INAUGURAL-DISSERTATION

zur

Erlangung der Doktorwürde

der

Gesamtfakultät für Mathematik, Ingenieur- und Naturwissenschaften

der

Ruprecht – Karls – Universität, Heidelberg

vorgelegt von

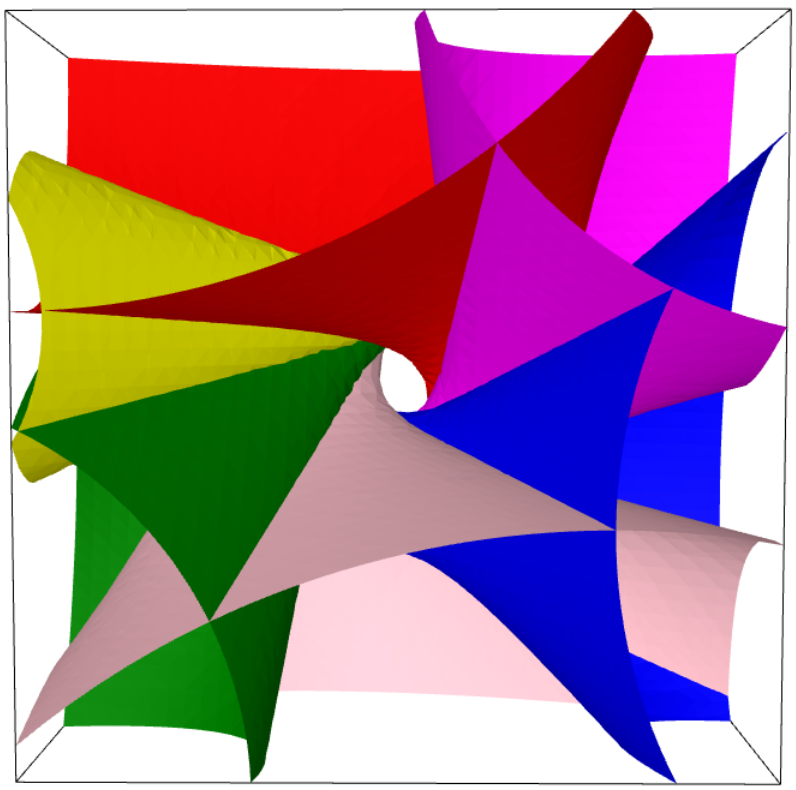
COLIN DAVALO

aus

PARIS

Tag der mündlichen Prüfung:

Geometric structures and representations of surface groups



Supervisor:

Prof.Dr. Beatrice Pozzetti

Zusammenfassung

Die Darstellungstheorie hyperbolischer Gruppen in Lie-Gruppen höheren Ranges ist in den letzten Jahren ein aktives Forschungsthema gewesen. Insbesondere die Charaktervarietät, die mit einer Flächengruppe für einige halbeinfache Lie-Gruppen nichtkompakten Typs assoziiert ist, weist bemerkenswerte Zusammenhängskomponenten auf, die nur diskrete und treue Darstellungen enthalten. Eine Vereinigung solcher zusammenhängender Komponenten wird als Teichmüller-Raum höheren Rangs bezeichnet. In allen bekannten Fällen erfüllen die Darstellungen in diesen Komponenten alle eine Anosov-Eigenschaft, welche eine dynamische Eigenschaft ist, die stärker als diskret und treu ist. Einige dieser Räume können als Räume geometrischer Strukturen interpretiert werden: beispielsweise als konvexe projektive Strukturen auf Flächen oder als gefaserte Photonstrukturen.

In dieser Dissertation leisten wir originelle Beiträge zu diesem Bereich, wobei wir uns insbesondere auf den lokal symmetrischen Raum und parabolische Strukturen konzentrieren, die mit Anosov-Darstellungen zusammenhängen sind. Der erste Teil dieser Dissertation ist eher allgemein und diskutiert parabolische Strukturen, die unter Verwendung eines Diskontinuitätsbereichs konstruiert wurden, sowie deren Beziehung zum lokal symmetrischen Raum für bestimmte Anosov-Darstellungen. Wir untersuchen genauer die Diskontinuitätsbereiche, die als Bereiche geeigneter Busemann-Funktionen interpretiert werden können.

Der zweite Teil konzentriert sich auf maximale Darstellungen in $\mathrm{Sp}(2n,\mathbb{R})$, eine besondere Klasse von Teichmüller-Räumen höheren Ranges. Wir charakterisieren maximale Darstellungen, im Hinblick auf geometrische Strukturen, die eine spezielle Faserung zulassen. Schließlich untersuchen wir maximale Darstellungen, die auch Borel Anosov sind, und zeigen insbesondere, dass in $\mathrm{Sp}(4,\mathbb{R})$ diese Darstellungen Hitchin sind, was eine Frage von Canary beantwortet.

Diese Dissertation umfasst die Ergebnisse der Arxiv-Preprints *Nearly geodesic immersions and domains of discontinuity* [Dav23] und *Finite-sided Dirichlet domains for Anosov subgroups* [DR24] , eines zukünftigen Preprints *Geometric structures for maximal representations and pencils* , und schließlich des Artikels *Maximal und Borel Anosov Darstellungen in* $\mathrm{Sp}(2n,\mathbb{R})$ [Dav24]. Der Preprint [DR24] ist eine gemeinsame Arbeit mit Max Riestenberg.

Abstract

Representations of hyperbolic groups into higher rank Lie groups has been an active topic of study in recent years. In particular the character variety associated with a surface group for some semi-simple Lie group of non-compact type admits remarkable connected components containing only discrete and faithful representations. A union of such connected components is called a higher rank Teichmüller space. In all the known cases, the representations in these components all satisfy an Anosov property, which is a dynamical property stronger than being discrete and faithful. Some of these spaces can be interpreted as spaces of geometric structures: as for instance convex projective structures on surfaces, or fibered photon structures.

In this thesis, we bring original contributions to this area, focusing in particular on the locally symmetric space and parabolic structures associated to Anosov representations. The first part of this thesis is rather general, and discuss parabolic structures constructed using a domain of discontinuity as well as their relation with the locally symmetric space for certain Anosov representations. We study more precisely the domains of discontinuity that can be interpreted as domains of proper Busemann functions.

The second part focuses on maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$, a particular class of higher rank Teichmüller spaces. We characterize maximal representations in terms of geometric structures that admit a special fibration. Finally we study maximal representations that are also Borel Anosov, and show in particular that in $\mathrm{Sp}(4, \mathbb{R})$ these representations are Hitchin, answering a question from Canary.

This thesis encompasses the results of the arxiv preprints *Nearly geodesic immersions and domains of discontinuity* [Dav23] and *Finite-sided Dirichlet domains for Anosov subgroups* [DR24] , a future preprint *Geometric structures for maximal representations and pencils* , and finally the article *Maximal and Borel Anosov representations in* $\mathrm{Sp}(2n, \mathbb{R})$ [Dav24]. The preprint [DR24] is joint work with Max Riestenberg

Acknowledgments

I would like to begin by warmly thanking Beatrice. Your support during these years was amazing, and it made this work possible. I learned a lot from your advice and all our discussions, and not only about math. Moreover your enthusiasm made it very enjoyable working with you.

I would like to thank Gabriele. Thank you for the cool things I learned with you from cool math to very nice cooking recipes during these years in Heidelberg, and for the cool hikes we did.

Thank you also Max with whom I had a lot of interesting and fun math discussions, that were very helpful for this thesis. Thank you also Thi and Johannes for our discussions in Heidelberg.

Thank you to all the members of the research station with whom I had a very pleasant stay in Heidelberg, and in particular thank to Marta, Arnaud, Fernando, Merik, , Mareike, Enja, Jiajun, Maurice, Erik, Mithul, Dai, Zach and Mengxue. I also thank the RTG members from Karlsruhe with whom I spend fun retreats.

Thank you Alex N., Xenia, Jacques, Alex M., Samuel and Konstantinos for our inspiring discussions and good times we had during conferences and workshops.

Thank you Anna, James, Peter for the discussions we had in Heidelberg.

I would like to thank Nora and Evelyn. Thank you for your support in administrative tasks, and for your patience.

Thank you Paolo and Mathias for being such nice flatmates. Ich danke Mathias, Frau Kreuz und den Mitgliedern der Capella Carolina für die Möglichkeit, mein Deutsch durch die Übung mit ihnen zu verbessern.

Je voudrais remercier Andrea et Gérard pour nos discussions lors de mon séjour à Grenoble. Je remercie également les doctorants et aggrégatifs de l'institut Fourier avec qui j'ai passé un séjour génial.

Enfin merci à mes parents qui m'ont soutenu tout le long de ma thèse.

This work was funded by the DFG Emmy Noether project 427903332 of B. Pozzetti and the RTG 2229 "Asymptotic Invariants and Limits of Groups and Spaces".

Contents

Chapter 1

Introduction

We begin this introduction with a section that reviews the notions of Fuchsian representations, Higher rank Teichmüller spaces and Anosov representations, with some historical background.

We then consider geometric structures for Anosov representations. We first discuss Klein geometries in some generality and in particular locally symmetric spaces. We then discuss parabolic structures and how Guichard-Wienhard [GW12] and Kapovich-Leeb-Porti [KLP17] construct such structures using domains of discontinuity for Anosov representations.

We then begin to introduce the results of the thesis. We introduce the notion of ω-undistorded subgroups and their associated domain of proper Busemann functions. We then present the results from Chapter 2 on nearly geodesic immersions and their application to construct fibrations of domains of discontinuity, and more precisely of domains of proper Busemann functions. Finally we present the results form Chapter 3, a joint work with Max Riestenberg, on the finite-sidedness of Dirichlet domains for Anosov representations, and more precisely for ω-undistorded representations.

The last part of this introduction focuses on maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$. We first present the results of Chapter 4 which characterize maximal representations by the existence of an equivariant surface of pencils of quadrics fibering the domain of discontinuity in projective space. Finally we present the results from Chapter 5 on Borel Anosov and maximal representations into $\mathrm{Sp}(2n, \mathbb{R})$.

1.1 Fuchsian representations and beyond

In recent years, a lot of focus have been put on the study of spaces of representations that generalize Fuchsian representations for higher rank Lie groups. These spaces are called higher rank Teichmüller spaces.

1.1.1 Fuchsian representations

Let S_g be a closed oriented surface of genus at least 2. Let Γ_g be the fundamental group of S_g. A *Fuchsian representation* $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is a representation that is discrete and faithful, i.e. a group homomorphism that is injective and whose image is a discrete subgroup.

The space of Fuchsian representations has the remarkable property that it forms a union of connected components for the compact-open topology of the representation variety $\mathrm{Hom}\left(\Gamma_g, \mathrm{PSL}(2,\mathbb{R})\right)$, which are in turn path connected components since the representation variety is locally an algebraic variety. In other words no deformation of a Fuchsian representation can degenerate to a non-discrete or non-faithful representation. Goldman [Gol80] showed that the connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ are completely determined by an invariant called the *Euler number*:

$$e : \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R})) \to [\![-2g+2, 2g-2]\!].$$

The union of the two connected components corresponding to the extremal values of the Euler number $e^{-1}(\{-2g+2, 2g-2\})$ is exactly the set of Fuchsian representations. Representations with even Euler number are exactly the representations that can be lifted to representations $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{R})$, therefore Fuchsian representations can be lifted.

The *character variety* of representations of a surface group Γ_g into a Lie group G is the quotient of the representation variety by the action of G by conjugation. The two connected components of the character variety associated with the space of Fuchsian representations are smooth manifolds, that are diffeomorphic to $\mathbb{R}^{6g-6}$. Each of these components of the character variety can be identified with the Fricke space and the Teichmüller space. The Fricke space is the space of marked hyperbolic structures on S_g, and the Teichmüller space is the space of marked Riemann surface structures on S_g, in both cases the structures are considered up to isotopy.

These identifications are obtained the following way. Let $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$. Since it is discrete, faithful, and since Γ_g is torsion free the action of $\rho(\Gamma_g)$ on $\mathbb{H}^2$ is properly discontinuous and free. The classification of surfaces implies that the quotient $\mathbb{H}^2/\rho(\Gamma_g)$ is diffeomorphic to S_g. Moreover there exits a unique such diffeomorphism up to isotopy that preserves the marking on the fundamental group of S_g. This quotient is naturally endowed with an oriented hyperbolic structure, and a Riemann surface structure.

Conversely given an oriented hyperbolic structure on a surface or a Riemann surface structure the universal cover is naturally diffeomorphic to a disk by the Cartan-Hadamard theorem or respectively is biholomorphic to the unit disk $\mathbb{H}^2$ by the uniformization theorem. To these structures one can therefore associate a *holonomy* which is a repesentation of Γ_g into the group $\mathrm{PSL}(2,\mathbb{R})$ of orientation preserving isometries of $\mathbb{H}^2$ which is also the group of biholomorphism of $\mathbb{H}^2$.

Fuchsian representation ρ satisfy a strong dynamical property: they are *quasi-isometric embeddings* which mean that for every base point o in the hyperbolic plane, the orbit map $\gamma \in \Gamma \mapsto \gamma \cdot o \in \mathbb{H}^2$ is a quasi isometric embedding. This means that for one and hence any finite generating system F of Γ_g, if d_Γ denotes the induced word distance on Γ_g, there exist constants $C, D > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma_g$ one has:

$$C^{-1} d_\Gamma(\gamma_1, \gamma_2) - D \leqslant d(\gamma_1 \cdot o, \gamma_2 \cdot o) \leqslant C d_\Gamma(\gamma_1, \gamma_2) + D.$$

This property also implies that ρ admits a *boundary map* $\xi_\rho : \partial\Gamma_g \to \partial\mathbb{H}^2$ that is a ρ-equivariant map from the Gromov boundary of Γ_g into the visual boundary of $\mathbb{H}^2$, which can be identified with $\mathbb{RP}^1$. This map is characterized by the property of being *dynamics preserving*, i.e. the image of the attracting fixed point $x \in \partial\Gamma_g$ of an element $\gamma \in \Gamma$ is mapped to the attracting fixed point of $\rho(\gamma)$ in $\partial\mathbb{H}^2$.

An important feature of $\mathbb{H}^2$, and more generally of Gromov hyperbolic spaces and in particular rank one symmetric spaces is that the space of quasi-isometric embedding is open in the space of representations $\mathrm{Hom}\,(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$. This does not hold for higher rank symmetric spaces.

1.1.2 Higher rank Teichmüller spaces

A way to construct interesting representations of a surface group into a more general Lie group G is to compose Fuchsian representations by a Lie group embedding of $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{R})$ into G.

For instance if we compose a Fuchsian representation with the natural inclusion $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$ we get a discrete and faithful representation $\rho_0 : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ that admits a totally geodesic equivariant map $h : \widetilde{S_g} \to \mathbb{H}^3$. A representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{C})$ is called *quasi-Fuchsian* if it is a quasi-isometric embedding, or equivalently if it is convex cocompact. The space of quasi-Fuchsian representations is not a connected component of the representation variety: it is open but not closed. The representations in the boundary of the quasi-Fuchsian locus are still discrete and faithful, but they can be continuously deformed into non-discrete representations.

An interesting phenomenon arises in higher rank Lie groups. For some simple Lie groups G one can find a union of connected components of representations of Γ_g into G that contains only discrete and faithful representations. A union of such connected connected components is called a *Higher rank Teichmüller space* [Wie18], as it generalizes one of the remarkable properties of the space of Fuchsian representations.

Hitchin [Hit92] discovered an exceptional component in the representation variety $\mathrm{Hom}(\Gamma_g, G)$ for any split simple Lie group G. He noticed in particular that the corresponding connected component of the character variety was diffeomorphic to $\mathbb{R}^{\dim(G)(2g-2)}$. He noticed that this property was a generalization of a property of Teichmüller space. He asked if one can understand the geometric significance of such representations.

Labourie showed that Hitchin representations are all discrete and faithful [Lab06]. He introduced the notion of Anosov representations, which is a generalization to higher rank Lie groups of the property of being a quasi-isometric embedding. He showed that Hitchin representations are all *Borel Anosov.* In particular these representations admit a boundary curve in the space of full flags. Later Guichard-Wienhard defined the notion of Θ-Anosov representation of word hyperbolic groups, where Θ is a set of simple roots [GW12]. Anosov representations can be characterized as uniformly regular undistorted representations, due to a result of Kapovich-Leeb-Porti [KLP18b] and Bochi-Potrie-Sambarino [BPS19]. To a Θ-Anosov representation one can associate a boundary map $\xi_\rho^\Theta : \partial\Gamma \to \mathcal{F}_\Theta$ into the flag manifold of G associated with Θ. Such boundary maps can be characterized as in the Fuchsian case by the property of being dynamics preserving.

Other higher rank Teichmüller spaces can be constructed for Lie groups G of Hermitian type and tube type. To a representation $\rho : \Gamma_g \to G$ one can associate an invariant $T(\rho)$ called the *Toledo number* that generalizes the Euler number. This invariant satisfies a generalization of the Milnor-Wood inequality [Mil58],[Woo71]:

$$|T(\rho)| \leqslant \mathrm{rank}(G)(2g-2).$$

Representations with maximal Toledo number are called *maximal representation* they form a union of connected components of the space of representations. Maximal representation can be characterized by the existaence of a maximal equivariant map from the circle $\partial\Gamma_g$ into the Shilov boundary [BIW03]. Burger-Iozzi-Labourie-Wienhard showed that maximal representations are Anosov with respect to the longest simple root [BILW05]. In particular these representations are all discrete and faithful and therefore form higher rank Teichmüller spaces.

Guichard-Wienhard introduced the notion of Θ-positive representation for some pairs (G, Θ) of a simple Lie group together with a set of simple roots [GW22]. For each such pair they construct a special union of connected component of the space of triples of transverse flags in the flag manifold $\mathcal{F}_\Theta$. These triples are called *positive triples.* A representation is called Θ*-positive* if it admits a continuous and equivaraint map $\xi^\Theta : \partial\Gamma \to \mathcal{F}_\Theta$ such that for all distinct $x, y, z \in \partial\Gamma$, the triple $\left(\xi^\Theta(x), \xi^\Theta(y), \xi^\Theta(z)\right)$ forms a positive triple. Hitchin representations, as well are representation with maximal or minimal Toledo number can be characterized as Θ-positive representations for a suitable choice of Θ.

Guichard-Labourie-Wienhard showed that Θ-positive representations are always Θ-Anosov and that for every notion of Θ-positivity there is at least one connected component consisting only of Θ-positive representations [GLW21].

One can generalise Hitchin's question as follows:

Question 1.1.1. Can the spaces of Hitchin representations, maximal representations, or even more generally Θ-positive representations be characterized as a space of geometric structures ?

1.2 Geometric structures

In this section we discuss geometric structures in the sense of Klein, or (G, X)-*structures*. To a discrete representation into a higher rank Lie group one can associate a locally symmetric space on a manifold M, but this manifold will not be compact in general. One can also associate to some discrete representations a parabolic structure on a compact manifold, i.e. a geometric structure modeled on a flag manifold.

1.2.1 Geometric structures in the sense of Klein

There are several notions of geometric structures. Klein introduced a general notion of (G, X) structures, which are in some sense rigid geometries, modeled on a space with a finite dimensional space of symmetries [Kle93]. A modern overview of this notion can be found in [Gol22].

Let G be a Lie group acting analytically, transitively and faithfully on a space X. Let M be a manifold of the same dimensions as X. A (G, X)-*structure* on M describes a way to model M on X with transitions in G, i.e. it is the data of a maximal atlas of charts between open sets of M into X such that the domains of definition of the charts cover M and the transition between any two charts is locally equal to the action of some element on G.

The manifold X admits a natural (G, X)-structure. We say that a map between manifolds equipped with (G, X)-structures is a local isomorphism if locally in the charts it is equal to the action of an element of G.

Given such a structure on M, one can define a pair $(\mathrm{dev}, \mathrm{hol})$ where $\mathrm{hol} : \pi_1(M) \to G$ is th *holonomy* representation and $\mathrm{dev} : \widetilde{M} \to X$ is the *developing map* which is a hol-equivariant local diffeomorphism that preserves the (G, X)-structure. This pair is unique up to the action of G, acting by conjugation on the holonomy and acting on the left on the developing map.

A particularly nice situation is when the developing map is a covering map. However this is not always the case, as for instance it can fail to be surjective. When G preserves a Riemannian metric on X, one can define the notion of a *complete* (G, X) structure on M. Indeed such a (G, X)-structure induces a Riemannian metric on M and we say that the structure is complete if this metric is complete. Any such (G, X)-structure on a closed manifold M is complete. The developing map of such a complete (G, X)-structure is a covering map.

If G is a semi-simple Lie group of non-compact type, every manifold X that admits a transitive action of G preserving a Riemannian metric fibers equivariantly over the *symmetric space* $\mathbb{X}$ associated to G, which is a simply connected non-positively curved Riemannian manifold on which G acts transitively.

Discrete subgroups $\Gamma \subset G$ up to conjugation correspond exactly to manifolds with a complete locally symmetric structure modeled on $\mathbb{X}$, i.e. a complete $(G, \mathbb{X})$ structure, up to isometry.

However if one is interested in the case when Γ is a surface subgroup and G is not locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, then the locally symmetric space cannot be compact. One can study compactifications of this locally symmetric space. These compactifications can be related in some cases to a parabolic structure on a compact manifold with the same holonomy.

1.2.2 Parabolic structures

A (G, X) structure is *parabolic* if X is a flag manifold associated to G with its standard action, i.e. if $X = G/P$ where P is a parabolic subgroup of G. If G is a semi-simple Lie group of non-compact type, then the flag manifolds never admit an invariant Riemannian metric.

Every flag manifold can be interpreted geometrically as the G-orbits of a point in the visual boundary of the symmetric space associated to $\mathbb{X}$. One can also interpret each flag manifold as special points of some horofunction compactification of the symmetric space.

In the following two examples of rank 2 Lie groups, one can identify a higher rank Teichmüller space with a space of parabolic structures.

Choi-Goldman [CG05] showed that the component of the character variety corresponding to Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$ can be identified with the space of *convex projective structures* on the associated surface up to isotopy, i.e. $(\mathrm{PSL}(2,\mathbb{R}), \mathbb{RP}^2)$-structures such that the image of the developing map is a properly convex set.

Collier-Tholozan-Toulisse [CTT19] showed that every maximal representation in the Hermitian Lie group of tube type $\mathrm{SO}_o(2, n+1)$ is the holonomy of a photon structure on a fiber bundle M over the surface S_g. A photon structure is a $(\mathrm{SO}_o(2, n+1), \mathbf{Pho}(\mathbb{R}^{2,n+1}))$ structure where $\mathbf{Pho}(\mathbb{R}^{2,n+1})$ is the space of isotropic planes in $\mathbb{R}^{2,n+1} \simeq \mathbb{R}^{n+3}$ equipped with a bilinear form of signature $(2, n+1)$. The structures arising this way are exactly the structures that admit a fibration over S_g whose fiber are mapped via the developing map into a translate by $\mathrm{SO}_o(2, n+1)$ of the codimension 2 subset $\mathbf{Pho}(\mathbb{R}^{2,n}) \subset \mathbf{Pho}(\mathbb{R}^{2,n+1})$.

In both of these cases, the structures are obtained by taking the quotient of a domain Ω in the flag manifold and taking its quotient by the action of $\rho(\Gamma_g)$. We say that a domain Ω with an action of a discrete group Γ is a *cocompact domain of discontinuity* if Γ acts properly discontinuously and with compact quotient on Ω.

Guichard-Wienhard and Kapovitch-Leeb-Porti constructed cocompact domains of discontinuity in flag manifolds for Anosov representations [GW12], [KLP18a]. Guichard-Wienhard used these domain to show in particular that the space of Hitchin representations $\rho : \Gamma_g \to \mathrm{PSL}(2n, \mathbb{R})$ can be identified with a connected component of the space of projective structures on a fiber bundle M over S_g. The fiber of this bundle is diffeomorphic to a *Stiefel manifold* [ADL21], which is diffeomorphic to the projective tangent bundle of the projective space $\mathbb{RP}^{n-1}$. In general, for representations that can be deformed in the space of Anosov representations from a Fuchsian representation in a subgroup of G locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, the quotient of any of the domains of discontinuity constructed by Kapovich-Leeb-Porti is a fiber bundle M over S_g, which was showed by Alessandrini-Maloni-Tholozan-Wienhard [AMTW23].

These results leave two questions open, which are the main focus our work:

- How can one describe the fiber bundle M?
- How can one characterize the geometric structures that are obtained by this procedure for representations in a higher rank Teichmüller space?

1.3 Fibered geometric structures

We now introduce the results of the thesis. We begin by discussing the notions of domains of proper Busemann functions and ω-undistorted representations. These are the domains of discontinuity and Anosov representations for which we will apply geometric methods to describe the associated geometric structures.

1.3.1 Domains of proper Busemann functions

Our construction depends on the choice of a linear form $\omega \in \mathfrak{a}^*$ on the restricted Cartan subalgebra of the semi-simple Lie group G, up to the action of the Weyl group and up to multiplication by a positive scalar. We fix such a choice of ω throughout the section.

We denote by $\tau \in \mathfrak{a}$ the element corresponding to ω via the Killing bilinear form, and up to acting by the Weyl group we assume that its belongs to the positive Weyl chamber. A G-orbit in the visual boundary of the symmetric space $\mathbb{X}$ is naturally associated with τ. We denote it by $\mathcal{F}_\omega$, and it consists of the classes of geodesic rays such that the Cartan projection of their derivative is equal to τ. This G-orbit is diffeomorphic to a flag manifold whose type depends on the set of walls of the Weyl chamber containing τ.

We consider representations satisfying the following condition, that we introduce with Max Riestenberg, and which is closely related to the Anosov property.

Definition 1.3.1. A representation of a finitely generated group Γ into G is *ω-undistorted* if for some $A, B \geqslant 0$ and some word metric on Γ, the Cartan projection $\boldsymbol{\mu}(\gamma) \in \mathfrak{a}$ of any $\gamma \in \Gamma$ satisfies for all w in the Weyl group:

$$\mid \omega(w \cdot \boldsymbol{\mu}(\gamma)) \mid \geqslant A|\gamma| - B.$$

In other words Γ is ω-undistorded if it is a quasi isometric embedding and its limit cone (Definition 2.2.1) avoids the hyperplane $w \cdot \mathrm{Ker}(\omega) \subset \mathfrak{a}$ for all w in the Weyl group.

Representations that are ω-undistorted and are not virtually infinite cyclic always satisfy some Anosov condition, see Lemma 3.2.5. When ω is in the

Weyl group orbit of a positive multiple of a root α, then a representation is ω-undistorted if and only if it is Θ-Anosov where Θ is the set of simple roots in the Weyl group orbit of α. For each simple Lie group, the set of simple roots intersects one or two Weyl group orbits, see Figure 2.4. Coincidentally, every notion of Θ-positivity corresponds to a set of simple roots Θ that intersects a single Weyl group orbit, or intersects two orbits and $\Theta = \Delta$. Since Θ-positive representations are Θ-Anosov [GLW21], Θ-positive representations are ω-undistorted for one or two Weyl group orbit of roots $\omega \in \mathfrak{a}^*$.

An ω-undistorted representation $\rho : \Gamma \to G$ always admit a cocompact domain of discontinuity in the flag manifold $\mathcal{F}_\omega$, that is a *domain of proper Busemann functions*. More precisely, to a point in the visual boundary of $\mathbb{X}$ and in particular to a point $a \in \mathcal{F}_\omega$ and a basepoint $o \in \mathbb{X}$ one can associate a *Busemann function* $b_{a,o} : \mathbb{X} \to \mathbb{R}$ based at o as the limit of $d(\cdot, x) - d(\cdot, o)$ for x in a geodesic ray converging to the point $a \in \partial_{\mathrm{vis}}\mathbb{X}$. A change of the basepoint only changes the Busemann function by an additive constant.

Theorem 1.3.2 (Proposition 3.4.8)**.** *Let ρ be an ω-undiostorded representation. The domain $\Omega^\omega_{\mathit{flag}} \subset \mathcal{F}_\omega$ of flags whose associated Busemann functions are bounded from below on one and hence any $\rho(\Gamma)$-orbit in $\mathbb{X}$ is a cocompact domain of discontinuity for the action of $\rho(\Gamma)$.*

Moreover in this case all the Busemann functions corresponding to points outside of the domain are not bounded from below on any $\rho(\Gamma)$-orbit in $\mathbb{X}$.

Remark 1.3.3. This domain always coincide with some domain constructed by Kapovich-Leeb-Porti in [KLP18a] using Tits-Bruuhat ideal, and in particular associated to an ideal constructed as a metric thickening. This domain is still defined and is still a cocompact domain of discontinuity if we deform ρ so that it is no longer ω-undistorted as long as it still satisfies the adequate Anosov property. However the domain cannot always be characterized as the domain of Busemann functions that are proper and bounded from below in this case.

Flag manifolds appear in other compactifications of the symmetric space. If G is not simple, we require from now on that the Weyl groups orbit of ω spans $\mathfrak{a}^*$, and we make this assumption from now on. To $\omega \in \mathfrak{a}^*$ we can associate the Finsler distance d_ω on the symmetric space $\mathbb{X}$ whose unit ball in the model flat identified with $\mathfrak{a}$ is a polygon whose sides are described by the equation $w \cdot \omega = 1$ for w in the Weyl group. Figure 1.1 illustrates in a few rank 2 examples the unit ball for the distance d_ω in the model flat identified with $\mathfrak{a}$. Note that in general the associated Finsler distance fails to be symmetric.

Using this metric one can embed $\mathbb{X}$ in the space of convex and 1-Lipshitz functions defined up to an additive constant on $\mathbb{X}$. The closure of the image of this embedding is called the *horofunction compactification* of $\mathbb{X}$ with respect to d_ω. The flag manifold $\mathcal{F}_\omega$ can be viewed as a subset of this compactification, since the associated Busemann functions are horofunction for d_ω. In general horofunctions for the Finsler distance d_ω were described by Kapovich-Leeb-Porti [KL18].

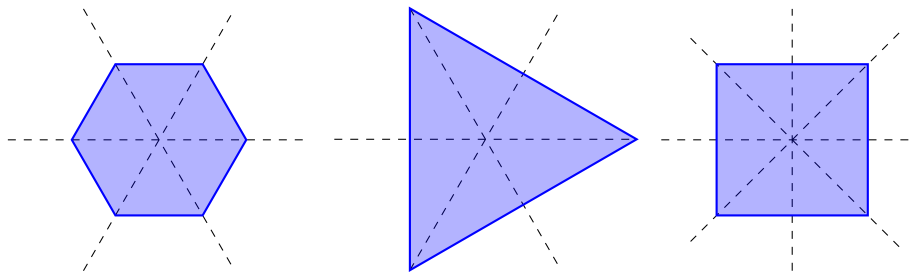


Figure 1.1: The unit ball in $\mathfrak{a}$ of the metric d_ω in 3 examples.

Theorem 1.3.4 (Proposition 3.5.5)**.** *For an ω-undistorted representation ρ, the domain Ω^ω_{horo} of horofunctions that are proper and bounded from below is a cocompact domain of discontinuity.*

Note that the domain of proper Busemann functions $\Omega^\omega_{\mathrm{flag}}$ correspond to the intersection of the domain of proper horofunctions with the image of the embedding of $\mathcal{F}_\omega$.

Remark 1.3.5. Kapovich-Leeb-Porti [KLP18a] constructed cocompact domains of discontinuity in this horofunction compactification for Anosov representations. The domain $\Omega^\omega_{\mathrm{horo}}$ for an ω-undistorted representation is a special instance of their construction. The fact that for ω-undistorted the domain can be characterized as a domain of proper horofunctions makes it easier to prove the properness and compactness of the action.

Let us assume now that ω is the highest restricted weight of an irreducible linear representations V of G. The horofunction compactification associated with the distance d_ω can be identified with a generalized Satake compactification, due to a work of Haettel-Schilling-Walsh-Wienhard [HSWW18]. The symmetric space $\mathbb{X}$ can be embedded in the symmetric space of $\mathrm{SL}(V)$ which itself can be embedded as a properly convex domain in $\mathbb{P}(S^2V)$, where S^2 is the space of symmetric tensors in $V \otimes V$. The Satake compactification of $\mathbb{X}$ is the closure in $\mathbb{P}(S^2V)$ of the image of this embedding.

In this case we define an invariant on pairs of points $\mathbb{X}$ that is at bounded distance from the Finsler distance d_ω: the *restricted Selberg invariant*, see Section 3.6. The horofunction compactification of $\mathbb{X}$ obtained using invariant can naturally be identified with the horofunction compactification for the Finsler metric. Moreover we show the following.

Theorem 1.3.6 (Theorem 3.6.3)**.** *Let $\rho : \Gamma \to G$ be an ω-undistorted representation of a torsion-free group Γ. The horofunction compactification of $\mathbb{X}/\rho(\Gamma)$ for the restricted Selberg invariant is naturally identifed with the quotient of the domain $\mathbb{X} \cup \Omega^\omega_{horo}$.*

In particular for ω-undistorted representations the associated parabolic structure and locally symmetric structure are related as follows: the compact manifold with a parabolic structure $\Omega^{\omega}_{\mathrm{flag}}/\rho(\Gamma)$ can naturally be identified with a subset of the horofunction compactification of the associated locally symmetric space $\mathbb{X}/\rho(\Gamma)$ for the restricted Selberg invariant.

1.3.2 Nearly geodesic immersions

We now introduce the results from Chapter 2, which is an adaptation of [Dav23]. We once again fix some non-zero linear form $\omega \in \mathfrak{a}^*$ on the model restricted Cartan subalgebra of a semi-simple Lie group G. In order to study the topology of the quotient of the domains of discontinuity Ω^{ω}_{ρ} of proper Busemann functions, or the locally symmetric space $\mathbb{X}/\rho(\Gamma)$, we construct for some representations a fibration of these quotients onto a compact manifold N such that $\pi_1(N) = \Gamma$.

We introduce and study a generalization of the condition of having principal curvature in $(-1, 1)$ for an embedding in $\mathbb{H}^n$ in the setting of symmetric spaces $\mathbb{X}$ associated to higher rank semi-simple Lie groups of non-compact type G.

Definition 1.3.7 (Definition 2.4.1)**.** Let $\omega \in \mathfrak{a}^*$ be a non-zero linear form. An immersion $u : M \to \mathbb{X}$ is called *ω-nearly geodesic* if for all $a \in \mathcal{F}_{\omega} \cup \mathcal{F}_{-\omega}$ the function $b_{a,o} \circ u : M \to \mathbb{R}$ has positive Hessian in any critical direction $\mathrm{v} \in TM$, for the metric on M induced by the immersion.

This notion depends on ω only up to the action of the Weyl group and up to scaling by a positive number.

Remark 1.3.8. In Chapter 2 we consider the element $\tau \in \mathfrak{a}$ corresponding to ω via the Killing pairing, and we write $\mathcal{F}_{\tau}$ for $\mathcal{F}_{\omega}$ and τ-nearly geodesic immersion for ω-nearly geodesic immersion. Up to scaling by a positive number and acting by the Weyl group we assume that this element has norm one and is in the positive Weyl chamber, i.e. $\tau \in \mathbb{S}\mathfrak{a}^+$.

This property is satisfied for totally geodesic immersions whose tangent vectors are ω-regular, namely whose Cartan projection does not lie in $w \cdot \mathrm{Ker}(\omega)$ for any w in the Weyl group. It is equivalent to an open bound on the second fundamental form that depends on the Cartan projection of the image of the differential of the immersion (see Proposition 2.4.3). When $G = \mathrm{PSL}(2, \mathbb{C})$ an ω-nearly geodesic immersion for the only $\omega \in \mathbb{S}\mathfrak{a}^+$ is exactly an immersion with principal curvature in $(-1, 1)$ in $\mathbb{X} = \mathbb{H}^3$ (see Proposition 2.4.6).

If the immersion is complete and *uniformly* ω-nearly geodesic, namely if the Hessian of Busemann functions in critical directions are uniformly bounded from below, we show moreover that it is an ω-regular embedding, a quasi isometric embedding (see Proposition 2.4.17) and that the nearest point projection is well defined for the Finsler distance d_{ω} (see Proposition 2.4.18).

When ω is a root, we also prove a sufficient condition for a surface to be ω-nearly geodesic. Let Θ be the set of simple roots in the Weyl group orbit of ω.

Theorem 1.3.9 (Theorem 2.4.24)**.** *Let $u : S \to \mathbb{X}$ be an immersion that satisfies for all $\mathrm{v} \in TS$ and $\alpha \in \Theta$:*

$$\|\mathrm{II}_u(\mathrm{v}, \mathrm{v})\| < c_\Theta \alpha \left(\boldsymbol{\mu}\left(\mathrm{d}u(\mathrm{v})\right)\right)^2. \tag{1.1}$$

Then u is an ω-nearly geodesic immersion.

Here $\boldsymbol{\mu} : T\mathbb{X} \to \mathfrak{a}^+$ denotes the *Cartan projection*. The constant c_Θ depends on the scaling of the metric chosen on $\mathbb{X}$, and on Θ.

In Section 2.5 we introduce and study *pencils of tangent vectors*, or d-pencils, which are vector subspaces $\mathcal{P} \subset T_x\mathbb{X}$ of dimension d for some $x \in \mathbb{X}$. When $G = \mathrm{PSL}(n, \mathbb{R})$ these can be thought of as *pencils of quadrics* with zero trace with respect to some scalar product (see Proposition 2.5.5). To a pencil we associate a subset of the flag manifold $\mathcal{F}_\omega$ that we call its *base*, which is a smooth submanifold if the pencil is ω-regular, i.e. if all non-zero vectors $\mathrm{v} \in \mathcal{P}$ are ω-regular, see Lemma 2.5.7. When $G = \mathrm{PSL}(n, \mathbb{R})$ and $\mathcal{F}_\omega \simeq \mathbb{RP}^{n-1}$ the base of the pencil corresponds to the intersection of all the quadric hypersurfaces defined by the elements of the corresponding pencil of quadrics.

