(B^{sym}, \circ)$ is formally real.*

Proof. Since for all $x, y \in B^{sym}$, $xy \in B$, we get $x \circ y \in B^{sym}$. Also $x \circ y = y \circ x$ is clear. The Jordan identity:

$$(x \circ y) \circ (x \circ x) = \frac{xy + yx}{2} \circ x^2 = \frac{xyx^2 + yx^3 + x^3y + x^2yx}{4} =$$

$$= \frac{xyx^2 + x^3y + yx^3 + x^2yx}{4} = x \circ \frac{yx^2 + x^2y}{2} = x \circ (y \circ (x \circ x)).$$

So $(B^{sym}, \circ)$ is a Jordan algebra.

Assume now B to be weakly Hermitian. Let $a_1, a_2 \in B^{sym}$, then $a_i^2 \in B^{sym}_{\geq 0}$. The convex cone $B^{sym}_{\geq 0}$ is proper, so $a_1^2 + a_2^2$ vanishes if and only if $a_1^2 = a_2^2 = 0$. Therefore, $a_1 = a_2 = 0$ by (3) in the Definition 2.6.11. $\square$

2.6.2 Classification of simple formally real Jordan algebras

In this section, we remind the well-known classification of simple formally real Jordan algebras (for more details, see [10, 17]).

Fact 2.6.15. *Every simple formally real Jordan algebra is isomorphic to one of the following Jordan algebras:*

1. $(\mathrm{Sym}(n, \mathbb{R}), \circ)$ *where* $a \circ b = \frac{ab+ba}{2}$ *for* $a, b \in \mathrm{Sym}(n, \mathbb{R})$*;*
2. $(\mathrm{Herm}(n, \mathbb{C}), \circ)$ *where* $a \circ b = \frac{ab+ba}{2}$ *for* $a, b \in \mathrm{Herm}(n, \mathbb{C})$*;*
3. $(\mathrm{Herm}(n, \mathbb{H}), \circ)$ *where* $a \circ b = \frac{ab+ba}{2}$ *for* $a, b \in \mathrm{Herm}(n, \mathbb{H})$*;*
4. $(B^{sym}(1, n), \circ)$ *where* $a \circ b = \frac{ab+ba}{2}$ *for* $a, b \in B^{sym}(1, n)$*;*
5. $(\mathrm{Herm}(3, \mathbb{O}), \circ)$ *where* $a \circ b = \frac{ab+ba}{2}$ *for* $a, b \in \mathrm{Herm}(3, \mathbb{O})$

where $B^{sym}(1, n)$ is the Jordan algebra defined in the Section 2.10, $\mathrm{Herm}(3, \mathbb{O})$ *is the space of* 3×3 *Hermitian octonionic matrices.*

Fact 2.6.16 ([17, Corollary 2.8.5]). *The Jordan algebra* $(\mathrm{Herm}(3, \mathbb{O}), \circ)$ *is exceptional. This means that there is no associative real algebra A that contains* $\mathrm{Herm}(3, \mathbb{O})$ *as a Jordan subalgebra.*

2.6.3 Spectral theorem

In this section, we assume B to be weakly Hermitian. As we have seen, $(B^{sym}, \circ)$ is a formally real Jordan algebra.

We are going to state the first versions of the spectral theorem for formally real Jordan algebras. But before we do it, first, we give some necessary definitions:

Definition 2.6.17.
- An element $c \in B^{sym}$ is called an *idempotent* if $c^2 = c$.
- Two idempotents $c, c' \in B^{sym}$ are called *orthogonal* if $c \circ c' = 0$.
- A tuple $(c_1, \dots, c_k)$ of pairwise orthogonal idempotents is called a *complete orthogonal system of idempotents* if $c_1 + \dots + c_k = 1$.

Remark 2.6.18. Every idempotent $c \in B^{sym}_{\geq 0}$.

Theorem 2.6.19 (Spectral theorem, first version [10, Theorem III.1.1])**.** *For every $b \in B^{sym}$, there exist a unique $k \in \mathbb{N}$, unique real numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, all distinct, and a unique complete system of orthogonal idempotents $c_1, \dots, c_k \in B^{sym}$ such that*

$$b = \sum_{i=1}^{k} \lambda_i c_i.$$

Corollary 2.6.20. *For $b \in B^{sym}_{\geq 0}$, the numbers $\lambda_1, \dots, \lambda_k \geq 0$. For $b \in B^{sym}_{+}$, the numbers $\lambda_1, \dots, \lambda_k > 0$. In particular,*

$$B^{sym}_{+} = \theta((B^{sym})^{\times}),\ B^{sym}_{\geq 0} = \theta(B^{sym}),$$

Corollary 2.6.21. *The set of all invertible elements $(B^{sym})^{\times}$ of B^{sym} consists of elements such that all $\lambda_i \neq 0$. If all $\lambda_i \neq 0$, then*

$$\left(\sum_{i=1}^{k} \lambda_i c_i\right)^{-1} = \sum_{i=1}^{k} \lambda_i^{-1} c_i.$$

Corollary 2.6.22. *B^{sym}_{+} is connected, open in B^{sym}, open and closed in $(B^{sym})^{\times}$. $B^{sym}_{\geq 0}$ is connected and closed in B^{sym}.*

Corollary 2.6.23. *For every (continuous/smooth) function $f\colon \mathbb{R} \to \mathbb{R}$, the (continuous/smooth) map*

$$\hat{f}\colon B^{sym} \to B^{sym}$$

can be defined: if

$$b = \sum_{i=1}^{k} \lambda_i c_i,$$

then

$$\hat{f}(b) := \sum_{i=1}^{k} f(\lambda_i) c_i.$$

This map is well defined because the spectral decomposition is unique. Analogously, for any function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ *or* $f\colon \mathbb{R}_+ \to \mathbb{R}$, $\hat{f}\colon B^{sym}_{\geq 0} \to B^{sym}$ *resp.* $\hat{f}\colon B^{sym}_+ \to B^{sym}$ *can be defined.*

In particular, for every $b \in B^{sym}_{\geq 0}$*, the element* $b^t \in B^{sym}_{\geq 0}$ *for* $t > 0$ *is well-defined. This definition is compatible with integer powers of element.*

Corollary 2.6.24. B^{sym}_+ *is homeomorphic to* B^{sym}*. In particular,* B^{sym}_+ *is open in* B^{sym} *and contractible.* $\{1\} \subset B^{sym}_+$ *is a deformation retract of* B^{sym}_+*.*

Proof. B^{sym} is a $\mathbb{R}$-vector space, so it is contractible. $\{0\} \subset B^{sym}$ is a deformation retract of B^{sym}. Take $f(t) = \log(t)$. □

To state the second version of the spectral theorem, we need to give some additional definitions:

Definition 2.6.25.
- An idempotent $0 \neq c \in B^{sym}$ is called *primitive* if it cannot be written as a sum of two orthogonal non-zero idempotents.
- A complete orthogonal system of primitive idempotents $(c_1, \dots, c_k)$ is called a *Jordan frame.*

Theorem 2.6.26 (Spectral theorem, second version [10, Theorem III.1.2])**.** *Suppose,* B^{sym} *has rank* n*. For every* $b \in B^{sym}$ *there exist a Jordan frame* $(e_1, \dots, e_n)$ *and real numbers* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *such that*

$$b = \sum_{i=1}^{n} \lambda_i e_i.$$

The numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *(with their multiplicities) called* eigenvalues *of* b *are uniquely determined by* b*. In particular, they do not depend (up to permutations) on the Jordan frame* $e_1, \dots, e_n \in B^{sym}$*.*

Remark 2.6.27. The Jordan frame $e_1, \dots, e_n \in B^{sym}$ associated to the element $b \in B^{sym}$ as in the Theorem 2.6.26 is, in contrast to the complete system of orthogonal idempotents from the Theorem 2.6.19, in general not unique.

Definition 2.6.28. Let $b \in B^{sym}$ and $\lambda_1, \dots, \lambda_n$ are all its eigenvalues (with multiplicities). We define the *trace* and the *determinant* of b:

$$\operatorname{tr}(b) := \sum_{i=1}^{n} \lambda_i, \ \det(b) := \prod_{i=1}^{n} \lambda_i.$$

Proposition 2.6.29 ([10, Proposition III.1.5])**.** *The function*

$$\begin{array}{cccc} \beta\colon & B^{sym} \times B^{sym} & \to & \mathbb{R} \\ & (b_1, b_2) & \mapsto & \operatorname{tr}\left(\frac{b_1 b_2 + b_2 b_1}{2}\right) \end{array}$$

is an ($\mathbb{R}$*-vector space) inner product on* B^{sym}*.*

2.6.4 Lie group corresponding to weakly Hermitian Lie algebra

As before, let (A, σ) be an $\mathbb{R}$-algebra with an anti-involution. The space $A^\times$ of invertible elements of A is a Lie group and its Lie algebra is A with the Lie bracket given by $[x, y] = xy - yx$. We take $G_0 < A^\times$ a connected Lie subgroup of $A^\times$ closed under σ. We denote $B := \mathrm{Lie}(G_0)$ the Lie algebra of G_0. Notice that G_0 is uniquely defined by B, and it is generated by $\exp(B)$. Since G_0 is closed under σ, B is closed under σ as well. We define

$$G_0^{sym} := G_0 \cap A^{sym},\ B^{sym} := B \cap A^{sym}.$$

Definition 2.6.30. A weakly Hermitian Lie subalgebra B is called *Hermitian*, if:

1. the group $U(G_0, \sigma) := \{g \in G_0 \mid \sigma(g)g = 1\}$ is compact;

Proposition 2.6.31. *If B is weakly Hermitian, then $B_+^{sym} \subseteq G_0$.*

Proof. Let $b \in B_+^{sym}$. Take its spectral decomposition: $b = \sum_{i=1}^k \lambda_i c_i$ where $\lambda_1, \ldots, \lambda_k > 0$, $(c_1, \ldots, c_k)$ is a complete system of orthogonal idempotents. Then $\log(b) = \sum_{i=1}^k \log(\lambda_i) c_i \in B^{sym}$ and $\exp(\log(b)) = b \in G_0$ because the map exp defined on $\mathbb{R}$ extended to B^{sym} and the exponential map $\exp\colon B \to G$ restricted to B^{sym} defined by the same power series and thus they agree. □

Corollary 2.6.32.

- *For every $b \in B_+^{sym}$ and for every $g \in G_0$, $\sigma(g)bg \in B_+^{sym}$. In particular, $\sigma(g)g \in B_+^{sym}$.*
- *For every $b \in B_{\geq 0}^{sym}$ and for every $g \in G_0$, $\sigma(g)bg \in B_{\geq 0}^{sym}$.*

Proof. It is clear that $\sigma(g)bg \in (B^{sym})^\times$ for $g \in G_0$ and $b \in B_+^{sym}$. Since G_0 is connected, $\sigma(g)g$ is in the connected component of $1 \in (B^{sym})^\times$ which is B_+^{sym}.

The second one follows from the fact that $B_{\geq 0}^{sym}$ is a topological closure of B_+^{sym} in B^{sym}. □

Let us restrict the action ψ from the Remark 2.6.10 to the subgroup $U(G_0, \sigma) < G_0$. Then the action $\psi|_{U(G_0,\sigma)}$ preserves Jordan frames.

Corollary 2.6.33. *Assume, $\psi|_{U(G_0,\sigma)}$ is transitive on Jordan frames of B^{sym}. Suppose, B^{sym} has rank n. For every Jordan frame $e_1, \ldots, e_n \in B^{sym}$ and for every $b \in B^{sym}$ there exist $u \in U(G_0, \sigma)$ such that*

$$\psi(u)b = \sum_{i=1}^n \lambda_i e_i.$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are all eigenvalues of b (with their multiplicities).

Remark 2.6.34. In general, for a fixed Jordan frame $e_1, \ldots, e_n$ and $b \in B^{sym}$, the element $u \in U(G_0, \sigma)$ is not unique.

Corollary 2.6.35. *For $b \in B_{\geq 0}^{sym}$, the numbers $\lambda_1, \ldots, \lambda_n \geq 0$. For $b \in B_+^{sym}$, the numbers $\lambda_1, \ldots, \lambda_n > 0$.*

2.6.5 Disconnected extension of G_0

Let $B \subseteq A$ be semi Hermitian Lie subalgebra of A. As we have seen, the group G_0 we considered before is supposed to be connected and $B_+^{sym} \subseteq G_0$. In this section, we study the disconnected extension G of G_0 generated by G_0 and $(B^{sym})^\times$.

Proposition 2.6.36. *The group G_0 is a normal subgroup of G.*

Proof. It is enough to show that $b^{-1}gb \in G_0$ for all $b \in B^{sym}$, $g \in G_0$. Since G_0 is generated by $\exp(B)$, it is enough to check it for all $g = \exp(b')$ for $b' \in B$. Notice that in this case, $b^{-1}gb = \exp(b^{-1}b'b)$.

By Proposition 2.6.5, $b'b + bb' = \tilde{b} \in B^{sym}$. Therefore, $b^{-1}b'b = b^{-1}\tilde{b} - b'$. Since B is of Jordan type, $b^{-1}\tilde{b} \in B$. Therefore, $b^{-1}b'b \in B$ and $\exp(b^{-1}b'b) \in G_0$. □

From now on, we assume that $U(G_0, \sigma)$ acts transitively on Jordan frames of B^{sym}.

Theorem 2.6.37. *The factor group G/G_0 is finite. Moreover, G has finitely many connected components and G_0 is one of them containing* 1. *In every connected component of G, there is an element of B^{sym}.*

Proof. Let $g \in G$, then by Definition of G, there exist $g_0, g_1, \dots, g_r \in G_0$, $b_1, \dots, b_r \in (B^{sym})^\times$ such that $g = g_0 b_1 g_1 \dots b_r g_r$. We take such presentation with minimal r. We choose a Jordan frame $(e_1, \dots, e_n)$ of B^{sym} and take a spectral decomposition according Corollary 2.6.33:

$$b_i = u_i^{-1} \sum_{j=1}^{n} \varepsilon_{ij} \lambda_{ij} e_j u_i = (u_i^{-1} \sum_{j=1}^{n} \varepsilon_{ij} e_j u_i)(u^{-1} \sum_{j=1}^{n} \lambda_{ij} e_j u_i)$$

where all $\lambda_{ij} > 0$, $\varepsilon_{ij} \in \{1, -1\}$, $u_i \in U(G_0, \sigma)$. We denote:

$$b_i' := \sum_{j=1}^{n} \lambda_{ij} e_j u_i, \; s_i := \operatorname{sgn}(b_i) := \sum_{j=1}^{n} \varepsilon_{ij} e_j.$$

Notice, $b_i' \in B_+^{sym} \subseteq G_0$, $s_i \in (B^{sym})^\times$. So we obtain:

$$g = g_0 b_1 g_1 \dots g_{r-1} b_r g_r = g_0 b_1 g_1 \dots g_{r-2} u_{r-1}^{-1} b_{r-1}' s_{r-1} u_{r-1} g_{r-1} u_r^{-1} b_r' s_r u_r g_r.$$

We denote $g_r' := u_r g_r$, $g_{r-1}' := u_{r-1} g_{r-1} u_r^{-1} b_r' \in G_0$, $g_{r-2}' := g_{r-2} u_{r-1}^{-1} b_{r-1}' \in G_0$. Then

$$g = g_0 b_1 g_1 \dots g_{r-2}' s_{r-1} g_{r-1}' s_r g_r' = b_1 g_1 \dots g_{r-2}' s_{r-1} s_r s_r^{-1} g_{r-1}' s_r g_r'.$$

Since G_0 is a normal subgroup in G, $g''r - 1 := s_r^{-1} g_{r-1}' s_r g_r \in G_0$. Moreover, $s_{r-1}' := s_{r-1} s_r = \sum_{j=1}^{n} \varepsilon_{r-1,j} \varepsilon_{rj} e_j \in B^{sym}$. So we obtain:

$$g = g_0 b_1 g_1 \dots g_{r-2}' s_{r-1}' g_{r-1}''.$$

So we reduced the number r. Therefore, g can be written as

$$g = g_0 b_1 g_1 = g_0 u_1^{-1} b_1' s_1 u_1 g_1 = g_0' s_1 g_1'$$

where $g_0' = g_0 u_1^{-1} b_1'$, $g_1' = u_1 g_1$. Further,

$$g = g_0' s_1 g_1' = s_1 s_1^{-1} g_0' s_1 g_1' = s_1 g'$$

where $g' := s_1^{-1} g_0' s_1 g_1' \in G_0$ because G_0 is a normal subgroup in G. Therefore, $gG_0 = s_1 G_0$.Consider the group

$$S := \left\{ \sum_{i=1}^{n} \varepsilon_i e_i \mid \varepsilon_i \in \{1, -1\} \right\} \subset (B^{sym})^\times.$$

This is a finite abelian subgroup of $A^\times$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. The map $S \ni s \mapsto sG_0 \in G/G_0$ is a surjective group homomorphism. Therefore, G/G_0 is finite. In particular, $\dim(G) = \dim(G_0)$, so G_0 is open in G. G_0 is closed in $A^\times$ as a Lie subgroup. Therefore, G_0 is also closed in G, i.e. G_0 is a connected component of G. Every connected component of G has form sG_0 for $s \in S$, so G has finitely many connected components, and, since $d \in B^{sym}$, in every connected component of G there is an element from B^{sym}. □

Corollary 2.6.38. *If B is Hermitian, then $U(G, \sigma)$ is compact.*

Proposition 2.6.39. *The group G acts on B^{sym} in the following way:*

$$\begin{array}{cccc} \psi\colon & G \times B^{sym} & \mapsto & B^{sym} \\ & (g, b) & \to & \sigma(g) b g \end{array}$$

preserving B_+^{sym} and $B_{\geq 0}^{sym}$.

Proof. First, we note that every element $g \in G$ can be written as $g = sg_0$ for $g_0 \in G_0$ and $s \in S$ from the proof of the previous theorem.

Since the construction of G does not depend on the choice of the Jordan frame $(e_1, \dots e_n)$ from the proof of the previous theorem, we assume this basis correspond to the spectral decomposition of b, i.e. $b = \sum_{i=1}^n \lambda_i e_i$. Then $g = sg_0$ for $s = \sum_{i=1}^n \varepsilon_i e_i$, $g_0 \in G_0$.

Then $\sigma(g) b g = \sigma(g_0)(sbs) g_0$. But $sbs = \sum_{i=1}^n \varepsilon_i^2 \lambda_i e_i = b \in B$. Therefore $\sigma(g) b g = \sigma(g_0) b g_0 \in B^{sym}$ because G acts on B^{sym} in this way.

From the same reason, if $b \in B_+^{sym}$ or $b \in B_{\geq 0}^{sym}$, then $\sigma(g) b g \in B_+^{sym}$ resp. $\sigma(g) b g \in B_{\geq 0}^{sym}$. □

2.6.6 Polar decomposition in G and maximal compact subgroup of G

Assume $B \subseteq A$ to be weakly Hermitian Lie subalgebra. In this section, we assume G to be either the connected group G_0 or the the extension G from the previous section. Notice that G acts on B^{sym} in the following way:

$$\begin{array}{cccc} \psi\colon & G \times B^{sym} & \mapsto & B^{sym} \\ & (g, b) & \to & \sigma(g) b g \end{array}$$

preserving B_+^{sym}.

Theorem 2.6.40 (Polar decomposition, first version)**.** *The map*

$$\begin{array}{rccl} \mathrm{pol}\colon & U(G,\sigma)\times B_+^{sym} & \to & G \\ & (u,b) & \mapsto & ub \end{array}$$

is a homeomorphism, i.e. for every $g\in G$ there exist unique $b\in B_+^{sym}$ and $u\in U(G,\sigma)$ such that $g=ub$.

Proof. The map pol is well-defined because $B_+^{sym}\subseteq G$. First, we prove the surjectivity. Take $g\in G$, then $\sigma(g)g=\psi(g)(1)\in B_+^{sym}$. Take $b:=(\sigma(g)g)^{\frac{1}{2}}$, then $u:=g(\sigma(g)g)^{-\frac{1}{2}}\in U(G,\sigma)$. Indeed,

$$\sigma(u)u=(\sigma(g)g)^{-\frac{1}{2}}\sigma(g)g(\sigma(g)g)^{-\frac{1}{2}}=1.$$

Now, we prove the injectivity. Let $g=ub=u'b'$ where $u,u'\in U(G,\sigma)$, $b,b'\in B_+^{sym}$. Then $\sigma(g)g=(b')^2=b^2\in B_+^{sym}$. We take the spectral decompositions of b and b':

$$b=\sum_{i=1}^{k}\lambda_i c_i,\ b'=\sum_{i=1}^{k'}\lambda_i' c_i'$$

where all $k,k'\in\mathbb{N}$, $\lambda_i,\lambda_i'>0$ and $\{c_i\}$, $\{c_i'\}$ complete orthogonal systems of idempotents of B^{sym}. Then

$$b^2=\sum_{i=1}^{k}\lambda_i^2 c_i=\sum_{i=1}^{k'}(\lambda_i')^2 c_i'=(b')^2.$$

Because of the uniqueness of the spectral decomposition, $k=k'$ and, up to reordering, all $\lambda_i^2=(\lambda_i')^2$, $c_i=c_i'$. But all $\lambda_i>0$, therefore, $\lambda_i=\lambda_i'$, i.e. $b=b'$ and $u=gb^{-1}=g(b')^{-1}=u'$.

Finally, by definition, pol is continuous. Moreover,

$$\mathrm{pol}^{-1}(g)=(g(\sigma(g)g))^{-\frac{1}{2}},(\sigma(g)g))^{\frac{1}{2}})$$

is continuous as well. Therefore, pol is a homeomorphism. □

Corollary 2.6.41. *Similarly can be proved that the map*

$$\begin{array}{ccl} B_+^{sym}\times U(G,\sigma) & \to & G \\ (b,u) & \mapsto & bu \end{array}$$

is a homeomorphism, i.e. for every $g\in G$ there exist unique $b\in B_+^{sym}$ and $u\in U(G,\sigma)$ such that $g=bu$.

Corollary 2.6.42. *For any $g\in G$, $\theta(g)=\sigma(g)g\in B_+^{sym}$.*

Corollary 2.6.43. *The group $U(G,\sigma) < G$ is a deformation retract of G. In particular, if B is Hermitian, it is a maximal compact subgroup of G.*

Corollary 2.6.44. *The polar decomposition 2.6.40 as well as the Corollary 2.6.43 hold also for any Lie subgroup $G \leq A^\times$ such that* $\mathrm{Lie}(G) = B$*. In particular, it holds in the case $B = A$ for a Hermitian algebra A.*

Theorem 2.6.45 (Polar decomposition, second version)**.** *Let G be the extension of G_0 as in the previous section. The map*

$$\begin{array}{cccc} \mathrm{pol}'\colon & U(G_0,\sigma)\times(B^{sym})^\times & \to & G \\ & (u,b) & \mapsto & ub \end{array}$$

is surjective and continuous.

Proof. Let $g \in G$. We take its polar decomposition as in the Theorem 2.6.40: $g = ub_0$ for an $u \in U(G,\sigma)$, $b_0 \in B_+^{sym}$. We take a Jordan frame $(e_i)_{i=1}^n$ such that $b_0 = \sum_{i=1}^n \lambda_i e_i$ and take a group

$$S := \left\{ \sum_{i=1}^n \varepsilon_i e_i \mid \varepsilon_i \in \{1,-1\} \right\} \subset (B^{sym})^\times \cap U(G,\sigma).$$

Then, as we have seen in the proof of the Theorem 2.6.37, every connected component of G contains an element form S. Moreover, since $U(G,\sigma)$ is a deformation retract of G, and $S \in U(G,\sigma)$, every connected component of $U(G,\sigma)$ contains an element of S. Therefore, there exists $s \in S$ such that $u = u_0 s$ for an $u_0 \in U(G_0,\sigma)$. Then

$$g = ub_0 = u_0 s b_0 =: u_0 b$$

for $b := sb_0 = \sum_{i=1}^n \varepsilon_i\lambda_i e_i \in (B^{sym})^\times$. □

Remark 2.6.46. The map pol′ is in general not injective. For example, if we take

$$B = A = \mathrm{Mat}(2,\mathbb{R})$$

with σ to be the transposition, then

$$U(G_0,\sigma) = \mathrm{SO}(2,\mathbb{R}),\ B^{sym} = \mathrm{Sym}(2,\mathbb{R}).$$

Then the matrix

$$\mathrm{Id} = u_1 b_1 = u_2 b_2$$

for $u_1 = b_1 = \mathrm{Id}$, $u_2 = b_2 = -\mathrm{Id}$.

The reason for that is the fact that $U(G_0,\sigma) \cap B^{sym} \neq \{1\}$. If $B = A$, then $U(G_0,\sigma) \cap B^{sym} = \{1\}$, if and only if $A = \mathbb{R}$ with $\sigma = \mathrm{Id}$ (follows from the Theorem 2.7.26).

Consider the topological closure $\bar{G}$ of G in A.

Proposition 2.6.47. *$\bar{G}$ is a monoid.*

Proof. Let $g, g' \in \bar{G} \subseteq A$, then $gg' \in A$. We want to show that there exists $\{h_i\} \subset G$ such that $\lim h_i = gg'$. Since $g, g' \in \bar{G}$, there exist $\{g_i\}, \{g_i'\} \subset G$ such that $\lim g_i = g$, $\lim g_i' = g'$. Take $h_i = g_i g_i' \in G$, then

$$\lim h_i = \lim g_i g_i' = \lim g_i \lim g_i' = gg'. \qquad \square$$

By taking closure in the polar decomposition, we get the following map:

$$\begin{array}{rccc} \overline{\mathrm{pol}}\colon & U(G,\sigma) \times B_{\geq 0}^{sym} & \to & \bar{G} \\ & (u,b) & \mapsto & ub. \end{array}$$

This map is not a homeomorphism anymore, but it is surjective. If B is Hermitian, it is also proper because $U(G,\sigma)$ is compact. The map θ can be extended to the map

$$\begin{array}{rccc} \bar{\theta}\colon & \bar{G} & \to & B_{\geq 0}^{sym} \\ & g & \mapsto & \sigma(g)g. \end{array}$$

Proposition 2.6.48. *If B is Hermitian, the map $\bar{\theta}\colon \bar{G} \to B_{\geq 0}^{sym}$ is proper.*

Proof. Let $K \subset B_{\geq 0}^{sym}$ be a compact subset. Then

$$\bar{\theta}^{-1}(K) = \{ub_+^{\frac{1}{2}} \mid u \in U(G,\sigma), b_+ \in K\} = \overline{\mathrm{pol}}(U(G,\sigma) \times K).$$

Since $U(G,\sigma) \times K$ is compact in $U(G,\sigma) \times B_{\geq 0}^{sym}$ and $\overline{\mathrm{pol}}$ is continuous, $\bar{\theta}^{-1}(K)$ is compact. $\square$

Proposition 2.6.49. *If B is Hermitian, the set*

$$D := D(\bar{G},\sigma) := \{a \in \bar{G} \mid 1 - \sigma(a)a \in B_{\geq 0}^{sym}\} \subseteq \bar{G}$$

is compact.

Proof. First, we need the following Lemma:

Lemma 2.6.50. *Let C be a closed proper convex cone in some finite-dimensional $\mathbb{R}$-vector space V. Then for every $c \in V$, the set $K := C \cap (c - C)$ is compact.*

Proof. Assume K is not compact. We fix some norm $\|\cdot\|$ on V. There exists a sequence (x_n) such that $\|x_n\| \to \infty$. Since $y_n := \frac{x_n}{\|x_n\|} \in S^1$ and for finite-dimensional V, S_1 is compact, there exists a limit point y of (y_n). Since C is a closed cone, $y\mathbb{R}_+ \subseteq C \cap (c - C)$ and, therefore $c - y\mathbb{R}_+ \subseteq C \cap (c - C)$. Analogously, $c + y\mathbb{R}_+ \subseteq C \cap (c - C)$ and, therefore $-y\mathbb{R}_+ \subseteq C \cap (c - C)$. That means, $y\mathbb{R} \in C \cap (-C)$, so $y = 0$. This contradicts to $y \in S^1$. Therefore, K is compact. $\square$

By the Lemma 2.6.50, the set

$$K := \{x \in B_{\geq 0}^{sym} \colon 1 - x \in B_{\geq 0}^{sym}\} = B_{\geq 0}^{sym} \cap (1 - B_{\geq 0}^{sym})$$

is compact. Since $\bar{\theta}^{-1}(K) = D$ and $\bar{\theta}$ is proper, D is compact. $\square$

Corollary 2.6.51. *Let G be the extension of G_0 from the previous section. The map*

$$\begin{array}{rccc} \overline{\mathrm{pol}'}\colon & U(G_0,\sigma)\times B^{sym} & \to & \bar{G} \\ & (u,b) & \mapsto & ub \end{array}$$

is surjective and continuous. In particular, $\bar{G}$ is connected.

2.7 The symplectic group over G

2.7.1 The group $\mathrm{Sp}_2(G,\sigma)$ (first definition)

Consider an $\mathbb{R}$-algebra A with an anti-involution σ. Consider $G \leq A^\times$ a Lie subgroup of $A^\times$ which is closed under σ, we denote $B := \mathrm{Lie}(G)$ and assume $(B^{sym})^\times \subseteq G$. By G_0, we denote the connected component of 1 in G. Then $B = \mathrm{Lie}(G_0) = \mathrm{Lie}(G) \leq A$. If B is weakly Hermitian, we always take G to be the extension form the Section 2.6.5.

Consider

$$\mathfrak{sp}_2(B,\sigma) = \left\{ \begin{pmatrix} x & z \\ y & -\sigma(x) \end{pmatrix} \mid x \in B, y, z \in B^{sym} \right\} \subseteq \mathfrak{sp}_2(A,\sigma)$$

In general, it is not a Lie algebra. We need to take some additional assumption:

Proposition 2.7.1. *$\mathfrak{sp}_2(B,\sigma)$ is a Lie subalgebra of $\mathfrak{sp}_2(A,\sigma)$ if and only if B is of Jordan type.*

Proof. Matrixes

$$r(z) := \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix},\ l(x) := \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \text{ and } d(z) := \begin{pmatrix} x & 0 \\ 0 & -\sigma(x) \end{pmatrix}$$

generate $\mathfrak{sp}_2(B,\sigma)$ as a vector space. $\mathfrak{sp}_2(B,\sigma)$ is a Lie subalgebra of $\mathfrak{sp}_2(A,\sigma)$ if and only if all Lie bracket of these elements are in $\mathfrak{sp}_2(B,\sigma)$.