To a complete and uniformly ω-nearly geodesic immersion $u : M \to \mathbb{X}$, we associate an open domain $\Omega_u^\omega \subset \mathcal{F}_\omega$ of proper Busemann functions, i.e. consisting of points $a \in \mathcal{F}_\omega$ for which $b_{a,o} \circ u$ is proper and bounded from below for one and hence any base-point $o \in \mathbb{X}$. We define a projection $\pi_u : \Omega_u^\omega \to M$ that associates to $a \in \Omega_u^\omega$ the point in M at which $b_{a,o} \circ u$ is minimal. This point will be unique because $b_{a,o} \circ u$ is convex and strictly convex in critical directions.

Theorem 1.3.10 (Theorem 2.6.3)**.** *Let $u : M \to \mathbb{X}$ be a complete and uniformly ω-nearly geodesic immersion. The map $\pi_u : \Omega_u^\omega \to M$ is a fibration. The fiber $\pi_u^{-1}(x)$ at a point $x \in M$ is the base $\mathcal{B}_\omega(\mathcal{P}_x)$ of the ω-regular pencil of tangent vectors $\mathcal{P}_x = \mathrm{d}u(T_xM)$.*

Let N be a compact manifold with fundamental group Γ, and let us consider immersions from $\tilde{N}$ that are equivariant with respect to some representation $\rho : \Gamma \to G$.

Theorem 1.3.11 (Theorem 2.4.23)**.** *If a representation $\rho : \Gamma \to G$ is equivariant with respect to an ω-nearly geodesic immersion $u : \tilde{N} \to \mathbb{X}$, then ρ is ω-undistorded.*

In this case the domain $\Omega_\rho^\omega := \Omega_u^\omega$ is the domain of proper Busemann functions. The fibration π_u from Theorem 1.3.10 is ρ-equivariant so the quotient of Ω_ρ^ω fibers over N.

We prove in Section 2.6.4 that the diffeomorphism type of the domains of discontinuity obtained by Tits-Bruhat ideals is invariant under deformation in the space of Anosov representations. We therefore understand the topology of some domains of discontinuity for all representations in some connected component of Θ-Anosov representation.

As an application for instance let G Lie group with a notion of Θ-positivity. Let $\omega \in \mathfrak{a}^*$ be an element in the Weyl group orbit of a root in Θ.

We say that a representation is *generalized Fuchsian* if it stabilizes and acts cocompactly on a totally geodesic copy of $\mathbb{H}^2$ in the symmetric space $\mathbb{X}$. For every notion of Θ-positivity there exist Θ-positive representations that are generalized Fuchsian [GW22].

Corollary 1.3.12 (Corollary 2.7.11)**.** *If $\rho : \Gamma \to G$ is in a connected component of Θ-positive representations that contains a generalized Fuchsian representation, the quotient Ω_ρ^ω is a non-empty fiber bundle over S_g whose fiber is the base in $\mathcal{F}_\omega$ of a pencil of tangent vectors in $T\mathbb{X}$.*

1.3.3 Finite-sided Dirichlet domains.

We now introduce the results from Chapter 3, which is a joint work with Max Riestenberg.

To any group Γ acting properly on a metric space (X, d) by isometries one can associate a fundamental domain for the action called the *Dirichlet domain* for each base point $o \in X$ defined as :

$$\bigcap_{\gamma \in \Gamma} \mathcal{H}(o, \gamma \cdot o) = \bigcap_{\gamma \in \Gamma} \{x \in X \mid d(x, o) \leqslant d(x, \gamma \cdot o)\}.$$

If $X = \mathbb{H}^n$, these domains are hyperbolic polyhedra which are finite sided for geometrically finite subgroups of the isometry groups. Moreover if the subgroup is convex-cocompact these polyhedra are finite sided in a stronger sense, which we call *properly finite-sided*: there exist a neighborhood U of the closure $\mathcal{D}_\Gamma(o)$ of this domain in the compactification $\overline{\mathbb{H}^n}$ of $\mathbb{H}^n$ such that for all but finitely many $\gamma \in \Gamma$ one has $U \subset \mathcal{H}(o, \gamma \cdot o)$.

For semi-simple Lie groups of non-compact type G one can still consider Dirichlet domains for any G-invariant metrics on the corresponding symmetric space $\mathbb{X}$. For instance given $\omega \in \mathfrak{a}^*$ one can use the previously defined metric d_ω and consider the following domain:

$$\bigcap_{\gamma \in \Gamma} \mathcal{H}^\omega(o, \gamma \cdot o) = \bigcap_{\gamma \in \Gamma} \{x \in X \mid d_\omega(x, o) \leqslant d_\omega(x, \gamma \cdot o)\}.$$

This domain can be naturally extended to a domain $\mathcal{D}_\Gamma^\omega(o)$ in the horofunction compactification of $\mathbb{X}$ for the metric d_ω containing the horofunctions $[h]$ such that for all $\gamma \in \Gamma$, $h(o) \leqslant h(\gamma \cdot o)$. The closure of the Dirichlet domain in the horofunction compactification is equal to this domain $\mathcal{D}_\Gamma^\omega(o)$.

Theorem 1.3.13 (Theorem 3.5.2)**.** *Let $\omega \in \mathfrak{a}^*$ and let $\rho : \Gamma \to G$ be an ω-undistorted representation. For any point $o \in \mathbb{X}$, the Dirichlet-Selberg domain $\mathcal{D}_\Gamma(o)$ for the Finsler metric d_ω is properly finite-sided.*

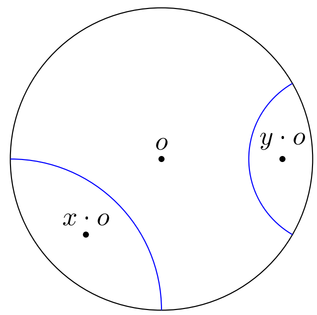


Figure 1.2: Illustration of the disjoint half space property.

The compact set $\mathcal{D}_\Gamma(o)$ is a compact fundamental domain for the action of $\rho(\Gamma)$ on the union $\mathbb{X} \cup \Omega^\omega_{\text{horo}}$, where $\Omega^\omega_{\text{horo}}$ is the domain of discontinuity in the horofunction compactification of $\mathbb{X}$ defined previously

Representation that are ω-undistorted can be characterized by a property that is more restrictive than having a properly finite-sided Dirichlet domain: one can check the ω-undistorted condition by verifying that finitely many half-spaces in $\mathcal{F}_\omega$ are disjoint.

Given $o_1, o_2 \in \mathbb{X}$ we define $\mathcal{H}^\omega(o_1, o_2)$ to be the set of elements $a \in \mathcal{F}_\omega$ such that the associated Busemann function satisfy $b_{a,o}(o_1) \leqslant b_{a,o}(o_2)$ for one and hence any $o \in \mathbb{X}$.

We fix a point $o \in \mathbb{X}$ and a word metric on Γ. We say that a representation $\rho : \Gamma \to G$ satisfies the *disjoint half space property* for ω if and only if for some integer D for every pair (x, y) at distance D from the identity e and such that (x, e, y) lie in a geodesic in this order, $\mathcal{H}(x \cdot o, o)$ and $\mathcal{H}(y \cdot o, o)$ are disjoint, as in Figure 1.2.

Theorem 1.3.14 (Theorem 3.5.12 and 3.5.11)**.** *A representation $\rho : \Gamma \to G$ is ω-undistorted if and only if is satisfies the disjoint half space property for ω and $-\omega$.*

This result is a discrete analog of Theorem 1.3.11. As a corollary, the space of ω-undistorted representations is open.

In general the bisectors for these metrics, i.e. the set of points $x \in \mathbb{X}$ for given points $o_1, o_2 \in \mathbb{X}$ such that $d(o_1, x) = d(o_2, x)$, are not totally geodesic, nor are they linear hyperplane in some suitable projective model for the symmetric space $\mathbb{P}(S^2V^{>0})$. For $G = \mathrm{SL}(n, \mathbb{R})$ Selberg introduced a *Dirichlet-Selberg domain*, which is a polyhedral fundamental domain on $\mathcal{X}$ for a discrete and faithful representation $\rho : \Gamma \to G$.

In order to define these domain one needs to introduce the *Selberg invariant* $\mathfrak{s} : \mathbb{P}(S^2V^{>0})^2 \to \mathbb{R}$ where S^2V is the space of positive symmetric tensors in $V \otimes V$. For a pair $(o, x) \in \mathbb{P}(S^2V^{>0})^2$ with representatives $O, X : V^* \to V$ such that $\det(O^{-1}X) = 1$:

$$\mathfrak{s}(o, x) = \log \frac{1}{\dim(V)} \operatorname{Tr}(O^{-1}X).$$

This invariant is linear in the second argument, hence for $o_1, o_2 \in \mathbb{P}(S^2V^{>0})$ the bisector $\{x \in \mathbb{P}(S^2V^{>0}) | \mathfrak{s}(o_1, x) = \mathfrak{s}(o_2, x)\}$ is the intersection of $\mathbb{P}(S^2V^{>0})$ with a linear hyperplane.

When o, x are close, this distance is close to the Riemannian distance. When they are far, this distance has a bounded difference with the Finsler distance d_{ω_1}, where $\omega_1 \in \mathfrak{a}^*$ is the first fundamental weight, which is the highest restricted weight of the standard representation on $\mathrm{SL}(V)$.

Let $\rho : \Gamma \to G$ be a discrete and faithful representation. The Dirichlet-Selberg domain associated to a basepoint $o \in \mathbb{P}(S^2V^{>0})$ is the domain :

$$\left\{x \in \mathbb{P}(S^2V^{>0}) \mid \forall \gamma \in \Gamma,\, \mathfrak{s}(o, x) \leqslant \mathfrak{s}(\gamma \cdot o, x)\right\}.$$

We denote by $\mathcal{DS}_\Gamma(o)$ the closure of this domain in $\mathbb{P}(S^2V^{\geqslant 0})$.

Theorem 1.3.15 (Corollary 3.5.4)**.** *If $\rho : \Gamma \to \mathrm{SL}(V)$ is ω_1-undistorted, then for all $o \in \mathbb{P}(S^2V^{>0})$ the Dirichlet-Selberg domain $\mathcal{DS}_\Gamma(o)$ is properly finite sided.*

It turns out that ω_1-undistorted representations exist only in even dimension,except if Γ is virtually cyclic. Actually we show :

Theorem 1.3.16 (Theorem 3.1.5)**.** *There exist Borel Anosov subgroups of* $\mathrm{SL}(3, \mathbb{R})$ *that admit infinite-sided Dirichlet-Selberg domains.*

For subgroups of $\mathrm{SL}(V)$ that are not ω_1-undistorted, we can sometimes prove that the intersection of the domain $\mathcal{DS}_\Gamma(o)$ with a Γ-invariant convex set in $\mathbb{P}(S^2V)$ is properly finite sided. In particular we consider a representation with finite kernel $G \to \mathrm{SL}(V)$ of a semi-simple Lie group G, whose highest restricted weight is ω. We consider representations ρ that factor through this representation.

Theorem 1.3.17 (Theorem 3.7.14)**.** *There exist a non-empty G-invariant compact convex subset $C \subset \mathbb{P}(S^2V^{\geqslant 0})$ such that the intersection of the Dirichlet-Selberg domain and C is properly finite-sided in C for all ω-undistorted representations $\rho : \Gamma \to G$.*

We describe explicitly the set C as the convex hull of a set of rank one points. One can apply this for instance to non-elementary Δ-Anosov subgroups $\Gamma \subset G$ for the adjoint representation $V = \mathfrak{g}$. Let $\mathfrak{n} \subset \mathbb{P}(\mathfrak{g})$ be the space of nilpotent elements in the Lie algebra $\mathfrak{g}$ of G and let $S^2\mathfrak{n} \subset \mathbb{P}\left(S^2\mathfrak{g}\right)$ be the corresponding space of rank one tensors.

Theorem 1.3.18 (Corollary 3.7.16)**.** *Suppose that $\Gamma \subset G$ is Δ-Anosov. Then every Dirichlet-Selberg domain $\mathcal{DS}^{\mathfrak{n}}_\Gamma(o)$ in $\mathbb{P}(S^2\mathfrak{g})$ obtained via the adjoint representation of G intersected with* $\mathrm{Hull}(S^2\mathfrak{n}) \subset \overline{\mathbb{P}(S^2V^{>0})}$ *is properly finite sided in* $\mathrm{Hull}(S^2\mathfrak{n})$.

1.4 Maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$

We now turn our attention to maximal representations into $\mathrm{Sp}(2n, \mathbb{R})$.

We first study the projective structures associated with maximal representations, which is an instance of a domain of proper Busemann functions, and characterize these structures by the existence of a fibration described by pencils of quadrics.

We then study a different question: we consider Borel Anosov representations in $\mathrm{Sp}(2n, \mathbb{R})$ that are also maximal, and prove in particular that for $n = 2$ these representations are necessarily Hitchin.

1.4.1 Fibration by pencils of quadrics

We characterize maximal representations by describing the associated projective structure using pencils of quadrics. This characterization builds on the notion of fitting maps of pencils, which are maps that define locally a fibration of projective space, as well as the symmetric space of $\mathrm{SL}(2n, \mathbb{R})$.

We begin by a digression on quasi-Fuchsian representations in $\mathrm{SL}(2, \mathbb{C})$ to illustrate the notion of fitting maps.

Let $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ be the composition of a Fuchsian representation and the inclusion $\mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SL}(2, \mathbb{C})$. The locally symmetric space $\mathbb{H}^3/\rho(\Gamma_g)$ is a fiber bundle over S_g with geodesics fibers. One can construct such a fibration by taking the geodesics orthogonal to the totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$ preserved by the action of $\mathrm{SL}(2, \mathbb{R})$. This fibration extends to a fibration of an open domain in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.

Such a fibration is described by ρ-equivariant map $u : \widetilde{S_g} \to \mathcal{G}$ where $\mathcal{G}$ is the space of geodesics in $\mathbb{H}^3$. We say that an immersion $u : \widetilde{S_g} \to \mathcal{G}$ is *fitting* if the corresponding geodesics locally define a smooth fibration of $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.

If $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ is *nearly Fuchsian*, i.e. if it admits an equivariant immersion $h : \widetilde{S_g} \to \mathbb{H}^3$ with principal curvature in $(-1, 1)$, the locally symmetric space $\mathbb{H}^3/\rho(\Gamma_g)$ admits a fibration described by the fitting immersion $\mathcal{G}h$ that associates to $x \in \widetilde{S_g}$ the geodesic orthogonal to $h(\widetilde{S_g})$ at $h(x)$. This was shown by Epstein [Eps86]. We say that $\mathcal{G}h$ is the *Gauss map* of h.

Nearly Fuchsian representations are quasi Fuchsian, i.e. are quasi-isometric embeddings [Eps86]. We generalize it to any representation that admits an equivariant fitting immersion.

Theorem 1.4.1. *Let $\rho : \Gamma_g \to \mathrm{SL}(2, \mathbb{C})$ be a representation that admits an equivariant fitting immersion $u : \widetilde{S_g} \to \mathcal{G}$. The representation ρ is quasi-Fuchsian.*

This theorem is a consequence of Theorem 4.4.4. There may a priori exist representations with equivariant fitting maps that are not nearly Fuchsian, see Remark 4.2.12. However Theorem 1.4.1 does not provide a characterization of quasi Fuchsian representations in general because of the following result.

Theorem 1.4.2 (Theorem 4.4.6)**.** *For a genus g large enough, there exist quasi Fuchsian representations $\rho : \Gamma_g \to S_g$ that admit no equivariant fitting immersions $u : \widetilde{S_g} \to \mathcal{G}$.*

Maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ satisfy a property that is not satisfied by quasi-Fuchsian representations : they form connected components of discrete and faithful representations. This raises the following question:

Question 1.4.3. Can maximal representations can be characterized by the existence of some fibration of the associated locally symmetric space?

We provide some affirmative answer to this question. In order to study the symmetric space for $\mathrm{Sp}(2n, \mathbb{R})$ we embed it into some projective space.

Let $V = \mathbb{R}^{2n}$. The symmetric space $\mathbb{X}$ associated to $\mathrm{Sp}(2n, \mathbb{R})$ can be identified with a submanifold of the projectivization $\mathbb{P}(S^2V^{>0})$ of the space S^2V of symmetric tensors on V that are positive, i.e. that define a positive bilinear forms on V^*.

A codimension 2 subspace of S^2V corresponds to a dimension 2 subspace of its dual $\mathcal{Q} = S^2V^*$, the space of quadrics on V. A plane in $\mathcal{Q}$ is a *pencil of quadrics*. We denote by $\mathrm{Gr}_2(\mathcal{Q})$ the space of such planes. We denote by $\mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ the set of pencils P that do not contain any positive element, or equivalently such that the corresponding codimension 2 subset $P^\circ \subset S^2V$ intersects $S^2V^{>0}$.

In this setting we say that an immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ is *fitting* is the corresponding codimension two subsets define locally a smooth fibration of the convex set $\mathbb{P}(S^2V^{\geqslant 0})$.

Let $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ be the set of pencils P such that every non-zero $q \in P$ is positive on some Lagrangian and negative on some Lagrangian of $\mathbb{R}^{2n}$. We show that the projectivization of the corresponding codimension 2 subspace P° of S^2V intersects transversely the symmetric space $\mathbb{X}$ associated to $\mathrm{Sp}(2n, \mathbb{R})$ (see Lemma 4.7.4). The set $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ is open in $\mathrm{Gr}_2(\mathcal{Q})$, but we show that it is disconnected. We select a special union of connected components that we denote by $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ and we show the following:

Theorem 1.4.4 (Theorem 4.5.5)**.** *Let $\rho : \Gamma_g \to G$ be a representation. If it admits a ρ-equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ it is maximal for some orientation of S_g.*

For $n = 2$ using results of [CTT19] we show that this is a characterization of maximal representations. For $n \geqslant 3$ we prove a weaker converse to this theorem. We construct equivariant maps $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ for maximal representations that locally define a fibration of $\mathbb{P}(S^2V^{\geqslant 0})$ but are only continuous.

These continuous maps of pencils admit a flow on an associated circle bundle over the surface with additional properties, that we call a *fitting flow*.

Let $\mathcal{E}$ be the tautological rank 2 vector bundle over $\mathrm{Gr}_2(\mathcal{Q})$, with a projection $\pi : \mathcal{E} \to \mathcal{Q}$. Given $u : \widetilde{S_g} \to \mathrm{Gr}_2(\mathcal{Q})$ we consider the induced circle bundle $u^*\mathbb{S}\mathcal{E}$

over $\widetilde{S_g}$ which is the quotient of $u^*\mathcal{E}$ by the action of positive scalars. A *fitting flow* is a flow Φ on $u^*\mathbb{S}\mathcal{E}$ such that for all $[q] \in u^*\mathbb{S}\mathcal{E}$ and all $t > 0$ one can find a representative of $[q'] = \Phi_t([q])$ such that $\pi(q') - \pi(q)$ is positive.

Equivariant continuous maps $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q})$ that admits an equivariant fitting flow also define locally a continuous fibration of $\mathbb{P}(S^2V^{\geqslant 0})$. Equivariant fitting immersions $u : \widetilde{S_g} \to \mathrm{Gr}_2(\mathcal{Q})$ always admit a fitting flow (see Proposition 4.3.8).

We obtain the following characterization of maximal representations of closed surface groups in $\mathrm{Sp}(2n, \mathbb{R})$.

Theorem 1.4.5 (Theorem 4.5.5)**.** *A representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ is maximal if and only if it admits a ρ-equivariant continuous map of pencils that admits an equivariant fitting flow:*

$$u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q}).$$

We also show that the quasi-Fuchsian representations from Theorem 1.4.2 do not admit continuous maps with an equivariant fitting flow.

A continuous map equivariant map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ for a representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ with a fitting flow defines a fibration of the symmetric space $\mathbb{X} \subset \mathbb{P}(S^2V^{>0})$ as $\mathbb{X}$ intersects the fibers transversely. Thus it defines a fibration of the locally symmetric space $\mathbb{X}/\rho(\Gamma_g)$. Moreover the intersection of the quadrics in a pencil of in $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ defines a codimension 2 subset of $\mathbb{P}(V)$ that is the intersection all the of the corresponding quadric hypersurfaces. A continuous map of pencils with a fitting flow defines a fibration of a domain in projective space, that is equal to the domain of discontinuity in projective space constructed by Guichard-Wienhard [GW12] when the map and the flow are equivariant with respect to a n-Anosov representation. Theorem 1.4.5 implies the following characterization of the contact projective structures corresponding to maximal representations.

Corollary 1.4.6. *A contact projective structure on a fiber bundle M with fiber F over S_g corresponds to a maximal representations by the construction of Guichard-Wienhard if and only if, up to homeomorphisms of M that stabilize $\pi_1(F)$ and act trivially on $\pi_1(M)/\pi_1(F) \simeq \Gamma_g$, the fibers are mapped via the developing map onto the bases of maximal pencils of quadrics parametrized by a continuous map that admits an equivariant fitting flow.*

More generally we define fitting immersions and continous maps with fitting flows for representations of the fundamental groups of closed d-manifolds in $\mathrm{SL}(2n, \mathbb{R})$. We show that the existance of such a map implies that the representation is n-Anosov property, see Theorem 4.4.4.

In Section 4.6 we discuss the case of $\mathrm{Sp}(4, \mathbb{R})$ and construct fitting immersions by using the maximal spacelike immersions from [CTT19]. Finally in Section 4.8 we briefly discuss how to decompose these projective structures into smaller pieces which are polygons of quadric hypersurfaces.

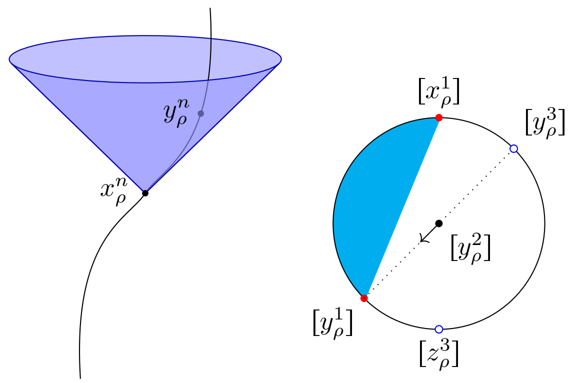


Figure 1.3: The second boundary map for a Hitchin representation, and the main argument of the proof of Theorem 5.6.5.

1.4.2 Maximal and Borel Anosov representations

Maximal representations $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ are always Anosov with respect to the longest root. The space of maximal representations $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$ contains several connected components. One of them is the Hitchin component, whose existence is due to the fact that $\mathrm{Sp}(2n, \mathbb{R})$ is split. Representations in the Hitchin component are Borel Anosov, i.e. Anosov with respect to all roots, which is stronger than being Anosov with respect to a single root. In general Hitchin representations are the only known representations of a closed surface group in $\mathrm{SL}(2n, \mathbb{R})$ that are Borel Anosov.

We prove that among representation into $\mathrm{Sp}(2n, \mathbb{R})$ that are $\{n-1, n\}$-Anosov, the ones that are maximal can be characterized by hyperconvexity.

Theorem 1.4.7. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{n-1, n\}$-Anosov representation. It is maximal if and only if it satisfies the hyperconvexisty condition H_n, i.e. if for all distinct $x, y, z \in \partial\Gamma$ the following sum is direct:*

$$\left(\xi^n_\rho(x) \cap \xi^{n+1}_\rho(z)\right) \oplus \left(\xi^n_\rho(y) \cap \xi^{n+1}_\rho(z)\right) \oplus \xi^{n-1}_\rho(z).$$

This result is a new link between positivity for maximal representations and hyperconvexity of boundary maps, for $\{n-1, n\}$-Anosov representations. On the one hand we see that maximality forces property H_n. On the other hand we show that property H_n implies positivity of the n-th boundary map combining the characterization of the tangents to the boundary maps [PSW21] together with the observation that a $\mathcal{C}^1$ curve whose derivative stays in a cone must also lie in a the cone.

Hyperconvexity conditions were studied extensively by Pozzetti-Sambarino-Wienhard [PSW21] and they showed in particular that hyperconvexity H_n implies that the ξ^n_ρ has $\mathcal{C}^1$-image with derivative given by ξ^{n-1}_ρ and ξ^{n+1}_ρ. Using this we show the following in the case $n = 2$.

Theorem 1.4.8. *Representations $\rho : \Gamma_g : \mathrm{Sp}(4, \mathbb{R})$ that are maximal and Borel Anosov are Hitchin.*

In order to prove Theorem 1.4.8 we use Theorem 1.4.7 and prove in Section 5.5 that Borel Anosov representations $\rho : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ which satisfy property H_2 also satisfy property H_1.

For that we project the boundary map onto the parallel tube in the symmetric space between two Lagrangians in the boundary curve. Concretely given 3 points (x, y, z) in the Gromov boundary of Γ_g we consider their full flags associated via the boundary map $(x^1_\rho, x^2_\rho, x^3_\rho)$, $(y^1_\rho, y^2_\rho, y^3_\rho)$, $(z^1_\rho, z^2_\rho, z^3_\rho)$ and construct 4 points in the circle $\mathbb{P}(x^2_\rho)$ by projecting the lines x^1_ρ, y^1_ρ and intersecting the hyperplanes z^3_ρ and y^3_ρ, yielding 4 points on the boundary of a copy of the hyperbolic plane. The Lagrangian y^2_ρ defines a point in the interior of this hyperbolic plane, see Figure 1.3.

We distinguish two possible configurations of these projections, one of which implies property H_1. To rule out the other configuration, we use again that the second boundary map has $\mathcal{C}^1$ image to show that the projection of the second boundary map must stay in a smaller convex cone, colored in the picture.

This leads to a contradiction as the point corresponding to y^2_ρ must lie in the geodesic joining the ideal points corresponding to y^1_ρ and y^3_ρ, since $y^1_\rho \subset y^2_\rho \subset y^3_\rho$. This geodesic is disjoint from the convex if the four points are ordered as in the picture.

In Section 5.6 we recall results from Labourie and Guichard to prove that a Borel Anosov representation in $\mathrm{Sp}(4, \mathbb{R})$ that satisfies property H_1 and H_2 is Hitchin.

We hope that such geometric argument will be useful to rule out the existence of other kinds of Anosov representations.

Let us say that a representation $\rho : \Gamma_g \to G$ into a split lie group G is *homotopy Hitchin* if it is Borel Anosov and if its boundary map in G/B where B is the Borel subgroup is freely homotopic to the boundary map of a fixed Hitchin representation, up to an orientation reversing homeomorphism of $\partial\Gamma$.

A consequence of Theorem 1.4.8 is that a representation $\rho : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ is Hitchin if and only if it is homotopy Hitchin. Indeed, because of Theorem 4.1.5 a homotopy Hitchin representation in $\mathrm{Sp}(4, \mathbb{R})$ is maximal for some orientation of $\partial\Gamma_g$.

One could therefore ask the following :

Question 1.4.9. Are all homotopy Hitchin representations $\rho : \Gamma_g \to G$ Hitchin?

A Borel-Anosov representation in $\mathrm{SL}(3, \mathbb{R})$ having a trivial boundary map in $\mathbb{RP}^2$ must preserve a properly convex domain, and hence be Hitchin due to [CG05]. Therefore the answer to this question is affirmative for $G = \mathrm{SL}(3, \mathbb{R})$.

1.5 Organisation of the thesis

This document is divided into 4 chapters that are mostly independent.

In Chapter 2 we introduce and discuss the notion of nearly geodesic immersions. Section 2.1, 2.2 and 2.3 review some definitions and properties of symmetric spaces, Anosov representations, and Busemann functions. In Section 2.4 we define and study the properties of nearly geodesic immersions. In Section 2.5 we study pencils of tangent vectors in the symmetric space, which determine the fibers of the fibrations associated to nearly geodesic immersions. In Section 2.6 we construct these fibrations. Finally we apply these results to construct fibration of parabolic structures associated to some representations in Section 2.7. This chapter is an adaptation of the preprint [Dav23].

In Chapter 3 we consider Dirichlet domains for Anosov subgroups of a semisimple Lie group. In Section 3.1 we explain Selberg's construction of a fundamental domain of discrete subgroups of $\mathrm{SL}(n, \mathbb{R})$, and discuss an example of such a domain that is infinite sided. In Section 3.2 we recall the necessary background on Anosov representations and symmetric spaces and we define the notion of ω-undistorted subgroup. In Section 3.3 we define the Finsler metric d_ω and the horofunction compactification of the symmetric space. In Section 3.4 we describe the domains of proper horofunctions for ω-undistorted subgroups. In Section 3.5 we discuss the finite-sidedness of Dirichlet-Finsler domains. In Section 3.7 we consider the restriction of Selberg's domain to smaller invariant convex sets, and provide a sufficient condition for the Dirichlet-Selberg domain to be finite sided in this convex set. In Section 3.6 we compare the horofunction compactification of the locally symmetric space for an ω-undistorted subgroup with the compactification of the symmetric space. Finally in Section 3.4.4 we show that the limit cone of any discrete subgroup of G that is not virtually cyclic is connected. This chapter is an adaptation of the preprint [DR24], which is a joint work with Max Riestenberg.

In Chapter 4 we study fitting maps of pencils of quadrics in order to characterize maximal representations by their geometric structures. In Section 4.1 we recall the definition of maximal and Anosov representations. In Section 4.2 we define pencils of quadrics, fitting pairs and directions, and finally fitting immersions of pencils of quadrics, that define fibrations of a convex domain in the symmetric power of $\mathbb{R}^{2n}$. In Section 4.3 we define fitting flows. In Section 4.4 we show that the existence of an equivariant map of pencils with a fitting flow implies the Anosov property. In Section 4.5 we show that maximal representations can be characterized by the existence of such a fibration. In Section 4.6 we show that photon structures provide such fibrations for maximal representations in $\mathrm{Sp}(4, \mathbb{R})$. Finally in Section 4.8 we briefly describe how one can decompose the projective structure associated to maximal representations into polygons of quadric hypersurfaces.

In Chapter 5 we study maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ that satisfy additional Anosov properties. We first recall in Section 5.1 the definition of Anosov representations in $\mathrm{Sp}(2n, \mathbb{R})$ and fix some notations. In Section 5.2 we describe a special chart of the space of Lagrangians. In Section 5.3 we discuss the link between hyperconvexity and smooth properties of the boundary maps.

In Section 5.4 we characterize maximal representations among $\{n-1, n\}$-Anosov representations by the hyperconvexity property H_n. In Section 5.5 and 5.6 we prove that maximal and Borel Anosov representations are Hitchin. This chapter is an adaptation of [Dav24].

Chapter 2

Nearly geodesic immersions

In this chapter we introduce and discuss the notion of nearly geodesic immersions, and apply it to construct fibrations of certain domains of discontinuity, that can be interpreted as domains of proper Busemann functions, for some Anosov representations.

Section 2.1, 2.2 and 2.3 review some definitions and properties of symmetric spaces, Anosov representations, and Busemann functions. In Section 2.4 we define and study the properties of nearly geodesic immersions.

In Section 2.5 we study pencils of tangent vectors in the symmetric space, which determine the fibers of the fibrations associated to nearly geodesic immersions. In Section 2.6 we construct these fibrations. Finally we apply these results to construct fibration of parabolic structures associated to some representations in Section 2.7.

This chapter is an adaptation of the preprint [Dav23].

2.1 Symmetric spaces of non-compact type.

In this section we recall the general theory of symmetric spaces of non compact type and fix some notations. References for the results mentioned can be found in [Hel78] and [Ebe96]. We then illustrate some of these notions for some families of Lie groups. Finally we introduce the notion of Weyl orbit of simple roots.

2.1.1 Symmetric space associated to a semi-simple Lie group.

Let G be a connected, semi-simple Lie group with finite center and no compact factors, i.e. of non-compact type.

Let $\mathfrak{g}$ be the Lie algebra of G, and let B be the *Killing form* on $\mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple it admits a *Cartan involution* i.e. an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $(\mathrm{v}, \mathrm{w}) \mapsto -B(\mathrm{v}, \theta(\mathrm{w}))$ is a scalar product on $\mathfrak{g}$. Any two Cartan involutions are conjugated by Ad_g for some $g \in G$.

Let $\mathbb{X}$ be the space of Cartan involutions of $\mathfrak{g}$. For any $x \in X$ we will write the corresponding Cartan involution $\theta_x : \mathfrak{g} \to \mathfrak{g}$. This involution determines a B-orthogonal decomposition $\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x$, where $\mathfrak{k}_x$ is the $+1$ eigenspace of θ, and by $\mathfrak{p}_x$ the -1 eigenspace.

For $x \in X$, define K_x to be the group of elements $k \in G$ such that Ad_k commutes with θ_x. This subgroup is a maximal compact subgroup of G. Given any $x \in \mathbb{X}$, one can identify $\mathbb{X}$ with the homogeneous space G/K_x. The Lie subalgebra $\mathfrak{k}_x$ is the Lie algebra of the compact K_x, and thus the space $\mathfrak{p}_x$ is naturally identified with $T_x\mathbb{X}$.

Let $\langle \cdot, \cdot \rangle_x$ be the scalar product defined for $\mathrm{v}, \mathrm{w} \in \mathfrak{g}$ as :

$$\langle \mathrm{v}, \mathrm{w} \rangle_x = B\left(\mathrm{v}, \theta_x(\mathrm{w})\right). \tag{2.1}$$

This scalar product restricted to $\mathfrak{p}_x \simeq T_x\mathbb{X}$ defines a Riemannian metric $g_{\mathbb{X}}$ on $\mathbb{X}$. We will denote by $d_{\mathbb{X}}$ the induced Riemannian distance on $\mathbb{X}$. With this metric the space $\mathbb{X}$ is a symmetric space in the sense that for all $x \in \mathbb{X}$ there is an isometry σ_x of X such that $\mathrm{d}_x\sigma = -\mathrm{Id}$.

The symmetric space $\mathbb{X}$ is of *non-compact type*. It is simply connected and has non-positive sectional curvature. In particular it is a *Hadamard manifold*.

Remark 2.1.1. We only consider symmetric spaces $\mathbb{X}$ associated to semi-simple Lie groups G, having their Riemannian metric defined via the Killing form.

2.1.2 Reduced root systems.

Fix a base point $o \in \mathbb{X}$. Let $\mathfrak{a}$ be a choice of a *maximal abelian subalgebra* of $\mathfrak{p}_o$. These maximal abelian subalgebras are all conjugated by elements of K_x. The dimension $\mathrm{rank}(\mathbb{X})$ of $\mathfrak{a}$ will be called the *rank* of $\mathbb{X}$.

Remark 2.1.2. In general $\mathrm{rank}(\mathbb{X}) \leqslant \mathrm{rank}(G)$, where $\mathrm{rank}(G)$ is the dimension of any Cartan subalgebra in $\mathfrak{g}$.

Let $\alpha \in \mathfrak{a}^*$ be a linear form. Let $\mathfrak{g}_\alpha$ be the set of elements $\mathrm{v} \in \mathfrak{g}$ such that for all $\tau \in \mathfrak{a}$:

$$\mathrm{ad}_\tau(\mathrm{v}) = \alpha(\tau)\mathrm{v}.$$

The *reduced root system* Σ is the set of linear forms $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$. An element $\tau \in \mathfrak{a}$ is *regular* if for all $\alpha \in \Sigma \backslash \{0\}$, $\alpha(\tau) \neq 0$.

Let us choose a regular element $\tau_0 \in \mathfrak{a}$. Let Σ^+ be the associated set *positive roots*, i.e. the set of $\alpha \in \Sigma$ such that $\alpha(\tau_0) > 0$. There exists a unique set Δ of linearly independent roots in Σ^+ such that any root if Σ^+ can be written as a linear combination of roots in Δ. The roots in Δ are called *simple roots*.

Let the *Weyl group* W be quotient of the subgroup of elements in K_x whose adjoint action stabilizes $\mathfrak{a}$ by the subgroup of elements who fix $\mathfrak{a}$ point-wise.

For any root $\alpha \in \Sigma \backslash \{0\}$ there is an element $\sigma_\alpha \in W$ such that its action on $\mathfrak{a}$ is the orthogonal symmetry with respect to $\mathrm{Ker}(\alpha)$ in $\mathfrak{a}$. The Weyl group acts linearly on $\mathfrak{a}$, and it is generated by the elements $(\sigma_\alpha)_{\alpha\in\Delta}$. The *model Weyl chamber* $\mathfrak{a}^+$ is the cone $\{\tau \in \mathfrak{a} | \forall \alpha \in \Delta, \alpha(\tau) \geqslant 0\}$. For any $\tau_0 \in \mathfrak{a}$ there is a unique $\tau \in \mathfrak{a}^+$ such that for some $w \in W$, $w \cdot \tau_0 = \tau$. An element $\tau \in \mathfrak{a}^+$ is Θ-regular for $\Theta \subset \Delta$ if for all $\alpha \in \Theta$, $\alpha(\tau) \neq 0$.

We denote by $\mathbb{S}\mathfrak{a}$ and $\mathbb{S}\mathfrak{a}^+$ reprectively the unit sphere in $\mathfrak{a}$ and the unit sphere intersected with the model Weyl chamber $\mathfrak{a}^+$, for the metric (2.1).

Let $w_0 \in W$ be the only element such that $w_0 \cdot \mathfrak{a}^+ = -\mathfrak{a}^+$. This element is called *the longest element* of the Weyl group. Let $\iota : \mathfrak{a}^+ \to \mathfrak{a}^+$ be the involution such that for $\tau \in \mathbb{S}\mathfrak{a}^+$, $\iota(\tau) = -w_0 \cdot \tau$. An element $\tau \in \mathbb{S}\mathfrak{a}^+$ is called *symmetric* if $\iota(\tau) = \tau$.

Given a set of simple roots $\Theta \subset \Delta$, we say that the *model Θ-facet* is the set of elements $\tau \in \mathbb{S}\mathfrak{a}^+$ such that for all $\alpha \in \Delta\backslash\Theta$, $\alpha(\tau) = 0$. The *open* model Θ-facet is the set of elements $\tau \in \mathbb{S}\mathfrak{a}^+$ such that for all $\alpha \in \Delta\backslash\Theta$, $\alpha(\tau) = 0$, and for all $\alpha \in \Theta$, $\alpha(\tau) > 0$. For an element $\tau \in \mathbb{S}\mathfrak{a}^+$ we will write $\Theta(\tau)$ the unique set of simple roots such that α lies in the open model $\Theta(\tau)$-facet.