For $y, z \in B^{sym}$

$$[r(z), l(y)] = \begin{pmatrix} zy & 0 \\ 0 & -yz \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma)$$

so $zy, yz \in B$ and we need the condition that B is of Jordan type.

For $a \in B$, $z \in B^{sym}$:

$$[d(x), r(z)] = \begin{pmatrix} 0 & xz + z\sigma(x) \\ 0 & 0 \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma)$$

so $xz + z\sigma(x) \in B^{sym}$. This holds for B of Jordan type by Proposition 2.6.5.

For $a \in B$, $y \in B^{sym}$:

$$[l(y), d(x)] = \begin{pmatrix} 0 & 0 \\ \sigma(x)y + yx & 0 \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma)$$

so $\sigma(x)y + yx \in B^{sym}$. This holds for B of Jordan type by Proposition 2.6.5. $\square$

We consider the following matrices:

$$D(x) := \begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1} \end{pmatrix},\ L(y) := \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},\ R(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

where $x \in G,\ y, z \in B^{sym}$. These matrices, acting on A^2, preserve the standard symplectic form $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore, the set $\{L(y)D(x)R(y) \mid y, z \in B^{sym},\ x \in G\}$ is contained in $\mathrm{Sp}_2(A, \sigma)$.

Definition 2.7.2. We denote by $\mathrm{Sp}_2(G, \sigma)$ the topological closure of $\{L(y)D(x)R(z) \mid y, z \in B^{sym},\ x \in G\}$ in $\mathrm{Sp}_2(A, \sigma)$.

Lemma 2.7.3. *If B is weakly Hermitian, $\mathrm{Sp}_2(G, \sigma)$ is connected.*

Proof. We show that for every generic element $g := L(y)D(x)R(z)$ such that $y, z \in B^{sym}$, $x \in G$ there exists a path $g_t \colon [0, 1) \to \mathrm{Sp}_2(G, \sigma)$ such that $g_0 = g$, $\lim_{t\to 1} g_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Using polar decomposition, $x = ub$ for some $u \in U(G_0, \sigma)$, $b \in (B^{sym})^\times$. Take $u_t \colon [0, 1] \to U(G_0, \sigma)$ such that $u_0 = u$, $u = 1$ for $t \geq \frac{1}{2}$. It is possible because $U(G_0, \sigma)$ is connected. Take $b_t \colon [0, 1) \to (B^{sym})^\times$ such that $\lim_{t\to 1} b_t = 0$. Take $y_t, z_t \colon [0, 1) \to B^{sym}$ such that $z_0 = z$ $y_0 = y$ and $x_t = -y_t = b_t^{-1}$ for $t \geq \frac{1}{2}$. It is possible since B^{sym} is connected. Define $g_t = L(y_t)D(x_t)R(z_t)$. Then:

$$\lim_{t\to 1} g_t = \lim_{t\to 1} \begin{pmatrix} u_t b_t & u_t b_t z_t \\ y_t u_t b_t & y_t u_t b_t z_t + \sigma(u_t b_t)^{-1} \end{pmatrix} =$$

$$= \lim_{t\to 1} \begin{pmatrix} b_t & b_t b_t^{-1} \\ b_t^{-1} b_t & b_t^{-1} b_t b_t^{-1} + b_t^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore,

$$\mathrm{Sp}_2(G, \sigma) = \overline{\{L(y)D(x)R(z) \mid y, z \in B^{sym},\ x \in G\} \cup \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}}$$

and $\{L(y)D(x)R(z) \mid y, z \in B^{sym},\ x \in G\} \cup \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ is connected. So $\mathrm{Sp}_2(G, \sigma)$ is connected. □

Theorem 2.7.4. *Let B be of Jordan type such that $1 \in B$. Then the space $\mathrm{Sp}_2(G, \sigma)$ is a group.*

Proof. First, we prove the two Lemmata:

Lemma 2.7.5. *Let $1+y$ is not invertible for a $y \in A$. Then there exists a neighborhood U of $0 \in \mathbb{R}$ such that for every $t \in U \setminus \{0\}$, $1 + y(1 + t)$ is invertible.*

Proof. As we have seen in the Proposition 2.1.5, A can be embedded as a subalgebra into $\mathrm{Mat}(r,\mathbb{R})$ for some $r \in \mathbb{N}$. We identify B with a Lie subalgebra of $\mathrm{Mat}(r,\mathbb{R})$. Since $1+y$ is not invertible, -1 is an eigenvalue of y. Since y has only finitely many eigenvalues, there exists a neighborhood U of $0 \in \mathbb{R}$ such that for every $t \in U \setminus \{0\}$, $1+y(1+t)$ is invertible. $\square$

We remind, $\bar{G}$ is the topological closure of G in A. It is a monoid.

Lemma 2.7.6. *Let $y, z \in B^{sym}$ then $1+zy \in \bar{G}$. In particular, if $1+zy$ is invertible, then $1+zy \in G$.*

Proof. First, assume $z \in (B^{sym})^\times \subset G$. Then $1+zy = (z+zyz)z^{-1} \in \bar{G}$ because $z^{-1} \in G$, $z, zyz \in B^{sym} \subseteq \bar{G}$.

If z is not invertible, take a sequence of invertible (z_i) such that $\lim z_i = z$. Then all $1+z_iy \in \bar{G}$ and $\lim(1+z_iy) = 1+zy$. Since $\bar{G}$ is closed, $1+zy \in \bar{G}$. $\square$

Let $a = \lim L(y_i)D(x_i)R(z_i)$, $b = \lim L(y_i')D(x_i')R(z_i')$ for some sequences $\{y_i\}, \{y_i'\}, \{z_i\}, \{z_i'\} \subset B^{sym}$, $\{x_i\}, \{x_i'\} \subset G$. We want to show that there exist sequences $\{y_i''\}, \{z_i''\} \subset B^{sym}$, $\{x_i''\} \subset G$ such that $ab = \lim L(y_i'')D(x_i'')R(z_i'')$.

Since limits for a and b exist, we can write:

$$ab = \lim L(y_i)D(x_i)R(z_i)L(y_i')D(x_i')R(z_i').$$

Consider the term

$$R(z_i)L(y_i') = \begin{pmatrix} 1 & z_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_i' & 1 \end{pmatrix} = \begin{pmatrix} 1+z_iy_i' & z_i \\ y_i' & 1 \end{pmatrix}.$$

If $1+z_iy_i'$ is invertible then by the Lemma 2.7.6, $1+z_iy_i' \in G$, then we can write

$$R(z_i)L(y_i') = L(z_i(1+z_iy_i')^{-1})D(1+z_iy_i')R((1+z_iy_i')^{-1}y_i').$$

If $1+z_iy_i'$ is not invertible in B then we take a sequence $\{t_i\} \subset \mathbb{R}$ such that $\lim t_i = 0$ and $t_i \in U_i$ where U_i is the neighborhood of $0 \in \mathbb{R}$ form the Lemma 2.7.5 for the element $1+z_iy_i'$. Then $b = \lim L(y_i'(1+t_i))D(x_i')R(z_i')$ and $1+z_iy_i'(1+t_i)$ is invertible in B.

To conclude the proof, note the following permutation rules:

$$D(x)L(y) = L(\sigma(x)^{-1}yx^{-1})D(x),$$
$$R(z)D(x) = D(x)R(x^{-1}z\sigma(x)^{-1})$$

that always make possible to reorder matrices and

$$L(y)L(y') = L(y+y'),\ R(x)R(x') = R(z+z'),\ D(x)D(x') = D(xx')$$

for all $x, x' \in G$, $y, y', z, z' \in B^{sym}$.

Now, we show that for $a \in \mathrm{Sp}_2(G,\sigma)$, $a^{-1} \in \mathrm{Sp}_2(G,\sigma)$. Indeed, $\mathrm{Sp}_2(G,\sigma) \subseteq \mathrm{Sp}_2(A,\sigma)$ and $\mathrm{Sp}_2(A,\sigma)$ is a group, therefore, a^{-1} exists in $\mathrm{Sp}_2(A,\sigma)$. As before, we take a sequence $a_i := L(y_i)D(x_i)R(z_i)$ where $x_i \in G$, $y_i, z_i \in B^{sym}$ such that $\lim a_i = a$. Then $a_i^{-1} = R(-z_i)D(x_i^{-1})L(-y_i) \in \mathrm{Sp}_2(G,\sigma)$. Therefore, $\lim a_i^{-1} = a^{-1} \in \mathrm{Sp}_2(G,\sigma)$. $\square$

Definition 2.7.7. Matrices of the form $L(y)D(x)R(z)$ for $y, z \in B^{sym},\ x \in G$ are called *generic.*

Proposition 2.7.8. *$\mathfrak{sp}_2(B,\sigma)$ is the Lie algebra of* $\mathrm{Sp}_2(G,\sigma)$.

Proof. The neighborhood of the identity in $\mathrm{Sp}_2(G,\sigma)$ consists only of generic matrices. Consider a smooth path $p(t) := L(y(t))D(x(t))R(z(t))$ such that $y, z\colon (-1,1) \to B^{sym}$ are smooth and $y(0) = z(0) = 0$ and $x\colon (-1,1) \to G$ smooth and $x(0) = 1$. Then

$$p'(0) = \begin{pmatrix} x'(0) & z'(0) \\ y'(0) & -\sigma(x'(0)) \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma)$$

Moreover, for every $m := \begin{pmatrix} x & z \\ y & -\sigma(x) \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma),\ y, x \in B^{sym},\ x \in B$ the path $p(t) := L(yt)D(\exp(xt))R(zt) \in \mathrm{Sp}_2(G,\sigma)$ and $p'(0) = m$. Therefore, $\mathrm{Lie}(\mathrm{Sp}_2(G,\sigma)) = \mathfrak{sp}_2(B,\sigma)$. □

2.7.2 Another definition of $\mathrm{Sp}_2(G,\sigma)$

We denote by $\mathrm{Sp}'_2(G,\sigma)$ the Lie subgroup of $\mathrm{Sp}_2(A,\sigma)$ generated by matrices:

$$D(x) := \begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1} \end{pmatrix},\ I := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ R(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

where $x \in G_0,\ z \in B^{sym}$. Since all generators of $\mathrm{Sp}'_2(G,\sigma)$ are in $\mathrm{Sp}_2(G,\sigma)$, $\mathrm{Sp}'_2(G,\sigma) \le \mathrm{Sp}_2(G,\sigma)$. In this subsection, we show that actually $\mathrm{Sp}'_2(G,\sigma) = \mathrm{Sp}_2(G,\sigma)$.

Lemma 2.7.9. *Matrices $L(z) := \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ are in $\mathrm{Sp}'_2(G,\sigma)$ for all $z \in B^{sym}$.*

Proof.

$$L(z) := \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

□

Proposition 2.7.10. *$\mathfrak{sp}_2(B,\sigma)$ is the Lie algebra of* $\mathrm{Sp}'_2(G,\sigma)$.

Proof. Let $x_0 \in B$. Since $\mathrm{Lie}(G) = B$, there exists a smooth path $x(t) \in G_0$ such that $x(0) = 1$ and $x'(0) = x_0$. Take a smooth path $g(t) = \begin{pmatrix} x(t) & 0 \\ 0 & \sigma(x(t))^{-1} \end{pmatrix} \in \mathrm{Sp}_2(G,\sigma)$. Then $g'(0) = \begin{pmatrix} x'(0) & 0 \\ 0 & -\sigma(x'(0)) \end{pmatrix} \in \mathfrak{sp}_2(B,\sigma)$.

Let $z_0 \in B^{sym}$. Since B^{sym} is a vector space, there exists a smooth path $z(t) \in B^{sym}$ such that $z(0) = 0,\ z'(0) = z_0$. Consider the smooth path $g(t) = \begin{pmatrix} 1 & z(t) \\ 0 & 1 \end{pmatrix} \in$

$\mathrm{Sp}_2(G,\sigma)$ where $z(t) \in B^{sym}$ for all t and $z(0) = 0$. Then $g'(0) = \begin{pmatrix} 0 & z'(0) \\ 0 & 0 \end{pmatrix}$. Since $z'(0) \in B^{sym}$, $g'(0) \in \mathfrak{sp}_2(B,\sigma)$.

Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix},$$

the similar argument as above shows that elements $\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$ for $z \in B^{sym}$ are also covered by derivations of paths in $\mathrm{Sp}_2(G,\sigma)$. □

Proposition 2.7.11. *If B is weakly Hermitian, then* $\mathrm{Sp}_2'(G,\sigma) = \mathrm{Sp}_2(G,\sigma)$

Proof. Let U be a small neighborhood of $0 \in \mathrm{Lie}(\mathrm{Sp}_2(G,\sigma))$ such that $\exp|_U : U \to \mathrm{Sp}_2(G,\sigma)$ is a diffeomorphism and $\exp(U)$ consists only of regular elements. Then $\exp(U) \subseteq \mathrm{Sp}_2'(G,\sigma)$.

Moreover, since $\mathrm{Sp}_2(G,\sigma)$ is connected, $\exp(U)$ generates $\mathrm{Sp}_2(G,\sigma)$. Therefore, $\mathrm{Sp}_2(G,\sigma) \leq \mathrm{Sp}_2'(G,\sigma)$. □

2.7.3 Center of $\mathrm{Sp}_2(G,\sigma)$ and the group $\mathrm{PSp}_2(G,\sigma)$

Proposition 2.7.12. *The center $Z(\mathrm{Sp}_2(G,\sigma))$ of $\mathrm{Sp}_2(G,\sigma)$ is isomorphic to $Z(G) \cap U(G,\sigma)$ where $Z(G)$ is the center of G. More precisely,*

$$Z(\mathrm{Sp}_2(G,\sigma)) = \{\mathrm{diag}(a,a) \mid a \in Z(G) \cap U(G,\sigma)\}.$$

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(\mathrm{Sp}_2(G,\sigma))$, then M commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This gives: $d = a$, $c = -b$. Also M commutes with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}_2(G,\sigma)$. This gives $b = 0$. Since $M = \mathrm{diag}(a,a) \in \mathrm{Sp}_2(A,\sigma)$, $\sigma(a)^{-1} = a$. Moreover, M commutes with all $\mathrm{diag}(g, \sigma(g)^{-1})$, i.e. $a \in Z(G)$. Therefore,

$$Z(\mathrm{Sp}_2(G,\sigma)) < \{\mathrm{diag}(a,a) \mid a \in Z(G) \cap U(G,\sigma)\}.$$

It is also easy to see that matrices $\mathrm{diag}(a,a)$ for $a \in Z(G) \cap U(G,\sigma)$ commute with all elements of $\mathrm{Sp}_2(G,\sigma)$. therefore,

$$Z(\mathrm{Sp}_2(G,\sigma)) = \{\mathrm{diag}(a,a) \mid a \in Z(G) \cap U(G,\sigma)\}.$$

□

Corollary 2.7.13. *For B Hermitian, $Z(\mathrm{Sp}_2(G,\sigma))$ is compact.*

Definition 2.7.14. The quotient group

$$\mathrm{PSp}_2(G,\sigma) := \mathrm{Sp}_2(G,\sigma)/Z(\mathrm{Sp}_2(G,\sigma))$$

is called *projective symplectic group.*

2.7.4 Maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$

In this section, we assume B to be Hermitian. We describe a maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$.

Definition 2.7.15. We denote:

$$\mathrm{U}_2(A,\sigma) := \{M \in \mathrm{Mat}_2(A) \mid \sigma(M)^T M = \mathrm{Id}\};$$

$$\mathrm{KSp}_2(G,\sigma) := \mathrm{Sp}_2(G,\sigma) \cap \mathrm{U}_2(A,\sigma).$$

Lemma 2.7.16. *For every $M \in \mathrm{Sp}_2(G,\sigma)$, all components $M_{ij} \in \bar{G}$.*

Proof. Since $M = \lim M^{(i)}$ such that $M^{(i)}$ are generic, i.e. $M_{11}^{(i)} \in G$, $M_{11} \in \bar{G}$. For every $M \in \mathrm{Sp}_2(G,\sigma)$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M \in \mathrm{Sp}_2(G,\sigma)$. Therefore, $\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M\right)_{11} = -M_{21} \in \bar{G}$. Similarly, $\left(M\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)_{11} = M_{12} \in \bar{G}$ and $\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)_{11} = -M_{22} \in \bar{G}$. □

Proposition 2.7.17. $\mathrm{KSp}_2(G,\sigma) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, \begin{matrix} \sigma(a)a + \sigma(b)b = 1 \\ \sigma(a)b - \sigma(b)a = 0 \end{matrix},\ a, b \in \bar{G} \right\}.$

Proof. Take $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{KSp}_2(G,\sigma)$. On one hand, $M \in \mathrm{Sp}_2(G,\sigma)$, therefore,

$$M^{-1} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma(M)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma(d) & -\sigma(b) \\ -\sigma(c) & \sigma(a) \end{pmatrix}.$$

On the other hand, $M \in \mathrm{U}_2(A,\sigma)$, therefore,

$$M^{-1} = \sigma(M)^T = \begin{pmatrix} \sigma(a) & \sigma(c) \\ \sigma(b) & \sigma(d) \end{pmatrix}.$$

So we obtain, $a = d$ and $b = -c$. □

Theorem 2.7.18. $\mathrm{KSp}_2(G,\sigma)$ *is a maximal compact subgroup of* $\mathrm{Sp}_2(G,\sigma)$.

Proof. By definition, $\mathrm{KSp}_2(G,\sigma)$ is closed subgroup of $\mathrm{Sp}_2(G,\sigma)$. Let $M := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{KSp}_2(G,\sigma)$. Then

$$\sigma(M)^T M = \begin{pmatrix} \sigma(a) & -\sigma(b) \\ \sigma(b) & \sigma(a) \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$= \begin{pmatrix} \sigma(a)a + \sigma(b)b & * \\ * & \sigma(b)b + \sigma(a)a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\sigma(a)a + \sigma(b)b = 1$, i.e.

$$a, b \in D = \{x \in \bar{G} \mid 1 - \sigma(x)x \in B^{sym}_{\geq 0}\} \subseteq \bar{G}$$

which is compact by 2.6.49, $\mathrm{KSp}_2(G,\sigma)$ can be seen as a closed subset of the compact D^4, so it is compact.

Now, we show that $\mathrm{KSp}_2(G,\sigma)$ is a maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$. Let K be some compact subgroup containing $\mathrm{KSp}_2(G,\sigma)$ as a proper subgroup. We consider the following decomposition of $\mathfrak{sp}_2(G,\sigma)$:

$$\mathfrak{sp}_2(G,\sigma) = \mathfrak{ksp}_2(G,\sigma) \oplus \mathrm{Sym}_2(G,\sigma)$$

where

$$\mathfrak{ksp}_2(G,\sigma) = \mathrm{Lie}(\mathrm{KSp}_2(G,\sigma)) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid \sigma(a) = -a \in B, b \in B^{sym} \right\},$$

$$\mathrm{Sym}_2(G,\sigma) = \left\{ \begin{pmatrix} c & d \\ d & -c \end{pmatrix} \mid c, d \in B^{sym} \right\}.$$

By our assumption, $\mathrm{Lie}(K)$ contains $\mathfrak{ksp}_2(G,\sigma)$ and has nontrivial intersection with $\mathrm{Sym}_2(G,\sigma)$. Take some $\begin{pmatrix} c & d \\ d & -c \end{pmatrix} \in \mathrm{Lie}(K) \cap \mathrm{Sym}_2(G,\sigma)$, $c, d \in B^{sym}$. The matrix $\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \in \mathfrak{ksp}_2(G,\sigma) \subset \mathrm{Lie}(K)$, therefore,

$$\begin{pmatrix} c & 2d \\ 0 & -c \end{pmatrix} = \begin{pmatrix} c & d \\ d & -c \end{pmatrix} + \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \in \mathrm{Lie}(K) \setminus \mathfrak{ksp}_2(G,\sigma).$$

Using the exponential map of $\mathfrak{sp}_2(G,\sigma)$ restricted to $\mathrm{Lie}(K)$, we obtain that there exits a matrix $M := \begin{pmatrix} g & gx \\ 0 & g^{-1} \end{pmatrix} \in K \setminus \mathrm{KSp}_2(G,\sigma)$ where $g = \exp(c) \in G^{sym}$, $x \in B^{sym}$. Consider the spectral decomposition of $g = \sum_{i=1}^{k} \lambda_i c_i$ for some $\lambda_i > 0$ and $(c_i)_{i=1}^{k}$ a complete orthogonal system of idempotents. Take a sequence $\{M^r\} \subseteq K$. Then

$$M^r_{11} = g^k = \sum_{i=1}^{k} \lambda_i^r c_i,$$

$$M^r_{22} = g^{-k} = \sum_{i=1}^{k} \lambda_i^{-r} c_i.$$

Assume, there exists $s \in \{1, \dots, k\}$ such that $\lambda_s \neq \pm 1$. Then either $0 < |\lambda_s| < 1$ or $0 < |\lambda_s^{-1}| < 1$. Without lost of generality, assume $0 < |\lambda_s| < 1$. Since K is compact, $\{M^r\} \subseteq K$ have a convergent subsequence $\{M^{r_j}\} \subseteq K$:

$$\lim M^{r_j}_{11} = \lim \sum_{i=1}^{k} \lambda_i^{r_j} c_i = \sum_{i=1}^{k} \hat{\lambda}_i c_i$$

where $\hat{\lambda}_i = \lim \lambda_i^{r_j}$. But $\hat{\lambda}_s = \lim \lambda_s^{r_j} = 0$ for any subsequence $\{r_j\}$. Therefore $\lim M_{11}^{r_j}$ is not invertible and do $\lim M^{r_j}$ is not invertible as well. Therefore, all $\lambda_i = \pm 1$ and $g^2 = 1$. The element $L := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in \mathrm{KSp}_2(G,\sigma) \subset K$. Then $ML = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in K$. Take $(ML)^r = \begin{pmatrix} 1 & rx \\ 0 & 1 \end{pmatrix} \in K$. This sequence does not have any convergent subsequence unless $x = 0$. So we get $M = L \in \mathrm{KSp}_2(G,\sigma)$. This contradicts to the assumption $M \notin \mathrm{KSp}_2(G,\sigma)$ and we obtain that $\mathrm{KSp}_2(G,\sigma)$ is a maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$. $\square$

2.7.5 Maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$ for complex G

In this section, we assume A to be a $\mathbb{C}$-algebra with a $\mathbb{C}$-linear anti-involution σ, $B \subseteq A$ a $\mathbb{C}$-Lie subalgebra, $G \subseteq A^\times$ a Lie subgroup such that $\mathrm{Lie}\, G = B$, B and G are closed under σ and (B,σ) to be of Jordan type. We denote by $\bar{\sigma}$ the composition of σ and the complex conjugation. To distinguish between symmetric elements with respect to different anti-involutions σ and $\bar{\sigma}$, we denote

$$B^\sigma := \mathrm{Fix}_B(\sigma),\ B^{\bar{\sigma}} := \mathrm{Fix}_B(\bar{\sigma}),$$

$$G^\sigma := \mathrm{Fix}_G(\sigma),\ G^{\bar{\sigma}} := \mathrm{Fix}_G(\bar{\sigma}).$$

We assume $(B^\sigma)^\times \subseteq G$, so the group $\mathrm{Sp}_2(G,\sigma)$ is well-defined. Assume also $(B,\bar{\sigma})$ to be Hermitian, so we have the proper convex cone $B^{\bar{\sigma}}_{\geq 0}$.

Definition 2.7.19. We denote:

$$\mathrm{KSp}_2^c(G,\sigma) := \mathrm{Sp}_2(G,\sigma) \cap \mathrm{U}_2(A,\bar{\sigma}).$$

Lemma 2.7.20. *For every $M \in \mathrm{Sp}_2(G,\sigma)$, all components $M_{ij} \in \bar{G}$.*

Proof. The proof is identic to the Lemma 2.7.16. $\square$

Proposition 2.7.21. $\mathrm{KSp}_2^c(G,\sigma) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \begin{array}{l} \bar{\sigma}(a)a + \sigma(b)\bar{b} = 1 \\ \bar{\sigma}(a)b - \sigma(b)\bar{a} = 0 \end{array},\ a, b \in \bar{G} \right\}.$

Proof. Take $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{KSp}_2(G,\sigma)$. On one hand, $M \in \mathrm{Sp}_2(G,\sigma)$, therefore,

$$M^{-1} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma(M)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma(d) & -\sigma(b) \\ -\sigma(c) & \sigma(a) \end{pmatrix}.$$

On the other hand, $M \in \mathrm{U}_2(A,\bar{\sigma})$, therefore,

$$M^{-1} = \bar{\sigma}(M)^T = \begin{pmatrix} \bar{\sigma}(a) & \bar{\sigma}(c) \\ \bar{\sigma}(b) & \bar{\sigma}(d) \end{pmatrix}.$$

So we obtain, $d = \bar{a}$ and $c = -\bar{b}$. $\square$

Theorem 2.7.22. $\mathrm{KSp}_2^c(G,\sigma)$ *is a maximal compact subgroup of* $\mathrm{Sp}_2(G,\sigma)$.

Proof. By definition, $\mathrm{KSp}_2^c(G,\sigma)$ is closed subgroup of $\mathrm{Sp}_2(G,\sigma)$. Let $M := \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{KSp}_2^c(G,\sigma)$. Then

$$\bar{\sigma}(M)^T M = \begin{pmatrix} \bar{\sigma}(a) & -\sigma(b) \\ \bar{\sigma}(b) & \sigma(a) \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\sigma}(a)a + \sigma(b)\bar{b} & * \\ * & \bar{\sigma}(b)b + \sigma(a)\bar{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\bar{\sigma}(a)a + \sigma(b)\bar{b} = \bar{\sigma}(a)a + \bar{\sigma}(\bar{b})\bar{b} = 1$, i.e.

$$a, \bar{b} \in D = \{x \in \bar{G} \mid 1 - \bar{\sigma}(x)x \in B^{\bar{\sigma}}_{\geq 0}\} \subseteq \bar{G}$$

which is compact by 2.6.49, $\mathrm{KSp}_2^c(G,\sigma)$ can be seen as a closed subset of the compact D^4, so it is compact.

Now, we show that $\mathrm{KSp}_2^c(G,\sigma)$ is a maximal compact subgroup of $\mathrm{Sp}_2(G,\sigma)$. Let K be some compact subgroup containing $\mathrm{KSp}_2^c(G,\sigma)$ as a proper subgroup. We consider the following decomposition of $\mathfrak{sp}_2(G,\sigma)$:

$$\mathfrak{sp}_2(G,\sigma) = \mathfrak{ksp}_2^c(G,\sigma) \oplus \mathrm{Herm}_2(G,\bar{\sigma})$$

where

$$\mathfrak{ksp}_2^c(G,\sigma) = \mathrm{Lie}(\mathrm{KSp}_2^c(G,\sigma)) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid \bar{\sigma}(a) = -a \in B, b \in B^{\bar{\sigma}} \right\},$$

$$\mathrm{Herm}_2(G,\sigma) = \left\{ \begin{pmatrix} c & d \\ \bar{d} & -\bar{c} \end{pmatrix} \mid c, d \in B^{\bar{\sigma}} \right\}.$$

By our assumption, $\mathrm{Lie}(K)$ contains $\mathfrak{ksp}_2^c(G,\sigma)$ and has nontrivial intersection with $\mathrm{Herm}_2(G,\sigma)$. Take some $\begin{pmatrix} c & d \\ \bar{d} & -\bar{c} \end{pmatrix} \in \mathrm{Lie}(K) \cap \mathrm{Herm}_2(G,\sigma)$, $c, d \in B^{\bar{\sigma}}$. The matrix $\begin{pmatrix} 0 & d \\ -\bar{d} & 0 \end{pmatrix} \in \mathfrak{ksp}_2^c(G,\sigma) \subset \mathrm{Lie}(K)$, therefore,

$$\begin{pmatrix} c & 2d \\ 0 & -\bar{c} \end{pmatrix} = \begin{pmatrix} c & d \\ \bar{d} & -\bar{c} \end{pmatrix} + \begin{pmatrix} 0 & d \\ -\bar{d} & 0 \end{pmatrix} \in \mathrm{Lie}(K) \setminus \mathfrak{ksp}_2^c(G,\sigma).$$

Using the exponential map of $\mathfrak{sp}_2(G,\sigma)$ restricted to $\mathrm{Lie}(K)$, we obtain that there exits a matrix $M := \begin{pmatrix} g & gx \\ 0 & \bar{g}^{-1} \end{pmatrix} \in K \setminus \mathrm{KSp}_2^c(G,\sigma)$ where $g = \exp(c) \in G^{\bar{\sigma}}$, $x \in B^{\bar{\sigma}}$. Consider the spectral decomposition of $g = \sum_{i=1}^k \lambda_i c_i$ for some $\lambda_i > 0$ and $(c_i)_{i=1}^k$ a complete orthogonal system of idempotents. Take a sequence $\{M^r\} \subseteq K$. Then

$$M^r_{11} = g^k = \sum_{i=1}^k \lambda_i^r c_i,$$

$$M_{22}^r = \bar{g}^{-k} = \sum_{i=1}^{k} \lambda_i^{-r} c_i.$$

Assume, there exists $s \in \{1, \dots, k\}$ such that $\lambda_s \neq \pm 1$. Then either $0 < |\lambda_s| < 1$ or $0 < |\lambda_s^{-1}| < 1$. Without lost of generality, assume $0 < |\lambda_s| < 1$. Since K is compact, $\{M^r\} \subseteq K$ have a convergent subsequence $\{M^{r_j}\} \subseteq K$:

$$\lim M_{11}^{r_j} = \lim \sum_{i=1}^{k} \lambda_i^{r_j} c_i = \sum_{i=1}^{k} \hat{\lambda}_i c_i$$

where $\hat{\lambda}_i = \lim \lambda_i^{r_j}$. But $\hat{\lambda}_s = \lim \lambda_s^{r_j} = 0$ for any subsequence $\{r_j\}$. Therefore $\lim M_{11}^{r_j}$ is not invertible and do $\lim M^{r_j}$ is not invertible as well. Therefore, all $\lambda_i = \pm 1$ and $g^2 = 1$. The element $L := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in \mathrm{KSp}_2^c(G, \sigma) \subset K$. Then $ML = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in K$. Take $(ML)^r = \begin{pmatrix} 1 & rx \\ 0 & 1 \end{pmatrix} \in K$. This sequence does not have any convergent subsequence unless $x = 0$. So we get $M = L \in \mathrm{KSp}_2^c(G, \sigma)$. This contradicts to the assumption $M \notin \mathrm{KSp}_2^c(G, \sigma)$ and we obtain that $\mathrm{KSp}_2^c(G, \sigma)$ is a maximal compact subgroup of $\mathrm{Sp}_2(G, \sigma)$. □

2.7.6 More on the algebra $A_{\mathbb{C}}$

In this section, we study some additional properties of the complexified algebra $A_{\mathbb{C}}$ of some Hermitian algebra (A, σ) that are connected to the spectral theorem. As we have seen, $(A_{\mathbb{C}}, \bar{\sigma})$ is Hermitian as well. First, we study the group $U(A_{\mathbb{C}}, \bar{\sigma})$. Later, we find out how $A_{\mathbb{C}}^\times$ acts on $(A_{\mathbb{C}}^\sigma)^\times$.

Theorem 2.7.23. *Let Y be a finite dimensional $\mathbb{C}$-algebra, $V \subseteq Y$ be a $\mathbb{C}$-vector subspace. Then $V^\times$ is connected.*

Proof. If $V^\times = \varnothing$ then $V^\times$ is connected.

Assume now that $1 \in V^\times \neq \varnothing$. As we have seen in the Proposition 2.1.5, Y can be embedded as a subalgebra into $\mathrm{Mat}(r, \mathbb{C})$ for some $r \in \mathbb{N}$. We identify Y as a subalgebra of $\mathrm{Mat}(r, \mathbb{C})$. Let $a \in V^\times \subseteq \mathrm{GL}(n, \mathbb{C})$. Since a has only finitely many eigenvalues, and 0 is not one of them, there is a point $z \in S^1 \subset \mathbb{C}$ such that the line in $\mathbb{C}$ through the origin containing z does not intersect any of the eigenvalues of a. Now, consider the path $f(t) = at + z(1-t)\,\mathrm{Id}$, $t \in [0, 1]$. It lies completely in V because it is a $\mathbb{C}$-vector space. This has determinant 0 if and only if $z(t-1)$ is an eigenvalue of at, which happens if and only if $z(1-t)/t$ is an eigenvalue of a (this does not work when $t = 0$, but then it is clear that the determinant is non-zero). By construction, it is not the case for any $t \in [0, 1]$, so this defines a path form a to $z\,\mathrm{Id}$.

Now, there is a path in $\mathbb{C}$ not passing through 0 from z to 1, and, since $\{z\,\mathrm{Id} \mid z \in S^1 \subset \mathbb{C}\} \subset V^\times$, this gives rise to a path in $A_{\mathbb{C}}^\times$ from $z\,\mathrm{Id}$ to Id, and so concatenating these two paths, we get a path from a to Id that lies in $V^\times$, showing that $V^\times$ is path connected.

Finally, if $1 \notin V^\times$ but there exists $v \in V^\times$, then $V^\times$ is connected if and only if $(v^{-1}V)^\times = v^{-1}(V^\times)$ is connected. Moreover, $v^{-1}V$ is also a $\mathbb{C}$-vector space and $1 \in (v^{-1}V)^\times$. So $(v^{-1}V)^\times$ is connected and, therefore, $V^\times$ is connected as well. $\square$

Corollary 2.7.24. *The space $A_{\mathbb{C}}^\times$ is path connected. In particular, $U(A_{\mathbb{C}}, \bar{\sigma})$ is path connected (as deformation retract of $A_{\mathbb{C}}^\times$).*

Later, we assume that $A_{\mathbb{C}}$ can be embedded as a subalgebra into $\mathrm{Mat}(r, \mathbb{C})$ for some $r \in N$.

Theorem 2.7.25 (Spectral theorem for $U(A_{\mathbb{C}}, \bar{\sigma})$, first version)**.** *For every $u \in U(A_{\mathbb{C}}, \bar{\sigma})$, there exist a unique $r \in \mathbb{N}$, a unique complete system of orthogonal idempotents $c_1, \dots, c_k \in A_{\mathbb{C}}^{\bar{\sigma}}$ and a unique sequence of elements $\theta_1, \dots, \theta_r \in \mathbb{R}/(2\pi\mathbb{Z})$ such that for all $i \neq j$, $\theta_i \neq \theta_j$*

$$u = \sum_{j=1}^{r} e^{i\theta_j} c_j.$$

Proof. Consider the Lie algebra of $U(A_{\mathbb{C}}, \bar{\sigma})$:

$$\mathrm{Lie}(U(A_{\mathbb{C}}, \bar{\sigma})) = \{z \in A_{\mathbb{C}} \mid \bar{\sigma}(z) = -z\}.$$

Consider the following map:

$$\begin{array}{cccc} \psi\colon & A_{\mathbb{C}}^{\bar{\sigma}} & \mapsto & \mathrm{Lie}(U(A_{\mathbb{C}}, \bar{\sigma})) \\ & a & \mapsto & ia \end{array}.$$

This map is injective. Moreover, because

$$\dim_{\mathbb{R}}(A_{\mathbb{C}}^{\bar{\sigma}}) = \dim_{\mathbb{R}} A = \dim_{\mathbb{R}}(\mathrm{Lie}(U(A_{\mathbb{C}}, \bar{\sigma}))),$$

this map is an isomorphism of $\mathbb{R}$-vector spaces. By the first version of the spectral theorem, for every $a \in A_{\mathbb{C}}^{\bar{\sigma}}$, there exists a unique complete system of orthogonal idempotents $c_1, \dots, c_r \in A_{\mathbb{C}}^{\bar{\sigma}}$ of $A_{\mathbb{C}}^{\bar{\sigma}}$ and a unique sequence of elements $\theta_1, \dots, \theta_r \in \mathbb{R}$ such that for all $i \neq j$, $\theta_i \neq \theta_j$ and

$$a = \sum_{j=1}^{r} \theta_j c_j.$$

Therefore,

$$\exp(ia) = \sum_{j=1}^{r} e^{i\theta_j} c_j \in U(A_{\mathbb{C}}, \bar{\sigma}).$$

The exponential map is surjective for compact connected groups, so every element of $U(A_{\mathbb{C}}, \bar{\sigma})$ admit such presentation.

Assume, there are two such presentations with minimal number of idempotents for an element $u \in U(A_{\mathbb{C}}, \bar{\sigma})$:

$$u = \exp(ia) = \exp(ia'),$$

$$a = \sum_{j=1}^{r} \theta_j c_j,\ a' = \sum_{j=1}^{r'} \theta'_j c'_j$$

for some $r, r' \in \mathbb{N}$, (θ_j), (θ'_j) in $\mathbb{R}$, (c_j), (c'_j) complete systems of orthogonal idempotents. Since the number of idempotents is minimal in both presentations, $e^{i\theta_j} \neq e^{i\theta_k}$ for all $i \neq k$ and also $e^{i\theta'_j} \neq e^{i\theta'_k}$ for all $i \neq k$.

We multiply u by c_k from the left and by c'_l from the right:

$$c_k u c'_l = e^{i\theta_k} c_k c'_l = e^{i\theta'_l} c_k c'_l.$$

So either $c_k c'_l = 0$ or $e^{i\theta_k} = e^{i\theta'_l}$, i.e.

$$\{e^{i\theta_k} \mid k \in \{1, \ldots, r\}\} = \{e^{i\theta'_l} \mid l \in \{1, \ldots, r'\}\},$$

in particular $r = r'$. We can reorder indices so that $e^{i\theta'_k} = e^{i\theta_k}$ for all $k \in \{1, \ldots r\}$. Therefore:

$$c_k u = e^{i\theta_k} c_k = e^{i\theta'_k} c_k c'_k,$$
$$u c'_k = e^{i\theta'_k} c'_k = e^{i\theta_k} c_k c'_k.$$

We get $c_k = c_k c'_k = c'_k$ for all $k \in \{1, \ldots, r\}$, i.e. $(c_j) = (c'_j)$. So we obtain that the presentation is unique. □

Theorem 2.7.26 (Spectral theorem for $U(A, \sigma)$)**.** *For every $u \in U(A, \sigma)$, there exist unique $r, s \in \mathbb{N}\{0\}$, $s \leq 2$, unique systems of idempotents $c_1, \ldots, c_r \in A_{\mathbb{C}}^{\bar{\sigma}}$ and $c'_1, \ldots, c'_s \in A^{\sigma}$ such that $c_1, \ldots, c_r, \bar{c}_1, \ldots, \bar{c}_r, c'_1, \ldots, c'_s$ is a complete orthogonal system of idempotents of $A_{\mathbb{C}}^{\bar{\sigma}}$ and unique sequences of elements $\theta_1, \ldots, \theta_r \in \mathbb{R}/(2\pi\mathbb{Z})$ and $\varepsilon_1, \ldots \varepsilon_s \in \{1, -1\}$ such that for all $i \neq j$, $\theta_i \neq \theta_j$ and $\varepsilon_i \neq \varepsilon_j$ and*

$$u = \sum_{j=1}^{r} (e^{i\theta_j} c_j + e^{-i\theta_j} \bar{c}_j) + \sum_{j=1}^{s} \varepsilon_j c'_j.$$

Proof. For an element $u \in U(A_{\mathbb{C}}, \bar{\sigma})$, $u \in U(A, \sigma)$ if and only if $u = \bar{u}$. We take the spectral decomposition of u and $\bar{u}$:

$$u = \sum_{j=1}^{k} e^{i\theta_j} c_j,$$

$$\bar{u} = \sum_{j=1}^{k} e^{-i\theta_j} \bar{c}_j.$$

Notice, all $\bar{c}_1, \ldots, \bar{c}_k$ is a complete orthogonal system of idempotents of $A_{\mathbb{C}}^{\bar{\sigma}}$ because $\bar{c}_i \bar{c}_j = \overline{c_i c_j}$. If $u = \bar{u}$, then, because of uniqueness of the spectral decomposition, for every $j \in \{1, \ldots, k\}$ there exists $j' \in \{1, \ldots, k\}$ such that $e^{i\theta_j} c_j = e^{-i\theta_{j'}} \bar{c}_{j'}$. There can be to cases:

1. $j = j'$ then $e^{i\theta_j} \in \mathbb{R}$ i.e. $e^{i\theta_j} \in \{1, -1\}$.

2. $j \neq j'$ then $c_{j'} = \bar{c}_j$.

Because all $e^{i\theta_j}$ are distinct, there can be at most one j such that $e^{i\theta_j} = 1$ and at most one j with $e^{i\theta_j} = -1$. For such j, $c_j \in A^\sigma$. So we obtain

$$u = \sum_{j=1}^{r} (e^{i\theta_j} c_j + e^{-i\theta_j} \bar{c}_j) + \sum_{j=1}^{s} \varepsilon_j c_j'.$$

for appropriate $r, s \in \mathbb{N}\{0\}$, $s \leq 2$. $\square$

Corollary 2.7.27. *In every connected component of $U(A, \sigma)$ there is an element from $(A^\sigma)^\times$.*

Proof. Take $u \in U(A, \sigma)$ and its spectral decomposition:

$$u = \sum_{j=1}^{r} (e^{i\theta_j} c_j + e^{-i\theta_j} \bar{c}_j) + \sum_{j=1}^{s} \varepsilon_j c_j'.$$

Since the circle $\mathbb{R}/(2\pi\mathbb{Z})$ is connected, take for every j a path $\theta_j(t)$ connecting θ_j and $0 \in \mathbb{R}/(2\pi\mathbb{Z})$. Then

$$u(t) = \sum_{j=1}^{r} (e^{i\theta_j(t)} c_j + e^{-i\theta_j(t)} \bar{c}_j) + \sum_{j=1}^{s} \varepsilon_j c_j'.$$

connects u and $\sum_{j=1}^{r} (c_j + \bar{c}_j) + \sum_{j=1}^{s} \varepsilon_j c_j' \in (A^\sigma)^\times$. $\square$

Corollary 2.7.28. *The group $A^\times$ is generated by $A_0^\times$ and $(A^\sigma)^\times$ where $A_0^\times$ is the connected component of $1 \in A^\times$.*

Proof. As we have seen using the polar decomposition, $U(G, \sigma)$ is a deformation retract of $A^\times$. In every connected component of $U(G, \sigma)$, there is an element of $(A^\sigma)^\times$. Therefore, every connected component $C \subseteq A$ can be written as $C = bA_0^\times$ where $A_0^\times$ is a connected component of $1 \in A^\times$, $b \in (A^\sigma)^\times$. Therefore $A^\times$ is contained in the group generated by $A_0^\times$ and A^σ. That the group generated by $A_0^\times$ and A^σ is contained in $A^\times$ is clear. $\square$

Corollary 2.7.29 (Spectral theorem for $U(A_\mathbb{C}, \bar{\sigma})$, second version)**.** *Suppose, $A_\mathbb{C}^{\bar{\sigma}}$ has rank n. For every $u \in U(A_\mathbb{C}, \bar{\sigma})$, there exist unique sequence $\theta_1, \ldots, \theta_n \in \mathbb{R}/(2\pi\mathbb{Z})$ and a Jordan frame $e_1, \ldots, e_n \in A_\mathbb{C}^{\bar{\sigma}}$ such that*

$$u = \sum_{j=1}^{n} e^{i\theta_j} e_j.$$

The elements $\theta_1, \ldots, \theta_n \in \mathbb{R}/(2\pi\mathbb{Z})$ (with their multiplicities) are uniquely determined by u. In particular, they do not depend (up to permutations) on the Jordan frame $e_1, \ldots, e_n \in B^{sym}$.

Definition 2.7.30. The *determinant map* on $U(A_{\mathbb{C}}, \bar{\sigma})$ is given by:

$$\det(u) = \prod_{j=1}^{n} e^{i\theta_j} \in S^1 \subset \mathbb{C}$$

where $u = \sum_{j=1}^{n} e^{i\theta_j} e_j$ for some Jordan frame $e_1, \dots, e_n \in A_{\mathbb{C}}^{\bar{\sigma}}$.

Proposition 2.7.31. *The fundamental group of $U(A_{\mathbb{C}}, \bar{\sigma})$ is infinite.*

Proof. The determinant map

$$\det\colon U(A_{\mathbb{C}}, \bar{\sigma}) \to S^1$$

is continuous and surjective. It induces the homomorphism of fundamental groups:

$$(\det)_*\colon \pi_1(U(A_{\mathbb{C}}, \bar{\sigma}), 1) \to \pi_1(S^1, 1)$$

that is surjective because the curve $u(t) = e^{it} e_1 + \sum_{j=2}^{n} e_j$ for $t \in [0, 2\pi]$ for some Jordan frame $e_1, \dots, e_n$ maps to the following generator of $\pi_1(S^1, 1)$: e^{it}, $t \in [0, 2\pi]$. So $\pi_1(U(A_{\mathbb{C}}, \bar{\sigma}), 1)$ is infinite. □

Corollary 2.7.32. *The fundamental group of $\mathrm{Sp}_2(A, \sigma)$ is infinite.*

Theorem 2.7.33. *The following action of $A_{\mathbb{C}}^{\times}$ on $(A_{\mathbb{C}}^{\sigma})^{\times}$*

$$\begin{array}{rccl} \psi_{\mathbb{C}}\colon & A_{\mathbb{C}}^{\times} \times (A_{\mathbb{C}}^{\sigma})^{\times} & \to & (A_{\mathbb{C}}^{\sigma})^{\times} \\ & (a, b) & \mapsto & \sigma(a) b a \end{array}$$

is transitive.

Proof. Take $b \in (A_{\mathbb{C}}^{\sigma})^{\times}$. We take a polar decomposition of $b = ub'$ where $u \in U(A, \bar{\sigma})$, $b' \in (A_{\mathbb{C}}^{\bar{\sigma}})^{\times}$. We take the spectral decomposition of b':

$$b' = \sum_{i=1}^{k} \lambda_i c_i$$

for a complete system of orthogonal idempotents $\{c_i\}_{i=1}^{k} \subset A_{\mathbb{C}}^{\bar{\sigma}}$ and all $\lambda_i > 0$.

The group $U(A_{\mathbb{C}}, \bar{\sigma})$ acts transitively on Jordan frame of $A_{\mathbb{C}}^{\bar{\sigma}}$. Therefore, if we fix a Jordan frame $\{x_i\}_{i=1}^{k} \subset A^{sym} \subset A_{\mathbb{C}}^{\bar{\sigma}}$, there exists $u' \in U(A_{\mathbb{C}}, \bar{\sigma})$ such that $x_i = \bar{\sigma}(u') c_i u'$. Then

$$\sigma(u') b u' = \sigma(u') u \bar{\sigma}(u')^{-1} \bar{\sigma}(u') b' u' = \sigma(u') u \bar{\sigma}(u')^{-1} \sum_{i=1}^{k} \lambda_i x_i =: u'' b''$$

where $u'' = \sigma(u') u \bar{\sigma}(u')^{-1}$, $b'' = \sum_{i=1}^{k} \lambda_i x_i$.

Since $\sigma(u') b u' = u'' b'' \in A^{\sigma}$, $u'' b'' = \sigma(u'' b'') = b'' \sigma(u'') = b'' (\bar{u}'')^{-1}$. Therefore, $b'' = u'' b'' \bar{u}''$. By induction, we obtain $b'' = (u'')^n b'' (\bar{u}'')^n$ for all $n \in \mathbb{Z}$, or equivalently

$(u'')^n b'' = b''(\bar{u}'')^{-n}$. Since it holds for every $n \in \mathbb{Z}$, the following holds: $f(u'')b'' = b'' f((\bar{u}'')^{-1})$ for every holomorphic function in small enough neighborhood of u''. Since $u'' \in U(A_{\mathbb{C}}, \bar{\sigma})$, $u'' \neq 0$. By the Theorem 2.7.25, there exists $w \in U(A_{\mathbb{C}}, \bar{\sigma})$ such that $(u'') = w^2$. So we can take as f the branch of square root such that $f(u'') = w$. Then we obtain $wb'' = b''(\bar{w})^{-1}$. Therefore,

$$u''b'' = wb''\bar{w}^{-1} = (wb^{\frac{1}{2}})b^{\frac{1}{2}}\bar{w}^{-1} = \sigma(b^{\frac{1}{2}}\bar{w}^{-1})b^{\frac{1}{2}}\bar{w}^{-1}.$$

Therefore, $\sigma(u')bu' = u''b''$ is in the orbit of 1 under the action $\psi_{\mathbb{C}}$ and also b is in this orbit because $b = \sigma((u')^{-1})u''b''(u')^{-1}$. $\square$

2.8 Invariants of G-isotropic lines

2.8.1 G-isotropic lines

In this section, we assume B to be weakly Hermitian. We consider $\mathrm{Is}_G(\omega) = \mathrm{Sp}_2(G, \sigma)(1, 0)^T$ the orbit of $(1, 0)^T \in A^2$ under the action of $\mathrm{Sp}_2(G, \sigma)$. Note, if $x \in \mathrm{Is}_G(\omega)$, then $xc \in \mathrm{Is}_G(\omega)$ for all $c \in G$.

Remark 2.8.1. The element $(0, 1)^T$ is in $\mathrm{Is}_G(\omega)$ since $-I(1, 0)^T = (0, 1)$ where $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark 2.8.2. The space $\mathrm{Is}_G(\omega)$ is closed in the space $\mathrm{Is}(\omega)$ of all isotropic elements of A^2. It follows from the fact that $\mathrm{Sp}_2(G, \sigma)$ is closed in $\mathrm{Sp}_2(A, \sigma)$ and $\mathrm{Sp}_2(A, \sigma)$ acts transitively on $\mathrm{Is}(\omega)$.

Definition 2.8.3. A line $l \subset A^2$ is called *G-isotropic* if $l = yA$ for some $y \in \mathrm{Is}_G(\omega)$.

Proposition 2.8.4. *Let $x = (x_1, x_2)^T \in \bar{G}^2$ be a regular element, then*

$$\sigma(x_1)x_1 + \sigma(x_2)x_2 \in B_+^{sym}.$$

In particular, this holds for all elements of $\mathrm{Is}_G(\omega)$.

Proof. First, we prove the following Lemma:

Lemma 2.8.5. *Let $b \in B^{sym}$ is not invertible, then there exists $b' \in B_{\geq 0}^{sym} \setminus \{0\}$ such that $bb' = 0$*

Proof. Assume b to be not invertible and consider its spectral decomposition $b = \sum_{i=1}^{k} \lambda_i c_i$ for some (c_i) complete system of orthogonal idempotents. Since b is not invertible, there exist $j \in \{1, \dots, k\}$ such that $\lambda_j = 0$. Take $b' = c_j \in B_{\geq 0}^{sym}$. $\square$

Since for every $g \in \bar{G}$, $\sigma(g)g \in B_{\geq 0}^{sym}$, $b := \sigma(x_1)x_1 + \sigma(x_2)x_2 \in B_{\geq 0}^{sym}$. Assume, b is not invertible for some regular $x \in \bar{G}^2$. Take b' as in Lemma, then

$$0 = b'bb' = \sigma(x_1 b')x_1 b' + \sigma(x_2 b')x_2 b'$$

and $\sigma(x_1b')x_1b', \sigma(x_2b')x_2b' \in B^{sym}_{\geq 0}$. Since $B^{sym}_{\geq 0}$ is a proper convex cone, $\sigma(x_1b)x_1b = \sigma(x_2b)x_2b = 0$. Since $x_1b, x_2b \in \bar{G}$, take its polar decomposition: $x_1b = u_1y_1$, $x_2b = u_2y_2$ where $y_1, y_2 \in B^{sym}$, $u_1, u_2 \in U(G,\sigma)$. Then $\sigma(x_1b)x_1b = y_1^2$, $\sigma(x_2b)x_2b = y_2^2$. Since B^{sym} does not contain nilpotents, $y_1 = y_2 = 0$. Therefore, $x_1b = x_2b = 0$, i.e. $x = (x_1, x_2)^T$ is not regular. This contradicts to our assumption that x is regular. $\square$

2.8.2 Action of $\mathrm{Sp}_2(G,\sigma)$ on G-isotropic lines

Proposition 2.8.6. $\mathrm{Sp}_2(G,\sigma)$ *acts transitively on the space of G-isotropic lines.*

$$\mathrm{Stab}_{\mathrm{Sp}_2(G,\sigma)}(1,0)^T A := \left\{ \begin{pmatrix} x & xy \\ 0 & \sigma(x)^{-1} \end{pmatrix} \mid x \in G, y \in B^{sym} \right\}$$

$$\mathrm{Stab}_{\mathrm{Sp}_2(G,\sigma)}(0,1)^T A := \left\{ \begin{pmatrix} x & 0 \\ zx & \sigma(x)^{-1} \end{pmatrix} \mid x \in G, z \in B^{sym} \right\}$$

Proof. $\mathrm{Sp}_2(G,\sigma)$ acts transitively on the space of G-isotropic lines since it acts transitively on $\mathrm{Is}_G(\omega)$.

We prove only the statement for the first stabilizer. The second one can be proved analogously.

Since
$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix},$$
$x \in A^\times$ and $z = 0$. Therefore, the matrix is generic. So it has the form $D(x)R(y)$. $\square$

2.8.3 Action of $\mathrm{Sp}_2(G,\sigma)$ on pairs of G-isotropic lines

Proposition 2.8.7. *Two elements $u, v \in \mathrm{Is}_G(\omega)$ are linearly independent if and only if, up to action of $\mathrm{Sp}_2(G,\sigma)$, $u = (1,0)^T$, $v = (a,b)^T$ with $b \in G$. Moreover, if $\omega(u,v) = 1$, then $a \in B^{sym}$, $b = 1$.*

Proof. $\mathrm{Sp}_2(G,\sigma)$ acts transitively on $\mathrm{Is}_G(\omega)$, therefore, up to $\mathrm{Sp}_2(G,\sigma)$-action, we can assume $u = (1,0)^T$.

Since u and v are linearly independent, $b \in A^\times$ and $v = g(1,0)^T$ for some $g \in \mathrm{Sp}_2(G,\sigma)$. If $g = L(y)D(x)R(z)$ for some $x \in G$, $y, z \in B^{sym}$, then $v = (x, yx)^T = (1, y)^T x$. Therefore, $y \in (B^{sym})^\times \subseteq G$ and so $b = yx \in G$.

If g is not generic, take a sequence $\{g_n\}$ of generic elements such that $g_n \to g$. Then $G \ni y_nx_n \to b \in A^\times$. Since G is closed in $A^\times$ and $x_n, y_n \in G$, $b \in G$.

Let now $\omega(u,v) = 1$, then $1 = \omega(u,v) = yx$. So if g generic, then $a = x = y^{-1} \in B^{sym}$. If g is not generic, then $a = \lim(x_n) = \lim(y_n^{-1})$. But all $y_n^{-1} \in B^{sym}$ and B^{sym} is closed in A, so $a \in B^{sym}$. $\square$

Corollary 2.8.8. *If $x, y \in \mathrm{Is}_G(\omega)$ linearly independent, then $\omega(x,y) \in G$.*

Definition 2.8.9. A symplectic basis (x,y) of (A^2,ω) is called (G,σ)-*symplectic* if $x, y \in \mathrm{Is}_G(\omega)$.

Proposition 2.8.10. *If (x, y) is a (G, σ)-symplectic basis then there exists the unique $g \in \mathrm{Sp}_2(G, \sigma)$ such that $g(1, 0)^T = x$, $g(0, 1)^T = y$. In particular, $\mathrm{Sp}_2(G, \sigma)$ acts transitively on (G, σ)-symplectic bases.*

Proof. We can assume, $x = (1, 0)^T$, $y = (a, 1)^T$ and $a \in B^{sym}$. Take $g := R(-a)$, then $R(-a)x = x$, $R(-a)y = (0, 1)^T$. □

Corollary 2.8.11. *Let xA, yA be two transverse isotropic lines with $x, y \in \mathrm{Is}_G(\omega)$. Then there exist $M \in \mathrm{Sp}_2(G, \sigma)$ and $y' \in \mathrm{Is}_G(\omega)$ such that $y'A = yA$ and $Mx = (1, 0)^T$, $My' = (0, 1)^T$. In particular, $\omega(x, y') = 1$.*

Proposition 2.8.12. $\mathrm{Sp}_2(G, \sigma)$ *acts transitively on pairs of transverse G-isotropic lines.*

$$\mathrm{Stab}_{\mathrm{Sp}_2(G,\sigma)}((1, 0)^T A, (0, 1)^T A) := \left\{ \begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1} \end{pmatrix} \mid x \in G \right\} \cong G.$$

Proof. By the Corollary 2.8.11, every pairs of transverse G-isotropic lines can be mapped to $((1, 0)^T A, (0, 1)^T A)$ by an element of $\mathrm{Sp}_2(G, \sigma)$. So $\mathrm{Sp}_2(G, \sigma)$ acts transitively on pairs of transverse G-isotropic lines.

By the Proposition 2.8.6, for every $M \in \mathrm{Stab}_{\mathrm{Sp}_2(G,\sigma)}((1, 0)^T A, (0, 1)^T A)$, $M = D(x)R(y)$ for $x \in G$, $y \in B^{sym}$. Moreover, $D(x)R(y)(0, 1)^T = (xy, \sigma(x)^{-1})$. Therefore, $y = 0$. □

2.8.4 Action of $\mathrm{Sp}_2(G, \sigma)$ on positive triples of G-isotropic lines

Let $(x_1 A, x_3 A, x_2 A)$ be a triple of pairwise transverse G-isotropic lines where all $x_i \in \mathrm{Is}_G(\omega)$. Because of transversality of $x_1 A$ and $x_2 A$, we can assume $\omega(x_1, x_2) = 1$. Up to action of $\mathrm{Sp}_2(G, \sigma)$, we can assume $x_1 = (1, 0)^T$, $x_2 = (0, 1)^T$. We can also normalize x_3 so that $\omega(x_3, x_2) = 1$. Then $x_3 = (1, b)^T$, $b = \omega(x_1, x_3) \in (B^{sym})^\times$.

Definition 2.8.13. A triple of pairwise transverse G-isotropic lines $(x_1 A, x_3 A, x_2 A)$ is called *positive* if $\omega(x_1, x_2) = \omega(x_3, x_2) = 1$ and $\omega(x_1, x_3) \in B^{sym}_+$.

Proposition 2.8.14. *The definition of positivity of a triple of G-isotropic lines does not depend on the choice of x_1, x_2, x_3.*

Proof. Let $y_i \in \mathrm{Is}_G(\omega)$ such that $y_i A = x_i A$ for all $i \in \{1, 2, 3\}$. Then $y_i = x_i g_i$ for some $g_i \in G$. Since $1 = \omega(y_1, y_2) = \sigma(g_1)\omega(x_1, x_2)g_2 = \sigma(g_1)g_2$, $g_2 = \sigma(g_1)^{-1}$. Similarly $g_2 = \sigma(g_3)^{-1}$. Therefore, $g_1 = g_3$.

$$\omega(y_1, y_3) = \sigma(g_1)\omega(x_1, x_3)g_1 = \sigma(g_1)bg_1 \in B^{sym}_+$$

if and only if $b \in B^{sym}_+$. □

Remark 2.8.15. For every transverse triple $(x_1 A, x_3 A, x_2 A)$, up to action of $\mathrm{Sp}_2(G, \sigma)$, we can write $x_1 = (1, 0)^T, x_2 = (0, 1)^T$ and:

$$x_3 A = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} A = \begin{pmatrix} 1 \\ b \end{pmatrix} A$$

for $b \in B^{sym}$. The triple is positive if and only if $b \in B^{sym}_+$. Matrices of the form $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ for $b \in B^{sym}_+$ form a subsemigroup of $\mathrm{Sp}_2(G,\sigma)$ which we denote by $U^{>0}$.