2.1.3 Maximal Flats, visual boundary and parabolic subgroups.

A *flat* in $\mathbb{X}$ is a complete totally geodesic subspace of $\mathbb{X}$ on which the sectional curvature completely vanishes. A flat F is maximal if $\dim(F) = \mathrm{rank}(\mathbb{X})$. Flats passing through a point $x \in \mathbb{X}$ are in one to one correspondence with abelian subalgebras of $\mathfrak{p}_x$, and maximal flats correspond to maximal subalgebras. As a consequence the action of G on the space of maximal flats is transitive. The maximal flat corresponding to $\mathfrak{a}$ will be called the *model flat*. Moreover for any $x \in X$ and $\mathrm{v} \in T_x\mathbb{X}$, there is a maximal flat F such that $x \in F$ and $\mathrm{v} \in T_xF$.

We say that two geodesic rays parametrized with unit length $\eta_1, \eta_2 : \mathbb{R}_{\geqslant 0} \to \mathbb{X}$ are *asymptotic* if there exist a positive constant C such that for all $t > 0$, $d_{\mathbb{X}}(\gamma_1(t), \gamma_2(t)) \leqslant C$. This defines an equivalence relation on the space of rays.

The *visual boundary* $\partial_{\mathrm{vis}}\mathbb{X}$ of the symmetric space $\mathbb{X}$ is the space of classes of asymptotic geodesic rays parametrized with unit speed. The group G acts by isometries on $\mathbb{X}$, hence it acts on $\partial_{\mathrm{vis}}\mathbb{X}$.

A unit vector $\mathrm{v} \in T_x\mathbb{X}$ at a point $x \in \mathbb{X}$ *points towards* $a \in \partial_{\mathrm{vis}}\mathbb{X}$ if the geodesic ray γ such that $\gamma(0) = x$, $\gamma'(0) = \mathrm{v}$ is in the class corresponding to a. Since $\mathbb{X}$ is a Hadamard manifold, for any $x \in \mathbb{X}$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$ there is a unique unit vector that points towards a. We will denote this vector by $\mathrm{v}_{a,x}$ throughout the paper. There exist a unique topology on $\partial_{\mathrm{vis}}\mathbb{X}$ such that for any $x \in X$ the map $\phi_x : a \mapsto \mathrm{v}_{a,x}$ is an homeomorphism between $\partial_{\mathrm{vis}}\mathbb{X}$ and $T^1_x\mathbb{X}$.

The visual boundary of the model flat can be identified with $\mathbb{S}\mathfrak{a}$, and is included in the visual boundary of $\mathbb{X}$. The G-orbit of a point $\tau \in \mathbb{S}\mathfrak{a}^+$ will be denoted by $\mathcal{F}_\tau$. A Θ-*facet* is a subset of $\partial_{\mathrm{vis}}\mathbb{X}$ that is the image of the model Θ-facet by the action of an element of G. We define similarly the notion of *open Θ-facet*. The stabilizer of the open model Θ-facet will be denoted by P_Θ, and we will denote by $\mathcal{F}_\Theta$ the associated *flag manifolds*, i.e. the quotient G/P_Θ. Since

P_Θ is also the stabilizer of any point in the open Θ-facet, there exist a natural G-equivariant diffeomorphism between the G-homogeneous spaces $\mathcal{F}_\Theta$ and $\mathcal{F}_\tau$ for any τ in the open model Θ-facet.

Example 2.1.3. Let us consider the case when $G = \mathrm{PSL}(n, \mathbb{R})$ to illustrate these notions. A parabolic subgroup is this case is the stabilizer of a partial flag f. Any point in $\partial_{\mathrm{vis}}\mathbb{X}$ belongs to a unique open facet, which corresponds to a partial flag. The type of the partial flag, i.e. the dimensions of the subspaces that form the flag, determine a set of roots Θ_f. The points in $\partial_{\mathrm{vis}}\mathbb{X}$ are in 1 to 1 correspondence with partial flags decorated with a point in the open Θ_f-model facet. This decoration can be interpreted as a collection of weights associated to the subspaces of the partial flag.

Given any two points $a, a' \in \partial_{\mathrm{vis}}\mathbb{X}$, one can define their *Tits angle* $\angle_{\mathrm{Tits}}(a, a')$ as the minimum of $\angle(\mathrm{v}_{a,x}, \mathrm{v}_{a',x})$ for $x \in \mathbb{X}$. This minimum is obtained when $x \in \mathbb{X}$ lies in a common flat with a and a'.

2.1.4 Cartan and Iwasawa decomposition.

The *Cartan projection* $\boldsymbol{\mu} : T\mathbb{X} \to \mathfrak{a}^+$ is the function that maps any vector $\mathrm{w} \in T_x\mathbb{X}$ to the unique element $\boldsymbol{\mu}(\mathrm{w}) \in \mathfrak{a}^+$ of the model Weyl Chamber such that for some $g \in G$, $g \cdot \mathrm{w} = \boldsymbol{\mu}(\mathrm{w})$.

The *generalised distance* $\mathbf{d}_\mathfrak{a}(x, y)$ based at $x \in \mathbb{X}$ of a point $y \in \mathbb{X}$ is the Cartan projection $\boldsymbol{\mu}(\mathrm{w})$ of the unique vector $\mathrm{w} \in T_x\mathbb{X} \simeq \mathfrak{p}_x$ such that $\exp(\mathrm{w})\cdot x = y$. This generalized distance is 1-Lipshitz in the following sense:

Lemma 2.1.4. *Let $x, y, z \in \mathbb{X}$:*

$$|\mathbf{d}_\mathfrak{a}(x, z) - \mathbf{d}_\mathfrak{a}(x, y)| \leqslant d_\mathbb{X}(y, z).$$

A proof of this lemma can be found for instance in [Rie21], Corollary 3.8. Here $|\cdot|$ means the norm induced by the metric (2.1).

We say that a vector $\mathrm{v} \in T\mathbb{X}$ is Θ*-regular* for a set of simple root Θ if it's Cartan projection is Θ-regular, i.e. it avoids the walls of the Weyl chamber associated with elements of Θ. We will later introduce a similar notion of a τ-regular vector in Definition 2.4.7.

Let $T_{a,x} : P_a \to G$ be the map that associates to $g \in G_a$ the limit :

$$\lim_{t \to +\infty} \exp(-t\mathrm{v}_a) g \exp(t\mathrm{v}_a)$$

It is a well defined continuous morphism. Let $N_{a,x}$ be the kernel of $T_{a,x}$, and $\mathfrak{n}_{a,x}$ its Lie algebra. The generalised Iwasawa decomposition is useful to compute Busemann functions.

Theorem 2.1.5 (Generalized Iwasawa decomposition)**.** *Let $x \in \mathbb{X}$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$. then the following map:*

$$\begin{aligned} N_{a,x} \times \exp(\mathfrak{a}_{a,x}) \times K_x &\longrightarrow G \\ (n, \exp(\mathrm{v}), k) &\longmapsto n \exp(\mathrm{v}) k \end{aligned}$$

is a diffeomorphism. In particular for every $x \in \mathbb{X}$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$ there is a splitting :

$$\mathfrak{g} = \mathfrak{n}_{a,x} \oplus \mathfrak{a}_{a,x} \oplus \mathfrak{k}_x.$$

The sum $\mathfrak{n}_{a,x} \oplus \mathfrak{a}_{a,x}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_x$.

In this Theorem, $\mathfrak{a}_{a,x} \subset \mathfrak{p}_x$ is the centralizer of $\mathrm{v}_{a,x}$.

2.1.5 Examples.

In this subsection we consider the case when the semi-simple Lie group G is equal to $\mathrm{PSL}(n, \mathbb{R})$, $\mathrm{PSL}(n, \mathbb{C})$, $\mathrm{PSp}(2n, \mathbb{R})$ or $\mathrm{PSO}(p, q)$. The notations introduced here will be used in the examples throughout the paper.

Let $\boldsymbol{G} = \mathbf{PSL}(\boldsymbol{n}, \mathbb{R})$ and let us fix a volume form on $\mathbb{R}^n$. Let $\mathcal{S}_n$ for $n \geqslant 2$ be the space of all scalar products on $\mathbb{R}^n$ having volume one. The group $\mathrm{PSL}(n, \mathbb{R})$ acts transitively on $\mathcal{S}_n$ by changing the basis, i.e for $g \in \mathrm{PSL}(n, \mathbb{R})$, $q \in X$ and $v, w \in \mathbb{R}^n$: $g \cdot q(v, w) = q(g^{-1}(v), g^{-1}(w))$. For any $q \in \mathcal{S}_n$ the space $\mathcal{S}_n$ can be identified with the quotient $\mathrm{PSL}(n, \mathbb{R})/\mathrm{PSO}(q) \cong \mathrm{PSL}(n, \mathbb{R})/\mathrm{PSO}(n, \mathbb{R})$.

Let θ_q at a point $q \in X$ be the involutive automorphism of $\mathfrak{sl}(n, \mathbb{R})$ defined by $u \mapsto -u^T$ where u^T is the transpose of u with respect to the scalar product q. This is a Cartan involution. The space $\mathcal{S}_n$ is the symmetric space of non-compact type associated to $G = \mathrm{PSL}(n, \mathbb{R})$.

The space $\mathfrak{p}_q$ is the space of symmetric endomorphsisms with respect to q, and $\mathfrak{t}_q$ is the space of antisymmetric endomorphsisms with respect to q. The scalar product $\langle \cdot, \cdot \rangle_q$ at a point $q \in \mathcal{S}_n$ is equal to $\langle u, v \rangle_q = 2n \operatorname{Tr}(u^T v)$ for $u, v \in \mathfrak{sl}(n, \mathbb{R})$.

We choose the standard scalar product $q \in \mathcal{S}_n$ on $\mathbb{R}^n$ to be our base point of $\mathcal{S}_n$. A maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}_q \subset \mathfrak{sl}(n, \mathbb{R})$ is equal to the algrbra of diagonal matrices:

$$\mathfrak{a} = \left\{ \mathrm{Diag}(\sigma_1, \cdots, \sigma_n) | \sigma_1, \cdots, \sigma_n \in \mathbb{R}^n, \sum_{i=1}^{n} \sigma_i = 0 \right\}.$$

A choice of simple root is $\Delta = \{\alpha_1, \cdots, \alpha_{n-1}\}$ where for any $1 \leqslant i \leqslant n-1$ and any $\tau = \mathrm{Diag}(\sigma_1, \cdots, \sigma_n) \in \mathfrak{a}$, $\alpha_i(\tau) = \sigma_i - \sigma_{i+1}$.

The Weyl chamber associated to this choice is :

$$\mathfrak{a}^+ = \left\{ \mathrm{Diag}(\sigma_1, \cdots, \sigma_n) | \sigma_1 \geqslant \cdots \geqslant \sigma_n \in \mathbb{R}^n, \sum_{i=1}^{n} \sigma_i = 0 \right\}.$$

The Weyl group W is isomorphic to $\mathfrak{S}_n$. It acts on $\mathfrak{a}$ by permuting the entries.

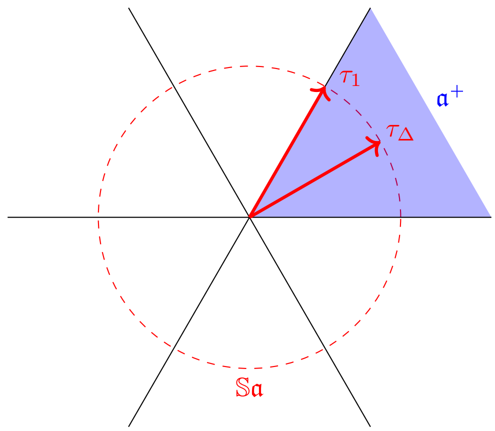


Figure 2.1: The model restricted Cartan algebra $\mathfrak{a}$ and its Weyl chamber $\mathfrak{a}^+$ for $\mathrm{PSL}(3, \mathbb{R})$.

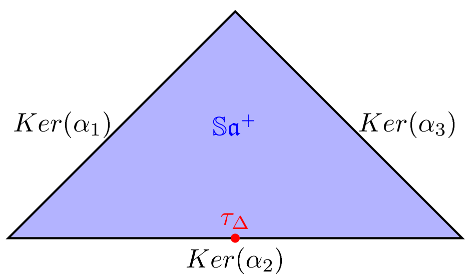


Figure 2.2: The projectivization of the Weyl chamber $\mathbb{S}\mathfrak{a}^+$ for $G = \mathrm{PSL}(4, \mathbb{R})$ in an affine chart.

Let $G = \mathrm{PSL}(n, \mathbb{C})$ The space $\mathcal{H}_n$ of all definite positive Hermitian bilinear forms of $\mathbb{C}^n$ having volume one can identified with $\mathrm{PSL}(n, \mathbb{C})/\mathrm{PSU}(n, \mathbb{C})$. It can be given in a similar way a Riemannian metric that makes it a symmetric space of non-compact type associated to $G = \mathrm{PSL}(n, \mathbb{C})$.

The subalgebra $\mathfrak{a} \subset \mathfrak{sl}(n, \mathbb{R}) \subset \mathfrak{sl}(n, \mathbb{C})$ defined previously is still a maximal abelian subalgebra of $\mathfrak{p}_q$. One has $\mathrm{rank}(\mathcal{S}_n) = \mathrm{rank}(\mathcal{H}_n) = \mathrm{rank}(\mathrm{PSL}(n, \mathbb{R})) = n - 1$, but $\mathrm{rank}(\mathrm{PSL}(n, \mathbb{C})) = 2n - 2$.

Let $G = \mathrm{PSp}(2n, \mathbb{R})$ Let ω be a symplectic form on $\mathbb{R}^{2n}$. Let $\mathcal{X}_n$ be the space of endomorphisms J on $\mathbb{R}^{2n}$ such that $J^2 = -\mathrm{Id}$ and $(\mathrm{v}, \mathrm{w}) \mapsto \omega(\mathrm{v}, J(\mathrm{w}))$ is a scalar product on $\mathbb{R}^{2n}$. The semi-simple Lie group $\mathrm{PSp}(2n, \mathbb{R})$ acts on $\mathcal{X}_n$ by conjugation. The space $\mathcal{X}_n$ can be identified with $\mathrm{PSp}(2n, \mathbb{R})/\mathrm{PSU}(n, \mathbb{R})$. This is one of the models for the *Siegel space*, see for instance [BP17].

For $J \in \mathcal{X}_n$, let us write $\theta_J = \mathrm{Ad}_J$. This is Cartan involution of $\mathfrak{sp}(2n, \mathbb{R})$. The Siegel space is the symmetric space of non-compact type associated with $G = \mathrm{PSp}(2n, \mathbb{R})$.

Let ω and $J \in \mathcal{X}_n$ be such that for $\mathrm{x} = (x_1, \cdots, x_{2n})$ and $\mathrm{y} = (y_1, \cdots, y_{2n})$:

$$\omega(\mathrm{x}, \mathrm{y}) = \sum_{i=1}^{n} x_i y_{2n-i} - \sum_{i=n+1}^{2n} x_i y_{2n-i},$$

$$J(x) = (-x_{2n}, \cdots, -x_{n+1}, x_n, \cdots, x_1).$$

A maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}_q \subset \mathfrak{sp}(2n, \mathbb{R})$ is:

$$\mathfrak{a} = \{\mathrm{Diag}(\sigma_1, \cdots, \sigma_n, -\sigma_n, \cdots, -\sigma_1) | \sigma_1, \cdots, \sigma_n \in \mathbb{R}^n\}.$$

A choice of simple roots is $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ where $\alpha_i(\tau) = \sigma_i - \sigma_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n(\tau) = 2\sigma_n$.

The Weyl chamber associated to this choice is :

$$\mathfrak{a}^+ = \{\mathrm{Diag}(\sigma_1, \cdots, \sigma_n, -\sigma_n, \cdots, -\sigma_1) | \sigma_1 \geqslant \cdots \geqslant \sigma_n \geqslant 0 \in \mathbb{R}^n\}.$$

The Weyl group W is isomorphic to the subgroup of elements in $\mathfrak{S}_{2n}$ that commutes with the involution $\iota : i \mapsto 2n + 1 - i$. It acts on $\mathfrak{a}$ by permuting the entries.

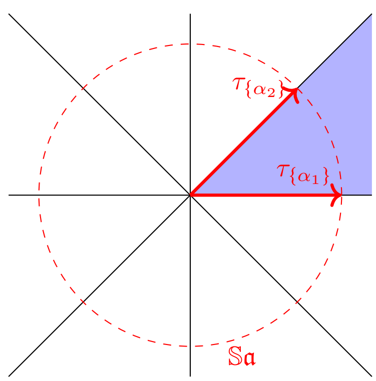


Figure 2.3: The model restricted Cartan algebra $\mathfrak{a}$ and its Weyl chamber $\mathfrak{a}^+$ for $\mathrm{PSp}(4, \mathbb{R})$.

Let $G = \mathrm{SO}(p, q)$ with $p < q$. Let $\mathbb{R}^{p,q}$ be the vector space $\mathbb{R}^{p+q}$ equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature (p, q) defined in the standard basis by:

$$\langle \mathrm{x}, \mathrm{y} \rangle = \sum_{i=i}^{p} (x_i y_{p+q-i} + x_{p+q-i} y_i) - \sum_{i=1}^{p-q} x_{p+i} y_{p+i}.$$

A model for the associated symmetric space $\mathbb{X}$ is the space of spacelike subspaces $U \subset \mathbb{R}^{p,q}$, i.e. subspaces on which $\langle \cdot, \cdot \rangle$ is definite positive.

A maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}_U \subset \mathfrak{so}(p,q)$ is the algebra :

$$\mathfrak{a} = \{\mathrm{Diag}(\sigma_1, \cdots, \sigma_p, 0, \cdots, 0, -\sigma_p, \cdots, -\sigma_1) | \sigma_1, \cdots, \sigma_p\}.$$

A choice of simple roots is $\Delta = \{\alpha_1, \cdots, \alpha_p\}$ where $\alpha_i(\tau) = \sigma_i - \sigma_{i+1}$ for $1 \leqslant i \leqslant p-1$, and $\alpha_p(\tau) = \sigma_p$.

The Weyl chamber associated to this choice is :

$$\mathfrak{a}^+ = \{\mathrm{Diag}(\sigma_1, \cdots, \sigma_p, 0, \cdots, 0, -\sigma_p, \cdots, -\sigma_1) | \sigma_1 \geqslant \cdots \geqslant \sigma_p \geqslant 0\}.$$

The Weyl group W is isomorphic to the subgroup of elements in $\mathfrak{S}_{2p}$ that commutes with the involution $\iota : i \mapsto 2p+1-i$. It acts on $\mathfrak{a}$ by permuting the first and last p entries.

2.1.6 Weyl orbits of simple roots.

In this subsection we introduce *Weyl orbits of simple roots*, which are special sets of simple roots. To a Weyl orbits of simple roots Θ one can associate a unit vector in the Weyl chamber $\tau_\Theta \in \mathbb{S}\mathfrak{a}^+$ which is colinear to a coroot.

We consider the restricted root system Σ associated with the semi-simple Lie group G, with a choice of a set of positive roots Σ^+ and of simple roots Δ. Two simple roots α and β are *conjugates* if there is an element w in the Weyl group W such that $\alpha = w \cdot \beta = \beta \circ w^{-1}$.

Definition 2.1.6. A set of simple roots $\Theta \subset \Delta$ is called a *Weyl orbit of simple roots* if it is an equivalence class for the conjugation relation on the set of simple roots Δ.

Proposition 2.1.7. *Let Θ be a Weyl orbit of simple roots. There exists a unique unit vector $\tau_\Theta \in \mathbb{S}\mathfrak{a}^+$ such that for any $\alpha \in \Theta$ there is some $w \in W$ such that $w \cdot \tau_\Theta$ is orthogonal to* $\ker(\alpha)$*. The vector $\tau_\Theta \in \mathbb{S}\mathfrak{a}^+$ will be called the* normalized coroot *associated to Θ.*

The normalized coroot associated to Θ is colinear to a coroot which is itself conjugate via the Weyl group to the corroot associated to any $\alpha \in \Theta$.

Proof. Let $\alpha \in \Theta$. Let $\tau_0 \in \mathbb{S}\mathfrak{a}$ be a unit vector orthogonal to $\ker(\alpha)$. Since every orbit for the action of the Weyl group on $\mathbb{S}\mathfrak{a}^+$ intersects exactly once the model Weyl chamber, there exist a unique vector $\tau_\Theta \in \mathbb{S}\mathfrak{a}^+$ such that $\tau_\Theta = w \cdot \tau_0$ for some $w \in W$.

This definition does not depend of the choice of $\alpha \in \Theta$, because if $\beta \in \Theta$ then for some $w_0 \in W$, $w_0 \cdot \beta = \alpha$ and hence any vector τ_0' orthogonal to $\ker(\beta)$ can be written $\tau_0' = w_0 \cdot \tau_0$ or $\tau_0' = (w_0 \sigma_\beta) \cdot \tau_0$, and hence $W \cdot \tau_0 = W \cdot \tau_0'$. Therefore $W \cdot \tau_0 \cap \mathbb{S}\mathfrak{a}^+ = W \cdot \tau_0' \cap \mathbb{S}\mathfrak{a}^+ = \{\tau_\Theta\}$. ☐

Note that in particular τ_Θ is symmetric.

Remark 2.1.8. If Θ is a Weyl orbit of simple roots, $\mathcal{F}_{\tau_\Theta}$ is not in general the same flag manifold as $\mathcal{F}_\Theta = G/P_\Theta$.

The *Dynkin diagram* associated to the restricted root system Σ is the graph with vertex set Δ such that for all $\alpha, \beta \in \Delta$ distinct roots there is a link between α and β of multiplicity depending of the order k of $\sigma_\alpha\sigma_\beta$ or $\sigma_\beta\sigma_\alpha$, where $\sigma_\alpha, \sigma_\beta \in W$ are the symmetries associated with the roots α, β. If $k = 2$ we consider that there is no link, there is a simple link if $k = 3$, a double link if $k = 4$ and a triple link if $k = 6$. These are the only cases that occur for spherical Dynkin diagrams. If two roots have different norms, we orient the edge towards the root with largest norm.

Proposition 2.1.9. *Consider the Dynkin diagram associated with the reduced root system Σ, and remove all the double or triple edges. A set $\Theta \subset \Delta$ is a Weyl orbit of simple roots if and only if it is a connected component of this graph.*

Proof. Let α, β be two simple roots such that $\sigma_\alpha = w\sigma_\beta w^{-1}$ for some $w \in W$. Then the simple root α and $w\cdot\beta$ are proportional, hence $\alpha = w\cdot\beta$ or $\alpha = -w\cdot\beta$. In any case α and β are conjugated. Reciprocally if α and β are conjugates, then σ_α and σ_β are conjugated.

The system $(W, (\sigma_\alpha)_{\alpha\in\Delta})$ is a Coxeter system. Generators of a Coxeter system are conjugated to one another if and only if there is a path of single edges between the corresponding vertices in the Dynkin diagram ([Gal05] Proposition 2.1). Hence Weyl orbits of simple roots correspond exactly to connected components for the modified Dynkin diagram. ☐

We can now describe the Weyl orbits of simple roots in Δ for the restricted system of roots associated to a simple group G. For this we use the classification of the Dynkin diagrams that occur as reduced root system for a symmetric space $\mathbb{X}$ associated to G.

Corollary 2.1.10. *If the restricted root system Σ is of type A_n, D_n for $n \geqslant 2$ or E_6, E_7, E_8, then the only Weyl orbit of simple roots in Δ is Δ.*

If the root system Σ is of type B_n, C_n for $n \geqslant 2$ or F_4, G_2, then Δ can be partitioned into its only two Weyl orbits of simple roots .

Example 2.1.11. We keep notations from Section 2.1.5. If $G = \mathrm{PSL}(n, \mathbb{R})$, with previous notations Δ is the only Weyl orbit of simple roots and:

$$\tau_\Delta = \frac{1}{2\sqrt{n}}\,\mathrm{Diag}(1, 0, \cdots, 0, -1).$$

The flag manifold $\mathcal{F}_{p_\Delta}$ can be identified with :

$$\mathcal{F}_{1,n-1} = \{(\ell, H) | \ell \subset H \subset \mathbb{R}^n, \dim(\ell) = 1, \dim(H) = n-1\}.$$

If $G = \mathrm{Sp}(2n, \mathbb{R})$, with previous notations $\Theta' = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ and $\Theta = \{\alpha_n\}$ are the two Weyl orbits of simple roots of Δ. One has :

$$\tau_\Theta = \frac{1}{2\sqrt{n}}\,\mathrm{Diag}(1, 0, \cdots, 0, -1).$$

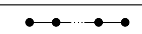
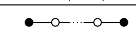
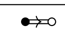
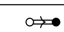
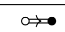
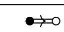
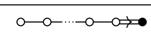
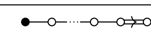
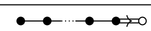
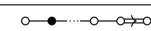
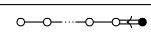
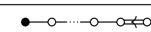
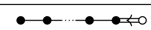
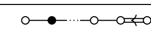
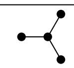
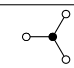
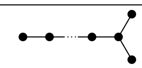
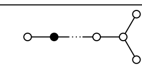
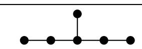
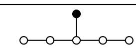
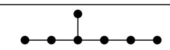
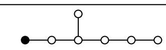
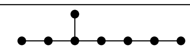
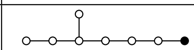
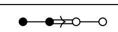
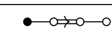
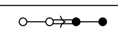
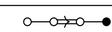
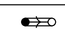
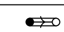
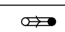
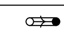
Δ	Θ	τ_Θ	$\Theta(\tau_\Theta)$
A_n		$\frac{1}{\sqrt{2}}(e_1 - e_{n+1})$	
BC_2		e_1	
		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
B_n, $n \geqslant 3$		e_1	
		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
C_n, $n \geqslant 3$		e_1	
		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
D_4		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
D_n, $n \geqslant 5$		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
E_6		$\sqrt{2}e$	
E_7		$\frac{1}{\sqrt{2}}(e_8 - e_7)$	
E_8		$\frac{1}{\sqrt{2}}(e_1 - e_9)$	
F_4		$\frac{1}{\sqrt{2}}(e_1 + e_2)$	
		e_1	
G_2		$\frac{1}{\sqrt{2}}(e_1 - e_3)$	
		$\frac{1}{\sqrt{6}}(2e_1 - e_2 - e_3)$	

Figure 2.4: Weyl orbits of simple roots, and their associated normalized coroots.

$$\tau_{\Theta'} = \frac{1}{2\sqrt{n}} \operatorname{Diag}(1, 1, 0, \cdots, 0, -1, -1).$$

The flag manifold $\mathcal{F}_{\tau_\Theta}$ can be identified with $\mathbb{RP}^{2n-1}$ and $\mathcal{F}_{\tau_{\Theta'}}$ can be identified with the Grassmannian of planes P in $\mathbb{R}^{2n}$ that are isotropic for ω, i.e. such that $\omega_{|P} = 0$.

In general, the Weyl orbits of simple roots for any root system are summarized in Figure 2.4. The table also includes an illustration of the set of roots $\Theta(\tau_\Theta)$ such that $\mathcal{F}_{\Theta(\tau_\Theta)} \simeq \mathcal{F}_{\tau_\Theta}$. The sets of roots are illustrated in the diagram as the set of filled vertices. Using notations from [OV90, Table 1, page 293], the basis (e_i) is an orthonormal basis such that $e_i^\vee = \epsilon_i$ for B_n, C_n, D_n, F_4 and $e_i^\vee - \frac{1}{n+1}\sum_{k=1}^{n+1} e_k^\vee = \epsilon_i$ for A_n, E_7, E_8 and G_2. For E_6, we write $e = \epsilon^\vee$.

The table can be checked as follows: for each Weyl orbit of simple root one can check that the vector τ_Θ is orthogonal to the kernel of a root conjugate to a root in Θ, and lies in the model Weyl chamber. Then one an check that the simple roots that do not vanish on τ_Θ are the one in $\Theta(\tau_\Theta)$, as depicted in Figure 2.4.

2.2 Representations of hyperbolic groups.

2.2.1 Gromov hyperbolic groups.

Let Γ be a finitely generated group. Let F be any finite generating system for Γ_g that is symmetric, i.e. such that $s^{-1} \in F$ for all $s \in F$. We can define the norm of an element $\gamma \in \Gamma_g$ as:

$$|\gamma|_F = \min \{n | n = s_1 s_2 \cdots s_n,\ s_i \in S\}.$$

This norm defines the word distance on Γ_g by taking $d_F(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|_F$ for $\gamma_1, \gamma_2 \in \Gamma_g$.

A map $f : Y \to X$ between two metric spaces X, Y is called a *quasi-isometric embedding* if there exist C, D such that for all $x_1, x_2 \in X$:

$$\frac{1}{C} d_Y(x_1, x_2) - D \leqslant d_X(f(x_1), f(x_2)) \leqslant C d_Y(x_1, x_2) + D.$$

By extension, we say that a representation ρ is a *quasi-isometric embedding* if some and hence any ρ-equivariant map $u_0 : \Gamma \to \mathbb{X}$ is a quasi-isometry, where Γ acts on itself by left multiplication.

This notion does not depend on the choice of F: indeed if F' is an other finite generating system, the identity map $(\Gamma, d_F) \to (\Gamma, d_{F'})$ is a quasi-isometric embedding.

The group Γ is called *hyperbolic* if as a metric space it is hyperbolic in the sense of Gromov. We denote by $\partial\Gamma$ the Gromov boundary of an hyperbolic group Γ, that we equip with the usual topology [Gro81].

Given a discrete representation, we will need to consider the limit cone of the Cartan projections of elements of the group.

Definition 2.2.1. The *limit cone* of a discrete representation $\rho : \Gamma \to G$ is the closed subset

$$\mathcal{C}_\rho = \bigcap_{n \in \mathbb{N}} \overline{\{[\mathbf{d}_\mathfrak{a}(o, \rho(\gamma) \cdot o)], |\gamma|_w \geqslant n\}} = \bigcap_{n \in \mathbb{N}} \overline{\left\{ \frac{\mathbf{d}_\mathfrak{a}(o, \rho(\gamma) \cdot o)}{d_\mathbb{X}(o, \rho(\gamma) \cdot o)}, |\gamma|_w \geqslant n \right\}} \subset \mathbb{S}\mathfrak{a}^+.$$

Recall that the generalized distance $\mathbf{d}_\mathfrak{a}$ was defined in Section 2.1. This definition does not depend on the choice of the base point $o \in \mathbb{X}$.

2.2.2 Anosov representations.

The Anosov properties are more restrictive for a representation than the property of being a quasi-isometric embedding. These notions are interesting in high rank because the Anosov properties hold for an open set of representations, whereas the property of being a quasi-isometric embedding is not necessarily open in $\mathrm{Hom}(\Gamma_g, G)$ when the rank of $\mathbb{X}$ is at least 2.

Definition 2.2.2 ([BPS19], Section 4)**.** Let $\Theta \subset \Delta$ be a non-empty set of simple roots. A representation $\rho : \Gamma \to G$ is Θ-*Anosov* if for every root $\alpha \in \Theta$ there exists some constants $b, c > 0$ such that for every $\gamma \in \Gamma$:

$$\alpha\left(\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o)\right) \geqslant b|\gamma|_F - c.$$

This definition does not depend on the choice of the generating set F and the base-point o.

A Δ-Anosov representation in the case when G is a split real simple Lie group is be called a *Borel-Anosov* representation.

Remark 2.2.3. A representation is P-Anosov for a parabolic subgroup P if it is Θ-Anosov for the corresponding set of simple roots $\Theta \subset \Delta$.

Let $\alpha \in \Delta$ and $\tau_0 \in \mathbb{S}\mathfrak{a}$ be orthogonal to $\mathrm{Ker}(\alpha)$. The evaluation of α to $\mathbf{d}_{\mathfrak{a}}(x, y)$ satisfies:

$$\alpha(\mathbf{d}_{\mathfrak{a}}(x, y)) = \alpha(\tau_0) d_{\mathbb{X}}(x, y) \cos\left(\angle(\mathbf{d}_{\mathfrak{a}}(x, y), \tau_0)\right).$$

Anosov representations are necessarily quasi-isometric embeddings. Reciprocally a quasi-isometric embedding is $\{\alpha\}$-Anosov if and only if the angle $\langle \mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o), \tau_0 \rangle$ is not too small in absolute value for $\gamma \in \Gamma_g$ large enough.

In particular we have the following characterization of Anosov representations:

Theorem 2.2.4 ([KLP17])**.** *A representation $\rho : \Gamma_g \to G$ is Θ-Anosov for $\Theta \subset \Delta$ if and only if it is a quasi-isometric embedding and if* $\mathrm{Ker}(\alpha) \cap \mathcal{C}_\rho = \varnothing$ *for all $\alpha \in \Theta$.*

Representations that are Θ-Anosov admit a natural continuous and equivariant map $\xi_\rho^\Theta : \partial \Gamma_g \to \mathcal{F}_\Theta = G/P_{\tau_\Theta}$, where $\partial \Gamma_g$ is the Gromov boundary of Γ_g.

In the proof of Theorem 2.6.11 we will use the following results about the boundary maps of Anosov representations. For two points $o, x \in \mathbb{X}$ let $\ell(o, x) \in \partial_{\mathrm{vis}} \mathbb{X}$ be the class of the unique geodesic ray with unit speed starting from o and passing through x.

Theorem 2.2.5 ([BPS19], Section 4)**.** *Let $\rho : \Gamma_g \to G$ be a Θ-Anosov representation for a non-empty set $\Theta \subset \Delta$. There exist a unique ρ-equivariant continuous and dynamic preserving map $\xi_\rho^\Theta : \partial \Gamma_g \to \mathcal{F}_\Theta$. This map is such that for any $o \in \mathbb{X}$ and any sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of Γ_g converging to $\zeta \in \partial \Gamma_g$, the Δ-facet containing any limit point of the sequence $(\ell(o, \rho(\gamma_n) \cdot o))_{n \in \mathbb{N}}$ also contains the Θ-facet $\xi_\rho^\Theta(\zeta)$.*

For instance, when $G = \mathrm{PSL}(n, \mathbb{R})$ and if $\Theta = \{\alpha_k\}$, one can associate a partial flag to any point in $\partial_{\mathrm{vis}} \mathbb{X}$. If the the representation ρ is $\{\alpha_k\}$-Anosov, the partial flag associated to any limit point of $(\ell(o, \rho(\gamma_n) \cdot o))_{n \in \mathbb{N}}$ contains the same k-dimensional plane, that will be denoted by $\xi_\rho^k(\zeta)$.

Kapovich, Leeb and Porti also proved a generalization of the Morse lemma. Here is a version of this result. Let us fix any metric on Γ quasi-isometric to a word metric.

Theorem 2.2.6 ([KLP18b], Theorem 1.3)**.** *Let $\rho : \Gamma \to G$ be a Θ-Anosov representation. Let $o \in \mathbb{X}$ be a base-point. There exist a constant $D > 0$ such that for every $\gamma \in \Gamma$, there exist a geodesic ray $\eta : \mathbb{R}_{>0} \to \mathbb{X}$ at distance at most D from $\rho(\gamma) \cdot o$ with $\eta(0) = o$, whose class $[\eta] \in \partial_{\mathrm{vis}}\mathbb{X}$ lies in a common Δ-facet with $\xi_\rho^\Theta(\zeta_\gamma)$. Here $\zeta_\gamma \in \partial\Gamma$ is the endpoint of any geodesicray in Γ starting at the identity and going through γ.*

2.3 Busemann functions on symmetric spaces.

Busemann functions are natural functions on Hadamard manifolds associated to points in the visual boundary. These functions will play a key role in the definition of τ-nearly geodesic immersions, and in the fibration of domains of discontinuity. In this section we prove the main properties of Busemann functions and compute their Hessian.

2.3.1 Main properties of Busemann functions.

Busemann functions can be interpreted as the distance of a point $x \in \mathbb{X}$ to a point a in the visual boundary relative to a base-point $o \in \mathbb{X}$.

Definition 2.3.1. The *Busemann function* associated to $a \in \partial_{\mathrm{vis}}\mathbb{X}$ and based at $o \in \mathbb{X}$ is the map $b_{a,o} : X \to \mathbb{R}$ that associates to $x \in \mathbb{X}$ the limit :

$$\lim_{t \to +\infty} d_{\mathbb{X}}(x, \gamma(t)) - d_{\mathbb{X}}(o, \gamma(t)),$$

for any geodesic ray $\gamma : \mathbb{R}^+ \to \mathbb{X}$ in the class of a.

This definition makes sense because $\mathbb{X}$ is a Hadamard manifold [Ebe96]. The definition implies that for any $x, o, o' \in X$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$, the Busemann cocycle holds:

$$b_{a,o'}(x) = b_{a,o}(x) + b_{a,o'}(o). \tag{2.2}$$

For symmetric spaces, this function can be computed using the generalized Iwasawa decomposition. First we prove that unipotent elements preserve the level lines of Busemann functions.

Lemma 2.3.2. *Let $x, o \in \mathbb{X}$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$ be two points. Let n be an element of the unipotent subgroup $N_{a,o}$ of G:*

$$b_{a,o}(n \cdot x) = b_{a,o}(x).$$

Proof. The Busemann cocycle implies that $b_{a,o}(n \cdot x) - b_{a,o}(x) = b_{a,x}(n \cdot x)$. This is by definition the limit when $t \to \infty$ of the difference:

$$\begin{aligned} d_{\mathbb{X}}\left(n \cdot x, \exp(t\mathrm{v}_{a,x}) \cdot x\right) - d_{\mathbb{X}}\left(x, \exp(t\mathrm{v}_{a,x}) \cdot x\right) \\ &= d_{\mathbb{X}}\left(x, n^{-1}\exp(t\mathrm{v}_{a,x}) \cdot x\right) - d_{\mathbb{X}}\left(x, \exp(t\mathrm{v}_{a,x}) \cdot x\right) \\ &\leqslant d_{\mathbb{X}}\left(n^{-1}\exp(t\mathrm{v}_{a,x}) \cdot x, \exp(t\mathrm{v}_{a,x}) \cdot x\right). \end{aligned}$$

But since $n \in N_{a,x}$, $\exp(-t\mathrm{v}_{a,x})n\exp(t\mathrm{v}_{a,x})$ converges to the identity when $t \to +\infty$, so this distance converges to 0. Hence $b_{a,x}(n \cdot x) = 0$.