Lemma 2.8.16. *For every positive triple (l_1, l_3, l_2) of isotropic lines, elements $y_1, y_2, y_3 \in \mathrm{Is}_G(\omega)$ can be chosen so that*

- *$l_i = y_i A$ for $i \in \{1,2,3\}$;*
- *$\omega(y_1, y_2) = 1$;*
- *$y_3 = y_1 + y_2$.*

Proof. Let $l_i = x_i A$ for some regular $x_i \in \mathrm{Is}_G(\omega)$. By transversality, (x_1, x_2) form a basis. As above, we can assume $\omega(x_1, x_2) = 1$, $x_3 = x_1 + x_2 a$ for $a \in B^{sym}_+$. Take $c := a^{\frac{1}{2}} \in B^{sym}_+ \subseteq G$. Consider a new basis $y_1 = x_1 c^{-1}$, $y_2 = x_2 \sigma(c)$. Then,

$$\omega(y_1, y_2) = \omega(x_1 c^{-1}, x_2 \sigma(c)) = \sigma(c)^{-1} 1 \sigma(c) = 1.$$

Moreover, $x_3 = y_1 c + y_2 \sigma(c)^{-1} \sigma(c) c = y_1 c + y_2 c$. If we take $y_3 := x_3 c^{-1} = y_1 + y_2$, we get $y_3 A = x_3 A$. □

Proposition 2.8.17. *$\mathrm{Sp}_2(G,\sigma)$ acts transitively on the space of positive triples of pairwise transverse isotropic lines.*

The stabilizer of the positive triple

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} A, \begin{pmatrix} 1 \\ 1 \end{pmatrix} A, \begin{pmatrix} 0 \\ 1 \end{pmatrix} A \right)$$

in $\mathrm{Sp}_2(G,\sigma)$ coincides with the following subgroup:

$$\hat{U} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in U(G,\sigma) \right\} \cong U(G,\sigma)$$

The stabilizer of every positive triple of isotropic lines is conjugated in $\mathrm{Sp}_2(G,\sigma)$ to $\hat{U}$.

Proof. Let (l_1, l_3, l_2) be a positive triple. By the Lemma 2.8.16, there exist $y_i \in l_i$, $i \in \{1,2,3\}$ such that $l_i = y_i A$, $\omega(y_1, y_2) = 1$ and $y_3 = y_1 + y_2$. By the Proposition 2.8.10, there exists $M \in \mathrm{Sp}_2(G,\sigma)$ such that $My_1 = (1,0)^T$, $My_2 = (0,1)^T$. Therefore, $Ml_1 = (1,0)^T A$, $Ml_2 = (0,1)^T A$, $Ml_3 = (1,1)^T A$ i.e. every positive triple can be mapped to the standard positive triple $((1,0)^T A, (1,1)^T A, (0,1)^T A)$.

By the Proposition 2.8.12, for every M that stabilizes $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} A, \begin{pmatrix} 1 \\ 1 \end{pmatrix} A, \begin{pmatrix} 0 \\ 1 \end{pmatrix} A \right)$, $M = D(x)$ for $x \in G$. Moreover, $M(1,1)^T = (x, \sigma(x)^{-1})$. Therefore, $x = \sigma(x)^{-1}$, i.e. $x \in U(G,\sigma)$. □

Proposition 2.8.18. *Positivity of triple is an invariant under cyclic permutations, i.e. if (l_1, l_3, l_2) is positive, then (l_2, l_1, l_3) is positive as well.*

Proof. The triple (l_1, l_3, l_2) is positive if and only if there exist $x_1, x_2, x_3 \in \mathrm{Is}_G(\omega)$ such that $l_i = x_i A$, $i \in \{1,2,3\}$ and $\omega(x_1, x_2) = \omega(x_3, x_2) = 1$, $\omega(x_1, x_3) = b \in B_+^{sym}$.

The triple (l_2, l_1, l_3) is positive if and only if there exist $y_1, y_2, y_3 \in \mathrm{Is}_G(\omega)$ such that $l_i = y_i A$, $i \in \{1,2,3\}$ and $\omega(y_2, y_3) = \omega(y_1, y_3) = 1$, $\omega(y_2, y_1) \in B_+^{sym}$.

Take $y_1 = x_1 b^{-1}$, $y_2 = -x_2$, $y_3 = x_3$, then $\omega(y_2, y_3) = \omega(-x_2, x_3) = 1$, $\omega(y_1, y_3) = \omega(x_1 b^{-1}, x_3) = 1$, $\omega(y_2, y_1) = \omega(-x_2, x_1 b^{-1}) = b^{-1} \in B_+^{sym}$. □

2.8.5 Invariant of a positive quadruple of G-isotropic lines

For this an next section, we fix some Jordan frame $(e_i)_{i=1}^n$ of B^{sym}.

Definition 2.8.19. A quadruple (l_1, l_3, l_2, l_4) of pairwise transverse G-isotropic lines is called *positive* if the triples (l_1, l_3, l_2), (l_2, l_4, l_1) are positive.

Proposition 2.8.20. *Let (l_1, l_3, l_2, l_4) be a positive quadruple of G-isotropic lines. Then there exist $y_1, \dots, y_4 \in \mathrm{Is}_G(\omega)$ such that $l_i = y_i A$, $y_3 = y_1 + y_2$, $y_4 = y_1 - y_2 a$, $a \in B_+^{sym}$.*

For any such choice of (y_1, y_2, y_3, y_4), there exists a Jordan frame $(e'_i)_{i=1}^n$ of B^{sym}, where $n = \mathrm{rk}(B^{sym})$ and a unique tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n > 0$ such that where $a = \sum_{i=1}^n \lambda_i e'_i$.

Up to action of $\mathrm{Sp}_2(B, \sigma)$ on G-isotropic lines, we can assume that all $e'_i = e_i$.

Proof. Follows directly from the spectral theorem and Proposition 2.8.17. □

Remark 2.8.21.
- The tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n > 0$ does not depend on the choice of (y_1, y_2, y_3, y_4). It is an invariant of quadruple of G-isotropic lines under the action of $\mathrm{Sp}_2(G, \sigma)$. We denote
$$[l_1, l_3, l_2, l_4] := (\lambda_1, \dots, \lambda_n).$$
- Although the tuple $(\lambda_1, \dots, \lambda_n)$ is unique, the Jordan frame $(e'_i)_{i=1}^n$ is in general not unique.

Proposition 2.8.22. *Positivity of quadruple is an invariant under cyclic permutations, i.e. if (l_1, l_3, l_2, l_4) is positive, then (l_3, l_2, l_4, l_1) is positive as well. Moreover, if $[l_1, l_3, l_2, l_4] = (\lambda_1, \dots, \lambda_n)$, then*
$$[l_3, l_2, l_4, l_1] = (\lambda_n^{-1}, \dots, \lambda_1^{-1}).$$

Proof. Let (l_1, l_3, l_2, l_4) be a positive quadruple of isotropic lines. Then up to action of $\mathrm{Sp}_2(G, \sigma)$, there exist $y_1, \dots, y_4 \in \mathrm{Is}_G(\omega)$ such that $l_i = y_i A$, $y_3 = y_1 + y_2$, $y_4 = y_1 - y_2 a$, $a = \sum_{i=1}^n \lambda_i e_i$ with $\lambda_1 \geq \dots \geq \lambda_n > 0$.

The triple (l_3, l_2, l_4) is positive. Indeed, since $a + 1 \in B_+^{sym}$, it is invertible. So we can take
$$x_3 := y_3 (a+1)^{-\frac{1}{2}} \in l_3,$$
$$x_4 := -y_4 (a+1)^{-\frac{1}{2}} \in l_4,$$

$$x_2 := y_2(a+1)^{\frac{1}{2}} \in l_2.$$

Therefore,

$$\omega(x_3, x_4) = \omega((y_1+y_2)(a+1)^{-\frac{1}{2}}, -(y_1-y_2a)(a+1)^{-\frac{1}{2}}) =$$

$$= -(a+1)^{-\frac{1}{2}}(-a-1)(a+1)^{-\frac{1}{2}} = 1,$$

$$\omega(x_2, x_4) = \omega(y_2(a+1)^{\frac{1}{2}}, -(y_1-y_2a)(a+1)^{-\frac{1}{2}}) = 1,$$

$$\omega(x_3, x_2) = \omega((y_1+y_2)(a+1)^{-\frac{1}{2}}, y_2(a+1)^{\frac{1}{2}}) = 1 \in B_+^{sym}.$$

Analogously, the triple (l_4, l_1, l_3) is positive as well. So we get that the quadruple (l_3, l_2, l_4, l_1) is positive.

Take $x_1 := y_1(a+1)^{\frac{1}{2}} \in l_1$. Then easy calculation shows that

$$x_2 = x_3 + x_4,\ x_1 = x_3 - x_4a^{-1}$$

where $a^{-1} = \sum_{i=1}^n \lambda_i^{-1}e_i$. Therefore, $[l_3, l_2, l_4, l_1] = (\lambda_n^{-1}, \dots, \lambda_1^{-1})$. $\square$

2.8.6 Invariant of a positive 5-tuple of G-isotropic lines: angles

Definition 2.8.23. A 5-tuple $(l_1, l_5, l_3, l_2, l_4)$ of pairwise transverse G-isotropic lines is called *positive* if the triples (l_1, l_3, l_2), (l_2, l_4, l_1) and (l_1, l_5, l_3) are positive.

Let $(l_1, l_5, l_3, l_2, l_4)$ be positive 5-tuple of isotropic lines, which we will think as the vertices of a pentagon, as in Figure 2.8.1.

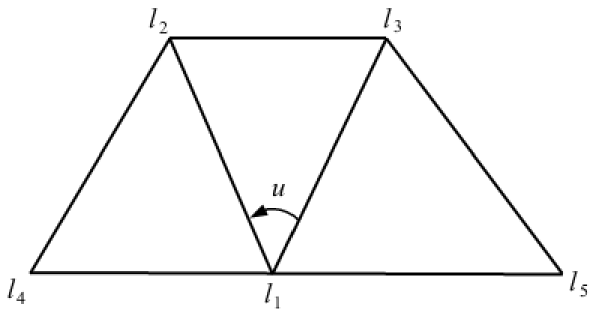

Figure 2.8.1:

Using Proposition 2.8.20, up to action of $\mathrm{Sp}_2(G,\sigma)$, we can find $y_1, \dots, y_4 \in \mathrm{Is}_G(\omega)$ such that $l_i = y_iA$, $i \in \{1,2,3,4\}$ and $y_3 = y_1 + y_2$, $y_4 = y_1 - y_2a$, and there exists a unique tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_k > 0$ such that where $a = \sum_{i=1}^n \lambda_ie_i$.

Since the triples (l_3, l_2, l_1) and (l_1, l_5, l_3) are positive and $y_2 = y_3 - y_1 = (y_1+y_2) - y_1$, by Proposition 2.8.20, there exists an element $y_5 \in \mathrm{Is}_G(\omega)$ such that $l_5 = y_5A$ and $y_5 = y_3 + y_1a'$, $a' \in B_+^{sym}$. By the second version of the spectral theorem, there exists an element $u \in U(G,\sigma)$ and a unique tuple $(\mu_1,\dots,\mu_n)$ with $\mu_1 \geq \dots \geq \mu_k > 0$ such that where $a' = \sigma(u)\sum_{i=1}^n \mu_i e_i u$.

We will call this element $u \in U(G,\sigma)$ an inner *angle* in the pentagon of isotropic lines $(L_1, L_4, L_2, L_3, L_5)$ (see Figure 2.8.1).

Remark 2.8.24. The element u is not uniquely defined. In general, u is only well defined as an element of the double coset space $\operatorname{Stab}(a) \setminus U(G,\sigma)/\operatorname{Stab}(a')$, where

$$\operatorname{Stab}(a) := \{v \in U(G,\sigma) \mid \ \sigma(v)av = a\},$$

$$\operatorname{Stab}(a') := \{v \in U(G,\sigma) \mid \ \sigma(v)a'v = a'\}.$$

This double coset is an invariant of 5-tuple of G-isotropic lines under the action of $\operatorname{Sp}_2(G,\sigma)$. It depends on the choice of the Jordan frame $(e_i)_{i=1}^n$.

2.9 Models for the symmetric space of $\operatorname{Sp}_2(G,\sigma)$ for real G

The goal of this Chapter is to construct different models of the symmetric space for $\operatorname{Sp}_2(G,\sigma)$ for $\operatorname{Lie} G = B$ where $B \subseteq A$ is a Hermitian Lie subalgebra.

2.9.1 Complex structures model

In this section, we assume (A,σ) to be an $\mathbb{R}$-algebra with an involution, $B \subseteq A$ be a Hermitian Lie subalgebra, $G \subseteq A^\times$ a Lie group such that $\operatorname{Lie} G = B$ as in the Section 2.6.5, so the group $\operatorname{Sp}_2(G,\sigma)$ is well-defined.

Definition 2.9.1. A *complex structure* on an right A-module V is an A-linear map $J\colon V \to V$ such that $J^2 = -\operatorname{Id}$.

Let $V = A^2$ and ω be the standard symplectic form in A^2. For every complex structure J on A^2, we can define the following σ-sesquilinear form

$$\begin{array}{cccc} h_J\colon & A^2 \times A^2 & \to & A \\ & (x,y) & \mapsto & \omega(J(x),y) \end{array}$$

Definition 2.9.2. A σ-sesquilinear form h on (A^2,ω) is called (G,σ)-*inner product* if h is σ-symmetric and for all $v \in \operatorname{Is}_G(\omega)$, $h(v,v) \in B_+^{sym}$.

We consider the following space:

$$\mathfrak{C} := \left\{ J \text{ complex structure on } A^2 \;\middle|\; \begin{array}{l} J(\operatorname{Is}_G(\omega)) = \operatorname{Is}_G(\omega), \\ h_J \text{ is a } (G,\sigma)\text{-inner product} \end{array} \right\}.$$

Definition 2.9.3. The *standard complex* structure on A^2 is the map

$$\begin{array}{cccc} J_0\colon & A^2 & \to & A^2 \\ & (x,y) & \mapsto & (y,-x) \end{array}$$

Remark 2.9.4. h_{J_0} is the standard (G,σ)-inner product on A^2, i.e. $[h_{J_0}] = \mathrm{diag}(1,1)$. By the Proposition 2.8.4, $J_0 \in \mathfrak{C}$.

Remark 2.9.5. If $G = A^\times$ and A is a Hermitian algebra, then every $(A^\times,\sigma)$-inner product on (A^2,ω) is a σ-inner product on A^2.

Proposition 2.9.6. *Let J be a complex structure on A^2. $J \in \mathfrak{C}$ if and only if there exists $w \in \mathrm{Is}_G(\omega)$ such that $(J(w),w)$ is a (G,σ)-symplectic basis.*

Proof. 1. Let $J \in \mathfrak{C}$ and $w' \in \mathrm{Is}_G(\omega)$. Since $h_J(w',w') = b \in B_+^{sym}$, we can take $w := w'b^{-\frac{1}{2}}$, then $h_J(w,w) = 1$. Then:

$$\omega(J(w),J(w)) = h_J(w,J(w)) = \sigma(h_J(J(w),w)) = \sigma(\omega(w,w)) = 0,$$

$$\omega(J(w),w) = h_J(w,w) = 1.$$

Therefore, $(J(w),w)$ is a (G,σ)-symplectic basis.

2. Let $w \in A^2$ such that $(J(w),w)$ is a (G,σ)-symplectic basis. Then,

$$h_J(w,w) = \omega(J(w),w) = 1$$

$$h_J(J(w),J(w)) = \omega(J^2(w),J(w)) = -\omega(w,J(w)) = 1,$$
$$h_J(J(w),w) = \omega(J^2(w),w) = -\omega(w,w) = 0.$$

Therefore, $(J(w),w)$ is an orthonormal basis for h_J and in this basis h_J is the standard σ-inner product, so h_J is an σ-inner product. $\square$

Theorem 2.9.7. $\mathrm{Sp}_2(G,\sigma)$ *acts on $\mathfrak{C}$ by conjugation. This action is transitive. The stabilizer of the standard complex structure is* $\mathrm{KSp}_2(G,\sigma)$.

In particular, $\mathfrak{C}$ is a model of the symmetric space of $\mathrm{Sp}_2(G,\sigma)$.

Proof. 1. First, we prove that $\mathrm{Sp}_2(G,\sigma)$ acts on $\mathfrak{C}$ by conjugation. Let $J \in \mathcal{S}'$, $g \in \mathrm{Sp}_2(A,\sigma)$. Consider $J' := g^{-1}Jg$. $(J')^2 = g^{-1}J^2g = -\mathrm{Id}$ so J' is a complex structure on A^2. For $x \in \mathrm{Is}_G(\omega)$, $g(x) \in \mathrm{Is}_G(\omega)$ and we obtain

$$h_{J'}(x,x) = \omega(J'(x),x) = \omega(g^{-1}Jg(x),x) =$$

$$= \omega(Jg(x),g(x)) = h_J(g(x),g(x)) \in B_+^{sym}.$$

Moreover, J' preserves $\mathrm{Is}_G(\omega)$, therefore, $h_{J'}$ is a (G,σ)-inner product on A^2, i.e. $J' \in \mathfrak{C}$.

2. Second, we prove that this action is transitive. Let $J \in \mathfrak{C}$, take a (G,σ)-symplectic basis $(J(w),w)$ from the Proposition 2.9.6. Since by Proposition 2.8.10, $\mathrm{Sp}_2(G,\sigma)$ acts transitively on (G,σ)-symplectic bases, there exists $g \in \mathrm{Sp}_2(G,\sigma)$ which maps the standard symplectic basis to $(J(w),w)$. That means, g maps the standard complex structure J_0 to J. So the action is transitive.

3. Finally, compute the stabilizer of J_0. $g \in \mathrm{Stab}_{\mathrm{Sp}_2(G,\sigma)}(J_0)$ if and only if $g \in \mathrm{Sp}_2(G,\sigma)$ and $g \in \mathrm{O}(h_{J_0}) = \mathrm{U}_2(A,\sigma)$, i.e.

$$g \in \mathrm{Sp}_2(G,\sigma) \cap \mathrm{U}_2(A,\sigma) = \mathrm{KSp}_2(G,\sigma). \qquad \square$$

2.9.2 Projective model

In this section, we assume $B \subseteq A$ to be a Hermitian Lie subalgebra of A. Let $B_{\mathbb{C}} := B \otimes_{\mathbb{R}} \mathbb{C} \subseteq A \otimes_{\mathbb{R}} \mathbb{C} = A_{\mathbb{C}}$ be the complexification of B and $G_{\mathbb{C}}$ the Lie subgroup of $A^{\times}$ as in the Section 2.6.5.

As usual, we denote by σ the $\mathbb{C}$-linear extension of σ, i.e.

$$\sigma(x+iy) = \sigma(x) + i\sigma(y)$$

for every $x, y \in B$ and by $\bar{\sigma}$ the $\mathbb{C}$-antilinear extension of σ, i.e.

$$\bar{\sigma}(x+iy) = \sigma(x) - i\sigma(y)$$

for every $x, y \in B$.

In this section, we do not assume $(B_{\mathbb{C}}, \bar{\sigma})$ to be Hermitian.

We extend ω in the complex linear way to $\omega_{\mathbb{C}}$ on $A^2_{\mathbb{C}}$. We also extend every complex structure on A^2 to a complex structure on $A^2_{\mathbb{C}}$ in the complex linear way. We denote the extension of J by $J_{\mathbb{C}}$.

Proposition 2.9.8. *For every complex structure $J \in \mathcal{S}'$, there exist regular isotropic $x, y \in A^2_{\mathbb{C}}$ such that $J_{\mathbb{C}}(x) = ix$, $J_{\mathbb{C}}(y) = -iy$. Lines $xA_{\mathbb{C}}$, $yA_{\mathbb{C}}$ are uniquely defined.*

Proof. Since $\mathrm{Sp}_2(G, \sigma)$ acts transitively on $\mathcal{S}'$, it is enough to prove the proposition for the standard complex structure J_0.

Since $J_0(a,b)^T = (b,-a)^T$, $(b,-a)^T = i(a,b)^T$ if and only if $b = ai$, i.e. $x = (1,i)^T a$ for $a \in A_{\mathbb{C}}$, i.e. $xA_{\mathbb{C}}$ is uniquely defined. Analogously, $y = (i,1)^T a$ for $a \in A_{\mathbb{C}}$, i.e. $yA_{\mathbb{C}}$ is uniquely defined. Moreover, $\omega_{\mathbb{C}}(x,x) = \omega_{\mathbb{C}}(y,y) = 0$, so $x, y \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ □

Remark 2.9.9. For every (G,σ)-symplectic basis (w,u) of (A^2, ω), elements $u + wi, w + iu$ are isotropic.

For a complex structure $J \in \mathfrak{C}$, we denote by l_J the line $yA_{\mathbb{C}}$ such that $J_{\mathbb{C}}(y) = -yi$.

Proposition 2.9.10. *The map*

$$\begin{array}{rccl} F\colon & \mathfrak{C} & \to & \mathfrak{P} := \{(u+wi)A_{\mathbb{C}} \mid (w,u) \textit{ is a } (G,\sigma)\textit{-symplectic basis of } A^2\} \\ & J & \mapsto & l_J \end{array}$$

defines is a homeomorphism that is equivariant under the action of $\mathrm{Sp}_2(G,\sigma)$.

Definition 2.9.11. We call the space $\mathfrak{P}$ the *projective model* of the symmetric space of $\mathrm{Sp}_2(G,\sigma)$.

Proof. 1. Show that $l_J \in \mathfrak{P}$. It is again enough to prove for the standard complex structure J_0. We take $v := (i,1)^T = (0,1)^T + (1,0)^T i$, then $l_{J_0} = vA_{\mathbb{C}}$, $((1,0)^T, (0,1)^T)$ is a (G,σ)-symplectic basis.

2. Show that F is surjective. Let $v = u + wi$, (w,u) is a (G,σ)-symplectic basis of (A^2, ω). We can define the following complex structure: $J(u) = w$, $J(w) = -u$. By Proposition 2.9.6, $J \in \mathfrak{C}$. Since

$$J_{\mathbb{C}}(v) = J_{\mathbb{C}}(u+iw) = w - iu = -i(u+iw) = -iv,$$

we obtain $F(J) = vA_\mathbb{C}$, i.e. F is surjective.

3. The map F is injective because if $l_J = l_{J'} = yA_\mathbb{C}$ for $J, J' \in \mathfrak{C}$ and $y = y_1 + y_2 i \in A^2_\mathbb{C}$. Then $J(y_1) = J'(y_1) = -y_2$, $J(y_2) = J'(y_2) = y_1$ and (y_1, y_2) is a basis of A^2, i.e. $J = J'$.

4. Now, show the equivariance of F. Let $M \in \mathrm{Sp}_2(G,\sigma)$, $J \in \mathfrak{C}$ and (w,u) be a (G,σ)-symplectic basis of (A^2,ω) such that $w := J(u)$, $J(w) = -u$. Then for $v = u + wi$, $J_\mathbb{C}(v) = -iv$.

Moreover, $MJM^{-1}(Mu) = Mw$, $MJM^{-1}(Mw) = -Mu$ where (Mw, Mu) is also a (G,σ)-symplectic basis of (A^2,ω). Then $(MJM^{-1})_\mathbb{C}(Mv) = -iMv$ and

$$F(MJM^{-1}) = (Mv)A_\mathbb{C} = M(vA_\mathbb{C}) = MF(J),$$

i.e. F is equivariant with respect to the $\mathrm{Sp}_2(G,\sigma)$-action. $\square$

Corollary 2.9.12. *The map*

$$\begin{array}{rccc} \tilde{\pi}'': & \mathrm{Sp}_2(G,\sigma)/\,\mathrm{KSp}_2(G,\sigma) & \to & \mathfrak{P} \\ & M\,\mathrm{KSp}_2(G,\sigma) & \mapsto & M(i,1)^T A_\mathbb{C} \end{array}$$

is an $\mathrm{Sp}_2(G,\sigma)$*-equivariant homeomorphism.*

Consider the following $\bar{\sigma}$-sesquilinear form on $A^2_\mathbb{C}$:

$$h(x,y) := i\omega_\mathbb{C}(\bar{x}, y).$$

It is indefinite. Indeed, it is σ-symmetric:

$$h(y,x) = i\omega_\mathbb{C}(\bar{y}, x) = \bar{\sigma}((-i)(-\omega_\mathbb{C}(\bar{x}, y))) = \bar{\sigma}(h(x,y)),$$

and in the basis $e_1 := \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)^T$, $e_2 := \left(\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right)^T$, h is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 2.9.13. *Assume* $(B_\mathbb{C}, \sigma)$ *to be of Jordan type and* $(B^\sigma_\mathbb{C})^\times \subseteq G_\mathbb{C}$*, so the group* $\mathrm{Sp}_2(G_\mathbb{C},\sigma)$ *is well defined. For* $v := u + wi \in A^2_\mathbb{C}$ *such that* (w,u) *is a basis of* A^2*, the basis* (w,u) *is* (G,σ)*-symplectic if and only if* $h(v,v) = 2 \in A^{\bar{\sigma}}_+$ *and* $v \in \mathrm{Is}_{G_\mathbb{C}}(\omega_\mathbb{C})$.

Proof. Direct computation. $\square$

Remark 2.9.14. $\mathrm{Sp}_2(G,\sigma)$ acts on $A^2_\mathbb{C}$ preserving $\omega_\mathbb{C}$ and, therefore, preserving h from the Section 2.3.2. So we can see $\mathrm{Sp}_2(G,\sigma)$ as a subgroup of $\mathrm{O}(h)$.

2.9.3 Precompact model

In this section, we assume $(B_{\mathbb{C}}, \bar\sigma)$ to be Hermitian. We want to see the symmetric space of $\mathrm{Sp}_2(G,\sigma)$ as a subset of some compact domain. We consider the following $\mathrm{Sp}_2(G_{\mathbb{C}},\sigma)$-transformation that maps h to the standard indefinite form:

$$T := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

i.e. $\bar\sigma(T)^T[h]T = \mathrm{diag}(-1,1) = [h_{st}]$. Since $T \in \mathrm{Sp}_2(G_{\mathbb{C}},\sigma)$, it stabilizes the set $\mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$.

Proposition 2.9.15. *The map*

$$\begin{array}{cccl} \Phi\colon & T^{-1}\mathfrak{P} & \to & \mathring{D}(B^{\sigma}_{\mathbb{C}},\bar\sigma) := \{c \in B^{\sigma}_{\mathbb{C}} \mid 1-\bar c c \in (B^{\bar\sigma}_{\mathbb{C}})_+\} \\ & (a,b)^T A_{\mathbb{C}} & \mapsto & ab^{-1} \end{array}$$

is a homeomorphism. The set $\mathring{D}(B^{\sigma}_{\mathbb{C}},\bar\sigma) \subseteq B^{\sigma}_{\mathbb{C}}$ is precompact.

Proof. First, we prove the following Lemma:

Lemma 2.9.16. *Let $(c,1)^T \in A^2_{\mathbb{C}}$. $(c,1)^T \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ if and only if $c \in B^{\sigma}_{\mathbb{C}}$.*

Proof. Let $(c,1)^T \in \mathrm{Is}_G(\omega)$, then $((1,0)^T,(c,1)^T)$ is a $(G_{\mathbb{C}},\sigma)$-symplectic basis. Therefore, the matrix $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}_2(G_{\mathbb{C}},\sigma)$ and so $c \in B^{\sigma}_{\mathbb{C}}$.

Let $(c,1)^T \in A^2_{\mathbb{C}}$ and $c \in B^{\sigma}_{\mathbb{C}}$. Consider $M := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}_2(G_{\mathbb{C}},\sigma)$, then $M(0,1)^T = (c,1)^T \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$. □

Let $v = (v_1,v_2)^T \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ such that $vA_{\mathbb{C}} \in \mathfrak{P}$ and $v = u + wi$ where (w,u) is a (G,σ)-symplectic basis of (A^2,ω). Then by the Lemma 2.9.13:

$$2 = h(v,v) = h_{st}(T^{-1}v, T^{-1}v) = -\bar\sigma(x_1)x_1 + \bar\sigma(x_2)x_2 \in B^{\bar\sigma}_+$$

where $T^{-1}v =: (x_1,x_2)^T$. Therefore, $\bar\sigma(x_2)x_2 = \bar\sigma(x_1)x_1 + 2 \in B^{\bar\sigma}_+$ because $B^{\bar\sigma}_+$ is a proper convex cone. This means that x_2 is invertible, i.e. $x_2 \in G$, $(c,1)^T := (x_1x_2^{-1},1)^T \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ and $(c,1)^T A_{\mathbb{C}} = (T^{-1}v)A_{\mathbb{C}}$.

By Lemma, $(c,1)^T \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ if and only if $c \in B^{\sigma}_{\mathbb{C}}$, and $h((c,1)^T,(c,1)^T) = 1 - \bar c c \in B^{\bar\sigma}_+$. Therefore, $\Phi(T^{-1}v) \in \mathring{D}(B^{\sigma}_{\mathbb{C}},\bar\sigma)$.

The map Φ is infective because T is injective and, if $x_1x_2^{-1} = \Phi(x_1,x_2)^T = \Phi(y_1,y_2)^T = y_1y_2^{-1}$, then $(y_1,y_2)^T = (x_1,x_2)^T x_2^{-1}y_2$, i.e. $(x_1,x_2)^T A_{\mathbb{C}} = (y_1,y_2)^T A_{\mathbb{C}}$.

The map Φ is surjective because for every $c \in \mathring{D}(B^{\sigma}_{\mathbb{C}},\bar\sigma)$, $(c,1)^T A_{\mathbb{C}} = (T^{-1}v)A_{\mathbb{C}}$ for $v := T(c,1)^T\sqrt{2}(1-\bar c c)^{-\frac{1}{2}}$. Then $v \in \mathrm{Is}_{G_{\mathbb{C}}}(\omega_{\mathbb{C}})$ and $h(v,v) = 2$. Therefore, by Lemma 2.9.13, $v = u + wi$ for (w,u) a (G,σ)-symplectic basis of (A^2,ω). Therefore, $vA_{\mathbb{C}} \in T^{-1}\mathfrak{P}$.