□

Recall that $\mathrm{v}_{a,o}$ is the unit vector in $T_o\mathbb{X}$ pointing towards $a \in \partial_{\mathrm{vis}}\mathbb{X}$. To compute a Busemann function one needs to understand it on maximal flats. Let $x = \exp(\mathrm{w}) \cdot o$ for $\mathrm{w} \in \mathfrak{p}_o$. Suppose that a, o, x lie in the same flat subspace, i.e. $[\mathrm{w}, \mathrm{v}_{a,o}] = 0$. The Busemann function on this Euclidean space is equal to:

$$b_{a,o}(x) = -d_{\mathbb{X}}(x, o)\cos(\angle_o(a, x)) = \langle -\mathrm{v}_{a,x}, \mathrm{w}\rangle_x.$$

Using these facts we can write Busemann functions in the symmetric space $\mathbb{X}$ explicitely. Let $o, x \in \mathbb{X}$ be a base point and $a \in \partial_{\mathrm{vis}}\mathbb{X}$.

Corollary 2.3.3. *Let $o, x \in X$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$. The Busemann function can be computed as :*

$$b_{a,o}(x) = \langle -\mathrm{v}_{a,o}, \mathrm{w}\rangle_o.$$

Where $\mathrm{w} \in \mathfrak{a}_{a,o}$ is given by the generalized Iwasawa decomposition, i.e. is the unique element such that one can write $x = n\exp(\mathrm{w})k \cdot o$ with $n \in N_{a,o}$ and $k \in K_o$.

Since G acts by isometries on $\mathbb{X}$, Busemann functions are G-equivariant in the following sense.

Corollary 2.3.4. *Let $o \in \mathbb{X}$ and $a \in \partial_{\mathrm{vis}}\mathbb{X}$. For any $g \in G$, and any $x \in \mathbb{X}$, $b_{g\cdot a, g\cdot o}(g \cdot x) = b_{a,o}(x)$.*

The gradient of Busemann functions is characterized as follows.

Proposition 2.3.5. *The gradient of the Busemann function based at any point $o \in \mathbb{X}$ associated to $a \in \partial_{\mathrm{vis}}\mathbb{X}$ is the vector field $(-\mathrm{v}_{a,x})_{x\in\mathbb{X}}$ of unit vectors pointing towards a.*

Proof. The differential $\mathrm{d}_x b_{a,o}$ of $b_{a,x}$ at x associates to an element $\mathrm{w} \in \mathfrak{p}_x$ the value $\langle \mathrm{w}', -\mathrm{v}_{a,x}\rangle_x$ where w' is the projection of w to $\mathfrak{a}_{a,x}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{n}_{a,x} \oplus \mathfrak{a}_{a,x} \oplus \mathfrak{k}_x$. Note that $\mathfrak{n}_{a,x}$ and $\mathfrak{k}$ are orthogonal to $\mathrm{v}_{a,o} \in \mathfrak{a}_{a,x}$ with respect to $\langle \cdot, \cdot \rangle_x$. Hence $\mathrm{w} = \mathrm{w}'$ so the gradient of $b_{a,o}$ at x is $-\mathrm{v}_{a,x}$. □

Busemann functions vary smoothly when the base flag varies in a flag manifold.

Lemma 2.3.6. *For any $o \in \mathbb{X}$, and $\tau \in \mathbb{S}\mathfrak{a}^+$. The map $\mathcal{F}_\tau \times \mathbb{X} \to \mathbb{R}$, $(a,x) \mapsto b_{a,o}(x)$ is smooth.*

Proof. Let P be the stabilizer of an element $a \in \mathcal{F}_\tau$. By Corollary 2.3.4, for $g \in G$ and $x, y \in \mathbb{X}$, $b_{g \cdot a_0, o}(y) = b_{a_0,o}(g^{-1} \cdot y) - b_{a_0,o}(g^{-1} \cdot o)$.

Hence the map $G \times \mathbb{X} \to \mathbb{R}$, $(g,x) \mapsto b_{g \cdot a_0, o}(x)$ is smooth, and defines a smooth map from the quotient $G/P \times X \simeq \mathcal{F}_\tau \times \mathbb{X}$.

∎

Example 2.3.7. Let $\mathbb{X} = \mathcal{S}_n$ or $\mathcal{H}_n$ the symmetric space associated with $\mathrm{PSL}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, as in Section 2.1.5. Let $(e_1, \cdots, e_n)$ be a basis of $\mathbb{K}^n$. The projective space $\mathbb{P}(\mathbb{K}^n)$ can be identified with the G-orbit $\mathcal{F}_{\tau_1}$ of the point $a \in \partial_{\mathrm{vis}}\mathbb{X}$ corresponding to the limit point where t goes to $+\infty$ of the geodesic ray:

$$t \mapsto \begin{pmatrix} e^{-t(n-1)} & 0 & \cdots & 0 \\ 0 & e^t & & 0 \\ \cdots & & \cdots & \\ 0 & 0 & & e^t \end{pmatrix}$$

The point $a \in \mathcal{F}_\tau \simeq \mathbb{RP}^{n-1}$ is identified with the first basis vector since the stabilizer of both points by the respective actions of $\mathrm{PSL}_n(\mathbb{K})$ on are equal.

The Busemann function $b_{[v],q_0}$ where $q_0 \in \mathbb{X}$ and $[v] \in \mathbb{P}(\mathbb{K}^n)$ associates to $q \in \mathbb{X}$ the value $\sqrt{\frac{n-1}{n}} \log(q(v,v))$ where v is a representative of $[v]$ such that $q_0(v,v) = 1$.

The asymptotic behavior of Busemann functions along geodesic rays is determined by the Tits angle between the endpoints.

Lemma 2.3.8. *Let $a \in \partial_{\mathrm{vis}}\mathbb{X}$ and $x \in \mathbb{X}$. Let η be a geodesic ray converging to $b \in \partial_{\mathrm{vis}}\mathbb{X}$. Then there exists a constant $C > 0$ such that for all $t \in \mathbb{R}$:*

$$|b_{a,x_0}(\eta(t)) + t\cos(\angle_{Tits}(a,b))| \leqslant C$$

Proof. There exist some element $g \in G$ such that $g \cdot a$ and $g \cdot b$ belong to $\partial_{\mathrm{vis}}F$ with F the model flat in $\mathbb{X}$. Moreover there exist a geodesic ray η' at bounded distance from $g \cdot \eta$ that belongs to the flat F. On the flat subspace F, the Busemann function can be computed:

$$b_{a,\eta'(0)}(\eta'(t)) = -t\cos(\angle_{\mathrm{Tits}}(g \cdot a, g \cdot b)).$$

This proves the lemma. ∎

2.3.2 Computation of the Hessian.

We compute here the Hessian of Busemann functions in the symmetric space $\mathbb{X}$. This computation will be used in the proof of Theorem 2.4.24.

Lemma 2.3.9. *Let $a \in \partial_{\mathrm{vis}}\mathbb{X}$, and $x, o \in \mathbb{X}$. The Hessian of the Busemann function $b_{a,o}$ at a point $x \in \mathbb{X}$ is given by the following quadratic form on $T_x\mathbb{X}$:*

$$\mathrm{v} \mapsto \left\langle \sqrt{\mathrm{ad}^2_{\mathrm{v}_{a,x}}}(\mathrm{v}), \mathrm{v} \right\rangle_x . \tag{2.3}$$

Here $\sqrt{\mathrm{ad}^2_{\mathrm{v}_{a,x}}}$ is the only root of the endomorphism $\mathrm{ad}_{\mathrm{v}_{a,x}} \circ \mathrm{ad}_{\mathrm{v}_{a,x}|\mathfrak{p}_x} : \mathfrak{p}_x \to \mathfrak{p}_x$ that is symmetric and semi-positive for the scalar product $\langle \cdot, \cdot \rangle_x$.

This quadratic form is semi-positive, and vanishes exactly on $\mathfrak{z}(\mathrm{v}_{a,x}) \cap \mathfrak{p}_x$. For $\mathrm{v} \in (\mathfrak{z}(\mathrm{v}_{a,x}) \cap \mathfrak{p}_x)^{\perp}$, it satisfies:

$$\mathrm{Hess}_x(\mathrm{v}, \mathrm{v}) \geqslant \|\mathrm{v}\|^2 \min_{\alpha \in \Sigma, \alpha(\boldsymbol{\mu}(a)) \neq 0} |\alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x}))|. \tag{2.4}$$

Recall that $\mathfrak{z}(\mathrm{v})$ for $\mathrm{v} \in \mathfrak{g}$ is the centralizer in $\mathfrak{g}$ of v.

Remark 2.3.10. The Hessian of a Busemann function is related to the sectional curvature of the symmetric space. When measured along a tangent plane spanned by two orthogonal unit vectors $\mathrm{v}, \mathrm{w} \in T_o\mathbb{X} \simeq \mathfrak{p}_o \subset \mathfrak{g}$ the sectional curvature of $\mathbb{X}$ is equal to:

$$\kappa_{\mathrm{v},\mathrm{w}} = -\langle [\mathrm{v}, [\mathrm{v}, \mathrm{w}]], \mathrm{w} \rangle_o = -\langle \mathrm{ad}^2_{\mathrm{v}}(\mathrm{w}), \mathrm{w} \rangle_o.$$

This Lemma implies that Busemann functions are strictly convex except on flats. This is a more general fact about Hadamard manifolds, see [Ebe96].

Proof. The Busemann function with respect to two different base points differ only by a constant. Hence we can assume here without any loss of generality that $x = o$.

Let $\mathrm{v} \in T_o\mathbb{X}$ be a vector. The generalized Iwasawa decomposition, and the fact that the exponential map is a local diffeomorphism on Lie group implies that there exists a neighborhood I of 0 in $\mathbb{R}$ such that for all $t \in I$:

$$\exp(t\mathrm{v}) = \exp(\mathrm{n}_t) \exp(\mathrm{w}_t) \exp(\mathrm{k}_t). \tag{2.5}$$

Here $\mathrm{n}_t \in \mathfrak{n}_{a,o}$, $\mathrm{w}_t \in \mathfrak{a}_{a,x}$ and $\mathrm{k}_t \in \mathfrak{k}_x$, and so that the map $t \mapsto (\mathrm{n}_t, \mathrm{w}_t, \mathrm{k}_t)$ is smooth. Let us denote by $(\dot{\mathrm{n}}, \dot{\mathrm{w}}, \dot{\mathrm{k}})$ and $(\ddot{\mathrm{n}}, \ddot{\mathrm{w}}, \ddot{\mathrm{k}})$ the first and second derivative of this map at $t = 0$.

The limited development at order 2 at $t = 0$ of (2.5) yields:

$$\exp(t\mathrm{v}) = \exp\left(\dot{\mathrm{n}}t + \frac{\ddot{\mathrm{n}}}{2}t^2\right) \exp\left(\dot{\mathrm{w}}t + \frac{\ddot{\mathrm{w}}}{2}t^2\right) \exp\left(\dot{\mathrm{k}}t + \frac{\ddot{\mathrm{k}}}{2}t^2\right) + o(t^2).$$

But the Baker–Campbell–Hausdorff formula [Hel78] implies that the right hand of this equality is equal to the exponential of:

$$\dot{\mathrm{n}}t + \dot{\mathrm{w}}t + \dot{\mathrm{k}}t + \frac{\ddot{\mathrm{n}}}{2}t^2 + \frac{\ddot{\mathrm{w}}}{2}t^2 + \frac{\ddot{\mathrm{k}}}{2}t^2 + \frac{1}{2}\left([\dot{\mathrm{n}}, \dot{\mathrm{w}}] + [\dot{\mathrm{n}}, \dot{\mathrm{k}}] + [\dot{\mathrm{w}}, \dot{\mathrm{k}}]\right)t^2 + o(t^2).$$

Hence we get the following two equalities:

$$\mathrm{v} = \dot{\mathrm{n}} + \dot{\mathrm{w}} + \dot{\mathrm{k}},$$

$$0 = \frac{\ddot{\mathrm{n}}}{2} + \frac{\ddot{\mathrm{w}}}{2} + \frac{\ddot{\mathrm{k}}}{2} + \frac{1}{2}\left([\dot{\mathrm{n}}, \dot{\mathrm{w}}] + [\dot{\mathrm{n}}, \dot{\mathrm{k}}] + [\dot{\mathrm{w}}, \dot{\mathrm{k}}]\right).$$

However since $\mathrm{v}, \dot{\mathrm{w}} \in \mathfrak{p}_x$, then $\theta_x(\dot{\mathrm{n}} + \dot{\mathrm{k}}) = -\dot{\mathrm{n}} - \dot{\mathrm{k}}$. Hence $\dot{\mathrm{k}} = -\frac{\dot{\mathrm{n}} + \theta_x(\dot{\mathrm{n}})}{2}$. This let us simplify the last part of the previous equation :

$$[\dot{\mathrm{n}}, \dot{\mathrm{w}}] + [\dot{\mathrm{n}}, \dot{\mathrm{k}}] + [\dot{\mathrm{w}}, \dot{\mathrm{k}}] = [\dot{\mathrm{n}}, \dot{\mathrm{w}}] - \frac{1}{2}[\dot{\mathrm{n}}, \theta_x(\dot{\mathrm{n}})] - \frac{1}{2}[\dot{\mathrm{w}}, \dot{\mathrm{n}} + \theta_x(\dot{\mathrm{n}})]$$

The metric on $\mathbb{X}$ can be written $\langle \cdot, \cdot \rangle_x = B(\cdot, \theta_x(\cdot))$ on $\mathfrak{p}_x$ with B the Killing form, defined on $\mathfrak{g}$.

Since $\mathrm{v}_{a,x}$ is orthogonal to $\mathfrak{n}_{a,o}$ and $\mathfrak{k}_x$ then $B(\mathrm{v}_{a,x}, \ddot{\mathrm{n}}) = B(\mathrm{v}_{a,x}, \ddot{\mathrm{k}}) = 0$. Moreover $[\dot{\mathrm{n}}, \dot{\mathrm{w}}] \in \mathfrak{n}_{a,o}$ so:

$$B(\mathrm{v}_{a,x}, [\dot{\mathrm{w}}, \dot{\mathrm{n}} + \theta_x(\dot{\mathrm{n}})]) = B(\mathrm{v}_{a,x}, [\dot{\mathrm{n}}, \dot{\mathrm{w}}]) = 0.$$

In particular one gets:

$$\mathrm{Hess}_x(b_{a,o})(\mathrm{v}, \mathrm{v}) = \langle -\mathrm{v}_{a,x}, \ddot{\mathrm{w}} \rangle_x = \frac{1}{2}B\left(-\mathrm{v}_{a,x}, [\dot{\mathrm{n}}, \theta_x(\dot{\mathrm{n}})]\right),$$

$$\mathrm{Hess}_x(b_{a,o})(\mathrm{v}, \mathrm{v}) = \frac{1}{2}B\left([-\mathrm{v}_{a,x}, \dot{\mathrm{n}}], \theta_x(\dot{\mathrm{n}})\right).$$

Let $\Sigma_a \subset \Sigma$ be the set of roots α such that $\alpha(\boldsymbol{\mu}(a)) \neq 0$. The Lie algebra decomposes into root spaces:

$$\mathfrak{g} = \mathfrak{z}(\mathrm{v}_{a,x}) \oplus \bigoplus_{\alpha \in \Sigma_a} \mathfrak{g}^{\alpha}_{a,x},$$

where $\mathrm{Ad}_g(\mathfrak{g}^{\alpha}_{a,x}) = \mathfrak{g}^{\alpha}$ the model root spaces with $g \in G$ any element such that $g \cdot \mathrm{v}_{a,x} = \boldsymbol{\mu}(\mathrm{v}_{a,x})$.

The restriction of $\mathrm{ad}_{\mathrm{v}_{a,x}}$ on $\mathfrak{g}^{\alpha}_{a,x}$ is an homothety of ratio $\alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x}))$. The vector v can be decomposed in this direct sum.

$$\mathrm{v} = \mathrm{v}^0 + \sum_{\alpha \in \Sigma_a} \mathrm{v}^{\alpha}.$$

The endomorphism $\sqrt{\mathrm{ad}^2_{\mathrm{v}_{a,x}}}$ associates to v the vector :

$$\sum_{\alpha \in \Sigma_a} |\alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x}))| \mathrm{v}^{\alpha} \in \mathfrak{p}_x.$$

Let Σ_a^+ be the set of roots α such that $\alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x})) > 0$. The vector $\dot{\mathrm{n}}$ can be expressed as :

$$\dot{\mathrm{n}} = 2 \sum_{\alpha \in \Sigma_a^+} \mathrm{v}^\alpha.$$

Hence, we get as desired:

$$\begin{aligned} \frac{1}{2} B([-\mathrm{v}_{a,x}, \dot{\mathrm{n}}], \theta_x(\dot{\mathrm{n}})) &= -2 \sum_{\alpha \in \Sigma_\alpha^+} \alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x})) B(\mathrm{v}^\alpha, \theta_x(\mathrm{v}^\alpha)) \\ &= \sum_{\alpha \in \Sigma_\alpha} |\alpha(\boldsymbol{\mu}(\mathrm{v}_{a,x}))| \langle \mathrm{v}^\alpha, \mathrm{v}^\alpha \rangle_x = \left\langle \sqrt{\mathrm{ad}^2_{\mathrm{v}_{a,x}}}(\mathrm{v}), \mathrm{v} \right\rangle_x . \end{aligned} \tag{2.6}$$

This is equal to zero if and only if $\mathrm{v} = \mathrm{v}^0$. □

2.4 Nearly geodesic immersions.

In this section we introduce a local condition for an immersion into the symmetric space of non-compact type $\mathbb{X}$ that generalizes the notion an immersion with principal curvature in $(-1, 1)$ inside $\mathbb{H}^n$.

2.4.1 Curvature bound and Busemann functions.

We introduce the key definition of a nearly geodesic immersion, which relies on Busemann functions (see Section 2.3). Let M be a smooth connected manifold, $u : M \to \mathbb{X}$ be an immersion, $o \in \mathbb{X}$ a base point and let $\tau \in \mathbb{S}\mathfrak{a}^+$ be a unit vector in the model Weyl chamber.

Definition 2.4.1. An immersion $u : M \to \mathbb{X}$ is called τ*-nearly geodesic* if for all $a \in \mathcal{F}_\tau \cup \mathcal{F}_{\iota(\tau)}$ and $\mathrm{v} \in TM$ such that $\mathrm{d}(b_{a,o} \circ u)(\mathrm{v}) = 0$, the function $b_{a,o} \circ u$ has positive Hessian in the direction v.

Remark 2.4.2. To a $\tau \in \mathfrak{a}$ we can associate its dual $\omega \in \mathfrak{a}^*$. We say that an imemrsion is ω-nearly geodesic if it is τ-nearly geodesic.

The Hessian considered in this definition is computed with the induced metric $u^* g_{\mathbb{X}}$ on M. Recall that $\mathcal{F}_{\iota(\tau)}$ is the opposite flag manifold to $\mathcal{F}_\tau$ for $\tau \in \mathbb{S}\mathfrak{a}^+$.

We will first show that the nearly geodesic condition can be written as a bound on the fundamental form $\mathrm{I\!I}_u$, depending on the Cartan projection of the surface tangent vectors.

Since the Hessian of a Busemann function $b_{a,o}$ on $\mathbb{X}$ does not depend on o, we will denote it by Hess_{b_a}. Recall that $\mathrm{v}_{a,o}$ is the unit vector in $T_o\mathbb{X}$ pointing towards $a \in \partial_{\mathrm{vis}}\mathbb{X}$. The second fundamental form $\mathrm{I\!I}_u$ for $x \in M$ of the immersion u is the difference $u^*\nabla^{\mathbb{X}} - \nabla^M$ where $\nabla^{\mathbb{X}}$ is the Levi-Civita connection on $T\mathbb{X}$ associated to $g_{\mathbb{X}}$ and ∇^M is the Levi-Civita connection on $TM \subset u^*T\mathbb{X}$

associated to the metric $u^* g_{\mathbb{X}}$. The second fundamental form is a symmetric 2-tensor with values in $u^*\mathrm{N}$ when $\mathrm{N} \subset T\mathbb{X}$ is the normal tangent bundle to $u(M)$.

Proposition 2.4.3. *An immersion $u : M \to \mathbb{X}$ is τ-nearly geodesic if and only if for all $y \in M$, for all $a \in \mathcal{F}_\tau \cup \mathcal{F}_{\iota(\tau)}$ and $\mathrm{v} \in T_y M$ such that $\langle \mathrm{d}u(\mathrm{v}), \mathrm{v}_{a,u(y)} \rangle_{u(y)} = 0$:*

$$\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v})) + \langle \mathrm{II}_u(\mathrm{v}, \mathrm{v}), \mathrm{v}_{a,u(y)} \rangle_{u(y)} > 0. \tag{2.7}$$

We will prove a sufficient condition that has a simpler form in Theorem 2.4.24 when $\tau = \tau_\Theta$ for a Weyl orbit of simple roots Θ.

Proof. Let $y \in M$ and $a \in \mathcal{F}_\tau \cup \mathcal{F}_{\iota(\tau)}$. The function $b_{a,o} \circ u$ is critical at y in the direction $\mathrm{v} \in T_y M$ if and only if $\langle \mathrm{d}u(\mathrm{v}), \mathrm{v}_{a,u(y)} \rangle_{u(y)} = 0$.

Let $\gamma : \mathbb{R} \to M$ be a geodesic for the metric $u^* g_{\mathbb{X}}$ on M such that $\gamma(0) = y$ and $\gamma'(0) = \mathrm{v}$. The Hessian of $b_{a,o} \circ u$ on M is equal to the derivative at $t = 0$ of the differential of the Busemann function, i.e. :

$$t \mapsto \langle \mathrm{v}_{a,u(\gamma(t))}, \mathrm{d}u(\gamma'(t)) \rangle_{u(y)}$$

The first term, $\langle \nabla^{\mathbb{X}}_{\mathrm{d}u(\mathrm{v})} \mathrm{v}_{a,u(\gamma(t))}, \mathrm{d}u(\gamma'(t)) \rangle_{u(y)}$, is equal to $\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v}))$. The second term can be written:

$$\nabla^{\mathbb{X}}_{\mathrm{d}u(\mathrm{v})} \mathrm{d}u(\gamma') = u_* \nabla^M_{\mathrm{v}} \mathrm{d}u(\gamma') + \mathrm{II}_u(\mathrm{v}, \mathrm{v})$$

But γ is a geodesic so $\nabla^M_{\mathrm{v}} \mathrm{d}u(\gamma') = 0$, therefore the Hessian of $b_{a,o} \circ u$ on M in the direction v is equal to :

$$\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v})) + \langle \mathrm{II}_u(\mathrm{v}, \mathrm{v}), \mathrm{v}_{a,u(y)} \rangle_{u(y)} > 0.$$

□

A consequence of Proposition 2.4.3 is that the property of being τ-nearly geodesic is locally an open property for the $\mathcal{C}^2$-topology, which is the topology associated with the uniform convergence over any compact set of the first two differentials.

Corollary 2.4.4. *Let $u_0 : M \to \mathbb{X}$ be a τ-nearly geodesic map for some $\tau \in \mathbb{S}\mathfrak{a}^+$. For all compact $K \subset M$, there exists a neighborhood U of u_0 for the $\mathcal{C}^2$-topology in the space of $\mathcal{C}^2$ maps from M to $\mathbb{X}$ and a neighborhood V of τ in $\mathbb{S}\mathfrak{a}^+$ such that for all $\tau' \in V$ and $u \in U$, u satisfies the τ'-nearly geodesic immersion condition on K.*

Let G be the isometry group of the n-dimensional hyperbolic space $\mathbb{H}^n$ for some $n \in \mathbb{N}$ with its usual metric with sectional curvature equal to -1. We prove that the notion of τ-nearly geodesic immersion generalizes the notion of immersion with principal curvatures in $(-1, 1)$ in $\mathbb{H}^n$. Principal curvatures are only defined for hypersurfaces, but the following definition allows to generalize the notion of having bounded principal curvature.

Definition 2.4.5. An immersion $u : M \to \mathbb{H}^n$ has *principal curvature in* $(-1,1)$ if and only if for all $\mathrm{v} \in TM$, $\|\mathrm{d}u(\mathrm{v})\|^2 > \|\mathrm{II}_u(\mathrm{v},\mathrm{v})\|$.

Since $\mathbb{H}^n$ is a rank one symmetric space, $\mathbb{S}\mathfrak{a}^+$ contains a single element.

Proposition 2.4.6. *An immersion* $u : M \to \mathbb{H}^n$ *is nearly geodesic for the only element* $\tau \in \mathbb{S}\mathfrak{a}^+$ *if and only u has principal curvature in* $(-1,1)$.

Proof. Let $x \in \mathbb{X} = \mathbb{H}^n$ and $a \in \mathcal{F}_\tau = \mathcal{F}_{\iota(\tau)} = \mathbb{CP}^1 = \partial\mathbb{H}^n$. For any $\mathrm{w} \in T_x\mathbb{X}$, $\mathrm{Hess}_{b_a}(\mathrm{w},\mathrm{w}) = \lambda\|\mathrm{w}^\perp\|^2$ where $\mathrm{w}^\perp$ is the orthogonal projection of w onto the orthogonal in $T_x\mathbb{X}$ of $\mathrm{v}_{a,x}$ by Proposition 2.3.9, with some constant λ which is equal to 1 for the metric of sectional curvature equal to -1 on $\mathbb{H}^n$ (see Remark 2.3.10).

If u is τ-nearly geodesic, then it is an immersion and for every $y \in M$ and $\mathrm{v} \in T_yM$ there exist $a \in \partial\mathbb{H}^n$ such that $\mathrm{v}_{a,u(y)}$ is positively collinear with $-\mathrm{II}_u\left(\mathrm{d}u(\mathrm{v}),\mathrm{d}u(\mathrm{v})\right)$. By Proposition 2.4.3, and since $\mathrm{v} \perp \mathrm{v}_{a,u(y)}$ one has:

$$\|\mathrm{d}u(\mathrm{v})\|^2 - \|\mathrm{II}_u(\mathrm{v},\mathrm{v})\| > 0.$$

Therefore the principal curvature of u is in $(-1,1)$.

Conversely if u is an immersion with principal curvatures in $(-1,1)$, let $a \in \partial\mathbb{H}^n$, $y \in M$ and $\mathrm{v} \in T_yM$ be such that $b_{a,o} \circ u$ is critical in the direction v. Hence $\mathrm{v}_{a,u(y)}$ is perpendicular to $\mathrm{d}u(\mathrm{v})$ so $\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v},\mathrm{d}u(\mathrm{v})) = \|\mathrm{d}u(\mathrm{v})\|^2$. Therefore the fact that u has principal curvature in $(-1,1)$ implies that the hypothesis of Proposition 2.4.3 hold, so u is τ-nearly geodesic. ☐

In general, the property of being τ-nearly geodesic implies that the surface is regular in the following sense.

Definition 2.4.7. A tangent vector $\mathrm{v} \in T\mathbb{X}$ is called *τ-regular* if its Cartan projection $\boldsymbol{\mu}(\mathrm{v})$ does not belong to $\bigcup_{w\in W}(w\cdot\tau)^\perp$.

We say that an immersion $u : M \to \mathbb{X}$ is τ regular if for all $\mathrm{v} \in TM$, $\mathrm{d}u(\mathrm{v})$ is τ-regular.

Being regular, namely having the Cartan projection in the interior of $\mathfrak{a}^+$, and being τ-regular is in general unrelated. However when $\tau = \tau_\Theta$ for a Weyl orbit of simple roots Θ, a τ-regular vector $\mathrm{v} \in T\mathbb{X}$ is exactly a Θ-regular vector, namely such that for all $\alpha \in \Theta$, $\alpha\left(\boldsymbol{\mu}(\mathrm{v})\right) \neq 0$.

Proposition 2.4.8. *Let* $\tau \in \mathbb{S}\mathfrak{a}^+$. *If u is a τ-nearly geodesic immersion, the tangent vectors* $\mathrm{d}u(\mathrm{v})$ *for* $\mathrm{v} \in TM$ *are τ-regular.*

Proof. Let $\mathrm{v} \in T_yM$ for some $y \in M$. Assume that $\mathrm{d}u(\mathrm{v})$ is not τ-regular, so its Cartan projection is orthogonal to $w\cdot\tau$ for some $w \in W$. Therefore there is a unit vector which lies in a common maximal flat with $\mathrm{d}u(\mathrm{v})$, and whose Cartan projection is equal to τ. This vector is equal to $\mathrm{v}_{a,u(y)}$ for some $a \in \mathcal{F}_\tau \cup \mathcal{F}_{\iota(\tau)}$.

Since $\mathrm{v}_{a,u(y)}$ and $\mathrm{d}u(\mathrm{v})$ are in a common flat, $\mathrm{Hess}_{b_a}\left(\mathrm{d}u(\mathrm{v}),\mathrm{d}u(\mathrm{v})\right) = 0$. One can assume that $\langle\mathrm{II}_u(\mathrm{v},\mathrm{v}),\mathrm{v}_{a,u(y)}\rangle_{u(y)} \leqslant 0$ up to exchanging a with its symmetric with respect to $u(y)$ which is still in $\mathcal{F}_\tau \cup \mathcal{F}_{\iota(\tau)}$. Moreover since $\langle \mathrm{v}_{a,u(y)}, \mathrm{d}u(\mathrm{v})\rangle_{u(y)} = 0$, this is a contradiction with the criterion from Proposition 2.4.3, so the immersion u cannot be τ-nearly geodesic. $\square$

The property of being τ-nearly geodesic is not necessarily satisfied for totally geodesic immersions, but it is satisfied for τ-regular totally geodesic immersions.

Proposition 2.4.9. *A totally geodesic immersion is τ-nearly geodesic if and only if it is τ-regular.*

Proof. An immersion u is totally geodesic if and only if $\mathrm{II}_u = 0$. If u is a τ-nearly geodesic immersion that is totally geodesic, for every $y \in M$, $\mathrm{v} \in T_yM$ and every $a \in \mathcal{F}_\tau$, $y \in M$ and $\mathrm{v} \in T_yM$ Proposition 2.4.3 implies that :

$$\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v})) > 0.$$

The implies that for no $a \in \mathcal{F}_\tau$ the vector $\mathrm{v}_{a,u(y)}$ lies in a common flat with $\mathrm{d}u(\mathrm{v})$ by Lemma 2.3.9. Hence the Cartan projection of $\mathrm{d}u(\mathrm{v})$ is not orthogonal to $w \in \tau$ for any $w \in W$.

Conversely if the totally geodesic immersion is τ-regular, the following Hessian $\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v}))$ is never equal to 0 for any $y \in M$, $\mathrm{v} \in T_yM$ and $a \in \mathcal{F}_\tau$ such that $\mathrm{v}_{a,u(y)}$ is orthogonal to v. Since Hess_{b_a} is non-negative, Proposition 2.4.3 implies that u is τ-nearly geodesic. $\square$

Proof. Consider $y_0 \in M$. The function $y \in M \mapsto \exp\left(\lambda d^\tau_{\mathbb{X}}(u(y), u(y_0))\right)$ is strictly convex for some $\lambda > 0$ and admits a minimum at $y = y_0$. The completeness of the metric $u^*(g_{\mathbb{X}})$ implies that there is a geodesic joining any two points. Hence the minimum of any strictly convex function is unique, so u is injective: it is an embedding. $\square$

2.4.2 Uniformly nearly geodesic immersions.

If the nearly geodesic condition for an immersion is satisfied uniformly, one can prove that the exponential of some multiple of Busemann functions are strictly convex on the image of the immersion.

Definition 2.4.10. Let $\tau \in \mathbb{S}\mathfrak{a}^+$. An immersion $u : M \to \mathbb{X}$ is *uniformly* τ-nearly geodesic if there exist $\epsilon > 0$ such that for all $\mathrm{v} \in TM$ such that $\|\mathrm{d}u(\mathrm{v})\| = 1$ one has for all $a \in \mathcal{F}_\tau$ satisfying $\mathrm{v}_{a,o} \perp \mathrm{v}$:

$$\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}), \mathrm{d}u(\mathrm{v})) + \langle \mathrm{II}_u(\mathrm{v}, \mathrm{v}), \mathrm{v}_{a,o}\rangle_o \geqslant \epsilon.$$

Remark 2.4.11. When $\mathbb{X} = \mathbb{H}^n$ being uniformly nearly geodesic is equivalent to having principal curvature in $(-\lambda, \lambda)$ for some $\lambda < 1$.

Suppose that $M = \widetilde{N}$ is the universal cover of a compact smooth manifold N. Let Γ be the fundamental group of N. A ρ-equivariant immersion $u : M \to \mathbb{X}$ for some representation $\rho : \Gamma \to G$ which is τ-nearly geodesic is necessarily uniformly τ-nearly geodesic since T^1N is compact.

If we consider a uniformly τ-nearly geodesic immersion u, not only are Busemann functions convex in critical directions, but for some $\lambda > 0$, $e^{\lambda b_{a,o} \circ u}$ is strictly convex on M.

Lemma 2.4.12. *Let $\tau \in \mathbb{S}\mathfrak{a}^+$. Let $u : M \to \mathbb{X}$ be a uniformly τ-nearly geodesic immersion. For some $\lambda > 0$, for all $a \in \mathcal{F}_\tau$ the function $\exp(\lambda b_{a,o} \circ u)$ has positive Hessian for the metric $u^*(g_{\mathbb{X}})$. Moreover there exists some $\epsilon > 0$ such that for any $a \in \mathcal{F}_\tau$ and any geodesic $\eta : \mathbb{R} \to M$ the functions $f_\eta = \exp(\lambda b_{a,o} \circ u \circ \eta)$ satisfy $f'' \geqslant \epsilon f$.*

Recall that the metric on M that we consider to define geodesics is the induced metric $u^*(g_{\mathbb{X}})$.

Proof. Let $o \in \mathbb{X}$ and let U_τ^ϵ be the compact set of pairs $(\mathrm{v}, \mathrm{II}) \in T^1\mathbb{X}_o \times T_o\mathbb{X}$ such that for all $a \in \mathcal{F}_\tau$ satisfying $\mathrm{v}_{a,o} \perp \mathrm{v}$:

$$\mathrm{Hess}_{b_a}(\mathrm{v}, \mathrm{v}) + \langle \mathrm{II}, \mathrm{v}_{a,o} \rangle_o \geqslant \epsilon.$$

Let us consider :

$$C = \inf_{a \in \mathcal{F}_\tau, (\mathrm{v}, \mathrm{II}) \in U_\tau^\epsilon} \frac{\mathrm{Hess}_{b_a}(\mathrm{v}, \mathrm{v}) + \langle \mathrm{II}, \mathrm{v}_{a,x} \rangle_o}{\langle \mathrm{v}_{a,o}, \mathrm{v} \rangle_o^2}.$$

This infimum is the infimum of a continuous function taking values in $\mathbb{R} \cup \{+\infty\}$ on a compact set. Indeed the numerator must be strictly positive whenever the denominator vanishes, and the denominator is always positive. Hence $C \in \mathbb{R} \cup \{+\infty\}$.

Let λ be any real number greater than $\max(1 - C, 0)$. Let η be any geodesic in M. Let us write $g = b_{a,o} \circ u \circ \eta$. Note that $g'' \geqslant C(g')^2$ by definition of C. Therefore:

$$\left(e^{\lambda g}\right)'' / e^{\lambda g} = \lambda g'' + \lambda^2 \left(g'\right)^2 \geqslant \left(C\lambda + \lambda^2\right) \left(g'\right)^2 + (\lambda - C) g'' \geqslant \lambda \left(g'\right)^2 \geqslant 0.$$

Note also that $\left(e^{\lambda g}\right)'' / e^{\lambda g} \geqslant \lambda g''$. Consider the following quantity:

$$M = \inf_{a \in \mathcal{F}_\tau, (\mathrm{v}, \mathrm{II}) \in U_\tau^\epsilon} \max\left(\mathrm{Hess}_{b_a}(\mathrm{v}, \mathrm{v}) + \langle \mathrm{II}, \mathrm{v}_{a,x} \rangle_o, \langle \mathrm{v}_{a,o}, \mathrm{v} \rangle_o^2\right).$$

Note that $M \leqslant \max\left(g'', (g')^2\right)$. Since $K \subset U_\tau$, this quantity is strictly positive as it is an infimum taken on a compact set of a positive function. Hence the function $f = e^{\lambda g}$ is strictly convex and satisfies $f'' > \lambda M f$. □

2.4.3 Convexity of a Finsler distance.

When $\mathbb{X} = \mathbb{H}^n$, and given $y \in \mathbb{H}^n$, for any nearly geodesic immersion $u : M \to \mathbb{H}^n$ the function $x \mapsto \exp(d_{\mathbb{H}^n}(u(x), y))$ is strictly convex. However for a general symmetric space of higher rank the τ-nearly geodesic condition doesn't imply the convexity for the Riemannian metric at critical points.

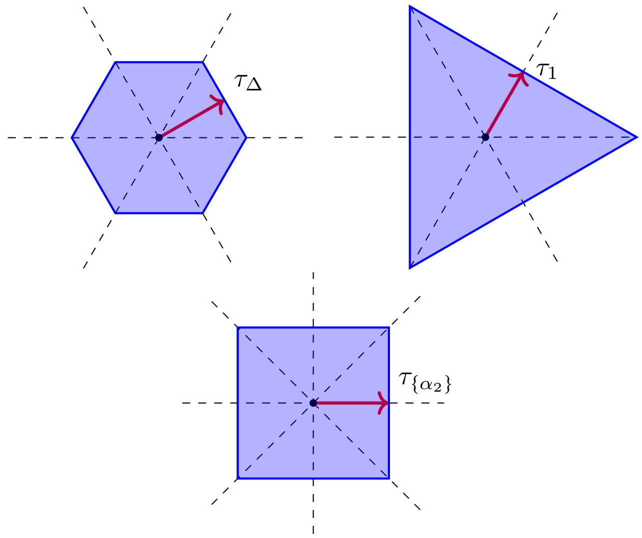


Figure 2.5: The unit ball in $\mathfrak{a}$ of $|\cdot|_{\tau_\Delta}$ and $|\cdot|_{\tau_1}$ for $G = \mathrm{SL}(3,\mathbb{R})$ and $|\cdot|_{\tau_{\{\alpha_2\}}}$ for $G = \mathrm{Sp}(4,\mathbb{R})$.

This leads us to consider a Finsler pseudo distance $d_{\mathbb{X}}^{\tau}$ on $\mathbb{X}$ associated to an element $\tau \in \mathbb{S}\mathfrak{a}^+$. We show in this section that this pseudo distance satisfies a similar convexity property for any τ-nearly geodesic immersion. This pseudo-distance is symmetric when τ is symmetric and it is equal to the Riemannian distance when $\mathrm{rank}(\mathbb{X}) = 1$. The convexity of this distance allows us to prove the injectivity and properness of complete τ-nearly geodesic immersions. This Finsler pseudo distance is studied in [KL18, Section 5].

Let us define for $\tau_0 \in \mathfrak{a}$:

$$|\tau_0|_\tau = \max_{w \in W} \langle w \cdot \tau, \tau_0 \rangle.$$

The map $\tau_0 \mapsto |\tau_0|_\tau$ is non-negative, homogeneous and subadditive, thus we call it in general a pseudo-norm.

This pseudo-norm is not necessarily symmetric: $|\tau_0|_\tau = |-\tau_0|_{\iota(\tau)}$. In particular it is symmetric if and only if τ is symmetric. Figure 2.5 illustrates the unit ball of this norm in $\mathfrak{a}$ for two examples of semi-simple Lie groups whose associated symmetric space has rank 2: on the left $G = \mathrm{SL}(3,\mathbb{R})$ and $\tau = \tau_\Delta$, in the middle $G = \mathrm{SL}(3,\mathbb{R})$ and $\tau = \tau_1$ (such that $\mathcal{F}_{\tau_1} \simeq \mathbb{RP}^2$) and on the right $G = \mathrm{Sp}(4,\mathbb{R})$ and $\tau = \tau_{\{\alpha_n\}}$ with the notations from Section 2.1.5.

This pseudo-norm isn't necessarily positive on non-zero vectors. However if a non-zero vector $\mathrm{v} \in \mathfrak{a}$ has zero norm, the Weyl group does not act irreducibly on $\mathfrak{a}$ since the W-orbit of τ is orthogonal to v, which means that the underlying Lie group G is not simple.

Example 2.4.13. Let $n \geqslant 2$ be an integer and $G = \mathrm{PSL}(2,\mathbb{R})^n$, $\mathbb{X} = \left(\mathbb{H}^2\right)^n$. A model flat in $\mathbb{X}$ is the product of a geodesic in each of the n copies of $\mathbb{H}^2$. Let τ_k be the tangent vector to the geodesic on the k-th copy of $\mathbb{H}^2$, the pseudo-distance on $\mathbb{X}$ defined by the pseudo-norm $|\cdot|_{\tau_k}$ is the distance in $\mathbb{H}^2$ of the k-th components, which is not a distance on $\mathbb{X}$.

However if G is simple and τ is symmetric $|\cdot|_\tau$ is a norm. This norm is W-invariant and hence it defines a G-invariant not necessarily symmetric Finsler metric on $\mathbb{X}$ such that for $\mathrm{v} \in T\mathbb{X}$, $\|\mathrm{v}\|_\tau = |\,\boldsymbol{\mu}(\mathrm{v})|_\tau$, where $\boldsymbol{\mu}$ is the Cartan projection ([Pla95] Theorem 6.2.1).

For any semi-simple Lie group G, we denote by $d^\tau_{\mathbb{X}} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geqslant 0}$ the corresponding pseudo-distance on $\mathbb{X}$, i.e. $d_{\mathbb{X}}(x,y)$ is the infimum for all piece-wise $\mathcal{C}^1$-path η from x to y of:

$$\int \|\eta'\|_\tau = \int |\,\boldsymbol{\mu}(\eta')|_\tau.$$

This distance can be characterized in terms of Busemann functions.

Proposition 2.4.14. *Let $x, y \in X$ be two points. The pseudo distance $d^\tau_{\mathbb{X}}$ between these two points satisfies:*

$$d^\tau_{\mathbb{X}}(x,y) = \max_{a \in \mathcal{F}_\tau} b_{a,x}(y).$$

Proof. Let $o \in \mathbb{X}$, $\mathrm{v} \in T_o\mathbb{X}$ and $a \in \mathcal{F}_\tau$. As usual $\mathrm{v}_{a,o}$ is the unit vector based at o pointing towards a. The maximum for $a \in \mathcal{F}_\tau$ of $\langle \mathrm{v}, \mathrm{v}_{a,o}\rangle$ is reached when v and $\mathrm{v}_{a,o}$ are in a common flat ([Ebe96] Proposition 24).

If we assume that v and $\mathrm{v}_{a,o}$ are in a common flat, the maximum is equal to $|\,\boldsymbol{\mu}(\mathrm{v})|_\tau$. Given two points $x, y \in \mathbb{X}$, any piece-wise $\mathcal{C}^1$ curve η such that $\eta(0) = x, \eta(1) = y$ satisfies for all $a \in \mathcal{F}_\tau$:

$$(b_a \circ \eta)' = \langle \mathrm{v}_{a,\eta(t)}, \eta'(t)\rangle_{\eta(t)} \leqslant \|\eta'\|_\tau.$$

Hence:

$$b_{a,x}(y) \leqslant \int \|\eta'\|_\tau.$$

Moreover equality is reached for the Riemannian geodesic such that $\eta(0) = x, \eta(1) = y$. Indeed there is a point $a \in \mathcal{F}_\tau$ that lies in a common flat with x and y such that $|\eta'(t)|_\tau = \langle \eta'(t), \mathrm{v}_{a,\eta(t)}\rangle_{\eta(t)}$ for all $t \in [0,1]$. Hence $b_{a,x}(y) = d^\tau_{\mathbb{X}}(x,y)$. Note that the curve reaching this minimum is not unique in general. □

This pseudo distance satisfies the desired convexity condition.

Proposition 2.4.15. *Let $u : M \to \mathbb{X}$ be a uniformly τ-nearly geodesic immersion. There exist $\lambda > 0$ such that for all $x \in \mathbb{X}$ the following function is strictly convex for the metric $u^* g_{\mathbb{X}}$:*

$$f : y \in M \mapsto \exp\left(\lambda d^\tau_{\mathbb{X}}(x, u(y))\right).$$

A continuous function is strictly convex on the manifold M for the metric $u^*(g_{\mathbb{X}})$ if it is strictly convex on any geodesic.

Proof. By the Lemma 2.4.12 there exist $\lambda > 0$ such that for any $a \in \mathcal{F}_\tau$, the function $\exp(\lambda b_{a,o} \circ u)$ is strictly convex on M. One can then write f as :

$$f(y) = \exp\left(\lambda d_{\mathbb{X}}^\tau(x, u(y))\right) = \sup_{a \in \mathcal{F}_\tau} \exp\left(b_{a,x} \circ u(y)\right).$$

Hence f is the supremum of a family of convex functions, so it is convex. Moreover the supremum is taken over a compact family of strictly convex functions, so it is strictly convex.

□

A consequence of the convexity of this Finsler distance is that the immersion u is injective, which is an interesting property of τ-nearly geodesic surface. We say that u is *complete* if M is complete for the induced metric $u^*(g_{\mathbb{X}})$.

Proposition 2.4.16. *Let $u : M \to \mathbb{X}$ be a complete uniformly τ-nearly geodesic immersion. Then u is an embedding.*

Proof. Consider $y_0 \in M$. The function $y \in M \mapsto \exp\left(\lambda d_{\mathbb{X}}^\tau(u(y), u(y_0))\right)$ is strictly convex for some $\lambda > 0$ and admits a minimum at $y = y_0$. The completeness of the metric $u^*(g_{\mathbb{X}})$ implies that there is a geodesic joining any two points. Hence the minimum of any strictly convex function is unique, so u is injective: it is an embedding. □

Moreover the immersion u cannot be too distorded: the metric induced by u is quasi-isometric to the ambient metric on $\mathbb{X}$. The notion of quasi-isometric embedding was recalled in Section 2.2.

Proposition 2.4.17. *Let $u : M \to \mathbb{X}$ be a complete uniformly τ-nearly geodesic immersion. Then u is a quasi-isometric embedding for the induced metric $u^* g_{\mathbb{X}}$ on M. In particular u is proper.*

Proof. Let $y_0 \in M$ and let $o = u(y_0)$. Let $\epsilon > 0$ and $\lambda > 0$ be the constants provided by Lemma 2.4.12. Let $\gamma : \mathbb{R}_{\geqslant 0} \to M$ be a geodesic ray parametrized with unit speed in M for the metric $u^* g_{\mathbb{X}}$ with $\gamma(0) = y_0$.

Let $a \in \mathcal{F}_\tau$ be such that $b_{a,o}\left(u \circ \gamma(1)\right) = d_{\mathbb{X}}^\tau\left(o, u \circ \gamma(1)\right)$. Consider the function:

$$f : t \in \mathbb{R}_{\geqslant 0} \mapsto \exp\left(\lambda b_{a, u\circ\gamma(t)}\left(u \circ \gamma(t)\right)\right).$$

It is strictly convex and satisfies $f(1) \geqslant f(0)$ so $f'(1) \geqslant 0$. Moreover $f'' > \epsilon f$ so $f(t) \geqslant \cosh\left(\epsilon(t-1)\right) > \frac{e^{\epsilon(t-1)}}{2}$. In particular:

$$d_{\mathbb{X}}^\tau\left(o, u \circ \gamma(t)\right) \geqslant b_{a,o}\left(u \circ \gamma(t)\right) \geqslant \frac{\epsilon}{\lambda}(t-1) - \frac{\log(2)}{\lambda}$$

For all $y \in M$ there exist a geodesic ray γ passing through y. If d_u is the Riemannian distance on M induced by $u^* g_{\mathbb{X}}$:

$$d^\tau_{\mathbb{X}}\left(u(x_0), u(x)\right) \geqslant \frac{\epsilon}{\lambda}\left(d_u\left(x, y\right) - 1\right)\frac{\log(2)}{\lambda}.$$

This Finsler metric is equivalent to the Riemannian metric $g_{\mathbb{X}}$ if G is simple, and in general it is dominated by the Riemannian metric. Moreover u is 1-Lipshitz with respect to the induced metric, so u is a quasi-isometric embedding. □

Using the convexity of this Finsler pseudo-distance one can define a continuous projection from the whole symmetric $\mathbb{X}$ to M. This projection will not be used in what follows, but the fibration of the domains in $\mathcal{F}_\tau$ constructed in Section 2.6 is an extension of it.

Proposition 2.4.18. *Let $u : M \to \mathbb{X}$ be a complete uniformly τ-nearly geodesic immersion. For every $x \in \mathbb{X}$, there exist a unique point $\pi^\tau_u(x) \in M$ that minimizes:*

$$y \in M \mapsto d^\tau_{\mathbb{X}}(x, u(y)).$$

The function $\pi^\tau_u : \mathbb{X} \to M$ is continuous, and $\pi^\tau_u(u(y)) = y$ for $y \in M$.

Proof. Let λ, ϵ be the two constants provided by the Lemma 2.4.12 and let $x \in \mathbb{X}$. The following function is strictly convex on M:

$$y \mapsto \exp\left(\lambda d^\tau_{\mathbb{X}}(x, u(y))\right).$$

It is moreover proper since u is proper by Proposition 2.4.17. Hence it has a unique minimum, so π^τ_u is well defined.

If we consider a sequence $(x_n) \in \mathbb{X}$ of points that converge to $x \in \mathbb{X}$, then the sequence $(\pi^\tau_u(x_n))$ is bounded since ρ is discrete. Moreover any of its limit points is a minimum of $d^\tau_{\mathbb{X}}(x, u(y))$, so the sequence converges to $\pi^\tau_u(x)$. The function π^τ_u is hence continuous. □

2.4.4 Anosov property for nearly Fuchsian representations.

Let N be a compact manifold with fundamental group Γ. We call a representation $\rho : \Gamma \to G$ that admits a τ-nearly geodesic equivariant immersion $u : \widetilde{N} \to \mathbb{X}$ a *τ-nearly Fuchsian* representation.

Proposition 2.4.19. *The set of τ-nearly Fuchsian representations is open in the space of representations $\rho : \Gamma \to G$, for the compact-open topology.*

Proof. One can continuously deform any ρ-equivariant immersion $u : \widetilde{N} \to \mathbb{X}$ to a ρ'-equivariant smooth map $u' : \widetilde{N} \to \mathbb{X}$ for ρ' close to ρ. Indeed fix a Riemannian metric on N let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function that is positive on $[0, R]$ for R large enough and vanishes on $[R', +\infty)$ for some $R' > R$. One can define $u'(y)$ for $y \in \widetilde{N}$ as the barycenter of the points $x^y_\gamma = \rho'(\gamma^{-1}) \cdot u(\gamma \cdot y)$

with weight $\lambda_\gamma^y = \eta(d(y, \gamma \cdot y))$ for $\gamma \in \Gamma$. Concretely this means that we consider the unique local minimum of the convex function :

$$D : x \in \mathbb{X} \mapsto \sum_{\gamma \in \Gamma} \lambda_\gamma^y d(x, x_\gamma^y)^2.$$

Note that $\rho'(\gamma_0) \cdot x_\gamma^y = x_{\gamma\gamma_0^{-1}}^{\gamma_0 \cdot y}$ and $\lambda_\gamma^y = \lambda_{\gamma\gamma_0^{-1}}^{\gamma_0 \cdot y}$ for $\gamma_0 \in \Gamma$. Therefore u' is ρ'-equivariant. Since $\mathbb{X}$ is a Hadamard manifold D is strictly convex so the barycenter map is well-defined and smooth. Therefore for ρ' close enough to ρ, u' is an immersion which is close to u for the $\mathcal{C}^2$-topology on any compact fundamental domain of the action of Γ on $\widetilde{N}$. In particular u' is a τ-nearly geodesic immersion for ρ close enough to ρ'. □

The condition that u is τ-nearly geodesic is local, but it will imply some coarse property on u and therefore on ρ. Recall that the limit cone $\mathcal{C}_\rho$ was defined in Section 2.2 (Definition 2.2.1). Due to flats Busemann functions are not strictly convex in critical directions on $\mathbb{X}$. However Busemann functions are strictly convex in critical directions on $u(\widetilde{N})$. We deduce that τ-nearly geodesic surfaces must coarsely avoid these flats, which in turn can be interpreted as a property of the limit cone $\mathcal{C}_\rho$ (see Definition 2.2.1).

Proposition 2.4.20. *Let $\rho : \Gamma \to G$ be a τ-nearly Fuchsian representation :*

$$\mathcal{C}_\rho \cap \bigcup_{w \in W} (w \cdot \tau)^\perp = \varnothing. \tag{2.8}$$

In other words ρ is ω-undistorded for the linear form $\omega \in \mathfrak{a}^*$ associated to $\tau \in \mathfrak{a}$, see Definition 1.3.1.

Recall that W is the Weyl group associated to G.

Proof. Let $x_0, x \in \widetilde{N}$ and $o = u(x_0)$. Let $w \in W$, there exist two points $a \in \mathcal{F}_\tau$ and $a' \in \mathcal{F}_{\iota(\tau)}$ that are opposite from o, i.e. $\mathrm{v}_{a,o} = -\mathrm{v}_{a',o}$, and such that :

$$b_{a,o}(u(x)) = \langle \mathbf{d}_\mathfrak{a}(o, u(x)), w \cdot \tau \rangle,$$

$$b_{a',o}(u(x)) = \langle \mathbf{d}_\mathfrak{a}(o, u(x)), -w \cdot \tau \rangle.$$

This holds for a, a' that lie in a maximal flat containing o and $u(x)$.

Let η be a geodesic parametrized with unit length in $\widetilde{N}$ for $u^*(g_\mathbb{X})$ such that $\gamma(0) = x_0$ and $\gamma(d_u(x_0, x)) = x$. Let $\lambda, \epsilon > 0$ be the constants given by Lemma 2.4.12. Consider the function:

$$f : t \mapsto \exp\left(\lambda b_{a,o} \circ u \circ \gamma(t)\right).$$

Since $\mathrm{v}_{a,o} = -\mathrm{v}_{a',o}$, up to exchanging a and a' we can assume that $f'(0) \geqslant 0$. By Lemma 2.4.12, one has $f'' > \epsilon f$. Together with the fact that $f(0) = 1$, this implies that for all $t \in [0, d_u(x_0, x)]$:

$$f(t) \geqslant \cosh(\epsilon t).$$

Hence $\langle \mathbf{d}_{\mathfrak{a}}(o,u(x)), w\cdot p\rangle \geqslant \frac{\epsilon}{\lambda} d_u(x_0,x) - \frac{\log(2)}{\lambda}$. Since u is a quasi-isometric embedding, there exist $c, D > 0$ such that for all $x \in \widetilde{N}$ the distance for the induced metric $u^*(g_{\mathbb{X}})$ between x and x_0 is at least:

$$cd_{\mathbb{X}}(o,u(x)) - D.$$

In conclusion:

$$\langle \frac{\mathbf{d}_{\mathfrak{a}}(o,u(x))}{d_{\mathbb{X}}(o,u(x))}, w\cdot p\rangle \geqslant \frac{c\epsilon}{\lambda} - \frac{\log(2)+\epsilon D}{\lambda d_{\mathbb{X}}(o,u(x))}.$$

Any element of $\mathcal{C}_\rho$ has therefore a scalar product at least $\frac{c\epsilon}{\lambda} > 0$ with $w\cdot\tau$, for any $w \in W$. This implies that the limit cone cannot intersect $\bigcup_{w\in W}(w\cdot\tau)^\perp$. $\square$

The set $\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W}(w\cdot\tau)^\perp$ contains a single connected component if and only if $(w\cdot\tau)^\perp$ is always a wall of the Weyl chamber decomposition of $\mathfrak{a}$, i.e. when $\tau = \tau_\Theta$ for a Weyl orbit of simple roots $\Theta \subset \Delta$ (this notion was defined Section 2.1.6). In this case:

$$\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W}(w\cdot\tau_\Theta)^\perp = \mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{\alpha\in\Theta}\mathrm{Ker}(\alpha).$$

Hence we get the following.

Theorem 2.4.21. *Let $\Theta \subset \Delta$ be a Weyl orbit of simple roots. A τ_Θ-nearly Fuchsian representation $\rho : \Gamma_g \to G$ is Θ-Anosov.*

If $\tau \in \mathbb{S}\mathfrak{a}^+$ does not correspond to a Weyl orbit of simple roots, let us assume that Γ is a finitely generated group that is not virtually cyclic, so that the limit cone of Γ is connected, (see Proposition 3.8.2).

To a τ-nearly Fuchsian representation $\rho : \Gamma \to G$ one can associate the connected component σ_ρ^τ in which $\mathcal{C}_\rho$ lies inside:

$$\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W}(w\cdot\tau)^\perp.$$

To a connected component of this space one can associate a non-empty set $\Theta(\sigma_\rho^\tau)$ of simple roots. Recall that for $\tau_0 \in \mathbb{S}\mathfrak{a}^+$, $\Theta(\tau_0) \subset \Delta$ is the set of simple roots α such that $\alpha(\tau_0) \neq 0$.

Lemma 2.4.22. *Let $\tau \in \mathbb{S}\mathfrak{a}^+$ and let $\sigma \subset \mathbb{S}\mathfrak{a}^+$ be a connected component of :*

$$\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W}(w\cdot\tau)^\perp. \tag{2.9}$$

Let $\Theta(\sigma) \subset \Delta$ be the set of simple roots α such that $\sigma \cap \mathrm{Ker}(\alpha) = \varnothing$. This set is non-empty, and there exist some $\tau_0 \in \sigma$ such that $\Theta(\tau_0) = \Theta(\sigma)$.

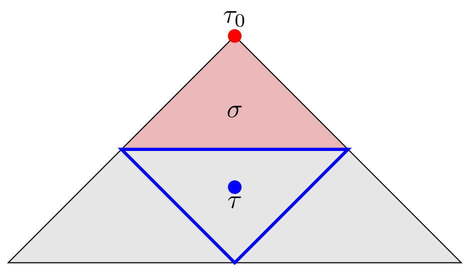


Figure 2.6: Illustration for $G = \mathrm{PSL}(4,\mathbb{R})$ of a connected component σ of $\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W}(w\cdot\tau)^\perp$ in an affine chart.

In other words there is $\tau_0 \in \sigma$ such that for any simple root $\alpha \in \Delta$, $\alpha(\tau_0) \neq 0$ if and only if for all $\tau_0{}' \in \sigma$, $\alpha(\tau_0{}') \neq 0$. Figure 2.6 illustrates the lines in $\mathbb{S}\mathfrak{a}^+$ corresponding to $\bigcup_{w\in W}(w\cdot\tau)^\perp$ for some $\tau \in \mathbb{S}\mathfrak{a}^+$, as well as some connected component of the complement σ. In this example $\Theta(\sigma)$ contains only one root.

Proof. Let $W_0 \subset W$ be the subgroup of the Weyl group generated by symmetries associated to $\alpha \in \Delta\backslash\Theta(\sigma)$. Let $\hat{\sigma} \subset \mathbb{S}\mathfrak{a}$ be the connected component of σ in:

$$\mathbb{S}\mathfrak{a}\backslash \bigcup_{w\in W}(w\cdot\tau)^\perp.$$

This connected component $\hat{\sigma}$ is stabilized by W_0. Indeed let α be in $\Delta\backslash\Theta(\sigma)$. By definition there is some $\mathrm{v} \in \sigma$ such that $\alpha(\mathrm{v}) = 0$, and hence that is fixed by the symmetry associated to α. Thus the connected component of v in $\mathbb{S}\mathfrak{a}$ is stabilized by this symmetry, and hence by the group W_0.

Let $\tau_0 \in \sigma$ be any element and let $\tau_0{}' \in \mathbb{S}\mathfrak{a}^+$ be the element that up to the action of W is positively colinear to:

$$\sum_{w\in W_0} w\cdot\tau_0.$$

This sum does not vanish since $\langle\tau, w\cdot\tau_0\rangle$ has constant sign for $w \in W_0$. This element $\tau_0{}'$ is W_0-invariant, hence for all $\alpha \in \Delta\backslash\Theta(\sigma)$, $\alpha(\tau_0{}') = 0$.

Moreover, since the Lie group considered G is semi-simple, the action of W has no global fixed point on $\mathbb{S}\mathfrak{a}$, and hence $W_0 \neq W$, which proves that $\Theta(\sigma) \neq \varnothing$. $\square$

If Γ is a finitely generated group that is not virtually infinite cyclic the limit cone of any discrete representation $\rho : \Gamma \to G$ is connected by Proposition 3.8.2. If ρ is τ-nearly Fuchsian then this limit cones lies to a connected component σ_ρ^τ of:

$$\mathbb{S}\mathfrak{a}\backslash \bigcup_{w\in W}(w\cdot\tau)^\perp.$$

Theorem 2.4.23. *A τ-nearly Fuchsian representation $\rho : \Gamma_g \to G$ from a finitely generated group Γ that is not virtually cyclic is $\Theta(\sigma_\rho^\tau)$-Anosov.*

In particular only hyperbolic groups admit τ-nearly Fuchsian representations (see [BPS19, Theorem 3.2]).

Note that $\Theta(\sigma_\rho^\tau) \neq \varnothing$, because of Lemma 2.4.22.

Proof. We use the characterization of Anosov representations from Theorem 2.2.4. We already proved that τ-nearly Fuchsian representations are quasi-isometric embeddings in Proposition 2.4.17.

Moreover the limit cone $\mathcal{C}_\rho$ lies inside σ_ρ^τ, which avoids $\mathrm{Ker}(\alpha)$ for $\alpha \in \Theta(\sigma_\rho^\tau)$. Hence ρ is $\Theta(\sigma_\rho^\tau)$-Anosov. ☐

The assumption that γ is not virtually cyclic is necessary: indeed the following representation $\rho : \mathbb{Z} \to \mathrm{SL}(3, \mathbb{R})$ is not Anosov for any set of roots:

$$n \mapsto \begin{pmatrix} 4^n & 0 & 0 \\ 0 & 2^{-n} & 0 \\ 0 & 0 & 2^{-n} \end{pmatrix}.$$

However this representation preserves a geodesic which is τ-regular for almost every $\tau \in \mathbb{S}\mathfrak{a}^+$.

2.4.5 A sufficient bound for an immersion to be nearly geodesic.

Let Θ be a Weyl orbit of simple roots as in Section 2.1.6. Let $\alpha \in \Theta$ be any root. We define the following constant:

$$c_\Theta = \min_{\beta \in \Sigma, \beta(\tau_\Theta) \neq 0} \frac{|\beta(\tau_\Theta)|}{\|\alpha\|^2}. \tag{2.10}$$

Here $\|\alpha\|$ for $\alpha \in \Theta$ denotes the maximum of $|\alpha(\tau)|$ for $\tau \in \mathbb{S}\mathfrak{a}$ a unit vector. This quantity is the same for any $\alpha \in \Theta$, since Θ is a Weyl orbit of simple roots.

A sufficient condition for the immersion u to be a τ_Θ-nearly geodesic surface is the following.

Theorem 2.4.24. *Let $u : S \to \mathbb{X}$ be an immersion that satisfies for all $\mathrm{v} \in TS$ and $\alpha \in \Theta$:*

$$\|\mathrm{I\!I}_u(\mathrm{v}, \mathrm{v})\|_{\tau_\Theta} < c_\Theta \alpha \left(\boldsymbol{\mu}\left(\mathrm{d}u(\mathrm{v})\right)\right)^2. \tag{2.11}$$

Then u is a τ_Θ-nearly geodesic immersion.

Note that $\|\cdot\|_{\tau_\Theta} < \|\cdot\|$ so having Inequality (2.11) with the Riemannian metric in the left hand side instead of the Finsler pseudo-distance from Section 2.4.5 is also a sufficient condition.

This property is a generalization of the property of having principal curvature in $(-1, 1)$, where the norm of the tangent vector is replaced by the evaluation of roots of the Cartan projection.

Proof. Let us show that (2.11) implies the condition of Proposition 2.4.3:

$$\mathrm{Hess}_{b_a}(\mathrm{d}u(\mathrm{v}),\mathrm{d}u(\mathrm{v})) + \langle \mathrm{II}_u(\mathrm{v},\mathrm{v}),\mathrm{v}_{a,u(y)}\rangle_{u(y)} > 0.$$

Let $x = u(y)$. Let us write $\mathrm{d}u(\mathrm{v}) = \mathrm{w}_0 + \mathrm{w}^\perp$ where $\mathrm{w}_0 \in \mathfrak{z}(\mathrm{v}_{a,x}) \cap \mathbf{p}_x$ and $\mathrm{w}^\perp \in (\mathfrak{z}(\mathrm{v}_{a,x}) \cap \mathbf{p}_x)^\perp$. Because of Lemma 2.3.9, one has :

$$\mathrm{Hess}_{b_a}\left(\mathrm{d}u(\mathrm{v}),\mathrm{d}u(\mathrm{v})\right) \geqslant \|\mathrm{w}^\perp\|^2 \min_{\beta\in\Sigma,\beta(\tau_\Theta)\neq 0} |\beta(\tau_\Theta)|. \tag{2.12}$$

Lemma 2.1.4 implies that for any $\alpha \in \Sigma$:

$$\alpha(\boldsymbol{\mu}(\mathrm{w}_0)) + \|\alpha\| \times \|\mathrm{w}^\perp\| \geqslant \alpha(\boldsymbol{\mu}(\mathrm{d}u(\mathrm{v}))).$$

We assumed that $\langle \mathrm{v}_{a,x}, \mathrm{d}u(\mathrm{v})\rangle_x = 0$. Since $\mathrm{v}_{a,x} \in \mathfrak{z}(\mathrm{v}_{a,x}) \cap \mathbf{p}_x$, one has therefore $\langle \mathrm{v}_{a,x}, \mathrm{w}^\perp\rangle_x = 0$ and hence $\langle \mathrm{v}_{a,x}, \mathrm{w}_0\rangle_x = 0$. Moreover $\mathrm{v}_{a,x}$ and w_0 are in a common flat. Since $\boldsymbol{\mu}(\mathrm{v}_{a,x}) = \tau_\Theta$, this implies that $\alpha(\boldsymbol{\mu}(\mathrm{w}_0)) = 0$ for some root $\alpha \in \Theta$. Therefore for this root α:

$$\|\mathrm{w}^\perp\| \geqslant \|\alpha\|^{-1}\alpha(\boldsymbol{\mu}(\mathrm{d}u(\mathrm{v}))). \tag{2.13}$$

Recall that for any $\mathrm{w} \in T_x\mathbb{X}$ and $a \in \mathcal{F}_{\tau_\Theta}$, $\langle \mathrm{w}, \mathrm{v}_{a,x}\rangle_x \leqslant \|\mathrm{w}\|_{\tau_\Theta}$, as a consequence of [Ebe96, Proposition 24].

Equation (2.12) and (2.13) imply together the following inequality, with c_Θ defined in (2.10):

$$\mathrm{Hess}_{b_a}\left(\mathrm{d}u(\mathrm{v}),\mathrm{d}u(\mathrm{v})\right) + \langle \mathrm{II}_u(\mathrm{v},\mathrm{v}),\mathrm{v}_{a,u(y)}\rangle_{u(y)} \geqslant c_\Theta \alpha(\boldsymbol{\mu}(\mathrm{d}u(\mathrm{v})))^2 - \|\mathrm{II}_u(\mathrm{v},\mathrm{v})\|_{\tau_\Theta}.$$

The rightmost term is strictly positive because of the condition (2.11). This concludes the proof. ☐

Example 2.4.25. Let $G = \mathrm{PSL}(n,\mathbb{R})$. We chose the standard metric on $\mathbb{X}$ that comes from the Killing form. In particular the Euclidean metric on $\mathfrak{a}$ is given by:

$$\langle \mathrm{Diag}(\lambda_1,\cdots,\lambda_n), \mathrm{Diag}(\mu_1,\cdots,\mu_n)\rangle = 2n\sum_{i=1}^n \lambda_i\mu_i.$$

In this case $\tau_\Delta = \mathrm{Diag}(\frac{1}{2\sqrt{n}},0,\cdots,0,-\frac{1}{2\sqrt{n}})$. The minimum non-zero value of $\beta(\tau_\Theta)$ for $\beta \in \Sigma$ is reached for the root $\alpha_1 : \mathrm{Diag}(\lambda_1,\cdots,\lambda_n) \mapsto \lambda_1 - \lambda_2$, and is equal to $\frac{1}{2\sqrt{n}}$ if $n \geqslant 3$ and $\frac{1}{\sqrt{n}}$ if $n = 2$.

The norm of any root α is equal to the norm of α_1. But $|\alpha_1(\tau)| \leqslant |\lambda_1|+|\lambda_2| \leqslant \sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2} \leqslant \frac{1}{\sqrt{n}}\|\tau\|$ with equality for some $\tau \in \mathbb{S}\mathfrak{a}$. Hence $\|\alpha_1\| = \frac{1}{\sqrt{n}}$, so if $n \geqslant 3$:

$$c_\Delta = 2\sqrt{n}.$$

And $c_\Delta = \sqrt{2}$ if $n = 2$. Note that if we rescale the metric on $\mathbb{X} = \mathbb{H}^2$ so that the sectional curvature is equal to -1, Equation (2.11) is exactly the condition of having principal curvature in $(-1,1)$.

2.5 Pencils of tangent vectors.

In this section we recall the classical notion of a pencil of quadrics, then we generalize it to the notion of a pencil of tangent vectors in a symmetric space of non compact type and its base in a flag manifold. Bases of pencils appear as the fibers of the fibration that will be constructed in Section 2.6.

2.5.1 Pencils of quadrics.

Some references for the notion of pencil of quadrics can be found in [FMS21]. Let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.5.1. A *pencil of quadrics*, or more precisely a *d-pencil of quadrics*[1] on V is a linear subspace $\mathcal{P}$ of dimension d in the space $\mathcal{S}(V)$ of symmetric bilinear forms on V if $\mathbb{K} = \mathbb{R}$, or in the space $\mathcal{H}(V)$ of Hermitian forms on V if $\mathbb{K} = \mathbb{C}$.

The *base* $b(\mathcal{P})$ of a d-pencil $\mathcal{P}$ is the set of points $[v] \in \mathbb{P}(V)$ such that for all $q \in \mathcal{P}$, $q(v, v) = 0$.

The following is a criterion for a pencil of quadrics to have a smooth base.

Lemma 2.5.2. *Let $\mathcal{P}$ be a pencil of quadrics such that all non-zero $q \in \mathcal{P}$ are non-degenerate bilinear forms. The map $p : V \to \mathcal{P}^*$, $v \mapsto (q \mapsto q(v, v))$ is a submersion at every $v \in V \backslash \{0\}$ such that $[v] \in b(\mathcal{P})$. In particular $b(\mathcal{P})$ is a smooth manifold of codimension d.*

Proof. Let $(q_1, \cdots, q_d)$ be a basis of $\mathcal{P}$. Let us consider some $v \in V\backslash\{0\}$ such that $q_1(v, v) = \cdots = q_d(v, v) = 0$. The kernel of the differential of p is the intersection of the orthogonal spaces $[v]^{\perp_{q_i}}$ with respect to q_i of the line generated by v for $1 \leqslant i \leqslant d$. Since the forms q_i are non-degenerate, these are hyperplanes.

Suppose that their intersection has not codimension d. In particular the linear forms $q_i(v, \cdot)$ for $1 \leqslant i \leqslant d$ are not linearly independent, so there exist a linear combination of the bilinear forms that is degenerate, but is a non-zero element of $\mathcal{P}$, contadicting our assumption.

Hence the the kernel of p has codimension d, so p is a submersion at v.

□

The base of a pencil of quadric is smooth and has codimension d around each of it's points which are non-singular, meaning that they are not degenerate points for any quadric in the pencil. We generalize this notion of singular points in the next section.

[1] In the literature, for instance in [FMS21], the term *pencil of quadrics* is often used only for 2-pencils of quadrics.

2.5.2 Pencils of tangent vectors in symmetric spaces.

In this subsection we consider pencils of tangent vectors in a symmetric space $\mathbb{X}$ of non compact type, which are related to pencils of quadrics when $G = \mathrm{PSL}(n, \mathbb{R})$.

Definition 2.5.3. A *pencil of tangent vectors* at $x \in \mathbb{X}$, or more precisely a *d-pencil*, is a vector subspace $\mathcal{P} \subset T_x\mathbb{X}$ of dimension d for some point $x \in \mathbb{X}$.

To a pencil one can associate some subsets of any G-orbit in the visual boundary. Recall that for $a \in \partial_{\mathrm{vis}}\mathbb{X}$ and $x \in \mathbb{X}$, the unit vector $\mathrm{v}_{a,x} \in T_x\mathbb{X}$ is the unit vector pointing towards a. Let $\tau \in \mathbb{S}\mathfrak{a}^+$.

Definition 2.5.4. The *τ-base of the pencil* $\mathcal{P}$, whose base-point is $x \in \mathbb{X}$, is the set $\mathcal{B}_\tau(\mathcal{P})$ of elements $a \in \mathcal{F}_\tau$ such that $\mathrm{v}_{a,x}$ is orthogonal to $\mathcal{P}$.

When $G = \mathrm{PSL}(n, \mathbb{R})$ and $\mathbb{X} = \mathcal{S}_n$, a pencil at $q \in \mathcal{S}_n$ corresponds to a subspace $\mathcal{P}'$ of symmetric bilinear forms on $\mathbb{R}^n$, i.e. a pencil of quadrics, that is compatible with q in the sense that the trace of the associated q-symmetric matrices vanishes.

Proposition 2.5.5. *Let $\tau \in \mathbb{S}\mathfrak{a}^+$ be such that $\mathcal{F}_\tau \simeq \mathbb{RP}^{n-1}$. The τ-base of the pencil $\mathcal{P}$ is identified via this identification with the base of the pencil of quadrics $\mathcal{P}'$.*

Proof. The τ-base of $\mathcal{P}$ is the space of lines $[C]$ where $C \in \mathbb{R}^n$ is a column vector satisfying $\mathrm{Tr}(CC^{\perp_q}M) = 0$ for all $M \in \mathcal{P}'$, since $CC^{\perp_q}$ is colinear to $\mathrm{v}_{[C],q}$. Hence the τ-base of the pencil is also the set of lines $[C]$ such that $C^{\perp_q}MC = 0$, i.e. the base of the pencil of quadrics $\mathcal{P}'$.

□

We now generalize Lemma 2.5.2 to general pencils of tangent vectors.

Definition 2.5.6. A point $a \in \mathcal{F}_\tau$ in the base $\mathcal{B}_\tau(\mathcal{P})$ of a pencil $\mathcal{P}$ at $x \in X$ is called *singular* if for some $\mathrm{w} \in \mathcal{P}$ one has $[\mathrm{w}, \mathrm{v}_{a,x}] = 0$.

We denote by $\mathcal{B}^*_\tau(\mathcal{P}) \subset \mathcal{B}_\tau(\mathcal{P})$ the set of non-singular points, that we will also call the *regular base*.

Lemma 2.5.7. *Let $\mathcal{P}$ be a pencil of tangent vectors at x in $\mathbb{X}$. The function which associates to $a \in \mathcal{F}_\tau$ the linear form $\mathrm{v} \mapsto \langle \mathrm{v}_{a,x}, \mathrm{v}\rangle_x$ on $\mathcal{P}$ is a submersion at $a \in \mathcal{B}_\tau(\mathcal{P})$ if and only if $a \in \mathcal{B}^*_\tau(\mathcal{P})$. In particular $\mathcal{B}^*_\tau(\mathcal{P})$ is always a smooth codimention d submanifold of $\mathcal{F}_\tau$.*

Proof. Let $\phi : \mathcal{F}_\tau \to \mathcal{P}^*$ be the map that associates to $a \in \mathcal{F}_\tau$ the linear form $\mathrm{v} \mapsto \langle \mathrm{v}_{a,x}, \mathrm{v}\rangle_x$.