The set $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ is precompact because it is a subset of the following domain:

$$D(\bar G_{\mathbb{C}},\bar\sigma):=\{a\in \bar G_{\mathbb{C}}\mid 1-\bar\sigma(a)a\in (B_{\mathbb{C}}^{\bar\sigma})_{\geq 0}\}$$

that is compact by Proposition 2.6.49. □

Remark 2.9.17. Assume that $(B_{\mathbb{C}},\bar\sigma)$ is not Hermitian but $(B_{\mathbb{C}},\sigma)$ is semi-Hermitian. Then we can define $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ in the following way:

$$\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma):=\{b\in B_{\mathbb{C}}^{\sigma}\mid 1-\bar bb\in \theta_{\mathbb{C}}((B_{\mathbb{C}}^{\sigma})^{\times}))\}$$

where $\theta_{\mathbb{C}}(b)=\bar bb$ for $b\in B_{\mathbb{C}}^{\sigma}$. If we assume additionally that $\theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma}))\subseteq B_{\mathbb{C}}^{\bar\sigma}$ and $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ is precompact, then the Proposition holds also in this case. We will also denote in this case

$$(B_{\mathbb{C}}^{\bar\sigma})_{+}:=\theta_{\mathbb{C}}((B_{\mathbb{C}}^{\sigma})^{\times}),$$
$$(B_{\mathbb{C}}^{\bar\sigma})_{\geq 0}:=\theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma}).$$

Remark 2.9.18. The group $T^{-1}\operatorname{Sp}_2(G,\sigma)T<\operatorname{Sp}_2(G_{\mathbb{C}},\sigma)$ acts on $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ by Möbius transformations.

2.9.4 Compactification and Shilov boundary

Let (B,σ) be a Hermitian Lie subalgebra of A such that

$$\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma):=\{c\in B_{\mathbb{C}}^{\sigma}\mid 1-\bar cc\in (B_{\mathbb{C}}^{\bar\sigma})_{+}\}$$

is precompact. As we have seen, it holds always if $(B_{\mathbb{C}},\bar\sigma)$ is Hermitian. It is also true for some other cases described in the Remark 2.9.17. Let us take the topological closure of $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ in B^{σ}:

$$D(B_{\mathbb{C}}^{\sigma},\bar\sigma):=\{c\in B_{\mathbb{C}}^{\sigma}\mid 1-\bar cc\in (B_{\mathbb{C}}^{\bar\sigma})_{\geq 0}\}$$

The boundary of $D(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ contains the following closed subspace:

$$\check S(B_{\mathbb{C}}^{\sigma},\bar\sigma):=\{c\in B_{\mathbb{C}}^{\sigma}\mid 1-\bar cc=0\}.$$

Definition 2.9.19. We call $\check S(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ *Shilov boundary* of the precompact model $\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar\sigma)$.

Note, that

$$\check S(B_{\mathbb{C}}^{\sigma},\bar\sigma)=U(G_{\mathbb{C}},\bar\sigma)\cap B_{\mathbb{C}}^{\sigma}$$

and it is compact.

Remark 2.9.20. The map Φ^{-1} extends to the boundary of $D(B_{\mathbb{C}}^{\sigma},\bar\sigma)$ and remains continuous and bijective. Since the boundary is compact, it is a homeomorphism. Therefore, we can see the boundary also in the projective model. In particular, we can see the Shilov boundary there.

The next Proposition describes the Shilov boundary in the projective model.

Proposition 2.9.21. *The preimage of the Shilov boundary $\check{S}(B_{\mathbb{C}}^{\sigma},\bar{\sigma})$ in $\mathrm{Is}_{G_{\mathbb{C}}}(\omega)$ under the map $\Phi\circ T^{-1}$ gives a compact subset of the boundary of the projective model. It consists of all lines of the form $xA_{\mathbb{C}}$ such that $x\in\mathrm{Is}_G(\omega)$.*

Proof. Note that the line $l\in\mathrm{Is}_{G_{\mathbb{C}}}(\omega)$ is of the form $xA_{\mathbb{C}}$ for some $x\in\mathrm{Is}_G(\omega)$ if and only if $\bar{l}=l$.

Assume $c\in\check{S}(B_{\mathbb{C}}^{\sigma},\bar{\sigma})$, i.e. $\bar{c}^{-1}=c$. Then

$$(\Phi\circ T^{-1})\overline{T\circ\Phi^{-1}(c)}=\Phi\left(\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}\bar{c}\\ 1\end{pmatrix}\right)=\Phi\left(\begin{pmatrix}1\\ \bar{c}i\end{pmatrix}\right)=\bar{c}^{-1}=c$$

i.e. for $l=(c,1)^TA_{\mathbb{C}}$, $\bar{l}=l$.

If we take a line $xA_{\mathbb{C}}$ for some $x=(x_1,x_2)^T\in\mathrm{Is}_G(\omega)$, then

$$c:=(\Phi\circ T^{-1})(xA_{\mathbb{C}})=(x_1-ix_2)(-ix_1+x_2)^{-1}.$$

Since $x\in\mathrm{Is}_G(\omega)$, $c\in B_{\mathbb{C}}^{\sigma}$

$$\bar{c}c=(x_1+ix_2)(ix_1+x_2)^{-1}(x_1-ix_2)(-ix_1+x_2)^{-1}=$$
$$=i(x_1+ix_2)(x_1-ix_2)^{-1}(x_1-ix_2)(-ix_1+x_2)^{-1}=$$
$$=i(x_1+ix_2)(-ix_1+x_2)^{-1}=(x_1+ix_2)(x_1+ix_2)^{-1}=1.$$

Therefore, $(\Phi\circ T^{-1})(xA)\in\check{S}(B_{\mathbb{C}}^{\sigma},\bar{\sigma})$. ☐

Corollary 2.9.22. *The space of G-isotropic lines of (A^2,ω) embedded into $\mathbb{P}(\mathrm{Is}_{G_{\mathbb{C}}}(\omega))$ as:*

$$xA\mapsto xA_{\mathbb{C}}$$

is a Shilov boundary in the projective model. This is a closed (even compact) orbit of the action of $\mathrm{Sp}_2(G,\sigma)$ on the boundary of the projective model.

2.9.5 Upperhalf space model

Let A be an $\mathbb{R}$-algebra, $G\subseteq A^{\times}$ be a Lie subgroup, $B:=\mathrm{Lie}(G)\subseteq A$ be a Hermitian Lie subalgebra of A.

We denote by $B_{\mathbb{C}}$ the complexification of B, i.e. $B_{\mathbb{C}}:=B\otimes_{\mathbb{R}}\mathbb{C}$. To be consistent with the next Chapter, we denote by j the imaginary unit in $\mathbb{C}$. We extend σ to $B_{\mathbb{C}}$ complex linearly, i.e. $\sigma(x+yj)=\sigma(x)+\sigma(y)j$. So $B_{\mathbb{C}}^{\sigma}=\mathrm{Fix}_{B_{\mathbb{C}}}(\sigma)=B^{sym}\oplus B^{sym}j$ is well defined.

Every element of $z\in B_{\mathbb{C}}^{\sigma}$ can be uniquely written as $z=x+yj$ where $x,y\in B^{sym}$. We denote by $\mathrm{Re}(z):=x$, $\mathrm{Im}(z):=y$. We also have a complex conjugation on $B_{\mathbb{C}}$ given by $\bar{z}=x-yj$.

Definition 2.9.23. The *B-upperhalf space* is

$$\mathfrak{U}:=\{z\in B_{\mathbb{C}}^{\sigma}\mid \mathrm{Im}(z)\in B_{+}^{sym}\}$$

Note, for Hermitian tube type A, $\mathcal{S}(A,\sigma)$ is open in $B_{\mathbb{C}}^{sym}$

Proposition 2.9.24. $\mathrm{Sp}_2(G,\sigma)$ *acts on* $\mathfrak{U}$ *via*

$$z \mapsto M.z = (az+b)(cz+d)^{-1}, \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This transformation is called Möbius transformation. *The kernel of this action is* $Z(\mathrm{Sp}_2(G,\sigma))$.

Proof. Since $\mathrm{Sp}_2(G,\sigma)$ is generated by by matrices

$$\begin{pmatrix} a & 0 \\ 0 & \sigma(a)^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where $a \in G$, $b \in B^{sym}$, we proof $M.z \in \mathcal{S}$ on these generators.

If $M := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in B^{sym}$, then $M.z = z + b \in B_{\mathbb{C}}^{\sigma}$ and $\mathrm{Im}(M.z) = \mathrm{Im}(z) \in B_{+}^{sym}$.

If $M := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $M.z = -z^{-1} \in B_{\mathbb{C}}^{\sigma}$. If $z = x + yj$, then

$$z^{-1} = y^{-1}x(y + xy^{-1}x)^{-1} - (y + xy^{-1}x)^{-1}j,$$

i.e. $\mathrm{Im}(M.z) = (y + xy^{-1}x)^{-1}$. For $y \in B_{+}^{sym}$, also $y^{-1} \in B_{+}^{sym}$.

Let $y^{-1} = \sigma(p)p$ for some $p \in B^{\times}$, then

$$y + xy^{-1}x = y + \sigma(px)px \in B_{+}^{sym}.$$

Therefore, $\mathrm{Im}(M.z) = (y + xy^{-1}x)^{-1} \in B_{+}^{sym}$.

If $M := \begin{pmatrix} a & 0 \\ 0 & \sigma(a)^{-1} \end{pmatrix}$ for $a \in G$, then $M.z = az\sigma(a) \in B_{\mathbb{C}}^{\sigma}$ because B^{sym} is closed by action of G. $\mathrm{Im}(M.z) = a\,\mathrm{Im}(z)\sigma(a) \in B_{+}^{sym}$ because B_{+}^{sym} is closed by action of G.

An direct calculation on matrices shows that this is an action and that the kernel of this action is exactly $Z(\mathrm{Sp}_2(G,\sigma))$. □

Proposition 2.9.25. *The map*

$$\begin{array}{rccc} \pi\colon & \mathrm{Sp}_2(G,\sigma) & \to & \mathfrak{U} \\ & M & \to & M.1j \end{array}$$

is continues, proper and surjective, i.e. $\mathrm{Sp}_2(G,\sigma)$ *acts transitively on* $\mathfrak{U}$. *The stabilizer of* $1j$ *is* $\mathrm{KSp}_2(G,\sigma)$.

Proof. Let $z = x + yj \in \mathfrak{U}$ then $y = u^2$ for some $u \in (B^{sym})^{\times}$. Then

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\right) = \pi\left(\begin{pmatrix} u & xu^{-1} \\ 0 & u^{-1} \end{pmatrix}\right) = x + yj = z$$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stabilizes $1j$ if and only if

$$1j = M.1j = (aj+b)(cj+d)^{-1} = (aj+b)(-c+dj)^{-1}j.$$

So, $a = d$ and $c = -b$, i.e. $M \in \operatorname{KSp}_2(G,\sigma)$. $\square$

Corollary 2.9.26. *The map π induces a homeomorphism*

$$\begin{array}{rlcl} \tilde{\pi}\colon & \operatorname{Sp}_2(G,\sigma)/\operatorname{KSp}_2(G,\sigma) & \to & \mathfrak{U} \\ & M\operatorname{KSp}_2(G,\sigma) & \mapsto & M.1j \end{array}$$

A Möbius transformation $z \mapsto M'.z$ corresponds under this homeomorphism to the left multiplication $M\operatorname{KSp}_2(G,\sigma) \mapsto M'M\operatorname{KSp}_2(G,\sigma)$.

2.10 Spin group as $\operatorname{Sp}_2(G,\sigma)$

2.10.1 $\mathfrak{spin}(2,n)$ as $\mathfrak{sp}_2(B,\sigma)$

Let V be a real vector space of dimension $m+n > 0$ with the standard symmetric bilinear form b of signature (m,n). We denote by $\operatorname{Cl}(b)$ the Clifford algebra generated by (V,b). We remind, $\operatorname{Cl}(b)$ is a unital algebra generated by all elements of V and the following relation $v^2 = b(v,v)$ for $v \in V$. From this relation follows that for $v, w \in V$, $vw + wv = 2b(v,w)$.

The Clifford algebra $\operatorname{Cl}(b)$ contains a subalgebra $\operatorname{Cl}_{even}(b)$ that is generated by elements $\{vw \mid v, w \in V\}$. It is called *even Clifford algebra*.

Fixing an orthonormal basis $(f_1,\dots,f_m,e_1,\dots,e_n)$, i.e. $b(e_i,e_j) = \delta_{ij}$, $b(f_i,f_j) = -\delta_{ij}$, $b(f_i,e_j) = 0$, we identify (V,b) with $\mathbb{R}^{m,n}$. The Clifford algebra corresponding to $\mathbb{R}^{m,n}$ is denoted by $\operatorname{Cl}(m,n)$.

We define the anti-involution σ on $\operatorname{Cl}(b)$ by the following rules: on V, it is defined as $\sigma(e_i) = e_i$, $\sigma(f_i) = -f_i$ and then extend it to $\operatorname{Cl}(b)$. We consider the following Lie subalgebra:

$$B(m,n) = \operatorname{Span}_{\mathbb{R}}(1, e_ie_j, f_ke_i, f_kf_l \mid i,j \in \{1,\dots,n\}, k,l \in \{1,\dots,m\}, i<j, k<l)$$

with the Lie bracket $[x,y] := xy - yx$. Notice that $B(m,n)$ is closed under σ, and

$$B^{sym}(m,n) = \operatorname{Fix}(\sigma|_{B(m,n)}) = \operatorname{Span}_{\mathbb{R}}(1, f_ke_i \mid i \in \{1,\dots,n\}, k \in \{1,\dots,m\}).$$

Proposition 2.10.1. *$B^{sym}(m,n)$ is of Jordan type if and only if $m \leq 1$ or $n \leq 1$. In particular, $\mathfrak{sp}_2(B(m,n),\sigma)$ is a Lie algebra if and only if $m \leq 1$ or $n \leq 1$.*

Proof. Assume $m > 1$ and $n > 1$ then $f_1e_1f_2e_2 \notin B$.

If $m = 0$, then $B^{sym} = \{1\}$ is of Jordan type.

If $m = 1$, then $f_1e_if_1e_j = -f_1^2e_ie_j = e_ie_j \in B$, so $\mathfrak{sp}_2(B(1,n),\sigma)$ so B is of Jordan type. $\square$

Definition 2.10.2. For an element $e \in \mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n)$, we denote:

$$\|e\| := \sqrt{b(e,e)} \geq 0.$$

This is a norm on $\mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n)$.

Proposition 2.10.3. *For every $x \in B^{sym}(1,n)$ there exist $a_0 \in \mathbb{R}$, $r \geq 0$ and $e \in \mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n)$ with $\|e\| = 1$ such that $x = a_0 + rf_1e$.*

Proof. Let $x \in B^{sym}(1,n)$, then $x = a_0 + \sum_{i=1}^n a_i f_1 e_i$. Take

$$v := \sum_{i=1}^n a_i e_i, \; r := \sqrt{b(v,v)}, \; e := \frac{v}{r}.$$

Then $x = a_0 + rf_1e$. ☐

Proposition 2.10.4.

$$B^{sym}_{\geq 0}(1,n) = \{t + uf_1e \mid t \geq 0, \, 0 \leq u \leq t, \, e \in \mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n), \, \|e\| = 1\}.$$

is a closed proper convex cone.

Proof. Let $x := a_0 + rf_1e \in B^{sym}(1,n)$ where $a_0 \in \mathbb{R}$, $r \geq 0$ and $e \in \mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n)$ with $\|e\| = 1$. Then $x^2 = a_0^2 + r^2 + 2a_0rf_1e$. Denote $t := a_0^2 + r^2 \geq 0$. Then $u := 2a_0r = 2\,\mathrm{sgn}(a_0)\sqrt{t - r^2}r$. For fixed $t \geq 0$, $u = u(r)$ takes all values between $-t$ and t. So we get

$$B^{sym}_{\geq 0}(1,n) = \{t + uf_1e \mid t \geq 0, \, |u| \leq t, e \in \mathrm{Span}_{\mathbb{R}}(e_1, \ldots, e_n), \, \|e\| = 1\}.$$

This is a closed cone. It is also proper because for

$$x \in B^{sym}_{\geq 0}(1,n) \cap (-B^{sym}_{\geq 0}(1,n)),$$

$t = 0$ and so $u = 0$. This cone is also convex. Indeed, take $x = t + uf_1e$, $x' = t' + u'f_1e'$. Then

$$x + x' = (t + t') + f_1(ue + u'e') = (t + t') + \tilde{v}f_1\tilde{e}$$

where $v = \|ue + u'e'\|$, $\tilde{e} = \frac{ue+u'e'}{v}$. Using the triangle inequality we get:

$$v = \|v\tilde{e}\| = \|ue + u'e'\| \leq \|ue\| + \|u'e'\| = |u| + |u'| < t + t'.$$

Therefore, $x + x' \in B^{sym}_{\geq 0}(1,n)$. ☐

Corollary 2.10.5. *$B(1,n)$ is weakly Hermitian.*

Proof. In the Proposition 2.10.1, we have seen that $B(1,n)$ is of Jordan type. By the Proposition 2.10.4, $B^{sym}_{\geq 0}(1,n)$ is a proper convex cone and $1 \in B^{sym}_{+}(1,n)$. Finally, show for $b \in B^{sym}$, $b = 0$ if and only if $b^2 = 0$. Let $b = a_0 + a_1fe \in B^{sym}$ with $b(e,e) = 1$. Then $b^2 = a_0^2 + a_1^2 + 2a_0a_1fe = 0$, therefore $a_0 = a_1 = 0$, i.e. $b = 0$ ☐

We are going to identify $\mathfrak{sp}_2(B(1,n),\sigma)$ with $\mathfrak{spin}(2,n+1)$. We recall the definition of $\mathfrak{spin}(m,n)$:

$$\mathfrak{spin}(m,n) = \mathrm{Span}_{\mathbb{R}}(e_ie_j, f_ke_i, f_kf_l \mid i,j \in \{1,\dots,n\}, k,l \in \{1,\dots,m\})$$

where e_i, f_j as above.

Theorem 2.10.6. *The following map is an isomorphism of Lie algebras:*

$$\begin{array}{cccc} \varphi\colon & \mathfrak{sp}_2(B(1,n),\sigma) & \to & \mathfrak{spin}(2,n+1) \\ & \mathrm{diag}(e_ie_j, e_ie_j) & \mapsto & e_ie_j \\ & \mathrm{diag}(f_1e_i, -f_1e_i) & \mapsto & f_1e_i \\ & \mathrm{diag}(1,-1) & \mapsto & f_2e_{n+1} \\ & S(f_1e_i) & \mapsto & f_2e_i \\ & A(f_1e_i) & \mapsto & e_ie_{n+1} \\ & S(1) & \mapsto & -f_1e_{n+1} \\ & A(1) & \mapsto & f_1f_2 \end{array}$$

where $S(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$, $A(x) = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$.

Proof. Direct computation of Lie brackets. □

2.10.2 Clifford group and its Lie algebra

In this section, we describe a Lie group which Lie algebra is $B(1,n)$.

The group of all invertible elements $\mathrm{Cl}(m,n)^\times$ of $\mathrm{Cl}(m,n)$ acts on $\mathrm{Cl}(m,n)$ in the following way

$$\begin{array}{cccc} \tau\colon & \mathrm{Cl}(m,n)^\times \times \mathrm{Cl}(m,n) & \to & \mathrm{Cl}(m,n) \\ & (x,y) & \mapsto & \alpha(x)yx^{-1} \end{array}$$

where α is the standard involution on $\mathrm{Cl}(m,n)$ induced by the automorphism $v \mapsto -v$ on $\mathbb{R}^{m,n}$.

Definition 2.10.7. The (even) *Clifford group* is the group of all invertible elements of $\mathrm{Cl}_{even}(m,n)$ that stabilize $V = \mathbb{R}^{m,n}$ under the action τ, i.e.

$$\mathrm{ClGr}(m,n) := \{x \in \mathrm{Cl}_{even}^\times(m,n) \mid \forall v \in V : \tau(x)v \in V\}.$$

We denote by $\mathrm{ClGr}_0(m,n)$ the connected component of 1 in $\mathrm{ClGr}(m,n)$. We call it also Clifford group.

Remark 2.10.8. The Lie algebra of $\mathrm{ClGr}(m,n)$ can be described as follows:

$$\mathfrak{clgr}(m,n) = \{x \in \mathrm{Cl}(m,n) \mid \forall v \in V : \alpha(x)v - vx \in V\}.$$

We recall the following well known properties of the Clifford group (for more details see [27]):

Fact 2.10.9. *The Clifford groups fit to the following exact sequence:*

$$1 \to \mathbb{R}^\times \xrightarrow{\subseteq} \mathrm{ClGr}(m,n) \xrightarrow{\tau} \mathrm{SO}(m,n) \to 1.$$

$$1 \to \mathbb{R}_+ \xrightarrow{\subseteq} \mathrm{ClGr}_0(m,n) \xrightarrow{\tau} \mathrm{SO}_0(m,n) \to 1.$$

In particular,

$$\dim_{\mathbb{R}} \mathrm{ClGr}(m,n) = \dim_{\mathbb{R}} \mathbb{R} + \dim_{\mathbb{R}} \mathrm{SO}(m,n) = 1 + \frac{(m+n)(m+n-1)}{2}.$$

Fact 2.10.10. *The following map is well defined:*

$$\begin{array}{rccc} N\colon & \mathrm{ClGr}(m,n) & \to & \mathbb{R}^\times \\ & x & \mapsto & x^t x \end{array}$$

where $(\cdot)^t$ *is the standard anti-involution on* $\mathrm{Cl}(m,n)$ *induced by the trivial automorphism on* $\mathbb{R}^{m,n}$*.*

We remind the definition of the spin group:

Definition 2.10.11. The *spin group* is the following subgroup of the Clifford group:

$$\mathrm{Spin}(m,n) = \{x \in \mathrm{ClGr}(m,n) \mid N(x) = 1\},$$

$$\mathrm{Spin}_0(m,n) = \{x \in \mathrm{ClGr}_0(m,n) \mid N(x) = 1\}.$$

Remark 2.10.12. If $m > 0$ and $n > 0$ then $\mathrm{Spin}(m,n)$ has two connected components. If $m = 0$ or $n = 0$, then $\mathrm{Spin}(m,n)$ is connected, i.e. $\mathrm{Spin}(m,n) = \mathrm{Spin}_0(m,n)$.

Corollary 2.10.13.

- *For* $x \in \mathrm{ClGr}_0(m,n)$, $N(x) > 0$;
- *The map*
$$x \mapsto \frac{x}{\sqrt{N(x)}}$$
maps $\mathrm{ClGr}_0(m,n)$ *surjectively to* $\mathrm{Spin}_0(m,n)$;
- *The map*
$$x \mapsto \frac{x}{\sqrt{|N(x)|}}$$
maps $\mathrm{ClGr}(m,n)$ *surjectively to* $\mathrm{Spin}(m,n)$;

Proposition 2.10.14. $U(\mathrm{ClGr}_0(m,n),\sigma) = \mathrm{Spin}(m) \times \mathrm{Spin}(n)$.

Proof. Both groups are connected. So it is enough to show that their Lie algebras agree.

$$\begin{aligned} \mathrm{Lie}(U(\mathrm{ClGr}_0(m,n),\sigma)) &= \{x \in B \mid \sigma(x) + x = 0\} = \\ &= \mathrm{Span}_{\mathbb{R}}(f_i f_j, e_k e_l \mid i,j \in \{1,\dots,m\},\ k,l \in \{1,\dots,n\}) = \\ &= \mathfrak{spin}(m) \oplus \mathfrak{spin}(n). \end{aligned}$$

□

Lemma 2.10.15. $\mathfrak{clgr}(m,n) = B(m,n)$.

Proof. Easy calculation on the basis $(f_i, e_j)_{i,j=1}^{m,n}$ shows $B(m,n) \subseteq \mathfrak{clgr}(m,n)$. Moreover,

$$\dim_{\mathbb{R}} B(m,n) = 1 + \frac{m^2-m}{2} + \frac{n^2-n}{2} + mn = 1 + \frac{m^2-m+n^2-n+2mn}{2} =$$

$$= 1 + \frac{(m+n)^2-(m+n)}{2} = \dim_{\mathbb{R}} \mathfrak{clgr}(m,n).$$

So we have, $B(m,n) = \mathfrak{clgr}(m,n)$ □

Proposition 2.10.16. *$B(1,n)$ is Hermitian.*

Proof. In Corollary 2.10.5, we have seen that $B(1,n)$ is weakly Hermitian. By Proposition 2.10.14, $U(\mathrm{ClGr}(1,n),\sigma)$ is compact. □

2.10.3 Spectral decomposition in $B^{sym}(1,n)$

To be able to use the Corollary 2.6.33 for $B^{sym}(1,n)$, we have to prove that $U(\mathrm{ClGr}_0(1,n),\sigma)$ on Jordan frames of $B^{sym}(1,n)$. In this section, we find out what all Jordan frames in $B^{sym}(1,n)$ look like and show that $U(\mathrm{ClGr}_0(1,n),\sigma)$ acts transitively on them.

Proposition 2.10.17. *Every nontrivial idempotent $c \in B^{sym}(1,n)$ is of the following form: $c = \frac{1+e}{2}$ for some $e \in \mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_n)$, $b(e,e) = 1$*

Proof. Let $c = x + yf_1e$ be an idempotent, $b(e,e)=1$. Then

$$c^2 = (x^2+y^2) + 2xyf_1e = x + yf_1e = c.$$

So $2xy = y$. If $y = 0$, then $c = 1$ is trivial idempotent. If $y \neq 0$, then $x = \frac{1}{2}$, $y = \frac{1}{2}$ because by 2.10.3, $y > 0$. □

Theorem 2.10.18. *Every Jordan frame is $B^{sym}(1,n)$ of the following form: (c_1,c_2) where $c_1 = \frac{1+e}{2}$, $c_2 = \frac{1-e}{2}$ for some $e \in \mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_n)$, $b(e,e) = 1$.*

Proof. Let $c_1 = \frac{1+f_1e_1}{2}$, $c_2 = \frac{1+f_1e_2}{2}$ be two orthogonal idempotents, $b(e_1,e_1) = b(e_2,e_2) = 1$. Then:

$$0 = c_1 \circ c_2 = \frac{1+f_1e_1}{2} \circ \frac{1+f_1e_2}{2} = \frac{(1+f_1e_1)(1+f_1e_2)+(1+f_1e_2)(1+f_1e_1))}{8} =$$

$$= \frac{1+f_1e_1+f_1e_2+f_1e_1f_1e_2+1+f_1e_1+f_1e_2+f_1e_2f_1e_1}{8} =$$

$$= \frac{1+b(e_1,e_2)+2f_1(e_1+e_2)}{8}.$$

Therefore, $e_1 = -e_2$.

So we have that every complete system of idempotents has at most two elements. In particular, all Jordan frames are of the form (c_1,c_2) where $c_1 = \frac{1+e}{2}$, $c_2 = \frac{1-e}{2}$. □

Theorem 2.10.19. *$U(\mathrm{ClGr}_0(1,n),\sigma)$ acts transitively on Jordan frame.*

Proof. Let $\left(c_1 = \frac{1+f_1e}{2}, c_2 = \frac{1-f_1e}{2}\right)$ and $\left(c_1' = \frac{1+f_1e'}{2}, c_2' = \frac{1-f_1e'}{2}\right)$ be two Jordan frame. Since $b(e,e) = b(e',e') = 1$, there exists an orthogonal transformation $u \in \mathrm{SO}(n)$ such that $u(e) = e'$. Take some preimage v of u in $\mathrm{Spin}(n) = U(\mathrm{ClGr}_0(1,n),\sigma)$. Then

$$\sigma(v)c_1v = v^{-1}c_iv = \frac{1+(v^{-1}fv)(v^{-1}ev)}{2} = \frac{1+u(f)u(e)}{2} = \frac{1+fe'}{2} = c_1'.$$

Similarly, $\sigma(v)c_2v = c_2$. $\square$

2.10.4 The group $\mathrm{Spin}_0(2,n+1)$ as $\mathrm{Sp}_2(\mathrm{ClGr}(1,n),\sigma)$

In this section, we want to identify the groups $\mathrm{Spin}_0(2,n+1)$ and $\mathrm{Sp}_2(\mathrm{ClGr}(1,n),\sigma)$.

First, note that a generic matrix in $\mathrm{Sp}_2(\mathrm{ClGr}(1,n),\sigma)$ has the following shape:

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & \sigma(x)^{-1} \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathrm{Spin}(1,n)$, $\lambda > 0$, $y, z \in B^{sym}(1,n)$.

The embedding:

$$\mathbb{R}^{1,n} = \mathrm{Span}(f_1, e_1, \ldots, e_n) \subset \mathbb{R}^{2,n+1} = \mathrm{Span}(f_1, f_2, e_1, \ldots, e_{n+1})$$

induces the embedding of spin groups $\iota\colon \mathrm{Spin}(1,n) \hookrightarrow \mathrm{Spin}(2,n+1)$. Moreover, we can embed the entire $\mathrm{Spin}(1,n)$ into $\mathrm{Spin}_0(2,n+1)$ the following way: we define the map ι_0

- if $x \in \mathrm{Spin}_0(1,n)$, then $\iota_0(x) := \iota(x)$;
- if $x \in \mathrm{Spin}(1,n) \setminus \mathrm{Spin}_0(1,n)$, then $\iota_0(x) = xf_2e_{n+1}$.

The map ι_0 is a group homomorphism.

Theorem 2.10.20. *The following map is an isomorphism of Lie groups:*

$$\begin{array}{rccc}
\Phi\colon & \mathrm{Sp}_2(\mathrm{ClGr}(1,n),\sigma) & \to & \mathrm{Spin}_0(2,n+1) \\
& \mathrm{diag}(y,\sigma(y)^{-1}) & \mapsto & \iota_0(y) \\
& \mathrm{diag}(\lambda,\lambda^{-1}) & \mapsto & \frac{\lambda+\lambda^{-1}}{2} + f_2e_{n+1}\frac{\lambda-\lambda^{-1}}{2} \\
& R(f_1e_i) & \mapsto & 1 + \frac{f_2+e_{n+1}}{2}e_i \\
& L(f_1e_i) & \mapsto & 1 + \frac{f_2-e_{n+1}}{2}e_i \\
& R(1) & \mapsto & 1 - \frac{f_2+e_{n+1}}{2}f_1 \\
& L(1) & \mapsto & 1 - \frac{f_2-e_{n+1}}{2}f_1
\end{array}$$

where $y \in \mathrm{Spin}(1,n)$, $\lambda > 0$, $R(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $L(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. The map φ is the differential of Φ at Id.

Proof. The map Φ is well defined and it is a homeomorphism of groups. This can be seen on generators. Moreover, on generators one can also see that the map φ is the differential of Φ at Id. This implies that Φ is a local diffeomorphism. Since both groups are connected, the map Φ is surjective.

To see, that the map Φ is injective, first we can note that there is only one generic element that is mapped to $1 \in \mathrm{Spin}_0(2, n+1)$, namely $\mathrm{Id} \in \mathrm{Sp}_2(\mathrm{ClGr}(1,n), \sigma)$. Then we can take a sequence of generic elements, map it by Φ to $\mathrm{Spin}_0(2, n+1)$ and see that this sequence converges to $\mathrm{Id} \in \mathrm{Sp}_2(\mathrm{ClGr}(1,n), \sigma)$ if and only if the sequence of images converges to $1 \in \mathrm{Spin}_0(2, n+1)$. □

2.10.5 Models of symmetric space of $\mathrm{Spin}_0(2, n+1)$

In this section, we construct different models of the symmetric space of $\mathrm{Spin}_0(2, n+1) \cong \mathrm{Sp}_2(\mathrm{ClGr}(1,n), \sigma)$.

Example 16 (Upperhalf space model)**.** We remind:

$$B(1,n) = \mathrm{Span}_{\mathbb{R}}(1, e_i e_j, f_1 e_k \mid i,j,k \in \{1, \dots n\})$$

$$B^{sym}(1,n) = \mathrm{Span}_{\mathbb{R}}(1, f_1 e_k \mid k \in \{1, \dots n\})$$

$$B^{sym}_{+}(1,n) = \{t + u f_1 e \mid t > 0,\ u \in [0,t), e \in \mathrm{Span}_{\mathbb{R}}(e_1, \dots, e_n),\ \|e\| = 1\}.$$

So

$$\mathfrak{U}(\mathrm{Spin}_0(2, n+1)) = \{x + yi \mid x \in B^{sym}(1,n),\ y \in B^{sym}_{+}(1,n)\}.$$

Non-Example 1. Let us try to do the complexification for the previous example. Consider the real Lie algebra $B_{\mathbb{R}} = B(1,n) = \mathfrak{clgr}(1,n)$ and corresponding real Lie group $G_{\mathbb{R}} = \mathrm{ClGr}(1,n)$. The anti-involution σ acts in the following way on generators $\sigma(f) = -f$, $\sigma(e_i) = e_i$ and extends then to $\mathrm{Cl}(1,n)$.

Now, consider complexified algebra:

$$B_{\mathbb{C}} = B(1,n) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Span}_{\mathbb{C}}(1, e_l e_j, f_1 e_k \mid j,k,l \in \{1, \dots n\})$$

$$B^{\sigma}_{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}(1, f_1 e_k \mid k \in \{1, \dots n\}).$$

Therefore,

$$\mathfrak{sp}_2(B_{\mathbb{C}}, \sigma) = \mathfrak{sp}_2(B, \sigma) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{spin}(2, n, \mathbb{C}) \cong \mathfrak{spin}(n+2, \mathbb{C}).$$

We get the same on the level of Lie groups, i.e. $\mathrm{Sp}_2(G_{\mathbb{C}}, \sigma) \cong \mathrm{Spin}(n+2, \mathbb{C})$, where $G_{\mathbb{C}} = \mathrm{ClGr}(1, n, \mathbb{C}) \cong \mathrm{ClGr}(n+1, \mathbb{C})$.

But

$$B^{\bar{\sigma}}_{\mathbb{C}} = \mathrm{Span}_{\mathbb{R}}(1, f_1 e_k, i e_l e_j \mid j,k,l \in \{1, \dots n\}),$$

i.e. $(B_{\mathbb{C}}, \bar{\sigma})$ is not of Jordan type because

$$(f_1 e_k)(i e_l e_j) = i f_1 e_k e_l e_j \notin B_{\mathbb{C}}$$

if $k \neq l \neq j$. So our construction does not work in this case because the complexified Lie algebra $B_{\mathbb{C}}$ together with complex antilinear extension $\bar{\sigma}$ of σ is not Hermitian.

Example 17 (Precompact model)**.** We take the complexification

$$B_{\mathbb{C}} := B_{\mathbb{C}}(1,n) = \mathrm{Span}_{\mathbb{C}}\{1, f_1e_i, e_ie_j \mid i,j \in \{1,\dots,n\}\}$$

with complex linear extension of σ denoted also by σ and the complex antilinear extension of σ denoted by $\bar{\sigma}$. Then

$$B_{\mathbb{C}}^{\sigma} = \mathrm{Span}_{\mathbb{C}}\{1, f_1e_i \mid i \in \{1,\dots,n\}\}.$$

To construct the precompact model of the symmetric space for $\mathrm{Spin}_0(2,n+1)$, we use the Remark 2.9.17. We consider the set:

$$\begin{aligned}\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar{\sigma}) &= \{b \in B_{\mathbb{C}}^{\sigma} \mid 1-\bar{b}b \in \theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma})^{\times}\} = \\ &= \{x+yf_1e \mid 1-(\bar{x}+\bar{y}f_1e)(x+yf_1e) \in \theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma})^{\times},\ \|e\|=1\} = \\ &= \{x+yf_1e \mid 1-(\bar{x}x+\bar{y}y)-(\bar{x}y+\bar{y}x)f_1e \in \theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma})^{\times},\ \|e\|=1\}.\end{aligned}$$

Let us find out what the set $\theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma})$ is. Let $r_1\exp(i\phi_1)+r_1\exp(i\phi_1)f_1e \in B_{\mathbb{C}}^{\sigma}$, $r_1,r_2 \geq 0$, $\phi_1,\phi_2 \in \mathbb{R}$, $\|e\|=1$ then

$$\theta_{\mathbb{C}}(r_1\exp(i\phi_1)+r_1\exp(i\phi_1)f_1e) = (r_1^2+r_2^2)+2r_1r_2\cos(\phi_2-\phi_1)f_1e.$$

We denote $r_1^2+r_2^2 =: r$. then

$$\theta_{\mathbb{C}}(r_1\exp(i\phi_1)+r_2\exp(i\phi_2)f_1e) = r+2r_1\sqrt{r-r_1^2}\cos(\phi_2-\phi_1)f_1e$$

For fixed $r \geq 0$, the expression $2r_1\sqrt{r-r_1^2}\cos(\phi_2-\phi_1)$ can take every value in the interval $[-r,r]$, i.e.

$$\theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma}) = \{r+pf_1e \mid r \geq 0,\ p \in [-r,r],\ \|e\|=1\} = B_{\geq 0}^{sym}.$$

Analogously,

$$\theta_{\mathbb{C}}(B_{\mathbb{C}}^{\sigma})^{\times} = B_{+}^{sym}.$$

Therefore,

$$\mathring{D}(B_{\mathbb{C}}^{\sigma},\bar{\sigma}) = \{x+yf_1e \in B_{\mathbb{C}}^{\sigma} \mid 1-(\bar{x}x+\bar{y}y)-(\bar{x}y+\bar{y}x)f_1e \in B_{+}^{sym},\ \|e\|=1\}$$

is the precompact model of the symmetric space for $\mathrm{Spin}_0(2,n+1)$.

Example 18 (Projective model)**.** We take the upperhalf space model:

$$\mathfrak{U}(\mathrm{Spin}_0(2,n+1)) \cong \{x+yi \mid x \in B^{sym}(1,n),\ y \in B_{+}^{sym}(1,n)\}.$$

We know that the map $z \mapsto (z,1)^T A_{\mathbb{C}}$ is a homeomorphism between the upperhalf space models and projective model. So we obtain:

$$\mathfrak{P}(\mathrm{Spin}_0(2,n)) = \{(x+yi,1)^T A_{\mathbb{C}} \mid x \in B^{sym}(1,n),\ y \in B_{+}^{sym}(1,n)\} =$$

$$= \left\{ \begin{pmatrix} (x_1 + y_1 i) + f_1(ex_2 + e'y_2)i \\ 1 \end{pmatrix} A_{\mathbb{C}} \;\middle|\; \begin{array}{l} x_1 \in \mathbb{R},\, y_1 \in \mathbb{R}_+,\, x_2 \in \mathbb{R}_{\geq 0}, \\ y_2 \in [0, x_2),\, \|e\| = \|e'\| = 1 \end{array} \right\}.$$

We can also construct the projective model for $\mathrm{Spin}_0(2, n+1)$ in terms of lines in $\mathbb{C}^{n+3}$. First, we note that the stabilizer of the line $(i, 1)^T A_{\mathbb{C}} \subset A^2_{\mathbb{C}}$ corresponds under the (complexified) map Φ from the Theorem 2.10.20 to the stabilizer of the line $(f_2 + f_1 i)\mathbb{C} \subset \mathbb{C}^{n+3}$ where $\mathrm{Spin}_0(2, n+1)$ acts on $\mathbb{C}^{n+3}$ by (complexified) τ. So we can take the following injective map:

$$\begin{array}{rccl} F\colon & \mathfrak{P}(\mathrm{Spin}_0(2, n+1)) & \to & \{l \in \mathbb{C}P^{n+2} \mid l = v\mathbb{C},\, v \in \mathbb{C}^{n+3},\, b_{\mathbb{C}}(v, v) = 0\} \\ & g(i, 1)^T A_{\mathbb{C}} & \mapsto & \tau(\Phi(g))(f_2 + f_1 i)\mathbb{C} = \Phi(g)(f_2 + f_1 i)\Phi(g)^{-1}\mathbb{C} \end{array}$$

where $g \in \mathrm{Sp}_2(G, \sigma)$, $b_{\mathbb{C}}$ the complex bilinear extension of b.

Since $\mathrm{Spin}_0(2, n+1)$ acts on $\mathbb{R}^{n+3}$ preserving b, it acts on $\mathbb{C}^{n+3}$ preserving $b_{\mathbb{C}}$ and the sesquilinear extension $\tilde{b}$ of b, i.e.

$$\tilde{b}(v_1 + v_2 i, w_1 + w_2 i) := b(v_1, w_1) + b(v_2, w_2) + (b(v_1, w_2) - b(v_2, w_1))i$$

for $v, w \in \mathbb{R}^{n+3}$. Note, the form $\tilde{b}$ on $\mathbb{C}^{n+3}$ has signature $(2, n+1)$. Therefore, F maps injectively $\mathfrak{P}(\mathrm{Spin}_0(2, n+1))$ to

$$\mathfrak{P}'(\mathrm{Spin}_0(2, n+1)) := \left\{ l \in \mathbb{C}P^{n+2} \mid l = v\mathbb{C},\, v \in \mathbb{C}^{n+3},\, b_{\mathbb{C}}(v, v) = 0,\, \tilde{b}(v, v) < 0 \right\}.$$

Let $v = v_1 + v_2 i \in \mathbb{C}^{n+3}$ such that $b_{\mathbb{C}}(v, v) = 0$, $\tilde{b}(v, v) = -2$. Then $b(v_1, v_1) = b(v_2, v_2) = -1$, $b(v_1, v_2) = 0$. There exists an $\mathrm{SO}(2, n+1)$-transformation that maps (f_2, f_1) to (v_1, v_2). Therefore, $\mathrm{Spin}_0(2, n+1)$ acts transitively on $\mathfrak{P}'(\mathrm{Spin}_0(2, n+1))$, so it is a model of the symmetric space of $\mathrm{Spin}_0(2, n+1)$.

Example 19 (Complex structure model)**.** We consider the complex structure model for the symmetric space of $\mathrm{Sp}_2(G, \sigma) \cong \mathrm{Spin}_0(2, n+1)$:

$$\mathfrak{C}(\mathrm{Sp}_2(G, \sigma)) := \left\{ J \text{ complex structure on } A^2 \;\middle|\; \begin{array}{l} J(\mathrm{Is}_G(\omega)) = \mathrm{Is}_G(\omega), \\ h_J \text{ is a } (G, \sigma)\text{-inner product} \end{array} \right\}$$

where $h_J(x, y) = \omega(J(x), y)$.

Notice that $\mathfrak{C}(\mathrm{Spin}_0(2, n+1)) \subseteq \mathrm{Sp}_2(G, \sigma)$ because the standard complex structure $J_0 \in \mathrm{Sp}_2(G, \sigma)$ and $\mathrm{Sp}_2(G, \sigma)$ acts on $\mathfrak{C}(\mathrm{Spin}_0(2, n+1))$ transitively by conjugation.

If we take the standard complex structure $J_0 = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\Phi(J_0) = f_1 f_2$ where Φ is the isomorphism between $\mathrm{Spin}_0(2, n+1)$ and $\mathrm{Sp}_2(G, \sigma)$ from the Theorem 2.10.20. For every $v \in \mathbb{R}^{n+3}$ there exist unique $e \in \mathrm{Span}_{\mathbb{R}}(e_1, \dots, e_n)$, $f \in \mathrm{Span}_{\mathbb{R}}(f_1, f_2)$ such that $v = e + f$. Then

$$\Phi(J_0) v \Phi(J_0)^{-1} = e - f.$$

For another complex structure J, there exists $g \in \mathrm{Sp}_2(G, \sigma)$ such that $J = g^{-1} J_0 g$. Let $v = \Phi(g)^{-1}(e + f)\Phi(g)$, e, f as above,

$$\Phi(J) v \Phi(J)^{-1} = \Phi(g)^{-1}\Phi(J_0)(e + f)\Phi(J_0)^{-1}\Phi(g) = \Phi(g)^{-1}(e - f)\Phi(g).$$

Since $\mathrm{Spin}_0(2,n+1)$ acts preserving b on $(\mathbb{R}^{n+3}, b)$, the restriction of b to the subspace $\Phi(g)^{-1}\,\mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_{n+1})\Phi(g)$ is positive definite and the restriction of b to the subspace $\Phi(g)^{-1}\,\mathrm{Span}_{\mathbb{R}}(f_1,f_2)\Phi(g)$ is negative definite.

Consider the following space:

$$\mathcal{D} := \{(V_+,V_-) \mid \mathbb{R}^{n+3} = V_+ \oplus V_-,\ b|_{V_+} \text{ is positive definite, } b|_{V_-} \text{ is negative definite}\}.$$

We have a map $F\colon \mathrm{Sp}_2(G,\sigma) \to \mathcal{D}$,

$$F(g) := (\Phi(g)^{-1}\,\mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_{n+1})\Phi(g), \Phi(g)^{-1}\,\mathrm{Span}_{\mathbb{R}}(f_1,f_2)\Phi(g)).$$

Notice,

$$F^{-1}(\mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_n), \mathrm{Span}_{\mathbb{R}}(f_1,f_2)) = \mathrm{Spin}(2)\times\mathrm{Spin}(n+1).$$

Therefore, $\mathcal{D}$ is isomorphic to $\mathrm{Spin}_0(2,n+1)/(\mathrm{Spin}(2)\times\mathrm{Spin}(n+1))$, i.e. $\mathcal{D}$ is the model of the symmetric space of $\mathrm{Spin}_0(2,n+1)$. This is an analogous of the complex structures model for $\mathrm{Spin}_0(2,n+1)$ because, as we have seen, the complex structures model $\mathfrak{C}(\mathrm{Sp}_2(G,\sigma))$ can be mapped to $\mathcal{D}$ by taking a complex structure, mapping it by Φ to $\mathrm{Spin}_0(2,n+1)$ and the considering the decomposition of $\mathbb{R}^{n+3}$ in its 1-eigenspace and (-1)-eigenspace. As we have seen, b restricted to the 1-eigenspace is positive definite, b restricted to the (-1)-eigenspace is negative definite.

2.11 Maximal representations into $\mathrm{Sp}_2(G,\sigma)$

In the Section 1.2.4, we introduced the space of maximal representations of the fundamental group of a punctured surface into $\mathrm{Sp}(2n,\mathbb{R})$. The notion of maximality can be generalized for every Hermitian Lie group, in particular, for groups that can be seen as $\mathrm{Sp}_2(G,\sigma)$ where G is the Lie subgroup contained in some algebra with anti-involution (A,σ) such that $\mathrm{Lie}\,G = B$ and (B,σ) is a Hermitian Lie subalgebra of A.

In this Chapter, we generalize $\mathcal{X}$-coordinates we introduced for decorated maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$ for decorated maximal representations into $\mathrm{Sp}_2(G,\sigma)$ and describe some topological properties of the space of decorated maximal representations into $\mathrm{Sp}_2(G,\sigma)$ using them as we have done it for the group $\mathrm{Sp}(2n,\mathbb{R})$.

2.11.1 Decorated representations

Definition 2.11.1. A representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ will be called *peripherally parabolic* if for every $g \in \pi_1^{per}(S)$, the matrix $\rho(g)$ leaves invariant some isotropic line form $\mathbb{P}(\mathrm{Is}_G(\omega))$.

We will denote by $\mathrm{Hom}^P(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ the subset of $\mathrm{Hom}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ consisting of peripherally parabolic representations.

Definition 2.11.2. The quotient space

$$\operatorname{Rep}^P(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) := \operatorname{Hom}^P(\pi_1(S), \operatorname{Sp}_2(G,\sigma))/\operatorname{Sp}_2(G,\sigma)$$

is called the *moduli space of peripherally parabolic representations.*

For a peripherally parabolic representation there might be many ways to choose the invariant isotropic line. A decoration is a special way to make this choice.

Definition 2.11.3. A *decoration* of ρ is a map

$$D\colon \pi_1^{per}(S) \to \operatorname{Is}_G(\omega)$$

satisfying the following properties:

(a) $D(g)$ is invariant under $\rho(g)$ for all $g \in \pi_1^{per}(S)$.

(b) If $g_1, g_2 \in \pi_1^{per}(S)$, $h \in \pi_1(S)$ such that $hg_1h^{-1} = g_2$, then

$$\rho(h)(D(g_1)) = D(g_2).$$

(c) For every $k \in \mathbb{Z} \setminus \{0\}$ and for every $g \in \pi_1^{per}(S)$,

$$D(g) = D(g^k).$$

A *decorated representation* is a pair (ρ, D), where ρ is a representation and D a decoration of ρ.

Remark 2.11.4. By properties a), b), c) of decorations, for every puncture, one has to choose a Lagrangian for only one peripheral element going around the puncture. Then the Lagrangians associated to the other peripheral elements going around the same puncture are determined.

We denote by $\operatorname{Hom}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma))$ the set of all decorated representations. The action of $\operatorname{Sp}_2(G,\sigma)$ on $\operatorname{Hom}(\pi_1(S), \operatorname{Sp}_2(G,\sigma))$ and on $\operatorname{Is}_G(\omega)$ induces an action on $\operatorname{Hom}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma))$. We will study the quotient:

Definition 2.11.5. The quotient space

$$\operatorname{Rep}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) := \operatorname{Hom}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma))/\operatorname{Sp}_2(G,\sigma)$$

is called the *moduli space of decorated representations.* We denote by $[\rho, D]$ the class of (ρ, D) in the moduli space of decorated representation.

Remark 2.11.6. We have natural surjective maps

$$\begin{array}{ccc} \operatorname{Hom}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) & \to & \operatorname{Hom}^P(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) \\ (\rho, D) & \mapsto & \rho \end{array}.$$

$$\begin{array}{ccc} \operatorname{Rep}^d(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) & \to & \operatorname{Rep}^P(\pi_1(S), \operatorname{Sp}_2(G,\sigma)) \\ [\rho, D] & \mapsto & [\rho] \end{array}.$$

These maps are generically finite-to-one maps.

2.11.2 Transverse representations

We now fix an ideal triangulation $\mathcal{T}$ of S.

Definition 2.11.7. We say that $(\rho, D) \in \mathrm{Hom}^d(\pi_1(S,b), \mathrm{Sp}_2(G,\sigma)$ is *transverse* with respect to $\mathcal{T}$ if the following condition holds: for every edge e of $\mathcal{T}$ connecting punctures p_i and p_j, for every point $b' \in Int(e)$ and for every curve γ connecting b and b', we require that the isotropic lines $D(\gamma * \alpha_i * \gamma^{-1})$ and $D(\gamma * \alpha_j * \gamma^{-1})$ are transverse, where the curves α_i and α_j are as in Figure 2.11.1.

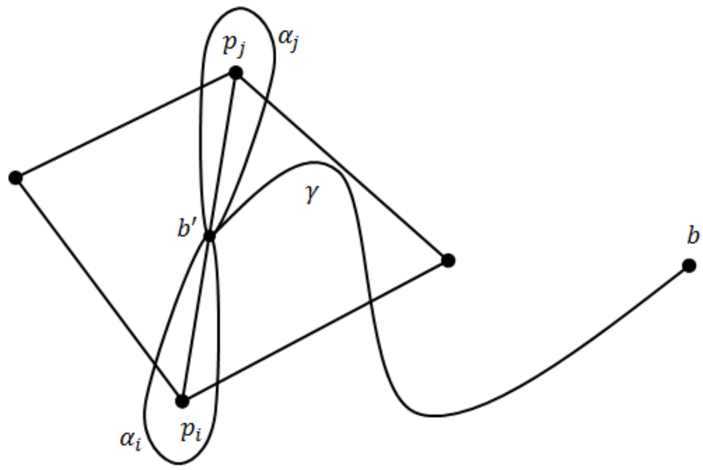

Figure 2.11.1:

We denote by $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}_2(G,\sigma))$ the set of all decorated representations which are transverse with respect to the triangulation $\mathcal{T}$.

Remark 2.11.8. The transversality property required in the previous definition does not depend on the choice of the path γ and the base point b. Moreover, this property is invariant under the action of $\mathrm{Sp}_2(G,\sigma)$, hence we can define the quotient:

$$\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G,\sigma)) := \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}_2(G,\sigma)) / \mathrm{Sp}_2(G,\sigma)$$

Remark 2.11.9. For each $\mathcal{T}$, the space $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ is an open dense subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$.

Let T be a triangle of $\mathcal{T}$ with boundary ∂T. Using the orientation of S, we can orient ∂T so that T is to the left from ∂T. This gives us a cyclic order on the vertices $\{p_1, p_2, p_3\}$ of T. We assume that (p_1, p_2, p_3) are in positive cyclic order.

Definition 2.11.10. Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$, and consider elements $g_1, g_2, g_3 \in \pi_1^{per}(S,b)$ that go around p_1, p_2, p_3 that are vertices of an oriented triangle T (see Figure 2.11.2). We can consider triple of isotropic lines $(D(g_1), D(g_2), D(g_3))$. We say that this triangle if *positive with respect to the decoration D* if the triple

$(D(g_1), D(g_2), D(g_3))$ is positive. Since the positivity is $\mathrm{Sp}(2n, \mathbb{R})$-invariant, it is a well defined invariant of $[\rho, D]$ if the triangle is positive with respect to D.

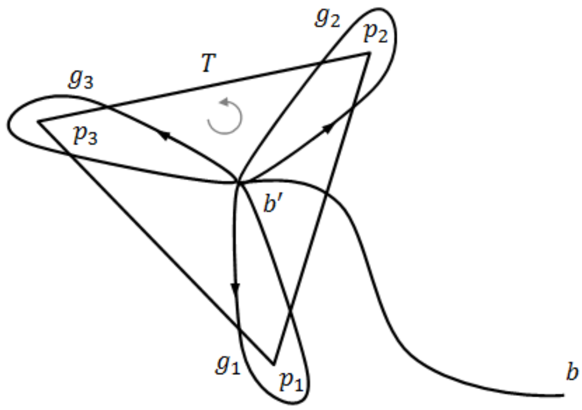

Figure 2.11.2:

2.11.3 Toledo number and maximal representations

We remind that the key invariant in the definition of maximality for representations $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ similarly to the case of $\mathrm{Sp}(2n, \mathbb{R})$ is the Toledo number, here denoted by T_ρ, which was defined in [7] using bounded cohomology. It is a real number which satisfies the Milnor–Wood inequality:

$$-n|\chi(S)| \leq T_\rho \leq n|\chi(S)|$$

where n is the rank of the Jordan algebra B^{sym}.

Moreover, for all representations $[\rho] \in \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$, this invariant takes only integer values.

Definition 2.11.11. A representation $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ is called *maximal* if $T_\rho = n|\chi(S)|$.

We denote by $\mathcal{M}(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ the subspace of $\mathrm{Rep}(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ consisting of all maximal representations. Similarly, we denote by $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ the subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ of all decorated maximal representations, and by $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$ the subspace of all decorated maximal representations which are transverse with respect to a chosen triangulation $\mathcal{T}$. The following facts are proven in [7].

Proposition 2.11.12. *[7]*

(a) $\mathcal{M}(\pi_1(S), \mathrm{Sp}_2(G, \sigma)) \subset \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}_2(G, \sigma))$. *In particular, the natural projection map*

$$\mathcal{M}^d(\pi_1(S), \mathrm{Sp}_2(G, \sigma)) \quad \to \quad \mathcal{M}(\pi_1(S), \mathrm{Sp}_2(G, \sigma)).$$

is surjective.

(b) *Maximal representations are transverse with respect to any ideal triangulation $\mathcal{T}$:*

$$\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G,\sigma)) = \mathcal{M}^d(\pi_1(S), \mathrm{Sp}_2(G,\sigma)).$$

(c) *All maximal representations are reductive, hence the spaces $\mathcal{M}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ and $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$ are Hausdorff.*

As for the representations into $\mathrm{Sp}(2n,\mathbb{R})$, we have the following Proposition:

Proposition 2.11.13. *Let $\mathcal{T}$ be an ideal triangulation of S and $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$. (ρ, D) is maximal if and only if all positive oriented triangles of $\mathcal{T}$ are positive with respect to the decoration D.*

The proof of this Proposition goes analogously to the proof of the Proposition 1.2.20.

2.11.4 Positive $\mathcal{X}$-coordinates

Let S be a surface with an oriented ideal triangulation $\mathcal{T}$. We use the notation introduced in Section 1.3.1.

Definition 2.11.14 (Positive $\mathcal{X}$-coordinates)**.** A *system of positive $\mathcal{X}$-coordinates* of type (G,σ) on $(S,\mathcal{T})$ is a map

$$x \colon E \sqcup W^+ \to \mathbb{R}^n_{>0} \sqcup U(G,\sigma)$$

such that

- the *edge invariant* $x(e)$ for an edge $e \in E$ is an n-tuple of positive real numbers $x(e) = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{>0}$ with $\lambda_i \geq \lambda_{i+1}$ where n is the rank of G ;
- the *angle invariant* $x(w)$ for a positive angle $w \in W^+$ is an element $x(w) \in U(G,\sigma)$. The angle coordinates are subject to the following relation: for each positive triple of positive angles (w_1, w_2, w_3) we require

$$x(w_3)x(w_2)x(w_1) = 1.$$

We denote by $\mathcal{X}^+(S,\mathcal{T},G,\sigma)$ the set of all positive systems of $\mathcal{X}$-coordinates of type (G,σ) on $(S,\mathcal{T})$.

Remark 2.11.15. As we have seen in the Chapter 1, the edge invariants and angle invariants are related to the invariants of quadruple of Lagrangians and 5-tuple of Lagrangians. In this more general situation, the edge and angle invariants are related in the same way to the invariants of quadruple and 5-tuple of G-isotropic lines, discussed in Sections 2.8.5 and 2.8.6.

As a convenient notation, if $x \in \mathcal{X}^+(S,\mathcal{T},G,\sigma)$ is a system of $\mathcal{X}$-coordinates and $w \in W^-$ is a negative angle, we will write $x(w) = x(w^{-1})^{-1}$.

Analogously to the case of $\mathrm{Sp}(2n,\mathbb{R})$, the map

$$[\mathrm{rep}^+] \colon \mathcal{X}^+(S,\mathcal{T},G,\sigma) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}_2(G,\sigma))$$

can be defined, and it is continuous, surjective and proper.

2.11.5 Topology of the space of maximal representations

Using positive $\mathcal{X}$-coordinates, we can understand the topology of the space of (decorated) maximal representation. In this section, we state the results we obtain for maximal representations into $\mathrm{Sp}_2(G,\sigma)$ and the consider examples for classical groups. All proofs go completely analogously to the case of $\mathrm{Sp}(2n,\mathbb{R})$.

Theorem 2.11.16. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S),\mathrm{Sp}_2(G,\sigma))$ is homotopically equivalent to $U(G,\sigma)^{2g+k-1}/U(G,\sigma)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $U(G,\sigma)$ on $U(G,\sigma)^{2g+k-1}$ by simultaneous conjugation.*

Theorem 2.11.17. *The space of decorated maximal representation $\mathcal{M}^d(\pi_1(S),\mathrm{Sp}_2(G,\sigma))$ is homeomorphic to*

$$(B_+^{sym})^{6g+3k-6} \times U(G,\sigma)^{2g+k-1}/U(G,\sigma)$$

where $U(G,\sigma)$ acts by simultaneous conjugation in every factor.

Now, we implement the Theorem 2.11.16 for classical Lie groups of tube type that we can see as $\mathrm{Sp}_2(A,\sigma)$ or $\mathrm{Sp}_2(G,\sigma)$.