Suppose that $a \in \mathcal{B}_\tau(\mathcal{P})\backslash\mathcal{B}^*_\tau(\mathcal{P})$. Then there exist some $\mathrm{w} \in \mathcal{P}$ such that $[\mathrm{w}, \mathrm{v}_{a,x}] = 0$. The map $\psi : k \in K_x \mapsto k \cdot a \in \mathcal{F}_\tau$ is a submersion, so for every tangent vector in $T_a\mathcal{F}_\tau$ the differential of $a \mapsto \mathrm{v}_{a,x}$ in this direction is $\mathrm{ad}_{\mathrm{k}}(\mathrm{v}_{a,x})$ for some $\mathrm{k} \in \mathfrak{k}_x$. The differential of $a \mapsto \langle \mathrm{v}_{a,x}, \mathrm{w}\rangle_x$ in this direction is equal to

$\langle \mathrm{ad}_{\mathrm{k}}(\mathrm{v}_{a,x}), \mathrm{w}\rangle_x = -B(\mathrm{ad}_{\mathrm{k}}(\mathrm{v}_{a,x}), \mathrm{w}) = -B(\mathrm{k}, [\mathrm{v}_{a,x}, \mathrm{w}]) = 0$. Hence the image of the differential of ϕ is not surjective: it is not a submersion.

Suppose that $a \in \mathcal{B}^*_\tau(\mathcal{P})$. Let $\mathrm{v} \in \mathcal{P}$ be any non-zero vector and consider $[\mathrm{v}_{a,x}, \mathrm{v}] = \mathrm{k} \in \mathfrak{k}_x$. The differential of $a \mapsto \langle \mathrm{v}_{a,x}, \mathrm{v}\rangle_x$ in the corresponding tangent direction is equal to $\langle \mathrm{ad}_{\mathrm{k}}(\mathrm{v}_{a,x}), \mathrm{v}\rangle_x = -B(\mathrm{ad}_{\mathrm{k}}(\mathrm{v}_{a,x}), \mathrm{v}) = -B([\mathrm{v}_{a,x}, \mathrm{v}], [\mathrm{v}_{a,x}, \mathrm{v}]) \neq 0$. Since for all $\mathrm{v} \in \mathcal{P}$ there is a direction in which the differential of $a \mapsto \langle \mathrm{v}_{a,x}, \mathrm{v}\rangle_x$ does not vanish, the map ϕ is therefore a submersion at a. □

A pencil of tangent vectors $\mathcal{P}$ at $x \in \mathbb{X}$ is called *τ-regular* if all its non-zero vectors are τ-regular as in Definition 2.4.7. In particular a τ-regular pencil satisfies $\mathcal{B}^*_\tau(\mathcal{P}) = \mathcal{B}_\tau(\mathcal{P})$, so the τ-base of a τ-regular vector is a smooth codimension d submanifold of $\mathcal{F}_\tau$.

Because of Lemma 2.5.7, the topology of the base of a regular pencil is not varying if the pencil is deformed continuously.

Corollary 2.5.8. *Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be two pencils at $x \in \mathbb{X}$ in the same connected component of the space of τ-regular pencils at x. Then $\mathcal{B}_\tau(\mathcal{P}_1)$ and $\mathcal{B}_\tau(\mathcal{P}_2)$ are diffeomorphic.*

Proof. Since the space of regular pencils is open is the Grassmanian of planes in $T_x\mathbb{X}$, there exist a smooth path $(\mathcal{P}_t)_{t\in[0,1]}$ of regular pencils between $\mathcal{P}_0$ and $\mathcal{P}_1$. Because of Lemma 2.5.7 the set $\{(a,t) | a \in \mathcal{F}_\tau, t \in [0,1]\}$ is a submanifold with boundary of $\mathcal{F}_\tau \times [0,1]$ that comes with a natural submersion $(a,t) \mapsto t$. Since this manifold is compact all the fibers are diffeomorphic by the Ehresmann fibration theorem. □

Example 2.5.9. Let $G = \mathrm{PSL}(3,\mathbb{R})$, $\mathbb{X} = \mathcal{S}_3$. We identify the tangent space $T_{q_0}\mathcal{S}_3$ at the point q_0 corresponding to the standard scalar product on $\mathbb{R}^3$ with the space of 3 by 3 symmetric matrices with real coefficients and zero trace. Consider the following two pencils:

$$\mathcal{P}_{\mathrm{irr}} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right\rangle, \ \mathcal{P}_{\mathrm{red}} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

Let $\tau_1 \in \mathbb{S}\mathfrak{a}^+$ be such that $\mathcal{F}_{\mathrm{p}_1}$ is diffeomorphic in a $\mathrm{PSL}(3,\mathbb{R})$-equivariant way to $\mathbb{RP}^{n-1}$, and let τ_Δ be the normalized coroot associated to the Weyl orbit of simple roots Δ. It satisfies $\mathcal{F}_{\tau_\Delta} \simeq \mathcal{F}_{1,2}$ the space of complete flags in $\mathbb{R}^3$.

The pencils $\mathcal{P}_{\mathrm{irr}}$ and $\mathcal{P}_{\mathrm{red}}$ are not τ_1-regular: $\mathcal{B}_{\tau_1}(\mathcal{P}_{\mathrm{irr}})$ is the disjoint union of a point and a line where $\mathcal{B}^*_{\tau_1}(\mathcal{P}_{\mathrm{irr}})$ contains only the point. In this particular case the regular base is a connected component of the base, so it is a smooth compact codimension 2 submanifold. The set $\mathcal{B}_{\tau_1}(\mathcal{P}_{\mathrm{red}})$ is a single point that is singular for the pencil. Here we see that a singular point can still be a point around which the base is a smooth codimension 2 submanifold.

Both pencils are τ_Δ-regular, but their τ_Δ-bases are different.

A flag $(\ell, H) = ([\mathrm{x}], [\mathrm{y}]^\perp)$ with $\mathrm{x} = (x_1, x_2, x_3)$ and $\mathrm{y} = (y_1, y_2, y_3)$ non-zero vectors such that $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$ and $x_1y_1 + x_2y_2 + x_3y_3 = 0$ belongs to $\mathcal{B}_{\tau_\Delta}(\mathcal{P}_{\mathrm{red}})$ if and only if:

$$x_1^2 - x_3^2 = y_1^2 - y_3^2,$$

$$2x_1x_3 = 2y_1y_3.$$

Up to replacing y by $-\mathrm{y}$, these equations are equivalent to $x_1 = y_1$, $x_2 = -y_2$, $x_3 = y_3$, $x_1^2 + x_3^2 = x_2^2$. The corresponding flags (ℓ, H) in the affine chart $(x, y) \mapsto [x, 1, y]$ of $\mathbb{RP}^2$ are the tangent point and tangent lines to the the circle of radius 1 centered at the origin.

A flag $(\ell, H) = ([\mathrm{x}], [\mathrm{y}]^\perp)$ with $\mathrm{x} = (x_1, x_2, x_3)$ and $\mathrm{y} = (y_1, y_2, y_3)$ non-zero vectors such that $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$ and $x_1y_1 + x_2y_2 + x_3y_3 = 0$ belongs to $\mathcal{B}_{\tau_\Delta}(\mathcal{P}_{\mathrm{irr}})$ if and only if:

$$2x_1^2 - 2x_3^2 = 2y_1^2 - 2y_3^2,$$

$$2x_1x_2 + 2x_2x_3 = 2y_1y_2 + 2y_2y_3.$$

Let $\ell_0 = \langle (1, 0, 1) \rangle$ and $H_0 = \langle (1, 0, -1), (0, 1, 0) \rangle$. The corresponding flags (ℓ, H) belong to one of the three circles in $\mathcal{F}_{1,2}$ defined by :

- $\ell = \ell_0$, H any plane through ℓ,
- $H = H_0$, ℓ any line in H,
- $\ell \subset H_0$, $\ell_0 \subset H$.

Indeed one can check that these flags satisfy the equations. In order to check that these are the only solutions, one can see that these are the fibers of a fibration over the surface with 3 connected components, see Section 2.7.3.2.

The τ_Δ-base $\mathcal{B}_{\tau_\Delta}(\mathcal{P}_{\mathrm{red}})$ is a circle whereas $\mathcal{B}_{\tau_\Delta}(\mathcal{P}_{\mathrm{irr}})$ is the union of 3 circles. Hence Corollary 2.5.8 implies that they must lie in different connected components of the space of τ_Δ-regular pencils.

Since the pencils will be the fibers of the domains of discontinuity that we will construct, proving that the domain is non-empty will be equivalent to having non-empty pencils. We present here a topological argument to prove that some pencils are non-empty.

Proposition 2.5.10. *Let $\tau \in \mathbb{S}\mathfrak{a}^+$ and $\mathcal{P}$ be a τ-regular pencil of tangent vectors based at $x \in \mathbb{X}$ of dimension d. If the τ-base of $\mathcal{P}$ is empty, then $\mathcal{F}_\tau$ fibers over the sphere S^{d-1}.*

If moreover $d = 2$ it implies that the fundamental group of $\mathcal{F}_\tau$ is infinite.

Proof. To $a \in \mathcal{F}_\tau$ we associate $\pi_0(a) \in \mathcal{P}$ the orthogonal projection of $\mathrm{v}_{a,x} \in T_x\mathbb{X}$ onto $\mathcal{P} \subset T_x\mathbb{X}$. Since the τ-base of $\mathcal{P}$ is empty, one can define a map $\pi : \mathcal{F}_\tau \to \mathbb{S}\mathcal{P}$ into the unit sphere of $\mathcal{P}$ where $\pi(a) = \frac{\pi_0(a)}{\|\pi_0(a)\|}$. This map is a submersion.

Indeed let $a \in \mathcal{F}_\tau$, let $\mathcal{P}_0 \subset \mathcal{P}$ be the orthogonal to $\pi(a)$ in $\mathcal{P}$. Lemma 2.5.7 applied to $\mathcal{P}_0$ implies that π is a submersion at a.

This submersion is proper since $\mathcal{F}_\tau$ is compact, hence it is a fibration. If $d = 2$, this fibration induces a long exact sequence, where F is the fiber.

$$\cdots \to \pi_1(\mathcal{F}_\tau) \to \pi_1(\mathbb{S}\mathcal{P}) \to \pi_0(F) \to \cdots .$$

Since F is compact, $\pi_0(F)$ is finite and $\pi_1(\mathbb{S}\mathcal{P}) \simeq \mathbb{Z}$, so $\pi_1(\mathcal{F}_\tau)$ is infinite. □

We conclude this section by the following remark that regular pencils cannot be tangent to flats.

Proposition 2.5.11. *If a 2-pencil is tangent to a flat, then it is not τ-regular for any $\tau \in \mathbb{S}\mathfrak{a}^+$.*

Proof. Up to the action of G one can identify $\mathcal{P}$ with a plane in $\mathfrak{a}$. But for any $\tau \in \mathbb{S}\mathfrak{a}^+$, the orthogonal of τ intersects this plane. Hence there is an element of $\mathcal{P}$ whose Cartan projection is orthogonal to $w \cdot \tau$ for some w in the Weyl group. □

2.6 Fibered domains in flag manifolds.

In this section we associate an open domain $\Omega_u^\tau \subset \mathcal{F}_\tau$ to any complete uniformly τ-nearly geodesic immersion $u : M \to \mathbb{X}$ with $\tau \in \mathbb{S}\mathfrak{a}^+$, and show that this domain is a smooth fiber bundle over M where the fibers are τ-bases of the pencils that are the tangent planes to the surface. This is the construction is the analog of the *Gauss map* for hypersurfaces in $\mathbb{H}^n$. We also mention what happens with our construction for totally geodesic immersions that are not τ-regular.

If $M = \tilde{N}$ for some compact manifold N with torsion-free fundamental group Γ, and if u is equivariant with respect to a representation ρ, we show that the domain Ω_u^τ is a co-compact domain of discontinuity for the action of ρ and its quotient fibers over N. This domain always coincides with some domain of discontinuity associated to Tits Bruhat ideals constructed by Kapovich-Leeb-Porti [KLP18a]. Finally we prove the invariance of the topology of the quotients of these domains of discontinuity.

2.6.1 A domain associated to a nearly geodesic immersion.

Let $\tau \in \mathbb{S}\mathfrak{a}^+$ be any unit vector and $u : M \to \mathbb{X}$ be a complete uniformly τ-nearly geodesic immersion.

We consider a particular domain of the flag manifold $\mathcal{F}_\tau$, defined for any nearly geodesic immersion $u : M \to \mathbb{X}$ using Busemann functions. For this we fix a base-point $o \in \mathbb{X}$, but the definition will dot depend on this choice.

Definition 2.6.1. Let Ω_u^τ be the set of elements $a \in \mathcal{F}_\tau$ such that the function $b_{a,o} \circ u$ is proper and bounded from below.

We have additional properties if u is a complete uniformly τ-nearly geodesic immersion.

Lemma 2.6.2. *Let $a \in \mathcal{F}_\tau$. There exist a critical point $x \in M$ for the function $b_{a,o} \circ u$ if and only if $a \in \Omega_u^\tau$. In this case this point is unique, and the Hessian of $b_{a,o} \circ u$ at this point is positive. The domain Ω_u^τ is open.*

Proof. Let $a \in \mathcal{F}_\tau$. Suppose that $b_{a,o} \circ u$ is critical at $y \in M$. Since u is τ-nearly geodesic the Hessian of $b_{a,o} \circ u$ at y is positive. Moreover, due to Lemma 2.4.12 there exist $\lambda > 0$ such that $\exp(\lambda b_{a,o} \circ u)$ has positive Hessian everywhere on M.

A convex function with positive Hessian on a complete connected Riemannian manifold has a unique minimum, and is proper. In particular the function $\exp(\lambda b_{a,o} \circ u)$, and hence the function $b_{a,o} \circ u$ are hence proper and have a unique minimum. In particular $a \in \Omega_\rho^\tau$.

Conversely if $a \in \Omega_\rho^\tau$, $b_{a,o} \circ u$ is proper so it admits a global minimum, which is a critical point.

If a function has a critical point with positive Hessian, every small deformation of the function for the $\mathcal{C}^0$-topology still admits a local minimum, and hence a critical point. Therefore Ω_ρ^τ is open. $\square$

We thus can define the projection $\pi_u : \Omega_u^\tau \to M$ associated to u as the map that associates to $a \in \Omega_u^\tau$ the unique critical point $\pi_u(a) \in M$ of $b_{a,o} \circ u$. This is an extension at infinity of the nearest point projection from Proposition 2.4.18.

Theorem 2.6.3. *Let $u : M \to \mathbb{X}$ be a complete and uniformly τ-nearly geodesic immersion. The map $\pi_u : \Omega_u^\tau \to M$ is a fibration. The fiber $\pi_u^{-1}(x)$ at a point $x \in M$ is the base $\mathcal{B}_\tau(\mathcal{P}_x)$ of the τ-regular pencil $\mathcal{P}_x = \mathrm{d}u(T_x M)$.*

Figure 2.7 illustrates this construction in the rank one case $G = \mathrm{PSL}(2, \mathbb{C})$, for a totally geodesic immersion u. The associated symmetric space $\mathbb{H}^3$ is depicted with Poincaré's ball model. Since $\mathbb{H}^3$ has rank 1, its visual boundary contains a single orbit $\mathcal{F}_\tau \simeq \mathbb{CP}^1$. The image of u is the disk bounded by the equator. The pencil $\mathcal{P}$ is depicted as a parallelogram. its τ-base is a fiber of the fibration, and is the co-dimension 2 submanifold $\mathcal{B}_\tau(\mathcal{P}) = \{a, a'\}$.

Remark 2.6.4. Note that if some element $g \in G$ preserves $u(M)$, then the map $\pi_u \circ u$ commutes with the action of g. In particular if $M = \tilde{N}$ for a compact manifold N with fundamental group Γ and if u is ρ-equivariant for some $\rho : \Gamma \to G$, then π_u is ρ-equivariant, and hence defines a fibration $\overline{\pi}_u : \Omega_u^\tau / \rho(\Gamma) \to N$.

The two important steps in the proof of Theorem 2.6.3 are to check that the fibers are distinct and far enough from one another using Lemma 2.6.2, and that these fibers are smooth manifolds using Lemma 2.5.7.

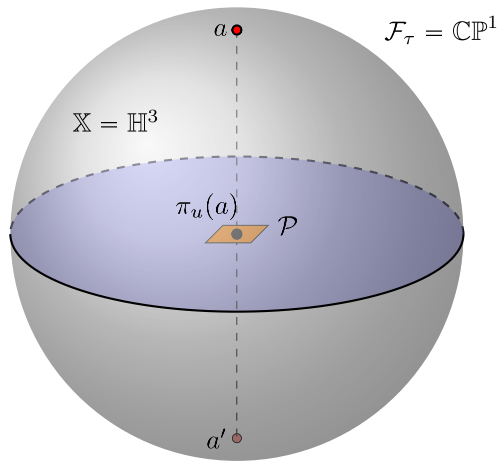


Figure 2.7: Fibration of the domain Ω_u^τ in the rank one case, $G = \mathrm{PSL}(2, \mathbb{C})$.

Proof of Theorem 2.6.3. Consider the set:

$$E = \{(a, x) \in \Omega_u^\tau \times M | \mathrm{d}_x(b_{a,o} \circ u) = 0\}.$$

Because of Proposition 2.6.2 the Hessian of $b_{a,o} \circ u$ is non-degenerate at critical points, hence E is locally the zero set of a submersion so it is a codimension 2 submanifold of $\Omega_u^\tau \times M$.

Let $\pi_1 : \Omega_u^\tau \times M \to \Omega_u^\tau$ and $\pi_2 : \Omega_u^\tau \times M \to M$ be respectively the projections onto the first and second factor.

Lemma 2.6.2 implies that π_1 restricted to E is a bijection. Moreover, again because of the non-degeneracy of the Hessian of $b_{a,o} \circ u$, the tangent space $T_{(a,x)}E$ at $(a, x) \in E$ intersects trivially $T_x M \subset T_{(a,x)} (\Omega_u^\tau \times M)$. Hence π_1 restricted to E is a local diffeomorphism, and therefore a diffeomorphism.

Let $(a, x) \in E$. By definition $\mathrm{d}_a(b_{a,o} \circ u) : \mathrm{v} \mapsto \langle \mathrm{d}u(\mathrm{v}), \mathrm{v}_{a,u(x)} \rangle_{u(x)}$ vanishes, so $\mathrm{v}_{a,u(x)} \perp \mathrm{d}u(T_x M) = \mathcal{P}_x$. Hence a belongs to the τ-base of $\mathcal{P}_x$. Because of Proposition 2.4.8, this pencil is τ-regular and hence its τ-base contains no singular points. Lemma 2.5.7 implies that the tangent space $T_{(a,x)}E$ at $(a, x) \in E$ intersects trivially $T_a\Omega_u^\tau \subset T_{(a,x)} (\Omega_u^\tau \times M)$. The map π_2 restricted to E is therefore a submersion at (a, x).

As a conclusion, $\pi_u = \pi_2 \circ \pi_1^{-1}$ is a smooth submersion. The τ-base of the pencil $\mathcal{P}_x$ is compact in $\mathcal{F}_\tau$, and it is included in Ω_u^τ because of Lemma 2.6.2. Hence π_u is a proper submersion over a connected manifold: by the Ehresmann fibration theorem it is a fibration. $\square$

2.6.2 Totally geodesic immersions that are not nearly geodesic.

In this subsection, let $u : M \to \mathbb{X}$ be a complete totally geodesic immersion. Let $\tau \in \mathbb{S}\mathfrak{a}^+$. We don't assume in this subsection that u is τ-regular, and hence τ-nearly geodesic.

One can still define Ω_u^τ as the set of $a \in \mathcal{F}_\tau$ such that $b_{a,o} \circ u$ is proper and bounded from below, but we can't always expect the domain to have compact fibers in this case. Lemma 2.6.2 can be adapted as follows:

Lemma 2.6.5. *A point $a \in \mathcal{F}_\tau$ belongs to Ω_u^τ if and only if the function $b_{a,o} \circ u$ admits a critical point $\pi_u(a) \in M$ at which the Hessian is positive. In this case the critical point is unique and is a global minimum of $b_{a,o} \circ u$. The domain Ω_u^τ is open.*

Proof. Let $a \in \Omega_\rho^\tau$, and let $y \in M$ a point at which $b_{a,o} \circ u$ has a global minimum. The function $b_{a,o} \circ u$ is convex, but not necessarily strictly convex. Assume that the Hessian of $b_{a,o}$ in the direction $\mathrm{d}u(\mathrm{v})$ vanishes for some $\mathrm{v} \in T_yM$.

There must exist a flat that contains a and $\mathrm{d}u(\mathrm{v})$ by Lemma 2.3.9. Let η be the geodesic ray starting at $\mathrm{d}u(\mathrm{v})$ in $\mathbb{X}$. The function $b_{a,o}$ is linear on η since a and η belong to a common flat. However the derivative of $b_{a,o}$ along η vanishes at $u(y)$, so $b_{a,o}$ is constant along η. Moreover u is totally geodesic and the whole geodesic ray starting at $\mathrm{d}u(\mathrm{v})$ in $\mathbb{X}$ belongs to the image of u, so $b_{a,o} \circ u$ is not proper.

For all $a \in \Omega_\rho^\tau$ the functions $b_{a,o} \circ u$ are convex and strictly convex at the critical point, which is therefore unique. The rest of the proof goes as in Lemma 2.6.2. $\square$

We define a map $\pi_u : \Omega_u^\tau \to M$, using Lemma 2.6.5. We show that this map is a fibration. Recall that the regular base $\mathcal{B}_\tau^*(\mathcal{P})$ defined in Section 2.5 is a subset of the base $\mathcal{B}_\tau(\mathcal{P})$ that is always a smooth codimension d submanifold.

Theorem 2.6.6. *Let $a \in \Omega_u^\tau$, the function $b_{a,o} \circ u$ admits a unique critical point denoted by $\pi_u(a) \in M$. The map $\pi_u : \Omega_u^\tau \to M$ is a smooth fibration, and the fiber of this map at $y \in M$ is the regular base $\mathcal{B}_\tau^*(Tu(T_yM))$.*

Proof. By the same argument as for Theorem 2.6.3, π_u is a smooth submersion.

However we need to proceed differently to prove that this map is a fibration, since the fiber is not necessarily compact. Let $g \in G$ be an element that stabilizes $u(M) \subset X$. The map π_u is equivariant with respect to g, i.e. for all $a \in \Omega_\rho^\tau$:

$$\pi_u(g \cdot a) = g \cdot \pi_u(a).$$

Let $y \in M$. Recall that the exponential map for the Lie group G defines a map $\exp : T_{u(y)}\mathbb{X} \simeq \mathfrak{p}_{u(y)} \subset \mathfrak{g} \to G$. Moreover since $u(M)$ is totally geodesic any element of $\exp(\mathrm{d}u(T_yM))$ is a transvection on this totally geodesic subspace hence it stabilizes $u(M)$. We consider the following map :

$$\begin{aligned}\phi : T_yM \times \mathcal{B}^*_\tau(Tu(T_yM)) &\longrightarrow \Omega^\tau_u \\ (\mathrm{v}, a) &\longmapsto \exp(\mathrm{d}u(\mathrm{v})) \cdot a\end{aligned}$$

This map is an immersion, between spaces of equal dimension. Moreover it is a bijection, hence it is a diffeomorphism. Through the identification $\exp : T_yM \to M$, this gives Ω^τ_u the structure of a fibration with projection π_u. $\square$

Since the regular base is open, it is compact if and only if the regular points form a union of connected component of $\mathcal{B}_\tau(\mathcal{P})$. This is for instance the case if $G = \mathrm{PSL}(3, \mathbb{R})$ and $\mathcal{P} = \mathcal{P}_{\mathrm{irr}}$ as in Example 2.5.9.

Example 2.6.7. Let $G = \mathrm{SL}(3, \mathbb{R})$, and ρ be a representation of the form $\rho = \iota_{\mathrm{irr}} \circ \rho_0$ for some Fuchsian representation $\rho_0 : \Gamma_g \to \mathrm{SO}(1,2) \simeq \mathrm{PSL}(2, \mathbb{R})$ of a surface group and the natural inclusion $\iota_{\mathrm{irr}} : \mathrm{SO}(1,2) \to \mathrm{SL}(3, \mathbb{R})$. This representation admits a ρ-equivariant totally geodesic map $u : \widetilde{S_g} \to \mathbb{X}$ (see Section 2.7.1 for more details). The pencil $\mathcal{P} = Tu(T_y\widetilde{S_g})$ for any $y \in \widetilde{S_g}$ is up to the action of G equal to the pencil $\mathcal{P}_{\mathrm{irr}}$ defined in Example 2.5.9.

This pencil is not τ_1 regular, so u is not τ_1-nearly geodesic. However because of Theorem 2.6.6 the domain $\Omega^{\tau_1}_u$ fibers over $\widetilde{S_g}$ with base $\mathcal{B}^*_{\tau_1}(\mathcal{P}_{\mathrm{irr}})$, which is a point in $\mathcal{F}_{\tau_1} \simeq \mathbb{RP}^2$. This domain is the disk of positive vectors for the chosen bilinear form of signature $(1,2)$ on $\mathbb{R}^3$. In this example the regular points of $\mathcal{B}_{\tau_1}(\mathcal{P}_{\mathrm{irr}})$ form a connected component so the fibration is proper. If we consider a point $\ell \in \mathbb{RP}^2$ outside of the closure of this disk, the associated Busemann function is minimal in $u(\tilde{S}_g)$ on a full geodesic line. If ℓ is in the boundary of the disk, the associated Busemann function is not bounded from below on $u(\tilde{S}_g)$.

2.6.3 Comparison with metric thickenings.

In this section we consider the case when $M = \widetilde{N}$ for some compact manifold N with fundamental group Γ and u is equivariant with respect to a representation $\rho : \Gamma \to G$. In other words ρ is a τ-nearly Fuchsian representation, as we defined in Section 2.4.4.

We show that if we have a τ-nearly Fuchsian ρ-equivariant map $u : \widetilde{N} \to \mathbb{X}$ for a representation $\rho : \Gamma \to G$ the domain Ω^τ_u coincides with a domain of discontinuity associated to Anosov representations constructed by Kapovich, Leeb, Porti [KLP18a].

The domain $\Omega^\tau_\rho := \Omega^\tau_u$ depends on τ and ρ but not on u. Indeed a 1-Lipshitz function on $\mathbb{X}$ is proper on the image of u if and only if it is proper on any $\rho(\Gamma)$ orbit in $\mathbb{X}$.

Even though this will be a consequence of Theorem 2.6.11, one can easily check that the existence of the fibration of Ω^τ_ρ implies that it is a cocompact domain of discontinuity.

Theorem 2.6.8. *Let $\rho : \Gamma \to G$ be a representation that admits an equivariant τ-nearly geodesic immersion $u : \widetilde{N} \to \mathbb{X}$. The action of Γ_g on Ω_ρ^τ via ρ is properly discontinuous and co-compact.*

Proof. Let π_u be the fibration from Theorem 2.6.3. Let A be a compact subset of Ω_ρ^τ. Its image $\pi_u(A) \subset \widetilde{N}$ is compact on $\widetilde{N}$. Since Γ acts properly on $\widetilde{N}$, all but finitely many $\gamma \in \Gamma$ satisfy $\pi_u(A) \cap \gamma\pi_u(A) = \varnothing$. Hence for all but finitely many $\gamma \in \Gamma :\ A \cap \rho(\gamma)A = \varnothing$. The action of Γ via ρ is therefore properly discontinuous.

Let D be a compact fundamental domain for the action of Γ on $\widetilde{N}$, i.e. a compact set that satisfies:

$$\bigcup_{\gamma\in\Gamma} D = \widetilde{N}.$$

The set $\pi_u^{-1}(D)$ is a fundamental domain for the action of Γ on Ω_ρ^τ by ρ by the equivariance of π_u. It is closed in Ω_ρ^τ. Moreover $\pi_u^{-1}(D)$ is closed in $\mathcal{F}_\tau$. Indeed if we consider a sequence (a_n) of elements of Ω_ρ^τ that converge to $a \in \mathcal{F}_\tau$ such that $\pi_u(a_n)$ always belong to D, one can assume that $\pi_u(a_n)$ converges to $y_0 \in D$ up to taking a subsequence. In the limit, one has $b_{a,o} \circ u(y_0) \leqslant b_{a,o} \circ u(y)$ for all $y \in \widetilde{N}$. Hence y_0 is a critical point for $b_{a,o} \circ u$ so by Lemma 2.6.2 $a \in \Omega_\rho^\tau$.

Hence Ω_ρ^τ admits a compact fundamental domain for the action of Γ via ρ, therefore this action is co-compact.

□

We consider the domains of discontinuity constructed by *metric thickenings*, which are particular instances of the domains of discontinuity associated with a *Tits-Bruhat ideal* defined in [KLP18a].

Let (τ, τ_0) be a pair of elements in $\mathbb{S}\mathfrak{a}^+$. This pair will be called *balanced* if τ_0 is τ-regular i.e. :

$$\tau_0 \notin \bigcup_{w\in W} (w \cdot \tau)^\perp.$$

Note that this is equivalent to τ being τ_0-regular. Using the Tits angle $\angle_{\mathrm{Tits}} : \partial_{\mathrm{vis}}\mathbb{X}^2 \to [0, \pi]$, see Section 2.1, we associate to any $b \in \partial_{\mathrm{vis}}\mathbb{X}$ a thickening $K_b \subset \mathcal{F}_\tau$ defined as:

$$K_b = \left\{ a \in \mathcal{F}_\tau | \angle_{\mathrm{Tits}}(a, b) \leqslant \frac{\pi}{2} \right\}.$$

Recall that the Tits angle was defined in Section 2.1, and is defined for points in $\partial_{\mathrm{vis}}\mathbb{X}$.

Lemma 2.6.9. *Let b_1, b_2 belong to a common maximal facet in $\partial_{\mathrm{vis}}\mathbb{X}$, and suppose that their Cartan projections lie in the same connected component of:*

$$\mathbb{S}\mathfrak{a}^+ \backslash \bigcup_{w\in W} (w \cdot \tau)^\perp.$$

Then $K_{b_1} = K_{b_2}$.

Proof. Let $f \in \mathcal{F}_\Delta$ be a maximal facet that contains b_1 and b_2. Let $a \in \mathcal{F}_\tau$, there exist a maximal flat of $\mathbb{X}$ such that f and a belong to its visual boundary. This flat can be identified with $\mathfrak{a}$ so that f corresponds to $\partial_{\mathrm{vis}}\mathfrak{a}^+$.

Hence as long as b lies in the visual boundary of this flat, $\angle_{\mathrm{Tits}}(a, b)$ is equal to the Euclidean angle in the flat, so the sign of its cosine does not vary as long as the Cartan projection of b does not lie in $(w \cdot \tau)^\perp$ for any $w \in W$. Therefore if the Cartan projections of b_1 and b_2 are in the same connected component of the complement, $K_{b_1} = K_{b_2}$. ☐

Recall that the manifold $\mathcal{F}_{\tau_0}$ is G-equivariantly diffeomorphic to the flag manifold $\mathcal{F}_{\Theta(\tau_0)} = G/P_{\Theta(\tau_0)}$ where $\Theta(\tau_0)$ is the set of simple root that do not vanish on τ_0. Hence given a flag $f \in \mathcal{F}_{\Theta(\tau_0)}$ one can define $K_f^{\tau_0} = K_b \subset \mathcal{F}_\tau$ for the unique $b \in \mathcal{F}_{\tau_0}$ corresponding to f.

Given a pair (τ, τ_0) and a $\Theta(\tau_0)$-Anosov representation (see Definition 2.2.2), Kapovich, Leeb and Porti define a domain of discontinuity in $\mathcal{F}_\tau$.

Theorem 2.6.10 ([KLP18a], Theorem 1.10)**.** *Let (τ, τ_0) be a pair of elements of $\mathbb{S}\mathfrak{a}^+$. Let $\rho : \Gamma_g \to G$ be a $\Theta(\tau_0)$-Anosov representation. The following is a domain of discontinuity for ρ:*

$$\Omega_\rho^{(\tau,\tau_0)} = \mathcal{F}_\tau \backslash \bigcup_{\zeta \in \partial\Gamma_g} K^{\tau_0}_{\xi_\rho^\Theta(\zeta)}.$$

Moreover if (τ, τ_0) is balanced, then the action of Γ_g via ρ on $\Omega_\rho^{(\tau,\tau_0)}$ is co-compact.

This theorem is a particular case of their result concerning Tits-Bruhat ideals. In [KLP18a] it is explained how a pair (τ, τ_0) yields a Tits-Bruhat ideal, defined via a *metric thickening.* This ideal is balanced if and only if the pair is balanced in our sense.

For a τ-nearly Fuchsian representation, the domain Ω_ρ^τ is always equal to some domain obtained by metric thickening. More precisely:

Theorem 2.6.11. *Let ρ be a τ-nearly Fuchsian representation of a hyperbolic group that is non-elementary. Recall that σ_ρ^τ and $\Theta(\sigma_\rho^\tau)$ were defined in Section 2.4.4. Let $\tau_0 \in \sigma_\rho^\tau$ be any element such that $\Theta(\tau_0) = \Theta(\sigma_\rho^\tau)$, whose existence is provided by Lemma 2.4.22.*

$$\Omega_\rho^\tau = \Omega_\rho^{(\tau,\tau_0)}.$$

The Theorem 2.6.10 from [KLP18a] is a domain of discontinuity since ρ is $\Theta(\sigma_\rho^\tau)$-Anosov by Theorem 2.4.23, and this domain is cocompact since the pair (τ, τ_0) is balanced.

Proof. Let us write $\Theta = \Theta(\sigma_\rho^\tau) = \Theta(\tau_0)$. Let $a \in \mathcal{F}_\tau \backslash \Omega_\rho^\tau$ and let $(y_n)_{n\in\mathbb{N}}$ be a diverging sequence of points in M such that $(b_{a,o}(u(y_n)))_{n\in\mathbb{N}}$ is bounded from above. Up to taking a subsequence let us assume that it converges to a point $\zeta \in \partial\Gamma \simeq \partial\widetilde{N}$. We consider the geodesic segments $[o, u(y_n)] \subset \mathbb{X}$ for $n \in \mathbb{N}$.

Since ρ is τ-nearly Fuchsian it is a quasi isometric embedding by Proposition 2.4.17 so in particular the length of these segments goes to $+\infty$. Up to taking a subsequence, we can assume that these geodesic segments converge to a geodesic ray $\eta : \mathbb{R}_{\geqslant 0} \to \mathbb{X}$ with $\eta(0) = o$. Let $[\eta] \in \partial_{\mathrm{vis}}\mathbb{X}$ be the point corresponding to the class of η.

Busemann functions are convex so the function $b_{a,o}$ is bounded from above on all the geodesic segments $[o, u(y_n)]$ for $n \in \mathbb{N}$ and hence $b_{a,o} \circ \eta$ is bounded from above. Therefore $\angle_{\mathrm{Tits}}(a, b) \leqslant \frac{\pi}{2}$ by Lemma 2.3.8, so $a \in K_{[\eta]}$. Let $b \in \mathcal{F}_{\tau_0}$ be an element such that b and $[\eta]$ belong to a common maximal facet and whose Cartan projection lies in σ_ρ^τ. Lemma 2.6.9 implies that $K_{[\eta]} = K_b$. Theorem 2.2.5 implies that $K_b = K^{\tau_0}_{\xi_\rho^\Theta(\zeta)}$. Therefore if $a \in \mathcal{F}_\tau \backslash \Omega_\rho^\tau$ then $a \in \mathcal{F}_\tau \backslash \Omega_\rho^{(\tau,\tau_0)}$.

Conversely let $a \in \Omega_\rho^\tau$ and let $\zeta \in \partial\Gamma$. Consider a geodesic ray $\eta : \mathbb{R}_{>0} \to \widetilde{N}$ for the metric $u^*(g_{\mathbb{X}})$ converging to ζ. Theorem 2.2.6 implies that there exist $D > 0$ such that for all $t > 0$, there exist a geodesic ray $\eta_t : \mathbb{R}_{>0} \to \mathbb{X}$ such that $\eta_t(0) = u \circ \eta(0)$, $\eta_t(t)$ is at distance at most D of $u \circ \eta(t)$ and $[\eta_t] \in \partial_{\mathrm{vis}}\mathbb{X}$ belongs to a common maximal facet with $\xi_\rho^\Theta(\zeta)$. Since $\mathcal{C}_\rho \subset \sigma_\rho^\tau$, for all t large enough, $K_{[\eta_t]} = K^{\tau_0}_{\xi_\rho^\Theta(\zeta)}$.

The Busemann function $b_{a,o}$ is proper on η hence for t large enough $b_{a,o} \circ \eta_t(t) > b_{a,o} \circ \eta_t(0)$. Since $b_{a,o}$ is convex, this implies that $b_{a,o}$ is growing at least linearly on η_t, so $a \notin K_{[\eta_t]}$ by Lemma 2.3.8. Therefore $a \in \Omega_\rho^{(\tau,\tau_0)}$: this concludes the proof. ☐

2.6.4 Invariance of the topology.

In this section we prove that the topology of the quotient of the domains of discontinuity considered by Kapovith-Leeb-Porti is not varying when the representation is deformed continuously. Guichard and Wienhard proved this already for the domains of discontinuity that they consider in [GW12].

Let Γ be a torsion-free finitely generated group and $\mathcal{F}$ a G-homogeneous space. Let $(\rho_t)_{t\in[0,1]}$ a smooth family of representations from Γ to G. Consider for every $t \in [0, 1]$ an open $\rho_t(\Gamma)$-invariant domain $\Omega_t \subset \mathcal{F}$.