Example 20. Let $A = \mathrm{Mat}(n,\mathbb{C})$ and $\bar{\sigma}$ be the transposition composed with the complex conjugation. This is a Hermitian algebra,

$$B_+^{sym} = \mathrm{Herm}^+(n,\mathbb{C}),\ U(G,\bar{\sigma}) = \mathrm{U}(n).$$

We take $G = A^\times$, then, as we have seen, $\mathrm{Sp}_2(G,\bar{\sigma}) \cong \mathrm{U}(n,n)$. So we obtain: $\mathcal{M}^d(\pi_1(S),\mathrm{U}(n,n))$ is homeomorphic to

$$(\mathrm{Herm}^+(n,\mathbb{C}))^{6g+3k-6} \times \mathrm{U}(n)^{2g+k-1}/\,\mathrm{U}(n),$$

ant it is homotopically equivalent to

$$\mathrm{U}(n)^{2g+k-1}/\,\mathrm{U}(n).$$

Example 21. Let $A = \mathrm{Mat}(n,\mathbb{H})$ and σ_1 be the transposition composed with the quaternionic conjugation. This is a Hermitian algebra,

$$B_+^{sym} = \mathrm{Herm}^+(n,\mathbb{H}),\ U(G,\sigma_1) = \mathrm{Sp}(n).$$

We take $G = A^\times$, then, as we have seen, $\mathrm{Sp}_2(G,\sigma_1) \cong \mathrm{SO}^*(4n)$. So we obtain: $\mathcal{M}^d(\pi_1(S),\mathrm{SO}^*(4n))$ is homeomorphic to

$$(\mathrm{Herm}^+(n,\mathbb{H}))^{6g+3k-6} \times \mathrm{Sp}(n)^{2g+k-1}/\,\mathrm{Sp}(n),$$

ant it is homotopically equivalent to

$$\mathrm{Sp}(n)^{2g+k-1}/\,\mathrm{Sp}(n).$$

Example 22. Let $A = \mathrm{Cl}(1,n)$ and σ be the anti-involution as in the Chapter 2.10. Then we take $G = \mathrm{ClGr}(1,n)$. We remind:

$$B(1,n) = \mathrm{Lie}(\mathrm{ClGr}(1,n)) = \mathrm{Span}_{\mathbb{R}}(1, e_i e_j, f_1 e_k \mid i,j,k \in \{1,\dots n\})$$

$$B^{sym}(1,n) = \mathrm{Span}_{\mathbb{R}}(1, f_1 e_k \mid k \in \{1,\dots n\})$$

$$B^{sym}_{+}(1,n) = \{t + u f_1 e \mid t > 0,\, u \in [0,t), e \in \mathrm{Span}_{\mathbb{R}}(e_1,\dots,e_n),\, \|e\| = 1\}\,.$$

where $(f_1, e_1, \dots, e_n)$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$ with the standard bilinear form of signature $(1,n)$.

Then, as we have seen, $\mathrm{Sp}_2(G,\sigma) \cong \mathrm{Spin}_0(2,n+1)$. So we obtain: $\mathcal{M}^d(\pi_1(S), \mathrm{Spin}_0(2,n+1))$ is homeomorphic to

$$(B^{sym}_{+}(1,n))^{6g+3k-6} \times \mathrm{Spin}(n)^{2g+k-1} / \mathrm{Spin}(n),$$

ant it is homotopically equivalent to

$$\mathrm{Spin}(n)^{2g+k-1} / \mathrm{Spin}(n).$$

A Appendix

A.1 Spectral theorem with signature

The well known spectral theorem from the linear algebra says that for two bilinear forms b_1, b_2 on a real vector space V such that b_1 is positive definite, there exists a basis $\mathbf{e}$ such that $[b_1]_{\mathbf{e}} = \mathrm{Id}_n$, $[b_2]_{\mathbf{e}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where $n = \dim V$ and $\lambda_1 \geq \dots \geq \lambda_n$. Therefore, the tuple $(\lambda_1, \dots, \lambda_n)$ defines the pair (b_1, b_2) up to change of basis of V. We can define the standard form of the pair of bilinear forms (b_1, b_2) to be the pair of matrices $(\mathrm{Id}_n, \mathrm{diag}(\lambda_1, \dots, \lambda_n))$ and say that the basis $\mathbf{e}$ puts (b_1, b_2) to the standard form. We use this standard form to define edge invariants for maximal representations in the Section 1.3.

In this section, we define the standard form for a pair of bilinear forms (b_1, b_2) assuming only nondegeneracy of b_1. This standard form will be used to define edge invariants for general representations in the Section 1.5.2.

A.1.1 Bilinear forms and symmetric linear maps

Let V be n-dimensional vector space over some field K, b_1, b_2 be symmetric bilinear forms on V and b_1 be not degenerate. We denote by $b_i^\sharp \colon V \to V^*$ the linear map corresponding to b_i, i.e. $b_i(x, y) = b_i^\sharp(y)(x)$ for $x, y \in V$. Then we can consider the linear isomorphism $f \colon V \to V$ such that $f := (b_1^\sharp)^{-1} \circ b_2^\sharp$.

Lemma A.1.1. *The map f is symmetric with respect to the form b_1 and for all $x, y \in V$*

$$b_2(x, y) = b_1(x, fy)$$

Proof.

$$b_1(x, fy) = b_1^\sharp((b_1^\sharp)^{-1}(b_2^\sharp(y)))(x) = b_2^\sharp(y)(x) = b_2(x, y)$$

$$b_1(fx, y) = b_1(y, fx) = b_2(y, x) = b_2(x, y) = b_1(x, fy)$$

□

A.1.2 Jordan blocks

In this section, we define a standard form for a pair of bilinear forms (b_1, b_2) if the map $f := (b_1^\sharp)^{-1} \circ b_2^\sharp$ is a Jordan block in some basis, i.e. there exists a basis $\mathbf{e}$ of V

such that

$$[f]_{\mathbf{e}} = J_n(l) = \begin{pmatrix} l & 1 & 0 & \dots & 0 & 0 \\ 0 & l & 1 & \dots & 0 & 0 \\ 0 & 0 & l & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & l & 1 \\ 0 & 0 & 0 & \dots & 0 & l \end{pmatrix}$$

for some $l \in K$. We also find out how unique the basis $\mathbf{e}$ is. Instead of $J_n(l)$, sometimes for simplicity, we will just write J.

Lemma A.1.2 (Jordan block over $\mathbb{R}$)**.** *Let $[f]_{\mathbf{e}} = J_n(l)$ in some basis $\mathbf{e}$ of V. Then there exists another basis $\mathbf{e}'$ of V such that $[f]_{\mathbf{e}'} = J_n(l)$ and either $[b_1]_{\mathbf{e}'} = \varepsilon C_n$ where $\varepsilon \in \{1, -1\}$,*

$$C_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Proof (Only idea). Let $\mathbf{e} = (e_1, \dots, e_n)$. Consider $\partial := f - l\,\mathrm{Id}$, then $\partial(e_i) = e_{i-1}$ and $\partial(e_1) = 0$. Since f is symmetric with respect to b_1, ∂ is it as well. We get

$$b(e_i, e_j) = b(\partial^{n-i} e_n, \partial^{n-j} e_n) = b(e_n, \partial^{2n-(i+j)} e_n).$$

Since $\partial^s = 0$ for $s > n-1$, $b(e_i, e_j) = 0$ for $i + j < n + 1$. Moreover, if $i + j \geq n + 1$, $b(e_i, e_j) = b(e_k, e_l)$ for $i + j = k + l$, $k, l > 0$, i.e.

$$[b_1]_{\mathbf{e}} = \begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_1 & a_2 \\ & & \dots & & \\ 0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}$$

We rescale the basis $\mathbf{e}$ such that $a_1 = \mathrm{sgn}(a_1)$. Then we take a basis $\mathbf{e}' := (\mathrm{Id} + b_1 \partial + \dots + b_{n-1} \partial^{n-1})\mathbf{e}$. Note that $[f]_{\mathbf{e}'} = [f]_{\mathbf{e}}$ for every b_i. Coefficients b_i can be successively chosen so that all $a_i = 0$ for $i > 1$.

□

Corollary A.1.3 (Jordan block over algebraically closed field)**.** *Over algebraically closed fields the basis $\mathbf{e}'$ in the previous lemma can be always chosen (by possible rescaling by i) so that*

$$[b_1]_{\mathbf{e}'} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Lemma A.1.4 (Over $\mathbb{R}$ or algebraically closed field). *The basis which was found in the previous lemma (in this lemma denoted by* $\mathbf{e}$*) is unique up to multiplication of all vectors with* ± 1.

Proof. Let $\mathbf{u} = (u_i)$ be another basis with necessary property.

Step 1. By induction, we will show that

$$u_k = \sum_{i=1}^{k} c_{k-i+1} e_i$$

1. $f(u_1) = \lambda u_1$, u_1 is an eigenvector of f. But all eigenvectors of f are ce_1, $c \in \mathbb{R}$. Therefore, $u_1 = c_1 e_1$ for some $c_1 \neq 0$.

2. We assume that $u_s = \sum_{i=1}^{s} c_{s-i+1} e_i$ for all $s < k$. $f(u_k) = au_k + u_{k-1}$, therefore $g(u_k) = f(u_k) - au_k \in \mathbb{R}u_{k-1} \leq \langle e_1, \ldots, e_{k-1} \rangle$. If we assume

$$u_k = \sum_{i=1}^{n} c_{ki} e_i,$$

then

$$g(u_k) = \sum_{i=2}^{n} c_{ki} e_{i-1} \in \langle e_1, \ldots, e_{k-1} \rangle .$$

Therefore $c_{ki} = 0$ for all $i > k$. Moreover

$$g(u_k) = u_{k-1} = \sum_{j=1}^{k-1} c_{k-1-j+1} e_j = [above] = \sum_{i=2}^{k} c_{ki} e_{i-1}$$

Therefore, $c_{ki} = c_{k-i+1}$, and so we have

$$u_k = \sum_{i=1}^{k} c_{k-i+1} e_i$$

Step 2. Now we show that $c_1 = \pm 1$ and $c_i = 0$ for $i > 1$. To do that we use the form b_1. By assumption

$$b_1(u_i, u_j) = b_1(e_i, e_j) = \delta_{i+j,n+1}$$

$$b_1(u_k, u_l) = \sum_{i=1}^{k} \sum_{j=1}^{l} c_{k-i+1} c_{l-j+1} b(e_i, e_j) = \sum_{i=1}^{k} c_{k-i+1} c_{l-n-1+i+1}$$

We assume here $c_i = 0$ for $i \leq 0$. If we take $l = n$, then we get

$$b_1(u_k, u_n) = \sum_{i=1}^{k} c_{k-i+1} c_i$$

For $k = 1$:

$$1 = b_1(u_1, u_n) = c_1 c_1$$

Therefore, $c_1 = \pm 1$. Further, we take $k = 2$,

$$0 = b_1(u_2, u_n) = c_2 c_1 + c_1 c_2$$

Therefore, $c_2 = 0$. And so on by induction, we assume $c_i = 0$ for all $1 < i < k$ for some k, then

$$0 = b_1(u_k, u_n) = c_k c_1 + c_{k-1} c_2 + \cdots + c_1 c_k$$

Therefore $c_k = 0$ for all $k \neq 1$. $\square$

Definition A.1.5. Let B be a symmetric $n \times n$ matrix over some field K. We denote by

$$\mathrm{O}(B) := \{X \in \mathrm{Mat}_n(K) \mid X^T B X = B\}.$$

the *orthogonal group* of B considered as a bilinear form on the vector space K^n.

Corollary A.1.6. *For every $l \in \mathbb{R}$*

$$\mathrm{O}(C_n) \cap \mathrm{O}(C_n J_n(l)) = \{\pm \mathrm{Id}_n\}$$

Remark A.1.7. If f is a Jordan block, then we have shown that there exists a basis $\mathbf{e}$ such that $[b_1]_{\mathbf{e}} = \varepsilon C_n$, $[b_2]_{\mathbf{e}} = \varepsilon C_n J_n$ for $\varepsilon \in \{1, -1\}$. We take this as the standard form for the pair of bilinear forms (b_1, b_2). The basis $\mathbf{e}$ is uniquely defined up to sign.

Dual bilinear forms

Let b_1, b_2 be two bilinear forms in some n-dimensional $\mathbb{R}$-vector space V such that $[b_1]_{\mathbf{e}} = \varepsilon C = \varepsilon C_n$, $[b_2]_{\mathbf{e}} = \varepsilon C J$ in some basis $\mathbf{e}$, where J is a Jordan block with eigenvalue $l \neq 0$, $w = \pm 1$.

In order to construct a representation by given coordinates in Section 1.5.3, we will need another basis $\mathbf{v}$ of V such that

$$[b_1^*]_{\mathbf{v}^*} = \mathrm{sgn}(l)[b_2]_{\mathbf{e}}$$

$$[b_2^*]_{\mathbf{v}^*} = \mathrm{sgn}(l)[b_1]_{\mathbf{e}}$$

where v^* is the dual basis for v, b_i^* are the bilinear forms on the dual space V^* corresponding to b_i, i.e. $(b_i^*)^\sharp = (b_i^\sharp)^{-1}$. We denote by Φ the change-of-basis matrix from $\mathbf{e}$ to $\mathbf{v}$.

Since $C = C^{-1}$, we get the following conditions for Φ:

$$\mathrm{sgn}(l)\Phi C \Phi^T = CJ$$

$$\mathrm{sgn}(l)\Phi (CJ)^{-1} \Phi^T = C$$

Lemma A.1.8. $\Phi = \pm \Phi^T$

Proof. We assume $l > 0$. The case $l < 0$ is similar.

$$\Phi C \Phi^T = CJ$$
$$\Phi (CJ)^{-1} \Phi^T = C$$

are equivalent to

$$\Phi C \Phi^T = CJ$$
$$\Phi^T C \Phi = CJ$$

Therefore,

$$\Phi \Phi^{-T} C \Phi^{-1} \Phi^T = C$$
$$\Phi \Phi^{-T} (CJ) \Phi^{-1} \Phi^T = CJ$$

So $\Phi^{-1}\Phi^T \in \mathrm{O}(C) \cap \mathrm{O}(CJ) = \{\pm \mathrm{Id}\}$ (A.1.6) and we have $\Phi = \pm \Phi^T$. $\square$

Lemma A.1.9. *If there exists $\Phi \in \mathrm{Sym}(n, \mathbb{K})$ such that*

$$\mathrm{sgn}(lw) \Phi C \Phi = CJ$$

then this Φ is unique up to sign.

Proof. We assume $l > 0$. The case $l < 0$ is similar. Assume, there are two $\Phi, \Psi \in \mathrm{Sym}(n, \mathbb{K})$ such that

$$\Phi C \Phi = \Psi C \Psi = CJ$$

Then we have

$$\Psi \Phi^{-1} C \Phi^{-1} \Psi = C$$
$$\Psi \Phi^{-1} (CJ) \Phi^{-1} \Psi = CJ$$

So $\Phi^{-1}\Psi \in \mathrm{O}(C) \cap \mathrm{O}(CJ) = \{\pm \mathrm{Id}\}$ and we have $\Phi = \pm \Psi$. $\square$

Lemma A.1.10. *There exists $\Phi \in \mathrm{Sym}(n, \mathbb{K})$ such that*

$$\mathrm{sgn}(lw) \Phi C \Phi = CJ$$

$$\Phi = \pm \begin{pmatrix} 0 & 0 & \dots & 0 & \sqrt{|l|} \\ 0 & 0 & \dots & \sqrt{|l|} & x_1 \\ 0 & 0 & \dots & x_1 & x_2 \\ & & \dots & & \\ 0 & \sqrt{|l|} & \dots & x_{n-2} & x_{n-1} \\ \sqrt{|l|} & x_1 & \dots & x_{n-1} & x_n \end{pmatrix}$$

where x_i are some rational functions in $\sqrt{|l|}$.

Proof. Put this matrix in the equation $\Phi C \Phi = CJ$ and calculate successively all coefficients. $\square$

Remark A.1.11. Φ is uniquely defined up to sign. To make the choice of Φ unique, we take plus sign in case $l > 0$. Otherwise, we take minus sign. At this point, it does not really matter how we choose the sign. It will be important later when we will consider degenerate representations.

A.1.3 Classification of symmetric maps

Over algebraically closed fields

In this section we want to show that over algebraically closed field K for every symmetric (with respect to some non-degenerate form b) linear map f there is an orthogonal basis $\mathbf{e}$ such that

$$[f]_{\mathbf{e}} = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix},$$

where J_k is a $n_k \times n_k$ Jordan block corresponding to the eigenvalue λ_k and

$$[b]_{\mathbf{e}} = \begin{pmatrix} I_1^* & 0 & \dots & 0 & 0 \\ 0 & I_2^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_k^* \end{pmatrix},$$

where

$$I_s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_s \times n_s}.$$

By the theorem of Jordan we already know that there exists a basis $\mathbf{e}$ such that $[f]_{\mathbf{e}}$ has a necessary form. We want to show that we can correct this basis to another basis $\mathbf{e}'$ such that $[b]_{\mathbf{e}'}$ is of the form as above and $[f]_{\mathbf{e}} = [f]_{\mathbf{e}'}$.

Lemma A.1.12. *Blocks with different eigenvalues are orthogonal.*

Proof. Let $v_1, \dots, v_l$ is a Jordan basis of a block with eigenvalue λ, i.e. $f(v_i) = \lambda v_i + v_{i-1}$, $f(v_1) = \lambda v_1$. Let $w_1, \dots, w_m$ is a Jordan basis of a block with eigenvalue μ, i.e. $f(w_i) = \mu w_i + w_{i-1}$, $f(v_1) = \mu w_1$ and $\mu \neq \lambda$.

1. First, prove that $b(v_1, w_1) = 0$:

$$\lambda b(v_1, w_1) = b(f(v_1), w_1) = b(v_1, f(w_1)) = \mu b(v_1, w_1).$$

Since $\mu \neq \lambda$, $b(v_1, w_1) = 0$.

2. Second, prove that $b(v_1, w_i) = 0$ for every $i \in \{2, \dots, m\}$ by induction assuming $b(v_1, w_r) = 0$ for all $1 \le r < i$:

$$\lambda b(v_1, w_i) = b(f(v_1), w_i) = b(v_1, f(w_i)) = b(v_1, \mu w_i + w_{i-1}) = \mu b(v_1, w_i).$$

Since $\mu \neq \lambda$, $b(v_1, w_i) = 0$ for every $i \in \{1, \dots, m\}$.

3. Finally, prove that $b(v_i, w_j) = 0$ for every $i \in \{2, \dots, n\}$, $j \in \{1, \dots, n\}$ by induction assuming $b(v_s, w_r) = 0$ for $1 \le s < i$ and $r \in \{1, \dots, m\}$, and $b(v_i, w_r) = 0$ for all $1 \le r < j$:

$$\lambda b(v_i, w_j) = b(f(v_i) - v_{i-1}, w_j) = b(f(v_i), w_j) = b(v_1, f(w_i)) =$$

$$= b(v_1, \mu w_i + w_{i-1}) = \mu b(v_1, w_i).$$

Since $\mu \neq \lambda$, $b(v_i, w_j) = 0$ for every $i \in \{1, \dots, l\}$, $j \in \{1, \dots, m\}$.

□

As we also have seen, if we restrict the form b to each Jordan block, then, if this form is not degenerate, then the basis of this block can be chosen so that this restriction of b has a form:

$$I^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We call such blocks "non-degenerate".

Therefore, we have to prove two things:

1. If the restriction of b to some block is degenerate, then there exists another block with the same eigenvalue. Using this block we will correct the "degenerate" block to "non-degenerate" block.

2. We can orthogonalize non-degenerate blocks with the same eigenvalue.

Lemma A.1.13. *Let J_s be a block, $m := n_s = \dim(J_s)$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$. Let J_p be a block with the same eigenvalue λ, $l := n_p = \dim(J_p)$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$.*

If $m > l$, then

$$B_{\mathbf{v},\mathbf{w}} := (b(v_i, w_j)) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & c_1 & c_2 \\ & & \dots & & \\ c_1 & c_2 & \dots & c_{n-1} & c_n \end{pmatrix}.$$

If $m < l$, then

$$B_{\mathbf{v},\mathbf{w}} := (b(v_i, w_j)) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & 0 & 0 & \dots & c_1 & c_2 \\ & & \dots & & & \dots & & \\ 0 & 0 & \dots & c_1 & c_2 & \dots & c_{m-1} & c_m \end{pmatrix}.$$

Proof. We proof the first case. The second is analogous. We use

$$b(f(v_i), w_j) = b(v_i, f(w_j)) = \lambda b(v_i, w_j) + b(v_i, w_{j-1})$$

$$b(f(v_i), w_j) = \lambda b(v_i, w_j) + b(v_{i-1}, w_j)$$

We get $b(v_i, w_{j-1}) = b(v_{i-1}, w_j)$ for all $i \in \{1, \dots, m\}$, $j \in \{1, \dots, l\}$ and $b(v_j, w_1) = 0$ for all $j \in \{1, \dots, m\}$. So we get $(b(v_i, w_j))$ as above inductively. □

Lemma A.1.14. *Let J_s be a block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of* $\mathbf{e}$ *for this block, $V = \mathrm{Span}(\mathbf{v})$. Let J_p be a block with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of* $\mathbf{e}$*, $W = \mathrm{Span}(\mathbf{w})$.*

Then $\mathbf{u} = \mathbf{v} + \mathbf{w}T$ is a basis of Jordan block with the same eigenvalue λ if and only if T has the following form: for $m \le l$

$$T = \begin{pmatrix} c_1 & c_2 & \dots & c_{m-1} & c_m \\ 0 & c_1 & \dots & c_{m-2} & c_{m-1} \\ & & \dots & & \\ 0 & 0 & \dots & c_1 & c_2 \\ 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

for $m \ge l$

$$T = \begin{pmatrix} 0 & \dots & 0 & c_1 & c_2 & \dots & c_{l-1} & c_l \\ 0 & \dots & 0 & 0 & c_1 & \dots & c_{l-2} & c_{l-1} \\ & & & & & \dots & & \\ 0 & \dots & 0 & 0 & 0 & \dots & c_1 & c_2 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & c_1 \end{pmatrix}$$

Matrices of this form we will call diagonal upper triangular.

Proof. For every basis $\mathbf{u} = (u_1, \dots, u_s)$ we denote by $\partial \mathbf{u} := (0, u_1, \dots, u_{s-1})$. Then for each basis of Jordan block we have $f(\mathbf{u}) = \lambda \mathbf{u} + \partial \mathbf{u}$. The map ∂ in basis $\mathbf{u}$ is given by matrix

$$P := [\partial]_{\mathbf{u}} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Now we want $f(\mathbf{u}) = \lambda \mathbf{u} + \partial \mathbf{u}$ for $\mathbf{u} = \mathbf{v} + \mathbf{w}T$. That means

$$f(\mathbf{v} + \mathbf{w}T) = f(\mathbf{v}) + f(\mathbf{w})T = \lambda \mathbf{v} + \partial \mathbf{v} + (\lambda \mathbf{w} + \partial \mathbf{w})T = \lambda(\mathbf{v} + \mathbf{w}T) + \partial \mathbf{v} + (\partial \mathbf{w})T =$$

$$= \lambda \mathbf{u} + \partial \mathbf{u} + (\partial \mathbf{w})T - \partial(\mathbf{w}T).$$

That means, $\mathbf{u}$ is Jordan basis if and only if $PT = TP$.

If $T = (t_{ij})$ then $PT = (t_{i-1,j})$, $TP = (t_{i,j+1})$ (to make this notation completely correct, we assume here $t_{ij} = 0$ for $i > l$ or $j > m$ or for $i, j < 1$). That means, $t_{i-1,j} = t_{i,j+1}$ and $t_{1j} = 0$ for $j > 1$, $t_{im} = 0$ for $i < m$. Therefore, T has a necessary form. □

Lemma A.1.15. *Let J_s be a block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of* $\mathbf{e}$ *for this block, $V = \mathrm{Span}(\mathbf{v})$. Let b_V be degenerate. Then* *there exists another block J_p with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$ and $b_{V \oplus W}$ is not degenerate.*

Moreover, there exists another basis $\mathbf{u} = (u_1, \dots, u_m)$ such that $U = \mathrm{Span}(\mathbf{u})$ is invariant by f, $U \oplus W = V \oplus W$, $[f|_U]_{\mathbf{u}} = J_s$ and b_U is not degenerate.

Proof. Without lost of generality, assume $\lambda = 0$. Otherwise, consider $f - \lambda \,\mathrm{Id}$ instead of f.

Since

$$b(f(v_i), v_j) = b(v_i, f(v_j)) = \lambda b(v_i, v_j) + b(v_i, v_{j-1})$$
$$b(f(v_i), v_j) = \lambda b(v_i, v_j) + b(v_{i-1}, v_j),$$

we get $b(v_i, v_{j-1}) = b(v_{i-1}, v_j)$ for all $i, j = 1, \dots, m$ and $b(v_1, v_j) = 0$ for all $j = 1, \dots, m-1$. Therefore,

$$B_{\mathbf{v}} := [b|_V]_{\mathbf{v}} = \begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_1 & a_2 \\ & & \dots & & \\ 0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}$$

This matrix is degenerate, that means that $a_1 = 0$ and v_1 is orthogonal to the whole block. v_1 is also orthogonal to all blocks with eigenvalues different form λ. But the form b is not degenerate. Therefore, there exists another block J_p with the eigenvalue λ and basis $\mathbf{w} = (w_1, \dots, w_l)$ such that $b|_{\mathrm{Span}(\mathbf{v},\mathbf{w})}$ is nondegenerate. Therefore, $b(v_1, w_l) \neq 0$ and $b(v_1, w_r) = 0$ for all $r < l$ since $b(v_1, w_r) = b(v_1, f^{l-r}(w_l)) = b(f^{l-r}(v_1), w_l) = b(0, w_l) = 0$.

Let $x := w_1 - \frac{b(w_1,w_l)}{b(v_1,w_l)} v_1 \neq 0$. Then $b(x, v_i) = b(x, w_j) = 0$ for all $i < m$, $j < l$. Moreover, $b(x, w_l) = b(w_1, w_l) - \frac{b(w_1,w_l)}{b(v_1,w_l)} b(v_1, w_l) = 0$. Since b is nondegenerate, $b(x, v_m) \neq 0$:

$$0 \neq b(x, v_m) = b(w_1, v_m) - \frac{b(w_1, w_l)}{b(v_1, w_l)} b(v_1, v_m) = b(w_1, v_m) = b(f^{l-1} w_l, v_m) =$$
$$= b(w_l, f^{l-1} v_m) = b(w_l, v_{m-l+1})$$

i.e. $m \geq l$.

We take $\mathbf{u} := (v_1, \dots, v_m) + (0, \dots, 0, v_1, \dots, w_l)$, it has all necessary properties. □

Using the last lemma we can always assume that the basis is chosen so that all Jordan blocks are non degenerate with respect to b. Now we want to correct this basis so that different blocks are orthogonal.

Lemma A.1.16. *Let J_s is a non degenerate with respect to b Jordan block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$.*

Let J_p be another block with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$. We assume $m \geq l$.

Then there exists a diagonal upper triangular matrix T such that $\mathbf{u} = \mathbf{w} + \mathbf{v}T$ is a basis of Jordan block which is orthogonal to J_s and $V \oplus W = U \oplus W$

Proof. That $\mathbf{u} = \mathbf{w} + \mathbf{v}T$ is a basis of Jordan block, we already know by the lemma A.1.14. We want orthogonality. That means

$$0 = b(\mathbf{v}, \mathbf{w} + \mathbf{v}T) = b(\mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{v}T) = B_{\mathbf{v},\mathbf{w}} + B_{\mathbf{v}}T$$

Because $B_{\mathbf{v}}$ is not degenerate, we have

$$T = B_{\mathbf{v},\mathbf{w}} B_{\mathbf{v}}^{-1}$$

This is a product of two diagonal upper triangular matrices, which is diagonal upper triangular. The new block is not degenerate because, otherwise, the form would be degenerate on $V \oplus U$, but this is not the case. □

Corollary A.1.17. *If we have many blocks with the same eigenvalue, then we do the process as in previous lemma successively as in Gram–Schmidt orthogonalization.*

Case $K = \mathbb{R}$ with real eigenvalues

Because $\mathbb{R}$ is not algebraically closed, we have to take care by the process which we did in the case K algebraic closed.

First, assume that all eigenvalues of f are real, otherwise the theorem of Jordan does not guarantee us that the Jordan basis exists.

Theorem A.1.18. *For every symmetric with respect to some non-degenerate form b linear map f with real eigenvalues there is an orthogonal basis $\mathbf{e}$ such that*

$$[f]_{\mathbf{e}} = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix}$$

where J_k is a $n_k \times n_k$ Jordan block corresponding to the eigenvalue λ_k and

$$[b]_{\mathbf{e}} = \begin{pmatrix} \sigma_1 I_1^* & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 I_2^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \sigma_k I_k^* \end{pmatrix}$$

where $\sigma_i = \pm 1$ and

$$I_s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_s \times n_s}$$

Moreover,

$$\operatorname{sgn}(I_s^*) = \begin{cases} 0 & \text{, for } n_s \text{ even} \\ 1 & \text{, for } n_s \text{ odd} \end{cases}$$

and, therefore,

$$\operatorname{sgn}(b) = \sum_{i=1}^{k} \sigma_i \operatorname{sgn}(I_i^*) = \sum_{\{i | n_i \text{ is odd}\}} \sigma_i$$

Proof. In this case, we only have to prove that in each Jordan block the basis can be chosen so that the restriction of b on this block is represented by a matrix $\pm I^*$.

To do this, first, we consider a complexification of f and find a complex basis $(v_i)_{i=1}^n$ for a fixed chosen Jordan block $n \times n$ as in the previous section. That means

$$f(v_i) = \lambda v_i + v_{i+1}$$

$$b(v_i, v_j) = \delta_{i+j,n+1}.$$

If we conjugate these equalities, we get (since $\lambda \in \mathbb{R}$):

$$f(\bar{v}_i) = \lambda \bar{v}_i + \bar{v}_{i+1}$$

$$b(\bar{v}_i, \bar{v}_j) = \delta_{i+j,n+1}.$$

Case 1. (v_i) and $(\bar{v}_i)$ define bases of different complex Jordan blocks. Therefore v_i and $\bar{v}_i$ are not collinear and there exist unique collections of non-zero vectors (u_i), (w_i) such that

$$v_i = \frac{u_i + i w_i}{\sqrt{2}}.$$

Therefore, (u_i) and (w_i) define real bases of two different Jordan blocks. We can correct these bases so that they are orthogonal and the restriction of b on corresponding subspaces is represented by a matrix $\pm I^*$ [see lemma A.1.2].

Case 2. (v_i) and $(\bar{v}_i)$ define bases of the same complex Jordan block. Because of uniqueness of basis $\bar{v}_i = \pm v_i$.

Case 2.1. $v_i = \bar{v}_i$. That means, $\mathbf{v} = (v_i)$ is a real basis of the chosen Jordan block with $[b|_{\operatorname{Span}_{\mathbb{R}}(\mathbf{v})}]_{\mathbf{v}} = I^*$.