Lemma 2.6.12. *Suppose that these domains are* uniformly *co-compact domains of discontinuity for (ρ_t), i.e. the domain $\Omega = \{(t, a) | a \in \Omega_t\} \subset [0, 1] \times \mathcal{F}$ is open and the action of Γ via ρ is properly discontinuous and co-compact where $\rho(\gamma) \cdot (t, a) = (t, \rho_t(\gamma) \cdot a)$. The quotient $\Omega_0/\rho_0(\Gamma)$ is diffeomorphic to $\Omega_1/\rho_1(\Gamma)$.*

Proof. The projection onto the first factor in $[0, 1] \times \mathcal{F}$ descends to a submersion $p : \Omega/\rho(\Gamma) \to \mathbb{R}$. Since $\Omega/\rho(\Gamma)$ is compact and the base is connected, Ehresmann's fibration theorem implies that the proper submersion p is a fibration. Hence $p^{-1}(0)$ and $p^{-1}(1)$ are diffeomorphic. ☐

Remarks 2.6.13. A concrete way to construct this diffeomorphism is to pick a Riemannian metric on $\Omega/\rho(\Gamma)$, and consider the flow of the gradient of p. If we consider two different Riemannian metrics, the diffeomorphisms obtained are isotopic, since the space of Riemannian metrics is path connected. Hence this operation constructs a unique diffeomorphism up to isotopy.

Note that a family of cocomact domains of discontinuity could be non-uniformly cocompact, for instance a family of representations so that $\rho_t : \mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$ is hyperbolic for $0 \leqslant t < 1$ and parabolic for $t = 1$. These representations admit a unique maximal domain of discontinuity in $\mathbb{RP}^1$ with two connected components for $t < 1$ and one for $t = 1$. The quotient of the corresponding domain Ω is homeomorphic to the non compact space $S^1 \times [0,1] \sqcup S^1 \times [0,1)$.

In order to apply Lemma 2.6.12, we need a slight adaptation of Theorem 2.6.10 from [KLP18a]. Let Γ be any Gromov-hyperbolic group. We check that the domains constructed by Kapovich, Leeb and Porti are uniformly cocompact domains of discontinuiy for any smooth path of Anosov representations.

Proposition 2.6.14 (Adaptation of [KLP18a], Theorem 1.10)**.** *Let (τ, τ_0) be a balanced pair as in Section 2.6.3. Let $\rho : [0,1] \to \mathrm{Hom}(\Gamma, G)$, $t \mapsto \rho_t$ be a continuous path such that the family $(\rho_t)_{t \in [0,1]}$ consists only of $\Theta(\tau_0)$-Anosov representations.*

The family of domains $\Omega_t = \Omega_{\rho_t}^{(\tau,\tau_0)}$ for $t \in [0,1]$ are uniformly co-compact domains of discontinuity for the family of representations ρ.

We check that the arguments from Kapovich, Leeb and Porti are uniform on neighborhoods of Anosov representations. The same proof holds if one considers more generally domains of discontinuity constructed with balanced Tits-Bruhat ideals as in [KLP18a].

Proof. The domain Ω is the complement in $[0,1] \times \mathcal{F}_\tau$ of:

$$K_\rho = \bigcup_{t \in [0,1]} \{t\} \times K_{\rho,t},$$

$$K_{\rho,t} = \bigcup_{x \in \partial\Gamma} K^{\tau_0}_{\xi^\Theta_{\rho_t}(x)}.$$

Since the boundary maps ξ^Θ_ρ are continuous and vary continuously when ρ varies continuously in the space of Θ-Anosov representations (see [BPS19, Section 6]), and since $K^{(\tau,\tau_0)}_{\xi^\Theta_\rho(x)}$ is compact, K_ρ is compact so $\Omega = \{(t,a) | a \in \Omega_t\} \subset [0,1] \times \mathcal{F}$ is open.

Let us fix a Riemannian distance d on $\mathcal{F}_\tau$. Let $A = \{(t,a) | t \in [0,1], a \in A_t \subset \Omega\}$ be a compact set and let $(\gamma_n) \in \Gamma$ be a diverging sequence. [KLP18a, Corollary 6.8] implies that given $t \in [0,1]$, for any $\epsilon > 0$ for all n large enough if $d(a, K_{\rho,t}) \geqslant \epsilon$, then $d(\rho_t(\gamma_n) \cdot a, K_{\rho,t}) \leqslant \epsilon$, where the minimal value of n needed depends on the constants b, c that come into play in the definition of Anosov representations (Definition 2.2.2).

Since we consider a compact set of Anosov representations, and since these constants can be chosen locally uniformly around a given Anosov representation (see [KLP17, Theorem 7.18]), these constants can be chosen uniformly for all representations $(\rho_t)_{t\in[0,1]}$. Moreover the compact sets A_t are at uniform distance from $K_{\rho,t}$ Therefore, for all $n \in \mathbb{N}$ large enough, for all $t \in \mathbb{R}$ $\rho_t(\gamma_n)\cdot A_t \cap A_t = \varnothing$, so for all n large enough $\rho(\gamma_n)\cdot A \cap A = \varnothing$. We have proven that the action of ρ on Ω is properly discontinuous.

In order to prove the cocompactness of the action on Ω, we will check that the transverse expansion holds uniformly. It follows from [KLP18a, Proposition 7.7] that for every $t \in [0,1]$, the action of ρ_t is *transversely expanding* at the limit set of ρ_t as in [KLP18a, Definition 5.21], i.e. for all $x \in \partial\Gamma$, there exist $\gamma \in \Gamma$, an open neighborhood U of $K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(x)}$ in $\mathcal{F}_\tau$ and a constant $\lambda > 1$ such that for all $a \in U$ and $y \in \partial_{\mathrm{vis}}\Gamma$ that satisfy $K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)} \subset U$ one has:

$$d\left(\rho_t(\gamma)\cdot a, \rho_t(\gamma)\cdot K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)}\right) \geqslant \lambda d\left(a, K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)}\right).$$

Let $g \in G$, $f \in \mathcal{F}_\Theta$ $\lambda > 1$ and $U \subset \mathcal{F}_\tau$ be an open set. We say that g *is exapnding* at $K^{\tau_0}_f$ over U with factor λ if for all $a \in U$:

$$d\left(g\cdot a, g\cdot K^{\tau_0}_f\right) \geqslant \mu d\left(a, K^{\tau_0}_f\right).$$

This property is open in the following sense: if g is expanding at $K^{\tau_0}_f$ over U with factor λ, then for any $1 < \lambda' < \lambda$ and any open subset E such that $\overline{E} \subset U$ there exist a neighborhood U_g of g in G and U_f of f in $\mathcal{F}_\Theta$ such that for all $g' \in U_g$ and $f' \in U_f$, g' is is expanding at $K^{\tau_0}_{f'}$ over E with factor λ'.

This implies that the action of ρ on $\mathcal{F}_\tau \times [0,1]$ satisfies the transverse expansion property where K_ρ is considered as a bundle over $[0,1]\times K$, i.e. for all $t \in [0,1]$, $x \in \partial\Gamma$, there exist $\gamma \in \Gamma$, an open neighborhood U_0 of $K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(x)}$ in $\mathcal{F}_\tau \times [0,1]$ and a constant $\lambda' > 1$ such that for all $a \in U$ and $y \in \partial\Gamma$ that satisfy $K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)} \times \{t\} \subset U_0$ one has:

$$d\left(\rho(\gamma)\cdot a, \rho_(\gamma)\cdot K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)} \times \{t\}\right) \geqslant \lambda' d\left(a, K^{\tau_0}_{\xi^{\Theta}_{\rho_t}(y)} \times \{t\}\right).$$

Therefore by [KLP18a, Proposition 5.26] the action of ρ is cocompact on Ω. ☐

From Proposition 2.6.14 and Lemma 2.6.12 we get the following corollary.

Corollary 2.6.15. *Assume that Γ is torsion-free. Let $\mathcal{C} \subset \mathrm{Hom}(\Gamma, G)$ be an open and connected set consisting only of Θ-Anosov representations for some $\Theta \subset \Delta$. Let (τ,τ_0) be a balanced pair such that $\forall\alpha \in \Delta\backslash\Theta$, $\alpha(\tau_0) = 0$. The diffeomorphism type of $\Omega^{(\tau,\tau_0)}_\rho/\rho(\Gamma)$ is independent of $\rho \in \mathcal{C}$*

If moreover $\mathcal{C}$ is simply connected, the diffeomorphism provided by Lemma 2.6.12 between $\Omega^{(\tau,\tau_0)}_{\rho_1}/\rho_1(\Gamma)$ and $\Omega^{(\tau,\tau_0)}_{\rho_2}/\rho_2(\Gamma)$ for $\rho_1, \rho_2 \in \mathcal{C}$ is uniquely determined up to isotopy.

An open set in $\mathrm{Hom}(\Gamma, G)$ is connected by paths that are piece-wise smooth if and only if it is connected since $\mathrm{Hom}(\Gamma, G)$ is locally a real algebraic variety.

2.7 Applications.

In this section we apply our results to prove that all representations in some connected components of Anosov representations are the restricted holonomy of a geometric structure on a fiber bundle over a manifold. For these applications we only consider nearly geodesic surfaces that are totally geodesic. We will mostly focus on surface groups, but in Section 2.7.5 we also describe two applications for representations of fundamental groups of higher dimensional compact hyperbolic manifolds.

2.7.1 Totally geodesic immersions.

Totally geodesic surfaces provide examples of τ-nearly geodesic surfaces if these surfaces are τ-regular (Proposition 2.4.9). The study of totally geodesic surfaces in $\mathbb{X}$ is related to the study of representation of semi-simple Lie algebras in $\mathfrak{g}$. We recall here a classical fact.

Proposition 2.7.1. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a semi-simple Lie subalgebra of non-compact type. Let H be the closed Lie subgroup of G with Lie algebra $\mathfrak{h}$ and $\mathbb{Y}$ it's associated symmetric space of non-compact type. There exist a H-equivariant and totally geodesic embedding $u_{\mathfrak{h}} : \mathbb{Y} \to \mathbb{X}$. The image of this embedding is unique up to the action of the centralizer:*

$$\mathcal{C}_G(\mathfrak{h}) = \{g \in G | \forall \mathrm{h} \in \mathfrak{h}, \, \mathrm{Ad}_g(\mathrm{h}) = \mathrm{h}\}.$$

If $y \in \mathbb{Y}$, let $K \subset G$ be the stabilizer of $u_{\mathfrak{h}}(y)$ in G. Every element in $\mathcal{C}_K(\mathfrak{h})$ of $\mathfrak{h}$ in G fixes $u_{\mathfrak{h}}(\mathbb{Y})$ pointwise. If the centralizer $\mathcal{C}_G(\mathfrak{h})$ in G is compact, then $\mathcal{C}_K(\mathfrak{h}) = \mathcal{C}_G(\mathfrak{h})$ so the totally geodesic submanifold $u_{\mathfrak{h}}(\mathbb{Y}) \subset \mathbb{X}$ is uniquely determined.

Proof. Let $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{h}$ associated with the Cartan involution θ_y for $y \in \mathbb{Y}$. Let $\widehat{\mathfrak{h}} = \mathfrak{k} + i\mathfrak{p}$ be the associated compact real form of $\mathfrak{h} \otimes \mathbb{C}$. Since $\widehat{\mathfrak{h}}$ is the Lie algebra of a semi-simple compact Lie group, it is the Lie algebra of a compact Lie subgroup of $G^{\mathbb{C}}$. In particular there exist a Cartan involution $\theta^{\mathbb{C}}$ of $\mathfrak{g} \otimes \mathbb{C}$. such that $\theta^{\mathbb{C}}(\mathrm{v}) = \mathrm{v}$ for all $\mathrm{v} \in \widehat{\mathfrak{h}}$.

Let $\overline{\theta^{\mathbb{C}}}$ be the Cartan involution conjugate to $\theta^{\mathbb{C}}$ in $\mathfrak{g} \otimes \mathbb{C}$. Let θ be the Cartan involution of $\mathfrak{g} \otimes \mathbb{C}$ corresponding to the midpoint in the symmetric space associated with $\mathfrak{g} \otimes \mathbb{C}$ of $\overline{\theta^{\mathbb{C}}}$ and $\theta^{\mathbb{C}}$. It is invariant by conjugation, so it descends to a Cartan involution on $\mathfrak{g}$, associated with a point $x \in \mathbb{X}$. There exist a transvection along the geodesic between $\overline{\theta^{\mathbb{C}}}$ and $\theta^{\mathbb{C}}$ in the symmetric space associated to $\mathfrak{g} \otimes \mathbb{C}$ that induces a unique inner automorphism ϕ of $\mathfrak{g} \otimes \mathbb{C}$ such that $\phi\theta^{\mathbb{C}}\phi^{-1} = \theta$, $\phi^2\theta^{\mathbb{C}}\phi^{-2} = \overline{\theta^{\mathbb{C}}}$. Moreover ϕ is symmetric positive with respect to $B^{\mathbb{C}}\left(\cdot, \theta^{\mathbb{C}}(\cdot)\right)$, where $B^{\mathbb{C}}$ is the Killing form on $\mathfrak{g} \otimes \mathbb{C}$ and the transvection ϕ^4 is equal to a composition of symmetries $\theta^{\mathbb{C}}\overline{\theta^{\mathbb{C}}}$. The transvection ϕ^4 stabilizes $\mathfrak{h} \otimes \mathbb{C}$, so ϕ also stabilizes $\mathfrak{h} \otimes \mathbb{C}$. Therefore $\theta = \phi\theta^{\mathbb{C}}\phi^{-1}$ stabilizes $\mathfrak{h} \otimes \mathbb{C}$. But θ is a real Cartan involution as it is preserved by conjugation, so it stabilizes $\mathfrak{h}$. Let $x \in \mathbb{X}$ be the point corresponding to θ in $\mathbb{X}$.

The map $u_{\mathfrak{h}} : \mathbb{Y} \to \mathbb{X}$ such that for all $h \in H$, $u_{\mathfrak{h}}(h \cdot y) = h \cdot x$ is well defined since $\mathfrak{t} \subset \mathfrak{t}_x$ so the image of the stabilizer of $y \in \mathbb{Y}$ by H lies in the stabilizer of $x \in \mathbb{X}$. Moreover it is totally geodesic, since $\eta(\mathfrak{p}) \subset \mathfrak{p}_x$ (see [Hel78] Ch 4, Section 7). This map is by definition H-equivariant.

Suppose that there is an other H-equivaiant and totally geodesic embedding $u'_{\mathfrak{h}}$ such that $u'_{\mathfrak{h}}(y) = x'$. Let θ' be the corresponding Cartan involution of $\mathfrak{g}$, then $\theta' \circ \theta$ is the identity on $\mathfrak{h}$ and is equal to the adjoint action of $\exp(\mathrm{z})$ for some $\mathrm{z} \in \mathfrak{p}_x$. Therefore z is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Hence $g = \exp(\frac{\mathrm{z}}{2}) \in \mathcal{C}_G(\mathfrak{h})$ satisfies $\mathrm{Ad}_g \circ u'_{\mathfrak{h}} = u_{\mathfrak{h}}$. Conversely let $g' \in \mathcal{C}_K(\mathfrak{h})$, then g' fixes $u_{\mathfrak{h}}(y) \in \mathbb{X}$ and it fixes $\mathfrak{h}$ so it fixes $\mathrm{d}u_{\mathfrak{h}}(T_y\mathbb{Y})$. Therefore it preserves and acts trivially on $u_{\mathfrak{h}}(\mathbb{Y})$.

Now assume that $\mathcal{C}_G(\mathfrak{h})$ is compact: there cannot be any element $\mathrm{z} \in \mathfrak{p}_x$ in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, so the totally geodesic and H-equivariant embedding $u_{\mathfrak{h}}$ is unique and $\mathcal{C}_K(\mathfrak{h}) = \mathcal{C}_G(\mathfrak{h})$. □

Remark 2.7.2. A totally geodesic embedding $u : \mathbb{Y} \to \mathbb{X}$ can only be a τ-nearly geodesic immersions if $\mathrm{rank}(\mathbb{Y}) = 1$, because otherwise it cannot be τ-regular for any $\tau \in \mathbb{S}\mathfrak{a}^+$ (see Proposition 2.5.11). When $\mathrm{rank}(\mathbb{Y}) = 1$, all the unit tangent vectors to this embedded surface have the same Cartan projection, so the embedding is τ-regular for all τ in the complement in $\mathbb{S}\mathfrak{a}^+$ of a finite collection of hyperplanes.

We illustrate Proposition 2.7.1 in the following example for some special $\mathfrak{sl}_2$ Lie subalgebras in $\mathfrak{sl}_n(\mathbb{R})$.

Example 2.7.3. In this example we construct representations ι from $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{K})$ that stabilize some totally geodesic hyperbolic planes inside the symmetric space $\mathbb{X} = \mathcal{S}_n$ associated with $G = \mathrm{SL}(n,\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $V_n(\mathbb{K})$ be the space of homogeneous polynomial with coefficients in $\mathbb{K}$ of degree $n-1$ in two variables X and Y. To an element $g \in \mathrm{SL}_2(\mathbb{R})$ one can associate an element $\iota_{\mathrm{irr}}(g) \in \mathrm{SL}(V_n(\mathbb{K}))$ that acts by a change of variable on $V_n(\mathbb{K})$ i.e that associates to $P \in V_n(\mathbb{K})$ the polynomial $P \circ g^{-1}$. Let $\mathfrak{h}$ be the corresponding $\mathfrak{sl}_2$-Lie subalgebra of $\mathfrak{sl}_n(\mathbb{R})$, note that $\iota_{\mathrm{irr}} = \iota_{\mathfrak{h}}$.

Let q the Euclidean or Hermitian metric on $V_n(\mathbb{K})$ such that for all $0 \leqslant a, b \leqslant n-1$:

$$q\left(X^a Y^{n-1-a}, X^b Y^{n-1-b}\right) = \begin{cases} \binom{n-1}{a}^{-1} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following basis of the lie algebra $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which satisfies $[\mathbf{g}, \mathbf{h}] = -2\mathbf{f}$, $[\mathbf{h}, \mathbf{f}] = 2\mathbf{g}$, $[\mathbf{f}, \mathbf{g}] = 2\mathbf{h}$. Fix the following orthonormal basis for q :

$$(e_a)_{0 \leqslant a \leqslant n-1} = \left(X^a Y^{n-1-a} \binom{n-1}{a}^{\frac{1}{2}} \right)_{0 \leqslant a \leqslant n-1}$$

.

For $0 \leqslant a \leqslant n-1$ one has :

$$\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{f})(e_a) = (2a - n + 1)e_a$$

$$\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{e})(e_a) = aX^{a-1}Y^{n-a}\binom{n-1}{a}^{\frac{1}{2}} - (n-1-a)X^{a+1}Y^{n-2-a}\binom{n-1}{a}^{\frac{1}{2}}$$

$$\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{e})(e_a) = \sqrt{a(n-a))}e_{a-1} - \sqrt{(a+1)(n-1-a)}e_{a+1}$$

and moreover $\mathbf{g} = \frac{1}{2}[\mathbf{f}, \mathbf{h}]$. In particular $\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{f})$ and $\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{g})$ are symmetric or Hermitian and $\mathrm{d}\iota_{\mathrm{irr}}(\mathbf{h})$ is anti-symmetric or anti-Hermitian with respect to q.

Hence as in the proof of Propostion 2.7.1, there is a totally geodesic map $u_{\mathfrak{h}} : \mathbb{H}^2 \to \mathcal{S}_n$ such that the image of the Cartan involution $M \mapsto -M^t$ is equal to the point in $\mathcal{S}_n$ corresponding to q.

This representation ι_{irr} is the unique irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$ up to conjugation by elements of $\mathrm{GL}(n,\mathbb{R})$. The image of ι_{irr} lies in the subgroup $\mathrm{Sp}(2k,\mathbb{R})$ for some symplectic form on $\mathbb{R}^{2k}$ when $n = 2k$ is even, and in $\mathrm{SO}(k,k+1)$ for some quadratic form on $\mathbb{R}^{2k+1}$ when $n = 2k+1$ is odd.

One can construct other totally geodesic hyperbolic planes in $\mathcal{S}_n$ by considering representations of $\mathrm{SL}(2,\mathbb{R})$ that can be decomposed into a direct sum of irreducible representations. Equivalently one can consider other reducible $\mathfrak{sl}_2$-subalgebras of $\mathfrak{sl}_n(\mathbb{R})$.

For instance one can define $\iota_{\mathrm{red}} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2n,\mathbb{R})$ that associates to a matrix $M \in \mathrm{SL}(2,\mathbb{R})$ the block diagonal matrix:

$$\iota_{\mathrm{red}}(M) = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}.$$

The image of ι_{red} lies in $\mathrm{Sp}(2n,\mathbb{R})$ for some symplectic form ω on $\mathbb{R}^{2n}$.

2.7.2 Geometric structures on fiber bundles.

Using the projection defined by Busemann functions from Theorem 2.6.3, one can show that Anosov deformations of representation that admit an equivariant nearly geodesic immersion are holonomies of (G, X) structures on a fiber bundle over S_g.

A (G, X)-structure on a manifold N, for a Lie group G and a G-homogeneous space X on which G acts faithfully, is a maximal atlas of charts of M valued in X whose transition functions are the restriction of the action of some elements in G. A more developed introduction to this notion can be found in [Ale19].

To a (G, X)-structure, one can associate a *developing map* $\mathrm{dev} : \widetilde{N} \to X$, that is a local diffeomorphism compatible with the atlas defining the (G, X)-structure on N, and a *holonomy* $\mathrm{hol} : \pi_1(N) \to G$ so that dev is hol-equivariant. This pair is unique up to the action of G by conjugation of the holonomy and post-composition of the developing map.

Let N be a manifold and Γ its fundamental group. In what follows, we say that a (G, X)-structure *on a fiber bundle* F over N is a (G, X)-structure on F for which the fundamental group of the fibers is included in the kernel of the holonomy. Hence one can define the *restricted holonomy* of the structure as the quotient map $\rho : \pi_1(N) \to G$ induced by the holonomy.

Constructing domains of discontinuity allows us to construct geometric structures.

Proposition 2.7.4. *Let $\rho : \Gamma \to G$ be a representation and $\Omega \subset X$ a co-compact non-empty domain of discontinuity which fibers ρ-equivariantly over $\widetilde{N}$. Any connected component of the quotient $\Omega/\rho(\Gamma)$ inherits a (G, X)-structure* on a fiber bundle*, with restricted holonomy ρ.*

Note that even if Ω is disconnected, the quotient $\Omega/\rho(\Gamma)$ can be connected.

From now on, we assume that G is center-free, so it acts faithfully on its flag manifolds. Let N be a compact manifold whose fundamental group Γ is Gromov hyperbolic and torsion-free. Let $\tau \in \mathbb{S}\mathfrak{a}^+$.

Theorem 2.7.5. *Let $\rho_0 : \Gamma \to G$ be a representation that admits an equivariant τ-nearly geodesic surface $u : \widetilde{N} \to \mathbb{X}$ such that $\Omega_u^\tau \neq \varnothing$. Let $\mathcal{C}$ be the connected component of the space of $\Theta(\sigma_{\rho_0}^\tau)$-Anosov representations in* $\mathrm{Hom}(\Gamma, G)$ *containing ρ_0. Every representation in $\mathcal{C}$ is the restricted holonomy of a $(G, \mathcal{F}_\tau)$-structure* on a fiber bundle *F over N.*

Proof. Theorem 2.6.8 implies that the domains Ω_u^τ admits a ρ_0-equivariant fibration over $\widetilde{N}$. The domain $\Omega_{\rho_0}^\tau$ coincides with a domain obtained as a metric thickening $\Omega_{\rho'}^{(\tau,\tau_0)}$ for some $\tau_0 \in \mathbb{S}\mathfrak{a}^+$ such that $\Theta(\tau_0) = \Theta(\sigma_{\rho_0}^\tau)$. Let F be a connected component of $\Omega_{\rho_0}^\tau/\rho_0(\Gamma)$, Corollary 2.6.15 implies that for every representation $\rho \in \mathcal{C}$, a connected component F_ρ of $\Omega_\rho^{(\tau,\tau_0)}/\rho(\Gamma)$ is diffeomorphic to F, which is a fiber bundle over M. The covering map $\Omega_\rho^{(\tau,\tau_0)} \to \Omega_\rho^{(\tau,\tau_0)}/\rho(\Gamma)$ induces the covering $\widehat{F_\rho} \to F_\rho \simeq F$ associated to the subgroup of the fundamental group of F corresponding to the fundamental group of the fiber of F over M.

Note that F is assumed to be non-empty. The $(G, \mathcal{F}_\tau)$-structure on $F_\rho \simeq F$ is such that the holonomy of the fundamental group of each fiber is trivial. Indeed the developing map $\mathrm{dev} : \widetilde{F_\rho} \to \mathcal{F}_\tau$ descends to the inclusion $\widehat{F_\rho} \to \mathcal{F}_\tau$, so the fundamental group of the fiber belongs to the kernel of the holonomy. $\square$

Let $M = S_g$ be a closed orientable surface of genus g and $\Gamma = \Gamma_g$ be its fundamental group. We apply Theorem 2.7.5 in cases when the nearly geodesic surface is totally geodesic and we describe the fibration that is obtained.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a $\mathfrak{sl}_2$ Lie subalgebra, i.e. a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Note that if G is a quotient of its adjoint form, which is the case since we assumed that G is center-free, the corresponding Lie group H is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$. We write $\iota_{\mathfrak{h}} : \mathrm{SL}(2,\mathbb{R}) \to G$ the corresponding Lie group representation, and $u_{\mathfrak{h}} : \mathbb{H}^2 \to \mathbb{X}$ a corresponding equivariant totally geodesic embedding.

Definition 2.7.6. We say that a representation $\rho : \Gamma_g \to G$ is $\mathfrak{h}$-*generalized Fuchsian* if it preserves and acts cocompactly on $u_{\mathfrak{h}}\left(\mathbb{H}^2\right)$.

If G is center-free and, a $\mathfrak{h}$-generalized Fuchsian representation can be written $\gamma \mapsto \iota_{\mathfrak{h}}(\rho_0(\gamma)) \times \chi(\gamma)$ for some Fuchsian representation $\rho_0 : \Gamma_g \to \mathrm{SL}(2,\mathbb{R})$ and some *associated character* $\chi : \Gamma_g \to \mathcal{C}_K(\mathfrak{h})$.

Let $y \in \mathbb{H}^2$ be a base-point, and let K be the stabilizer in G of $u_{\mathfrak{h}}(y)$. We write $\mathcal{B}^\tau(\mathfrak{h})$ for the τ-base of the pencil of tangent vectors $\mathrm{d}u_{\mathfrak{h}}(T_y\mathbb{H}^2)$. Note that this pencil is stabilized by $\iota_{\mathfrak{h}}\left(\mathrm{SO}(2,\mathbb{R})\right) \times \mathcal{C}_K(\mathfrak{h})$.

The quotient map $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) \simeq \mathbb{H}^2$ defines a principal $\mathrm{SO}(2,\mathbb{R})$-bundle $\widetilde{P}$ over $\mathbb{H}^2$. Let P_{S_g} be its quotient via some Fuchsian representation. Given a character $\chi : \Gamma_g \to \mathcal{C}_K(\mathfrak{h})$, let $P_{S_g,\chi} \to S_g$ be the $\mathrm{SO}(2,\mathbb{R}) \times \mathcal{C}_K(\mathfrak{h})$-principal bundle obtained as the product of P_{S_g} and the flat $\mathcal{C}_K(\mathfrak{h})$-bundle associated to χ.

Theorem 2.7.7. *Suppose that* $\mathfrak{h}$ *is* τ-regular*, i.e.* $u_{\mathfrak{h}}$ *is* τ*-regular, and that* $\Omega^\tau_{u_{\mathfrak{h}}} \neq \varnothing$*. Let* $\rho : \Gamma_g \to G$ *be a* $\mathfrak{h}$*-generalized Fuchsian representation with associated character* χ *and let* $\tau \in \mathbb{S}\mathfrak{a}^+$*.*

Let $\mathcal{C}$ *be the connected component of the space of* $\Theta(\sigma^\tau_\rho)$*-Anosov representations that contains* ρ*. Every representations in* $\mathcal{C}$ *is the restricted holonomy of a* $(G, \mathcal{F}_\tau)$*-structure* on a fiber bundle F *over* S_g*.*

The fiber bundle F *is a connected component of the reduction of the principal bundle* $P_{S_g,\chi}$ *with structure group* $\mathrm{SO}(2,\mathbb{R}) \times \mathcal{C}_K(\mathfrak{h})$ *over* S_g *via the action of* $\iota_{\mathfrak{h}}\left(\mathrm{SO}(2,\mathbb{R})\right) \times \mathcal{C}_K(\mathfrak{h})$ *on* $\mathcal{B}^\tau(\mathfrak{h})$*.*

Note that when $\Theta \subset \Delta$ is a Weyl orbit of simple roots and $\tau = \tau_\Theta$, a τ_Θ-regular immersion is just a Θ-regular immersion and $\Theta(\sigma^\tau_\rho) = \Theta$.

Remark 2.7.8. If we replace G and $\mathcal{F}_\tau$ by finite covers $\widehat{G}$ and $\widehat{\mathcal{F}_\tau}$ so that $\widehat{G}$ acts faithfully on $\widehat{\mathcal{F}_\tau}$, Theorem 2.7.7 still applies. In this case one should replace $\mathcal{B}^\tau(\mathfrak{h})$ by its pre-image $\widehat{\mathcal{B}^\tau(\mathfrak{h})}$ by the covering map $\widehat{\mathcal{F}_\tau} \to \mathcal{F}_\tau$.

The only part of Theorem 2.7.7 that is not already contained in 2.7.5 is the description of the fibration.

Proof. The embedding $u_{\mathfrak{h}}$ is totally geodesic. Moreover it is assumed to be τ-regular. Hence $u_{\mathfrak{h}}$ is τ-nearly geodesic, so one can apply Theorem 2.7.5. It only remains to describe the fibration.

The fibration $\pi : \Omega^{\tau}_{u_{\mathfrak{h}}} \to \mathbb{H}^2$ is $\iota_{\mathfrak{h}}(\mathrm{SL}(2,\mathbb{R}))$-equivariant. The corresponding fiber bundle can be identified in a $\mathrm{SL}(2,\mathbb{R})$-equivariant way as the reduction of the $\mathrm{SO}(2,\mathbb{R})$-principal bundle $\widetilde{P}$ by the action of $\iota_{\mathfrak{h}}(\mathrm{SO}(2,\mathbb{R}))$ on the fiber $\mathcal{B}^{\tau}(\mathfrak{h})$. The quotient of this fiber bundle by any Fuchsian representation $\rho_0 : \Gamma_g \to \mathrm{SL}(2,\mathbb{R})$ is hence the bundle induced by the principal bundle P_{S_g}. Once we quotient $\Omega^{\tau}_{u_{\mathfrak{h}}}$ by $\rho = \iota_{\mathfrak{h}} \circ \rho_0 \times \chi$ the quotient becomes the twisted fiber bundle induced by $P_{S_g,\chi}$. $\square$

2.7.3 Higher rank Teichmüller spaces.

In this section we apply Theorem 2.7.7 to Hitchin representations in $\mathrm{PSL}(n,\mathbb{R})$ and to maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$. Then we explain how in general one can apply it to the connected components of Θ-positive representations containing at least one generalized Fuchsian representation associated to a Θ-principal $\mathfrak{sl}_2$-Lie subalgebra.

2.7.3.1 Positive representations.

Let G be a connected simple Lie group of non-compact type with trivial center and Θ a set of its simple roots such that the pair (G,Θ) admits a notion of Θ-positivity in the sense of [GW22]. We moreover assume that G is not locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. This means that either :

(i) G is split real and $\Theta = \Delta$,

(ii) G is Hermitian of tube type,

(iii) G is locally isomorphic to $\mathrm{SO}(p,q)$ and $\Theta = \{\alpha_1, \cdots, \alpha_p\}$

(iv) G is a real form of the complex Lie group with Dynkin diagram F_4, E_6, E_7 or E_8 with restricted Dynkin diagram F_4 and Θ consists of the 2 larger roots.

For any of these pairs, Guichard and Wienhard [GW22] constructed a connected component U in the space and transverse triples of elements in G/P_Θ. They call such triples *positive triples*, and a representation $\rho : \Gamma_g \to G$ is called Θ-positive if it admits a continuous and ρ-equivariant map $\xi : \partial\Gamma \to G/P_\Theta$ so that for all distinct triple of points $(x,y,z) \in \partial\Gamma^{(3)}$, $(\xi(x),\xi(y),\xi(z)) \in U$. They prove in particular that such representations are Θ-Anosov. Moreover the space of Θ-positive representation is closed in the space of representation that do not virtually factor through a parabolic subgroup, by a work of Guichard, Labourie and Wienhard [GLW21].

A Θ-principal Lie subalgebra $\mathfrak{h}_\Theta$ for a pair (G,Θ) that admits a notion of Θ-positivity is a principal subalgebra of the split Lie subalgebra $\mathfrak{g}_\Theta \subset \mathfrak{g}$ generated by all the rootspaces associated to Θ, see [GW22]. These Lie subalgebra were introduced by Bradlow, Collier Gothen and Garci-Prada as *magical triples* in [BCGP+21]. They proved the following. Let $\mathfrak{h}_\Theta$ be a Θ-principal $\mathfrak{sl}_2$ Lie subalgebra of $\mathfrak{g}$.

Theorem 2.7.9. *[BCGP+21, Theorem 8.8] The exist a union of connected components of $\rho : \Gamma \to G$, the* Cayley components*, consisting only of representations that do not factor through any parabolic subgroup. All $\mathfrak{h}_\Theta$-generalized Fuchsian representations with respect to the principal $\mathfrak{sl}_2$ Lie subalgebra $\mathfrak{h}_\Theta$ lie in some Cayley component.*

Cayley components are conjectured to be all the connected components of Θ-positive representations [BCGP+21], but there exist components consisting of positive representations that do not contain $\mathfrak{h}_\Theta$-generalized representations, for instance the Gothen components for $G = \mathrm{Sp}(4, \mathbb{R})$. The results of Guichard, Labourie implies the following [GLW21].

Corollary 2.7.10. *Every connected component of representations $\rho : \Gamma_g \to G$ containing a $\mathfrak{h}_\Theta$-generalized Fuchsian representation consist only of Θ-positive representations.*

The sets of simple roots $\Theta \subset \Delta$ that admit a notion of Θ-positivity aways admit one or two subset which are Weyl orbits of simple roots, see Figure 2.4. Let $\Theta' \subset \Theta$ be a Weyl orbit of simple roots. Let $G \neq \mathrm{PSL}(2, \mathbb{R})$, Theorem 2.7.7 implies:

Corollary 2.7.11. *Let $\rho : \Gamma_g \to G$ be a representation in a connected component of Θ-positive representations containing a $\mathfrak{h}_\Theta$-generalized Fuchsian representation. It is the restricted holonomy of a $(G, \mathcal{F}_{\tau_{\Theta'}})$-structure on a fiber bundle F over S_g.*

Let χ be the character associated to one of the $\mathfrak{h}_\Theta$-generalized Fuchsian representations in the Cayley component. This fiber bundle is diffeomorphic to the reduction of the $\mathrm{SO}(2, \mathbb{R}) \times \mathcal{C}_K(\mathfrak{h})$-bundle $P_{S_g,\chi}$ via its action on the base $\mathcal{B}^\tau(\mathfrak{h})$ of the pencil of tangent vectors associated to $\mathfrak{h}$.

The proof of the fact that the associated domains are non empty as soon as G is not isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ is delayed to Section 2.7.4.

2.7.3.2 Hitchin representations in $\mathrm{PSL}(n, \mathbb{R})$.

Let $\mathfrak{h}$ be a principal $\mathfrak{sl}_2$ Lie sugalgebra in $\mathfrak{sl}_n(\mathbb{R})$. The associated representation $\iota_\mathfrak{h}$ is the representation ι_{irr} from Example 2.7.3.

Definition 2.7.12. A representation $\rho : \Gamma_g \to \mathrm{PSL}(n, \mathbb{R})$ is *Hitchin* if it is a deformation in $\mathrm{Hom}(\Gamma_g, G)$ of a $\mathfrak{h}$-generalized Fuchsian representaton.

The centralizer of $\mathfrak{h}$ in $\mathrm{PSL}(n, \mathbb{R})$ is trivial, so $\mathfrak{h}$-generalized Fuchsian representations can be written $\iota_{\mathrm{irr}} \circ \rho_0$.

Hitchin proved that the quotient of the space of Hitchin representations by conjugation in $\mathrm{PGL}(n, \mathbb{R})$ is a ball of dimention $(2g-2)(n^2-1)$ [Hit92]. Labourie proved that Hitchin representations are Borel Anosov, i.e. Δ-Anosov [Lab06].

The unique Weyl orbit of simple roots for $G = \mathrm{PSL}(n, \mathbb{R})$ is Δ. The flag manifold $\mathcal{F}_{\tau_\Delta}$ can be identified with the flag manifold $\mathcal{F}_{1,n-1}$ consisting of pairs of subspaces (ℓ, H) where $\ell \subset H \subset \mathbb{R}^2$, $\dim(\ell = 1)$, $\dim(H) = n-1$. The $\mathfrak{sl}_2$ Lie subalgebra $\mathfrak{h}$ is Δ-regular. Theorem 2.7.7 implies the following:

Corollary 2.7.13. *Let $n \geqslant 3$, every Hitchin representation $\rho : \Gamma_g \to \mathrm{PSL}(n, \mathbb{R})$ is the restricted holonomy of a $(\mathrm{PSL}(n, \mathbb{R}), \mathcal{F}_{1,n-1})$-structure on a fiber bundle over S_g.*

When $n = 3$, the boundary map of any $\mathfrak{h}$-generalized Fuhsian representation is an ellipsoid $\mathcal{E} \subset \mathbb{RP}^2$. The domain $\Omega_\rho^{\tau_\Delta} \subset \mathcal{F}_{1,2}$ admits 3 connected components: the set of $(\ell, H) \in \mathcal{F}_{1,2}$ with ℓ in the inside of the ellipsoid $\mathcal{E}$, with H completely outside of the ellipsoid, and finally with ℓ outside the ellipsoid and H crossing the ellipsoid in two points. One can see that the quotient of this domain is the union of three copies of the projectivization of the tangent bundle of S_g. If we apply Theorem 2.6.3 to $u_\mathfrak{h}$, we obtain a fibration where the model fiber $\mathcal{B}^\tau(\mathfrak{h})$ is the union of 3 circles described in Example 2.5.9. Also when $n = 3$ one can get a domain in projective space, as described in Example 2.6.7 that is the interior of an ellipse.