Case 2.2. $v_i = -\bar{v}_i = i w_i$. That means, $\mathbf{w} = (w_i)$ is a real basis basis of the chosen Jordan block with $[b|_{\operatorname{Span}_{\mathbb{R}}(\mathbf{w})}]_{\mathbf{w}} = -I^*$. □

Case $K = \mathbb{R}$ with complex eigenvalues. Generalized Jordan blocks

Remark A.1.19. For some technical reasons, we need some linear order on $\mathbb{C}$. It does not really matter which one, but to make some constructions unique we have to fix one. We will use the following order: we say $z > z'$ if $\operatorname{Re}(z) > \operatorname{Re}(z')$ or $\operatorname{Re}(z) = \operatorname{Re}(z')$ and $\operatorname{Im}(z) > \operatorname{Im}(z')$.

If the linear map f have a complex not real eigenvalue $\lambda = a + ib$ then it has an eigenvalue $\bar{\lambda} = a - ib$ as well because the characteristic polynomial is real. We consider some Jordan block J with eigenvalue λ of the size $m \times m$. Then we have automatically a Jordan block for $\bar{\lambda}$. Moreover, these both blocks have the same size because if

$$f(v_j) = \lambda v_j + v_{j-1}$$

then

$$f(\bar{v}_j) = \bar{\lambda}\bar{v}_j + \bar{v}_{j-1}$$

where (v_j) is a basis of the block J. So $(\bar{v}_j)$ is a basis of another Jordan block with eigenvalue $\bar{\lambda}$ which we denote by $\bar{J}$. We denote

$$v_j = \frac{u_j + iw_j}{\sqrt{2}}$$

We can also assume $b(v_j, v_k) = b(\bar{v}_j, \bar{v}_k) = \delta_{j+k,m+1}$

We consider another basis for pair of blocks $(J, \bar{J})$:

$$u_j = \frac{v_j + \bar{v}_j}{\sqrt{2}}, \quad w_j = \frac{v_j - \bar{v}_j}{i\sqrt{2}}$$

It is easy to see that

$$f(u_j) = au_j - bw_j + u_{j-1}$$

$$f(w_j) = bu_j + aw_j + w_{j-1}$$

Because of the discussion above we can assume that all complex Jordan blocks are orthogonal to each other. Therefore,

$$b(u_j, u_k) = -b(w_j, w_k) = b(v_j, v_k) = \delta_{j+k,m+1},$$

$$b(u_j, w_k) = 0.$$

So we get that in the real basis $(u_1, w_1, \dots, u_m, w_m)$ the pair of blocks $(J, \bar{J})$ is represented by the following matrix

$$K = \begin{pmatrix} a & b & 1 & 0 & \dots & 0 & 0 \\ -b & a & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & a & b & \dots & 0 & 0 \\ 0 & 0 & -b & a & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & 0 & \dots & -b & a \end{pmatrix}_{2m \times 2m}$$

which we will call generalized Jordan block. The restriction of b on $\mathrm{Span}(u_1, w_1, \dots, u_m, w_m)$ have the form I_{2m}^{2*}, where

$$I_{2m}^{2*} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ & & & & \dots & & \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{2m \times 2m}$$

This matrix has signature

$$\mathrm{sgn}(I_{2m}^{2*}) = 0$$

Moreover, because all complex Jordan blocks are orthogonal, this generalized block is orthogonal to other blocks.

Corollary A.1.20. *If f consists only on one generalized Jordan block then the basis above is unique up to simultaneous multiplication of all basis vectors with -1.*

Proof. The proof is identical to the proof of A.1.4. □

Lemma A.1.21. *There exists unique up to sign $\Phi \in \mathrm{Sym}(n, \mathbb{R})$ such that*

$$\Phi I^{2*} \Phi = I^{2*} K$$

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & c & d \\ 0 & 0 & 0 & 0 & \dots & d & -c \\ 0 & 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & 0 & \dots & * & * \\ & & & & \dots & & \\ 0 & 0 & c & d & \dots & * & * \\ 0 & 0 & d & -c & \dots & * & * \\ c & d & * & * & \dots & * & * \\ d & -c & * & * & \dots & * & * \end{pmatrix}$$

where $(c + id)^2 = a + ib$ and $*$ are some rational functions in c, d.

Proof. Similar to A.1.10. □

Remark A.1.22. The pair (c, d) is defined up to sign. To make Φ unique we choose (c, d) so that $c + id$ is the biggest square root of $a + ib$.

A.1.4 Standard form of a pair of bilinear forms

So we can summarize that for each bilinear form b and each linear operator f which is symmetric with respect to b there exists a basis $\mathbf{e}$ such that

$$[b]_{\mathbf{e}} = \begin{pmatrix} \mathcal{I}_1^* & 0 & 0 \\ 0 & -\mathcal{I}_2^* & 0 \\ 0 & 0 & \mathcal{I}^{2*} \end{pmatrix}, \quad [f]_{\mathbf{e}} = \begin{pmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ 0 & 0 & \mathcal{K} \end{pmatrix}$$

where for $r = 1, 2$

$$\mathcal{I}_r^* = \begin{pmatrix} I_{1r}^* & 0 & \dots & 0 & 0 \\ 0 & I_{2r}^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_{k_r r}^* \end{pmatrix}, \quad \mathcal{J}_r = \begin{pmatrix} J_{1r} & 0 & \dots & 0 & 0 \\ 0 & J_{2r} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_{k_r r} \end{pmatrix}$$

$$\mathcal{I}^{2*} = \begin{pmatrix} I_1^{2*} & 0 & \dots & 0 & 0 \\ 0 & I_2^{2*} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_s^{2*} \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} K_1 & 0 & \dots & 0 & 0 \\ 0 & K_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & K_s \end{pmatrix}$$

where $n_{ir} := \dim(I_{ir}^*) = \dim(J_{ir})$, $m_j := \dim(I_j^{2*}) = \dim(K_j)$.

Definition A.1.23 (Order on blocks)**.** For two (generalized) Jordan blocks J with eigenvalue l and J' with eigenvalue l' we will say that $J > J'$ if $\dim J > \dim J'$ or $\dim J = \dim J'$ and $l > l'$ (for generalized blocks we compare complex numbers using the order defined earlier).

Definition A.1.24 (Standard form of a pair of bilinear forms)**.** If the basis $\mathbf{e}$ is chosen as above and blocks in $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{K}$ are in order of decreasing then we will say that pair of forms $b_1 = b$ and $b_2 = b \circ f$ is in the *standard form*. We will use the following notation:

$$X(b_1, b_2) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$$

$$X^0(b_1, b_2) = [f]_{\mathbf{e}}, \ \ X^1(b_1, b_2) = [b_1]_{\mathbf{e}}, \ \ X^2(b_1, b_2) = [b_2]_{\mathbf{e}}$$

Remark A.1.25. Because

$$\operatorname{sgn}(I^*) = \begin{cases} 0 & \text{, for } \dim I^* \text{ even} \\ 1 & \text{, for } \dim I^* \text{ odd} \end{cases}, \quad \operatorname{sgn}(I^{2*}) = 0$$

we get

$$\operatorname{sgn}(b) = \#\{i | \dim I_{i1}^* \text{ is odd}\} - \#\{i | \dim I_{i2}^* \text{ is odd}\}$$

Remark A.1.26. The standard form is unique. Instead, the basis, which puts a pair of forms to the standard form, is not.

Remark A.1.27. $X(b_1, b_2)$ defines $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$ uniquely and defines b_1, b_2 uniquely up to change of basis.

$$X^0(b_1, b_2) = \operatorname{diag}(X(b_1, b_2))$$

$$X^0(b_1, b_2) = (X^1(b_1, b_2))^{-1} X^2(b_1, b_2)$$

We define the signature

$$\operatorname{sgn}(X(b_1, b_2)) := \operatorname{sgn}(b_1)$$

A.1.5 Back transformation

Definition A.1.28. We will say that matrix H is *consistent* to the pair of forms (b_1, b_2), if

$$H = \operatorname{diag}(H_1, H_2, H_3)$$

$$H_k = \operatorname{diag}(H_{1k}, \dots, H_{k_r k})$$

and $\dim H_{ij} = \dim J_{ij}$ for $j = 1, 2$, $\dim H_{i3} = \dim K_i$ for all possible i.

Definition A.1.29. Let $Y = \operatorname{diag}(Y_1, \dots, Y_s)$, $\sigma \in \operatorname{Sym}(\{1, \dots, s\})$. The matrix

$$T_\sigma = \begin{pmatrix} T_{11} & \dots & T_{1s} \\ & \dots & \\ T_{s1} & \dots & Tss \end{pmatrix}$$

is called *block permutation matrix* for Y if $T_{i,\sigma(i)} = \mathrm{Id}_{\dim Y_i}$ for all i and $T_{ij} = 0$ for all other (i, j).

Remark A.1.30. It is easy to see that

$$T_\sigma^T \operatorname{diag}(Z_1, \dots, Z_s) T = \operatorname{diag}(Z_{\sigma(1)}, \dots, Z_{\sigma(Z_s)})$$

for all $\operatorname{diag}(Z_1, \dots, Z_s)$ such that $\dim Z_i = \dim Y_i$ for all i.

Definition A.1.31 (Minimal ordering matrix)**.** Let

$$Y = \operatorname{diag}(Y_1, \dots, Y_s), \quad Z = \operatorname{diag}(Z_1, \dots, Z_s)$$

and $\dim Y_i = \dim Z_i$ for all $i \in \{1, \dots, s\}$. Moreover, assume

$$Y_i \in \{I_r^*, -I_r^*, I_r^{2*} \mid r \in \mathbb{N}\}$$

and Z_i is a Jordan block for all $i \in \{1, \dots, s\}$ such that $Y_i = \pm I_r^*$ and Z_i is a generalized Jordan block for all $i \in \{1, \dots, s\}$ such that $Y_i = I_r^{2*}$.

We will say that a block permutation matrix $T = T_\sigma$ for Y is *minimal ordering matrix* for (Y, Z) if

- $(Y', Y'Z') := (T^T Y T, T^T Y Z T)$ is the standard form for some pair of bilinear forms, where
$$Y' = \mathrm{diag}(Y_{\sigma(1)}, \dots, Y_{\sigma(s)})$$
$$Z' = \mathrm{diag}(Z_{\sigma(1)}, \dots, Z_{\sigma(s)})$$
- if $Y_i Z_i = Y_j Z_j$ for $i < j$ then $\sigma(i) < \sigma(j)$.

Remark A.1.32. For fixed pair (Y, Z) as above the minimal ordering matrix is unique because is well-defined by the corresponding permutation σ which is unique.

Proposition A.1.33. *There exist consistent to (b_1, b_2) matrix $\Phi \in \mathrm{Sym}(n, \mathbb{R})$ and the (unique) minimal ordering matrix T for $(\Phi X^2(b_1, b_2)^{-1}\Phi, \Phi X^1(b_1, b_2)\Phi)$ such that*

$$T^T \Phi X^1(b_1, b_2)\Phi T = X^2(b_2^*, b_1^*) =: \tilde{X}^2(b_1, b_2)$$

$$T^T \Phi X^2(b_1, b_2)^{-1}\Phi T = X^1(b_2^*, b_1^*) =: \tilde{X}^1(b_1, b_2).$$

We will call this transformation back transformation.

Proof. It follows from A.1.10 and A.1.21. We take $\Phi = \mathrm{diag}(\Phi_1, \dots, \Phi_p)$ where Φ_i are from A.1.10 or A.1.21 for corresponding pair of blocks of $(X^1(b_1, b_2), X^2(b_1, b_2))$. After that we do a minimal ordering. □

Remark A.1.34. In the previous proposition, Φ is unique up to sign of each block. But as we already have seen, this sign can be chosen in a canonical way. So we can assume that Φ and T are well defined by (b_1, b_2).

Remark A.1.35. The direct calculation shows that the back transformation applied twice gives the identity map.

Corollary A.1.36. *The last proposition can be reformulated in the following way:*

Let (b_1, b_2) is a pair of bilinear forms on a vector space V and in a basis $\mathbf{e}$:

$$[b_1]_{\mathbf{e}} = X^1(b_1, b_2),\ [b_2]_{\mathbf{e}} = X^2(b_1, b_2)$$

We consider a pair of bilinear forms (b_2^*, b_1^*) on the dual space V^*. In the dual basis f:

$$[b_1]_{\mathbf{f}} = X^1(b_1, b_2)^{-1},\ [b_2]_{\mathbf{f}} = X^2(b_1, b_2)^{-1}$$

The change of basis on V given by a matrix $\Phi^{-1}T^{-T} \colon \mathbf{e} \mapsto \mathbf{e}'$ induce change of basis on V^* by a matrix $\Phi T \colon \mathbf{f} \mapsto \mathbf{f}'$ so that

$$[b_1^*]_{\mathbf{f}'} = X^2(b_2^*, b_1^*),\ [b_2^*]_{\mathbf{f}'} = X^1(b_2^*, b_1^*)$$

This change-of-basis is determined by $X(b_1, b_2)$. We denote this transformation on bases of V by $\sigma_{X(b_1,b_2)}$, the corresponding dual transformation of bases of V^* is denoted by $\sigma^*_{X(b_1,b_2)}$. This transformation will be used to define the basis associated to the opposite oriented edge.

A.1.6 (p,q)-shape transformation

Let

$$\mathbf{n} := (n_1, \ldots, n_{k_1}), \;\; \mathbf{m} := (m_1, \ldots, m_{k_2}), \;\; \mathbf{r} := (r_1, \ldots, r_{k_3})$$

be three decreasing sequences of natural numbers.

$$I_{\mathbf{nmr}} := \operatorname{diag}(I^*_{n_1}, \ldots, I^*_{n_{k_1}}, -I^*_{m_1}, \ldots, -I^*_{m_{k_2}}, I^{2*}_{r_1}, \ldots, I^{2*}_{r_{k_3}})$$

We consider this matrix as a matrix of some bilinear form. Let (p,q) be the signature of this form. We fix one matrix $P_{\mathbf{nmr}}$ such that

$$P^T_{\mathbf{nmr}} I_{pq} P_{\mathbf{nmr}} = I_{\mathbf{nmr}}$$

and the corresponding P-matrix for I_{pq} is Id.

Definition A.1.37. We denote by $\mathcal{P}_{pq}$ the set of all matrices $P_{\mathbf{nmr}}$ such that $I_{\mathbf{nmr}}$ has signature (p,q).

Definition A.1.38. Let (b_1, b_2) be a pair of bilinear forms and b_1 has signature (p,q). We denote by $P_{b_1 b_2}$ the corresponding $P_{\mathbf{nmr}}$ as above such that

$$X^1(b_1, b_2) = P^T_{b_1 b_2} I_{pq} P_{b_1 b_2}$$

Definition A.1.39. Let $X = X(b_1, b_2)$ for some pair of forms (b_1, b_2). Then b_1 has signature (p,q). We denote by P_X the corresponding $P_{b_1 b_2}$ as above such that

$$X^1 = X^1(b_1, b_2) = P^T_X I_{pq} P_X$$

Remark A.1.40. As we have seen before, $X(b_1, b_2)$ defines $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$. So if we know $X(b_1, b_2)$, we do not need any information about (b_1, b_2). Therefore, sometimes we will write just X instead of $X(b_1, b_2)$ and also X^0, X^1, X^2 instead of $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$ (and correspondent expressions with $\tilde{}$) if forms (b_1, b_2) are not important.

A.2 Three isomorphisms of matrix algebras

In this section, we describe three well-known matrix algebras isomorphisms that we will use. For every algebra A and (anti-)involution σ, we denote by A^σ the set of fixed points of σ in A.

A.2.1 $\operatorname{Mat}(n, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\operatorname{Mat}(n, \mathbb{C}) \times \operatorname{Mat}(n, \mathbb{C})$

Fact A.2.1. *The following map is an isomorphism of $\mathbb{C}\{i\}$-algebras:*

$$\begin{array}{cccc} \chi\colon & \operatorname{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\} & \to & \operatorname{Mat}(n, \mathbb{C}\{i\}) \times \operatorname{Mat}(n, \mathbb{C}\{i\}) \\ & a + bI & \mapsto & (a+bi, a-bi) \end{array}$$

where $a, b \in \operatorname{Mat}(n, \mathbb{C}\{i\})$. In particular,

$$\chi(\operatorname{Id} I \otimes 1) = (i, -i), \;\; \chi(\operatorname{Id} \otimes i) = (i, i).$$

The induced by $\sigma \otimes \mathrm{Id}$ anti-involution

$$\chi \circ (\sigma \otimes \mathrm{Id}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (m_1^T, m_2^T).$$

The induced by $\bar{\sigma} \otimes \mathrm{Id}$ anti-involution

$$\chi \circ (\bar{\sigma} \otimes \mathrm{Id}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (m_2^T, m_1^T).$$

The induced by $\mathrm{Id} \otimes \bar{\sigma}$ involution

$$\chi \circ (\mathrm{Id} \otimes \bar{\sigma}) \circ \chi^{-1}$$

on $\mathrm{Mat}(n, \mathbb{C}\{i\}) \times \mathrm{Mat}(n, \mathbb{C}\{i\})$ acts in the following way:

$$(m_1, m_2) \mapsto (\bar{m}_2, \bar{m}_1).$$

Therefore:

$$\chi((\mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\})^{\bar{\sigma} \otimes \mathrm{Id}}) = \{(m, m^T) \mid m \in \mathrm{Mat}(n, \mathbb{C}\{i\})\},$$

$$\chi((\mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\})^{\bar{\sigma} \otimes \bar{\sigma}}) = \mathrm{Herm}(n, \mathbb{C}\{i\}) \times \mathrm{Herm}(n, \mathbb{C}\{i\}),$$

$$\chi(\mathrm{Mat}(n, \mathbb{C}\{I\})) = \chi((\mathrm{Mat}(n, \mathbb{C}\{I\}) \otimes_{\mathbb{R}} \mathbb{C}\{i\})^{\mathrm{Id} \otimes \bar{\sigma}}) = \{(m, \bar{m}) \mid m \in \mathrm{Mat}(n, \mathbb{C}\{i\})\}.$$

A.2.2 $\mathrm{Mat}(n, \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathrm{Mat}(2n, \mathbb{C})$

Fact A.2.2. *The following map is an isomorphism of $\mathbb{C}\{I\}$-$\mathbb{C}\{i\}$-algebras:*

$$\begin{array}{rccc} \psi\colon & \mathrm{Mat}(n, \mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\} & \to & \mathrm{Mat}(2n, \mathbb{C}\{i\}) \\ & (q_1 + q_2 j) + (p_1 + p_2 j) I & \mapsto & \begin{pmatrix} q_1 + p_1 i & q_2 + p_2 i \\ -\bar{q}_2 - \bar{p}_2 i & \bar{q}_1 + \bar{p}_1 i \end{pmatrix}. \end{array}$$

where $q_1, q_2, p_1, p_2 \in \mathrm{Mat}(n, \mathbb{C}\{i\})$. In particular,

$$\chi(\mathrm{Id}\, i \otimes 1) = \begin{pmatrix} \mathrm{Id}\, i & 0 \\ 0 & -\mathrm{Id}\, i \end{pmatrix},\ \chi(\mathrm{Id} \otimes j) = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix},$$

$$\chi(\mathrm{Id}\, k \otimes 1) = \begin{pmatrix} 0 & \mathrm{Id}\, i \\ \mathrm{Id}\, i & 0 \end{pmatrix},\ \chi(\mathrm{Id} \otimes I) = \mathrm{Id}\, i.$$

The induced by $\sigma_0 \otimes \mathrm{Id}$ anti-involution

$$\psi \circ (\sigma_0 \otimes \mathrm{Id}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} m^T \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}.$$

The induced by $\sigma_1 \otimes \mathrm{Id}$ anti-involution

$$\psi \circ (\sigma_1 \otimes \mathrm{Id}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The induced by $\mathrm{Id} \otimes \bar{\sigma}$ involution

$$\psi \circ (\mathrm{Id} \otimes \bar{\sigma}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \bar{m} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \bar{m} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The induced by $\sigma_0 \otimes \bar{\sigma}$ anti-involution

$$\psi \circ (\sigma_0 \otimes \bar{\sigma}) \circ \psi^{-1}$$

on $\mathrm{Mat}(2n, \mathbb{C})$ acts in the following way:

$$m \mapsto \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \bar{m}^T \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Therefore:

$$\psi((\mathrm{Mat}(n, \mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\})^{\sigma_1 \otimes \mathrm{Id}}) =$$
$$= \left\{ m \in \mathrm{Mat}(2n, \mathbb{C}\{i\}) \mid m = -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} m^T \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \right\} =$$
$$= \mathfrak{o}\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} = \mathfrak{sp}(2n, \mathbb{C}),$$

$$\psi((\mathrm{Mat}(n, \mathbb{H}\{i,j,k\}) \otimes_{\mathbb{R}} \mathbb{C}\{I\})^{\sigma_0 \otimes \bar{\sigma}}) =$$
$$= \left\{ m \in \mathrm{Mat}(2n, \mathbb{C}\{i\}) \mid m = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \bar{m}^T \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \right\},$$

$$\psi((\mathrm{Mat}(n,\mathbb{H}\{i,j,k\})\otimes_{\mathbb{R}}\mathbb{C}\{I\})^{\sigma_1\otimes\bar{\sigma}}) = \mathrm{Herm}(2n,\mathbb{C}),$$

$$\psi(\mathrm{Mat}(n,\mathbb{H}\{i,j,k\})) = \psi((\mathrm{Mat}(n,\mathbb{H}\{i,j,k\})\otimes_{\mathbb{R}}\mathbb{C}\{I\})^{\mathrm{Id}\otimes\bar{\sigma}}) =$$
$$= \left\{ m\in \mathrm{Mat}(2n,\mathbb{C}\{i\}) \mid m = -\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0\end{pmatrix} \bar{m} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0\end{pmatrix}\right\} =$$
$$= \left\{ \begin{pmatrix} q_1 & q_2 \\ -\bar{q}_2 & \bar{q}_1 \end{pmatrix} \mid q_1, q_2 \in \mathrm{Mat}(n,\mathbb{C}\{i\})\right\}.$$

A.2.3 $\mathrm{Mat}(n,\mathbb{H})\otimes_{\mathbb{R}}\mathbb{H}$ and $\mathrm{Mat}(4n,\mathbb{R})$

Fact A.2.3. *The following map:*

$$\phi\colon\ \mathrm{Mat}(n,\mathbb{H}\{I,J,K\})\otimes_{\mathbb{R}}\mathbb{H}\{i,j,k\}\to\mathrm{Mat}(4n,\mathbb{R})$$

defined on generators of $A_{\mathbb{H}}$ as follows:

$$\phi(a\otimes i) = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix},\ \phi(a\otimes j) = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ -a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix},$$

$$\phi(a\otimes k) = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \end{pmatrix},\ \phi(aI\otimes 1) = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix},$$

$$\phi(aJ\otimes 1) = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix},\ \phi(aK\otimes 1) = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

where $a\in\mathrm{Mat}(n,\mathbb{R})$ is an $\mathbb{R}$-algebra isomorphism.

The anti-involution $\sigma_1\otimes\sigma_0$ corresponds under ϕ to the following anti-involution

$$\phi\circ(\sigma_1\otimes\sigma_0)\circ\phi$$

on $\mathrm{Mat}(4n,\mathbb{R})$: $m\mapsto -\Xi m^T\Xi$ where

$$\Xi := \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n & 0 & 0 \\ -\mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix}.$$

The anti-involution $\sigma_1\otimes\sigma_1$ corresponds under ϕ to the transposition on $\mathrm{Mat}(4n,\mathbb{R})$.

Therefore:

$$\phi((\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})\otimes_{\mathbb{R}}\mathbb{H}\{i,j,k\})^{\sigma_1\otimes\sigma_0}) = \left\{m\in\mathrm{Mat}(4n,\mathbb{R}) \mid m = -\Xi m^T\Xi\right\} =$$

$$= \mathfrak{o}(\Xi)\cong\mathfrak{sp}(4n,\mathbb{R}),$$

$$\phi((\mathrm{Mat}(n,\mathbb{H}\{i,j,k\})\otimes_{\mathbb{R}}\mathbb{H}\{i,j,k\})^{\sigma_1\otimes\sigma_1}) = \mathrm{Sym}(4n,\mathbb{R}),$$

The real locus $\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})$ of $\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})\otimes_{\mathbb{R}}\mathbb{H}\{i,j,k\}$ is mapped by ϕ to:

$$\phi(\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})) =$$

$$= \left\{\begin{pmatrix} a & -b & -c & -d\\ b & a & -d & c\\ c & d & a & -b\\ d & -c & b & a\end{pmatrix} \mid a,b,c,d\in\mathrm{Mat}(n,\mathbb{R})\right\}.$$

A.3 Embeddings between matrix algebras

In this section, we consider the following two embeddings:

$$\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{C}\{j\}\hookrightarrow\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{H}\{i,j,k\},$$

$$\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})\otimes\mathbb{C}\{j\}\hookrightarrow\mathrm{Mat}(n,\mathbb{H}\{I,J,K\})\otimes\mathbb{H}\{i,j,k\}.$$

We are interested in this embedding because, for the first embedding, the restriction of $\bar\sigma\otimes\sigma_1$ corresponds to $\bar\sigma\otimes\bar\sigma$ and the restriction of $\bar\sigma\otimes\sigma_0$ corresponds to $\sigma\bar\otimes\mathrm{Id}$. For the second embedding, the restriction of $\sigma_1\otimes\sigma_1$ corresponds to $\sigma_1\otimes\bar\sigma$ and the restriction of $\sigma_1\otimes\sigma_0$ corresponds to $\sigma_1\otimes\mathrm{Id}$, and so we can use these embedding to see the symmetric space for the real group inside the symmetric space for complexified group.

A.3.1 Embedding $\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{C}\{j\}\hookrightarrow\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{H}\{i,j,k\}$

In the previous sections, we have seen isomorphisms:

$$\begin{array}{cccc}\chi\colon & \mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_{\mathbb{R}}\mathbb{C}\{j\} & \to & \mathrm{Mat}(n,\mathbb{C}\{j\})\times\mathrm{Mat}(n,\mathbb{C}\{j\})\\ & a+bI & \mapsto & (a+bj,a-bj)\end{array}$$

where $a,b\in\mathrm{Mat}(n,\mathbb{C}\{j\})$. And

$$\begin{array}{cccc}\psi\colon & \mathrm{Mat}(n,\mathbb{C}\{I\})\otimes_{\mathbb{R}}\mathbb{H}\{i,j,k\} & \to & \mathrm{Mat}(2n,\mathbb{C}\{i\})\\ & (q_1+q_2j)+(p_1+p_2j)I & \mapsto & \begin{pmatrix} q_1+p_1i & q_2+p_2i\\ -\bar q_2-\bar p_2 i & \bar q_1+\bar p_1 i\end{pmatrix}.\end{array}$$

where $q_1,q_2,p_1,p_2\in\mathrm{Mat}(n,\mathbb{C}\{i\})$. Since

$$\iota\colon\ \mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{C}\{j\}\hookrightarrow\mathrm{Mat}(n,\mathbb{C}\{I\})\otimes\mathbb{H}\{i,j,k\},$$

we want to describe the map $\psi \circ \iota \circ \chi^{-1}$.

Let $(a, b) := (a_1 + a_2 j, b_1 + b_2 j) \in \mathrm{Mat}(n, \mathbb{C}\{j\}) \times \mathrm{Mat}(n, \mathbb{C}\{j\})$ for $a_1, a_2, b_1, b_2 \in \mathrm{Mat}(n, \mathbb{R})$, then

$$\chi^{-1}(a, b) = \frac{a+b}{2} + \frac{a-b}{2j} I = \frac{a_1 + b_1 + (a_2 + b_2)j}{2} + \frac{a_2 - b_2 - (a_1 - b_1)j}{2} I.$$

Therefore,

$$\psi(\chi^{-1}(a, b)) = \frac{1}{2} \begin{pmatrix} a_1 + b_1 + (a_2 - b_2)i & a_2 + b_2 - (a_1 - b_1)i \\ -(a_2 + b_2) + (a_1 - b_1)i & a_1 + b_1 + (a_2 - b_2)i \end{pmatrix}$$

and

$$\mathrm{Im}(\psi \circ \iota \circ \chi^{-1}) = \left\{ \begin{pmatrix} q & p \\ -p & q \end{pmatrix} \mid p, q \in \mathrm{Mat}(n, \mathbb{C}\{i\}) \right\} =$$

$$= \left\{ m \in \mathrm{Mat}(2n, \mathbb{C}\{i\}) \mid m = - \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} m \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \right\}.$$

A.3.2 Embedding $\mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{C}\{j\} \hookrightarrow \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{H}\{i, j, k\}$

In the previous sections, we have seen isomorphisms:

$$\begin{array}{cccc} \psi\colon & \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes_{\mathbb{R}} \mathbb{C}\{j\} & \to & \mathrm{Mat}(2n, \mathbb{C}\{I\}) \\ & (q_1 + q_2 J) + (p_1 + p_2 J)j & \mapsto & \begin{pmatrix} q_1 + p_1 I & q_2 + p_2 I \\ -\bar{q}_2 - \bar{p}_2 I & \bar{q}_1 + \bar{p}_1 I \end{pmatrix}. \end{array}$$

where $q_1, q_2, p_1, p_2 \in \mathrm{Mat}(n, \mathbb{C}\{I\})$ and

$$\phi\colon \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes_{\mathbb{R}} \mathbb{H}\{i, j, k\} \to \mathrm{Mat}(4n, \mathbb{R})$$

defined as in the Section A.2.3. Since

$$\iota\colon \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{C}\{j\} \hookrightarrow \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{H}\{i, j, k\},$$

we want to describe the image of the map $\phi \circ \iota \circ \psi^{-1}$. Note that for $x \in \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{H}\{i, j, k\}$, $x \in \mathrm{Mat}(n, \mathbb{H}\{I, J, K\}) \otimes \mathbb{C}\{j\}$ if and only if x commutes with $1 \otimes j$. So we obtain:

$$\mathrm{Im}(\psi \circ \iota \circ \chi^{-1}) = \{m \in \mathrm{Mat}(4n, \mathbb{R}) \mid m = -\phi(\mathrm{Id}_n \otimes j) m \phi(\mathrm{Id}_n \otimes j)\}$$

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