When n is even, Hitchin representations arTe also the holonomy of projective structures. his domain was shown in [ADL21] to fiber over $\widetilde{S_g}$, and Theorem 2.7.7 gives a new proof of this Fact.

In general for any split simple Lie group G, Fock and Goncharov proved that all Δ-positive representations can be deformed into a $\mathfrak{h}_\Delta$-generalized Fuchsian representation [FG06], i.e. lie in a Hitchin component, so our method always applies.

2.7.3.3 Maximal representations in $\mathbf{PSp}(2n, \mathbb{R})$.

Given an orientation of the surface S_g, one can define the *Toledo invariant* $\mathrm{Tol} : \mathrm{Hom}(\Gamma_g, \mathrm{PSp}(2n, \mathbb{R})) \to \mathbb{Z}$. This continuous map can de defined as the pullback by ρ of an element of the continous group cohomology $H^2(G, \mathbb{Z})$ of G by ρ [BIW03]. Reversing the orientation of S_g reverses the sign of the Toledo invariant.

A representation $\rho : \Gamma_g \to \mathrm{PSp}(2n, \mathbb{R})$ is called *maximal* if its Toledo invariant is maximal among all representations, i.e. if $\mathrm{Tol}(\rho) = n(2g-2)$. A way to construct maximal representation is to use the representation $\iota_{\mathrm{red}} : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSp}(2n, \mathbb{R})$. Burger, Iozzi, Labourie and Wienhard proved that maximal representations are $\{\alpha_n\}$-Anosov [BILW05].

Let $\mathfrak{h}$ be the $\mathfrak{sl}_2$ Lie subalgebra of $\mathfrak{sp}_{2n}(\mathbb{R})$ which is the image of $\mathrm{d}\iota_\mathfrak{h}$. Every h-generalized representation is maximal for one of the two orientations of the surface S_g.

Theorem 2.7.14 ([GW10]). *If $n \geqslant 3$, every maximal representation $\rho : \Gamma_g \to \mathrm{PSp}(2n, \mathbb{R})$ can be deformed into a $\mathfrak{h}_{\{\alpha_n\}}$-generalized Fuchsian representation in the sense of Definition 2.7.6.*

Theorem 2.7.7 implies :

Corollary 2.7.15. *Let $n \geqslant 2$, every maximal representation $\rho : \Gamma_g \to \mathrm{PSp}(2n, \mathbb{R})$ is the holonomy of a contact projective structure i.e. a $\big(\mathrm{PSp}(2n, \mathbb{R}), \mathbb{RP}^{2n-1}\big)$-structure on a fiber bundle .*

Indeed one can consider the Weyl orbit of simple roots $\Theta = \{\alpha_n\}$. The corresponding flag manifold $\mathcal{F}_{\tau_{\{\alpha_n\}}}$ can be identified with $\mathbb{RP}^{2n-1}$. The Lie subalgebra $\mathfrak{h}$ is $\{\alpha_n\}$-regular, so one can apply Theorem 2.7.7.

It is not clear if our method applies to the *Gothen components* of representations $\rho : \Gamma_g \to \mathrm{PSp}(4, \mathbb{R})$ that contain only Zariski-dense representations [BGPG12]. However since $\mathrm{PSp}(4, \mathbb{R}) \simeq \mathrm{SO}_o(2, 3)$, the case $n = 2$ of Corollary 2.7.15 is a consequence of the work of Collier, Tholozan, Toulisse [CTT19].

The fiber obtained for Hitchin representations, that are also $\{\alpha_n\}$-positive, is a union of connected components of $\mathcal{B}^{\tau_{\{\alpha_n\}}}(\mathfrak{h}_\Delta)$ and for maximal representations one gets a union of connected components of $\mathcal{B}^{\tau_{\{\alpha_n\}}}(\mathfrak{h}_\Theta)$. These two submanifolds of $\mathbb{RP}^{2n-1}$ are diffeomorphic to the same Stiefel manifold, which is connected if $n \geqslant 3$.

2.7.3.4 Positive representations in $\mathbf{PSO}(p, q)$.

Let $G = \mathrm{PSO}(p, q)$ with $q > p$ and $\Theta = \Delta \backslash \{\alpha_p\}$. This pair admits a notion of Θ-positivity. The corresponding flag manifold $\mathcal{F}_{\tau_\Theta}$ can be identified with the Grassmanian of isotropic planes in $\mathbb{R}^{p,q}$.

Representations satisfying the Θ-positive property were studied by Beyer and Pozzetti [BP21]. In particular they show that all Θ-positive representations $\rho : \Gamma_g \to \mathrm{PSO}(p, q)$ can be deformed to a $\mathfrak{h}_\Theta$-generalized Fuchsian representation when $q > p + 1$, so in this case Corollary 2.7.11 applies to all Θ-positive representations. However when $q = p + 1$, there are connected components of Θ-positive representations that are conjectured to contain only Zariski-dense representations, it is not clear if our techniques can be applied to these components.

2.7.4 Non-empty domains.

In order to get a geometric structure associated to a domain of discontinuity, one need to ensure that the domain is non-empty. Kapovich, Leeb and Porti have a condition that ensures that there exist a thickening such that the domain is not empty [KLP18a], and Guichard and Wienhard proved that the domains they considered were not empty by computing the dimension of their complement.

We will use the following criterion to prove that some domains of discontinuity for surface groups are non-empty. Remember that the groups Γ_g that we consider here are surface groups.

Lemma 2.7.16. *Let $\rho : \Gamma_g \to G$ be a Θ-Anosov representation and (τ, τ_0) be a balanced type pair of elements in $\mathbb{S}\mathfrak{a}^+$, i.e. such that τ_0 is τ-regular, satisfying $\Theta(\tau_0) = \Theta$. Suppose that $\mathcal{F}_\tau$ has finite fundamental group. The domain of discontinuity $\Omega_\rho^{(\tau,\tau_0)}$ is non-empty.*

Note that it is however not a necessary condition to have a non-empty domain, as we see later for $\mathrm{SO}(2,3)$. This lemma is very similar to Proposition 2.5.10.

Proof. If the domain is empty, it means that the flag manifold $\mathcal{F}_\tau$ can be written as the union for $x \in \partial\Gamma_g$ of the thickenings $K^{\tau_0}_{\xi^\Theta_\rho(x)}$. This uniojn is disjoint since the limit map is transverse. Moreover the limit map ξ^Θ_ρ is continuous, see [BPS19] for instance, which implies that $\mathcal{F}_\tau$ fibers over the circle with a compact base. As in Proposition 2.5.10, this implies that $\mathcal{F}_\tau$ has infinite fundamental group. $\square$

We prove that some domains of discontinuity are non-empty for representations of a surface group Γ_g. Let (G, Θ) be a pair that admits a notion of positivity and let $\Theta' \subset \Theta$ be a Weyl orbit of simple roots.

Proposition 2.7.17. *The flag manifold $\mathcal{F}_{\tau_{\Theta'}}$ has finite fundamental group, except if G is locally isomorphic to* $\mathrm{PSL}(2,\mathbb{R})$ *or if G is locally isomorphic to* $\mathrm{SO}_o(2,3)$*, $\Theta = \Delta$, and $\Theta' = \{\alpha_2\}$.*

If G is locally isomorphic to $\mathrm{SO}_o(2,3)$, $\Theta = \Delta$, and $\Theta' = \{\alpha_2\}$ we work by hand using the notations from Section 2.1.5. The domain of discontinuity associated to a $\{\alpha_2\}$-Anosov representation $\rho : \Gamma_g \to \mathrm{SO}_o(2,3)$ obtained by metric thickening for any $\{\alpha_2\}$-regular $\tau_0 \in \mathbb{S}\mathfrak{a}^+$ is:

$$\Omega_\rho^{(\tau_{\{\alpha_2\}},\tau_0)} = \mathbf{Ein}(\mathbb{R}^{2,3}) \backslash \bigcup_{x \in \partial\Gamma_g} \mathbb{P}\left(\xi^2_\rho(x)^\perp\right).$$

For any $x \in \partial\Gamma_g$, The submanifold $\mathbb{P}\left(\xi^2_\rho(x)^\perp\right)$ has dimension 1 in $\mathbf{Ein}(\mathbb{R}^{2,3})$ which has dimension 3. Therefore the domain $\Omega_\rho^{(\tau_{\{\alpha_2\}},\tau_0)}$ is non-empty.

Lemma 2.7.16 together with Proposition 2.7.17 implies the following.

Corollary 2.7.18. *If G is not locally isomorphic to* $\mathrm{PSL}(2,\mathbb{R})$*, the domain of discontinuity obtained by metric thickening $\Omega_\rho^{(\tau_{\Theta'},\tau_0)} \subset \mathcal{F}_{\tau_{\Theta'}}$ for any Θ'-regular vector $\tau_0 \in \mathbb{S}\mathfrak{a}^+$ and any Θ-Anosov representation $\rho : \Gamma_g \to G$ is non-empty.*

The proof of Proposition 2.7.17 relies on a description of the fundamental groups of flag manifolds associated to real Lie groups.

Let us write $\Delta_0 \subset \Delta$ the set of roots whose associated root-space is a line. Let $\alpha, \beta \in \Delta$. We define $\epsilon(\alpha,\beta) = (-1)^{(\alpha,\beta^\vee)}$ i.e. $\epsilon(\alpha,\beta) = 1$ if α and β are linked by no edge or if they are liked by two edges and α is the longest root, and else $\epsilon(\alpha,\beta) = -1$.

Theorem 2.7.19. *[Wig98, Theorem 1.1] Let $A \subset \Delta$ be a set of simple roots. The fundamental group of $\mathcal{F}_A = G/P_A$ is the group generated by $(t_\alpha)_{\alpha \in \Delta_0}$, defined by the relations $t_\beta t_\alpha = t_\alpha (t_\beta)^{\epsilon(\alpha,\beta)}$ for $\alpha, \beta \in \Delta_0$ and $\alpha \neq \beta$, and $t_\alpha = e$ if $\alpha \in \Delta_0 \backslash A$.*

The following lemma will deal with most cases.

Lemma 2.7.20. *If the Dynkin Diagram restricted to Δ_0 of the restricted root system of G has no connected component of type C_n or A_1, every flag manifold of G has finite fundamental group.*

Proof. Let $A \subset \Delta$. Using the relations, one write any element of $\pi_1(\mathcal{F}_A)$ as a product of powers of generators so that each generator appears at most once. Therefore $\pi_1(\mathcal{F}_A)$ is a finite group if and only if for every $\alpha \in A$, t_α has finite order.

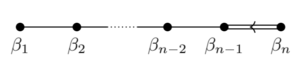
If no connected component of the Dynkin Diagram restricted to Δ_0 is of type C_n or A_1, then every $\alpha \in \Delta_0$ belongs to a sub-diagram inside the Dynkin diagram of Δ_0 of type A_2, G_2 or B_3. In each of this cases the relations between the generators imply that they have finite order: indeed the flag manifold associated to the Borel subgroup of $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SO}(3,4)$ and the real split Lie group associated to G_2 have finite fundamental group, see [Wig98]. $\square$

We now prove Proposition 2.7.17.

Proof. If G is a split Lie group $\Delta_0 = \Delta$. If G is locally isomorphic to $\mathrm{SO}(p,q)$ with $p \geqslant 3$ and $q > p+1$, the Dynkin diagram restricted to Δ_0 is of type A_{p-1}. Finally if G is the real forms of the complex Lie group associated to E_6, E_7 or E_8 whose restricted root system is of type F_4, the Dynkin diagram restricted to Δ_0 is of type A_2. This follows from [OV90, Table 9].

Therefore if (G,Θ) is a pair that admits a notion of Θ-positivity Lemma 2.7.20 applies and every flag manifold associated to G has finite fundamental group except in the following two cases: if G is of Hermitian type and of tube type with Θ consisting of only the longest simple root, or if G is split of type A_1 or C_n and $\Theta = \Delta$.

If G is of Hermitian type the Dynkin diagram of the associated root system is C_n, $n \geqslant 2$. Suppose that $\Theta' = \{\beta_n\}$, then $\mathcal{F}_{\tau_{\Theta'}} = \mathcal{F}_{\{\beta_1\}}$ as in Figure 2.4. Either $\beta_1 \notin \Delta_0$ in which case $\mathcal{F}_{\{\beta_1\}}$ is trivial, or G is split, by [OV90, Table 9]. If G is split, $\beta_1, \beta_2 \in \Delta_0$, so the generator t_{β_1} of $\pi_1(\mathcal{F})$ satisfies the relation $t_{\beta_1} t_{\beta_2} = t_{\beta_2} (t_{\beta_1})^{-1}$, and $t_{\beta_2} = e$ so $t_{\beta_1}^2 = e$. Therefore $\mathcal{F}_{\tau_{\Theta'}}$ has finite fundamental group.



It remains to consider the case where G is split with root system C_n for $n \geqslant 3$, $\Theta = \Delta$ and $\Theta' = \{\alpha_1, \cdots, \alpha_n\}$. But in this case $\mathcal{F}_{\tau_{\Theta'}}$ is the flag manifold associated to the root β_2, as shown in Figure 2.4. The root β_2 belongs to a subdiagram of type A_2, so the fundamental group of $\mathcal{F}_{\Theta'}$ is finite. $\square$

2.7.5 Other applications.

Theorem 2.7.5 can also be applied to Gromov hyperbolic groups that are not surface groups. In this subsection we consider representations of fundamental groups of hyperbolic manifolds.

For instance one can consider a compact hyperbolic 3-manifold M with fundamental group Γ and holonomy $\rho_0 : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$. Let $n \geqslant 3$ and let $\iota_{\mathrm{irr}} : \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{PSL}(n, \mathbb{C})$ be the irreducible representation as in Example 2.7.3 and let $\mathfrak{h} = \mathrm{d}\iota_{\mathrm{irr}}(\mathfrak{sl}_2(\mathbb{C}))$. As for the real case, $\mathfrak{h}$ is Δ-regular, so in particular $\iota_{\mathrm{irr}} \circ \rho_0$ is Δ-Anosov. Here Δ is the only Weyl orbit of simple roots, and the corresponding flag manifold $\mathcal{F}_{\tau_\Delta}$ can be identified with the space $\mathcal{F}^{\mathbb{C}}_{1,n-1}$ of pairs (ℓ, H) where $\ell \subset H \subset \mathbb{C}^n$, ℓ is a line and H a hyperplane.

The domain of discontinuity associated to a Δ-Anosov representation ρ for one and hence any balanced pair of the form (τ_Δ, τ_0) is the following, where the limit map of ρ decomposes as $\xi^{\Theta}_\rho = \left(\xi^1_\rho, \cdots, \xi^{n-1}_\rho\right)$:

$$\mathcal{F}^{\mathbb{C}}_{1,n-1} \backslash \bigcup_{x \in \partial\Gamma} \{(\ell, H) | \exists 1 \leqslant k \leqslant n-1 \, \text{s.t. } \ell \subset \xi^k_\rho \subset H\}.$$

The topological dimension of the thickening $K^\tau_f \subset \mathcal{F}^{\mathbb{C}}_{1,n-1}$ for any flag $f \in G/P_\Delta$ is the maximum for $1 \leqslant k \leqslant n-1$ of the real dimension of $\{(\ell, H) \in \mathcal{F}^{\mathbb{C}}_{1,n-1} | \ell \subset E \subset H\}$ for some $E \subset \mathbb{C}^n$ of dimension k. The dimension of the thickening equals $2n-4$ and the dimension of the flag manifold equals $4n-6$ so for $n \geqslant 3$ the domain is non-empty. Theorem 2.7.5 implies therefore the following:

Corollary 2.7.21. *The representation $\iota_{irr} \circ \rho_0 : \Gamma \to \mathrm{PSL}(n, \mathbb{C})$ is the restricted holonomy of a $(\mathrm{PSL}(n, \mathbb{C}), \mathcal{F}^{\mathbb{C}}_{1,n-1})$-structure* on a fiber bundle *over M.*

One can also consider a hyperbolic n-manifold M for $n \geqslant 2$ with fundamental group Γ and holonomy $\rho_0 : \Gamma \to \mathrm{SO}_o(1, n)$. Let $\iota : \mathrm{SO}(1, n) \to \mathrm{SO}(p, np)$ be the diagonal representation for $n \geqslant 1$ and let $\mathfrak{h}$ be the image of $\mathrm{d}\iota$. Here $\mathfrak{h}$ is $\{\alpha_p\}$-regular, so in particular $\iota_{\mathrm{irr}} \circ \rho_0$ is $\{\alpha_p\}$-Anosov. Note that $\{\alpha_p\}$ is a Weyl orbit of simple roots, and the corresponding flag manifold $\mathcal{F}_{\tau_{\{\alpha_p\}}}$ can be identified with the set of isotropic lines $\mathcal{I} \subset \mathbb{P}(\mathbb{R}^{p,np})$.

The domain of discontinuity associated to a $\{\alpha_p\}$-Anosov representation ρ for one and hence any balanced pair of the form $(\tau_{\{\alpha_p\}}, \tau_0)$ is the following:

$$\mathcal{I} \backslash \bigcup_{x \in \partial\Gamma} \{\ell | \ell \subset \xi^{\{\alpha_p\}}_\rho(x)\}.$$

The dimension of the complement equals $(p-1) + n - 1$ and the dimension of the flag manifold equals $n(p+1) - 2$ so for $n \geqslant 2$ and $p \geqslant 2$ the domain is non-empty. Theorem 2.7.5 implies therefore the following:

Corollary 2.7.22. *Let $\mathcal{C}$ be the connected component of $\iota \circ \rho_0$ in the space of $\{\alpha_p\}$-Anosov representations $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{C})$. Every representation in $\mathcal{C}$ is the restricted holonomy of a $(\mathrm{SO}(p, pn), \mathcal{I})$-structure* on a fiber bundle *over M.*

The fibers of this fiber bundle can be described as the set of isotropic lines in the intersection of p quadrics in $\mathbb{R}^{p,pn}$.

The centralizer of $\iota(\mathrm{SO}(1,n-1))$ in $\mathrm{SO}(p,pn)$ has a larger dimension than the centralizer of $\iota(\mathrm{SO}(1,n))$. Indeed let $E \subset \mathbb{R}^{p,np}$ be the p-dimensional subspace preserved by $\iota(\mathrm{SO}(1,n-1))$. Any element $g \in \mathrm{SO}(p,pn)$ acting trivially on $E^{\perp}$ centralizes $\iota(\mathrm{SO}(1,n-1))$ but only finitely many such elements centralize $\iota(\mathrm{SO}(1,n))$. Hence if the hyperbolic manifold contains a totally geodesic embedded hypersurface, there exist non-trivial deformations of $\iota \circ \rho$ that one can construct using bending.

Chapter 3

Dirichlet-Selberg and Finsler fundamental domains

In this chapter we consider Dirichlet domains for Anosov subgroups of a semi-simple Lie group. In Section 3.1 we explain Selberg's construction of a fundamental domain of discrete subgroups of $\mathrm{SL}(n, \mathbb{R})$, and discuss an example of such a domain that is infinite sided.

In Section 3.2 we recall the necessary background on Anosov representations and symmetric spaces and we define the notion of ω-undistorted subgroup. In Section 3.3 we define the Finsler metric d_ω and the horofunction compactification of the symmetric space. In Section 3.4 we describe the domains of proper horofunctions for ω-undistorted subgroups.

In Section 3.5 we discuss the finite-sidedness of Dirichlet-Finsler domains. In Section 3.7 we consider the restriction of Selberg's domain to smaller invariant convex sets, and provide a sufficient condition for the Dirichlet-Selberg domain to be finite sided in this convex set.

In Section 3.6 we compare the horofunction compactification of the locally symmetric space for an ω-undistorted subgroup with the compactification of the symmetric space. Finally in Section 3.4.4 we show that the limit cone of any discrete subgroup of G that is not virtually cyclic is connected.

This chapter is an adaptation of the preprint [DR24], and is a joint work with Max Riestenberg.

3.1 Dirichlet-Selberg domains

3.1.1 Selberg's construction

Let V be a d-dimensional real vector space. Let S^2V be the space of symmetric bilinear tensors $Q : V^* \to V$. We consider the subspace $\mathcal{X} = \mathcal{X}(V) = \mathbb{P}(S^2V^{>0})$ of $\mathbb{P}(S^2V)$ consisting of positive symmetric 2-tensors, i.e. positive definite symmetric bilinear forms on V^*. The Lie group $\mathrm{SL}(V) \simeq \mathrm{SL}(d, \mathbb{R})$ acts naturally on S^2V so that for $Q \in S^2V$ and $g \in \mathrm{SL}(V)$:

$$g \cdot Q = g \circ Q \circ g^*.$$

Hence $\mathrm{PSL}(V)$ acts on $\mathbb{P}(S^2V)$, and preserves $\mathcal{X}$. This action is moreover transitive on $\mathcal{X}$, and the stabilizer of an element $[Q] \in \mathcal{X}$ is equal to the subgroup $\mathrm{PSO}(Q) \simeq \mathrm{PSO}(d,\mathbb{R})$ of $\mathrm{PSL}(V) \simeq \mathrm{PSL}(d,\mathbb{R})$. Hence $\mathcal{X}$ can be identified with the symmetric space $\mathbb{X} = \mathrm{PSL}(d,\mathbb{R})/\mathrm{PSO}(d,\mathbb{R})$ associated to $\mathrm{PSL}(d,\mathbb{R})$. The space $\mathcal{X}$ is called the *projective model* for this symmetric space.

Given $x_1, x_2 \in \mathcal{X}$, we choose any representatives $Q_1, Q_2 \in S^2V$ of the corresponding lines so that $\det(Q_1^{-1}Q_2) = 1$. The *Selberg invariant* is given by:

$$\mathfrak{s}(x_1,x_2) := \log\left(\frac{1}{d}\operatorname{Tr}(Q_1^{-1}Q_2)\right).$$

It is asymmetric and fails the triangle inequality, but has other good properties in common with metrics.

Proposition 3.1.1. *Let $x_1, x_2 \in \mathcal{X}$ and $g \in \mathrm{PSL}(V)$:*

- $\mathfrak{s}(x_1,x_2) = 0$ *if and only if* $x_1 = x_2$.
- $\mathfrak{s}(x_1,x_2) \geqslant 0$.
- $\mathfrak{s}(gx_1,gx_2) = \mathfrak{s}(x_1,x_2)$.
- $\mathfrak{s}(x_1,x_2) = \log\left(\frac{1}{d}\sum_{i=1}^d e^{\lambda_i}\right)$, *where* $Q_1^{-1}Q_2 : V \to V$ *is conjugate to:*

$$\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_d} \end{pmatrix}.$$

The closure of the properly convex domain $\mathcal{X}$ in $\mathbb{P}(S^2V)$ is called the *Satake compactification* of $\mathcal{X}$, denoted by $\overline{\mathcal{X}}$ with boundary $\partial\mathcal{X}$. For $h \in \overline{\mathcal{X}} = \partial\mathcal{X} \cup \mathcal{X}$ and $o, x \in \mathcal{X}$ one can not always define $\mathfrak{s}(x,h)$ but one can make sense of the difference $\mathfrak{s}(x,h) - \mathfrak{s}(o,h)$. Indeed let S, Q_0, Q be representatives of h, o, x respectively such that $\det(Q_o^{-1}Q) = 1$:

$$\mathfrak{s}_o(x,h) := \log\left(\frac{1}{d}\operatorname{Tr}(Q^{-1}S)\right) - \log\left(\frac{1}{d}\operatorname{Tr}(Q_0^{-1}S)\right).$$

Note that this definition does not depend on the chosen representatives, and that it satisfies the cocycle condition $\mathfrak{s}_o(x,h) = \mathfrak{s}_{o'}(x,h) + \mathfrak{s}_o(o',h)$ for $o' \in \mathbb{X}$.

The main advantage of the Selberg invariant over the invariant Riemannian metric of $\mathcal{X}$ is that the bisectors (resp. half-spaces) of $\mathfrak{s}$ are projective hyperplanes (resp. half-spaces) intersected with $\mathcal{X}$.

For $x_1 \neq x_2$ be in $\mathcal{X}$, we let $\mathcal{H}(x_1, x_2)$ denote the *closed half-space* in $\overline{\mathcal{X}}$, defined as the set of $y \in \overline{\mathcal{X}}$ such that:

$$\operatorname{Tr}\left((X_1^{-1} - X_2^{-1})Y\right) \geqslant 0$$

where X_1 and X_2 are positive definite representatives of x_1 and x_2 respectively such that $\det(X_1 X_2^{-1}) = 1$ and Y is a positive semidefinite representative of y. It is easy to check that the half-space is also given by

$$\mathcal{H}(x_1, x_2) = \{x \in \overline{\mathcal{X}} \mid \mathfrak{s}_o(x_1, x) \leqslant \mathfrak{s}_o(x_2, x)\},$$

and note that this is independent of the basepoint $o \in \mathcal{X}$. For $x, y \in \mathcal{X}$, the *Selberg bisector* is the set

$$\operatorname{Bis}(x, y) \coloneqq \{z \in \overline{\mathcal{X}} \mid \mathfrak{s}_o(x, z) = \mathfrak{s}_o(y, z)\},$$

and can also be written as the subset consisting of all $z \in \overline{\mathcal{X}}$ satisfying

$$\operatorname{Tr}(X^{-1}Z) = \operatorname{Tr}(Y^{-1}Z)$$

where X, Y, Z are representatives of x, y, z respectively with $\det(X^{-1}Y) = 1$.

Given a discrete subgroup Γ of $\operatorname{PSL}(V)$ we may consider a variation of the Dirichlet domain associated to the Selberg invariant. Let $o \in \mathcal{X}$, and define the *Dirichlet-Selberg domain* based at o by:

$$\mathcal{DS}_\Gamma(o) \coloneqq \{x \in \overline{\mathcal{X}} \mid \forall \gamma \in \Gamma,\, \mathfrak{s}_o(o, x) \leqslant \mathfrak{s}_o(\gamma \cdot o, x)\} = \bigcap_{\gamma \in \Gamma \setminus \Gamma_o} \mathcal{H}(o, \gamma \cdot o).$$

This domain is in general a compact convex subset of $\overline{\mathcal{X}} \subset \mathbb{P}(S^2 V)$.

Definition 3.1.2. We call a Dirichlet-Selberg domain $\mathcal{DS}_\Gamma(o)$ *properly finite-sided* if there exists a neighborhood U of $\mathcal{DS}_\Gamma(o)$ in $\overline{\mathcal{X}}$ and a finite set $F \subset \Gamma$ such that for all $\gamma \in \Gamma \setminus F$, $U \subset \mathcal{H}(o, \gamma \cdot o)$.

In particular if a Dirichlet-Selberg domain is properly finite-sided then there exists a finite set $F \subset \Gamma$ such that

$$\mathcal{DS}_\Gamma(o) = \bigcap_{\gamma \in F} \mathcal{H}(o, \gamma \cdot o).$$

These definitions can be related to purely geometric notions of convex subsets of $\mathcal{X}$. We adapt the definitions of Ratcliffe [Rat19] for real hyperbolic space to $\mathcal{X}$. A *side* of a convex subset C of $\mathcal{X}$ is a nonempty maximal convex subset of $\partial C \subset \mathcal{X}$. A *convex polyhedron in* $\mathcal{X}$ is a nonempty closed convex subset of $\mathcal{X}$ such that the collection of its sides is locally finite in $\mathcal{X}$.

Proposition 3.1.3 ([Kap23]). *For any discrete* $\Gamma < \operatorname{PSL}(V)$*, the Dirichlet-Selberg domain* $\mathcal{DS}_\Gamma(o) \cap \mathcal{X}$ *is a convex polyhedron in* $\mathcal{X}$.

The proof follows from the fact that the Selberg invariant is comparable with the Riemannian metric or any G-invariant Finsler metric on the symmetric space $\mathcal{X}$, see Lemma 3.5.3. Proposition 3.1.3 may also be deduced from a result of Jaejeong Lee [Lee08] which applies to more general properly convex domains, see also [Mar09, Section 2.5].

Proposition 3.1.4. *The domain $\mathcal{DS}_\Gamma(o) \cap \mathcal{X}$ has finitely many sides if and only if there exists a finite subset $F \subset \Gamma$ such that:*

$$\mathcal{DS}_\Gamma(o) = \bigcap_{\gamma \in F} \mathcal{H}(o, \gamma \cdot o).$$

Proof. A convex polyhedron in $\mathcal{X}$ has finitely many sides if and only if it is the intersection of $\mathcal{X}$ with finitely many closed half-spaces. This is proved by Ratcliffe for hyperbolic space, see [Rat19, Theorems 6.3.2], but the proof goes through for properly convex domains in general. In particular if a Dirichlet-Selberg domain may be represented as a finite intersection of closed half-spaces, then its intersection with $\mathcal{X}$ has finitely many sides.

On the other hand, suppose that $\mathcal{DS}_\Gamma(o) \cap \mathcal{X}$ has finitely many sides. Each of its sides spans a bisector $\mathrm{Bis}(o, \gamma \cdot o)$, by the same proof as [Rat19, Theorem 6.7.4(1)]. Moreover, distinct sides span distinct bisectors by convexity. It follows that there is a finite set $F \subset \Gamma$ such that

$$\mathcal{DS}_\Gamma(o) \cap \mathcal{X} = \bigcap_{\gamma \in F} \mathcal{H}(o, \gamma \cdot o) \cap \mathcal{X}. \tag{3.1}$$

In general, if Y is a closed convex subset of $\overline{\mathcal{X}}$ and Y contains a point in $\mathcal{X}$, then $Y = \overline{Y \cap \mathcal{X}}$. So by taking closures in (3.1) we can conclude the proof. $\square$

3.1.2 Infinitely-sided Dirichlet Selberg domains.

In this subsection we study Dirichlet-Selberg domains for lattices Γ in the subgroup $\mathrm{SO}(1,n) < \mathrm{SL}(n+1,\mathbb{R})$ of elements that preserve a symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature $(1,n)$. We show that the Dirichlet-Selberg domain is infinitely-sided for some specific basepoints $o \in \mathcal{X}$.

Let q be the symmetric bilinear form of signature $(1,n)$ preserved by $\mathrm{SO}(1,n)$. The subgroup $\mathrm{SO}(1,n) \subset \mathrm{SL}(n+1,\mathbb{R})$ preserves a totally geodesic copy $\mathcal{H} \subset \mathcal{X}_{n+1} = \mathcal{X}(\mathbb{R}^{n+1})$ of the hyperbolic space $\mathbb{H}^n$. Indeed $\mathbb{H}^n$ can be seen as the space of lines on which the symmetric bilinear form q is positive. To such a line ℓ whose orthogonal for q is $\ell^\perp$, we associate the inverse $x_\ell = q_\ell^{-1} : V^* \to V$ of the symmetric bilinear form $q_\ell : V \to V^*$ for which ℓ and $\ell^\perp$ are orthogonal, and such that $q_\ell = q$ on ℓ and $q_\ell = -q$ on $\ell^\perp$.

Theorem 3.1.5. *Let Γ be a lattice in* $\mathrm{SO}(1,n)$ *and let $o \in \mathcal{H}$. The Dirichlet-Selberg domain $\mathcal{DS}_\Gamma(o) \cap \mathcal{X}$ has infinitely many sides.*

In particular, uniform lattices in $\mathrm{SO}(1,n)$ are projective Anosov subgroups of $\mathrm{SL}(n+1,\mathbb{R})$, and these admit Dirichlet-Selberg domains in $\mathcal{X}_{n+1} = \mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ with infinitely many sides. When $n = 2$, a uniform lattice in $\mathrm{SO}(1,2)$ includes as a Borel Anosov subgroup of $\mathrm{SL}(3,\mathbb{R})$.

The main ingredient of the proof is to understand the intersection of $\mathbb{RP}^n$ with the Selberg bisector between x_ℓ and $x_{\ell'}$ for $\ell \neq \ell' \in \mathbb{H}^n$. Recall that the subset consisting of rank one symmetric tensors in $\overline{\mathcal{X}_{n+1}}$ is in one-to-one correspondence with $\mathbb{RP}^n$. We identify these two spaces in all this subsection. When $x, y \in S^2(\mathbb{R}^{n+1})$ are positive of the same determinant, the intersection of the Selberg bisector $\mathrm{Bis}([x],[y])$ with $\mathbb{RP}^n$ is the zero set of the quadratic form $x^{-1} - y^{-1}$.

We can make the following two observations:

Lemma 3.1.6. *For every $\ell \neq \ell' \in \mathbb{H}^n$, the intersection of the half-space $\mathcal{H}(x_{\ell'}, x_\ell)$ and $\mathbb{P}(\ell^\perp) \subset \mathbb{RP}^n$ is the hyperplane $\mathbb{P}\left(\ell^\perp \cap (\ell')^\perp\right)$.*

If ℓ is fixed and ℓ' converges to $u \in \partial\mathbb{H}^n \subset \mathbb{RP}^n$, the bisectors $\mathrm{Bis}(x_\ell, x_{\ell'}) \cap \mathbb{RP}^n$ converge for the Hausdorff topology to $\mathbb{P}(u^\perp)$.

Proof. In the present case, the intersection of $\mathrm{Bis}(q_\ell, q_{\ell'})$ and $\mathbb{RP}^n$ is equal to the quadratic form $q_\ell - q_{\ell'}$. This bilinear form vanishes on $\ell^\perp \cap (\ell')^\perp$ but restricts to a form of signature $(1,1)$ on $\ell \oplus \ell'$. Hence this symmetric bilinear form has signature $(1,1,n-1)$, and the zero locus of the corresponding quadric is the intersection of two distinct projective hyperplanes whose intersection is $\ell^\perp \cap (\ell')^\perp$. Moreover neither ℓ nor ℓ' belongs to the intersection of $\mathrm{Bis}(q_\ell, q_{\ell'})$ and $\mathbb{RP}^n$.

The intersection of the half-space $\mathcal{H}(x_{\ell'}, x_\ell)$ and $\mathbb{P}(\ell^\perp) \subset \mathbb{RP}^n$ is therefore the set of lines on which $q_{\ell'} \geqslant q_\ell$, but $q_\ell = -q$ on $\ell^\perp$. Since $q_{\ell'} \geqslant -q$, it means that $\mathbb{P}(\ell^\perp)$ only intersects this half-space for lines on which $q_{\ell'} = -q$, i.e. on $\mathbb{P}\left(\ell^\perp \cap (\ell')^\perp\right)$.

The two hyperplanes that form $\mathrm{Bis}(q_\ell, q_{\ell'})$ are the hyperplanes H^+ and H^- which are generated by $\ell^\perp \cap \ell'^\perp$ and respectively $\mathrm{v} + \mathrm{w}$ and $\mathrm{v} - \mathrm{w}$ where $\mathrm{v} \in \ell$ and $\mathrm{w} \in \ell'$ satisfy $q(\mathrm{v},\mathrm{v}) = q(\mathrm{w},\mathrm{w}) = 1$. When ℓ' converges to $u \in \partial\mathbb{H}^n$, the intersection $\ell^\perp \cap \ell'^\perp$ converges to $\ell^\perp \cap u^\perp$ and the lines generated by $\mathrm{v} + \mathrm{w}$ and $\mathrm{v} - \mathrm{w}$ both converge to $u \subset u^\perp \backslash \ell^\perp$. Hence H^+ and H^- both converge to $u^\perp$, so $\mathrm{Bis}(q_\ell, q_{\ell'})$ converges for the Hausdorff topology to $\mathbb{P}(u^\perp)$.

□

Figure 3.1 illustrates the intersection of $\mathrm{Bis}(q_\ell, q_{\ell'})$ and $\mathbb{RP}^n$ for $n = 2$. The circle represents the isotropic lines for q. The two projective lines represent the intersection with $\mathrm{Bis}(q_\ell, q_{\ell'})$. They intersect the line at infinity $\ell^\perp$ at $\ell^\perp \cap (\ell')^\perp$. If ℓ' converges to u on the circle, then Lemma 3.1.6 tells us that the two blue lines converge to the tangent of the circle at u.

We can now prove Theorem 3.1.5.

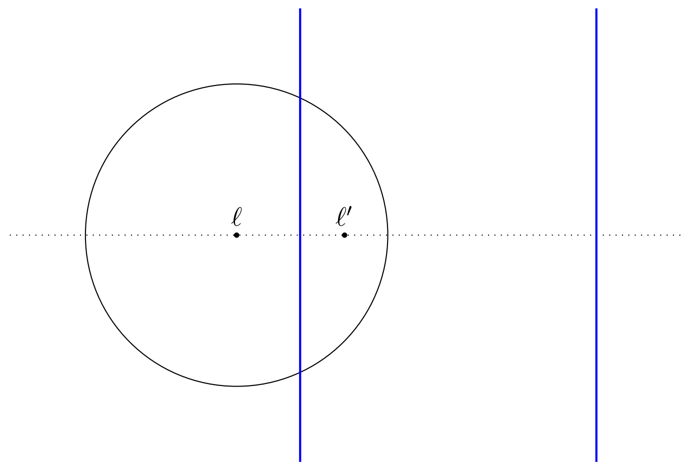


Figure 3.1: Illustration of the intersection of a Selberg bisector and $\mathbb{RP}^2$.

Proof. Suppose for the sake of contradiction that the Dirichlet-Selberg domain has finitely many sides. By Proposition 3.1.4 there is a finite set $F \subset \Gamma$ such that

$$\mathcal{DS}_\Gamma(o) = \bigcap_{\gamma \in F} \mathcal{H}(o, \gamma \cdot o).$$

The interior of $\mathcal{DS}_\Gamma(o)$ intersected with $\mathbb{P}(\ell^\perp) \subset \mathbb{RP}^n$ is the complement in $\mathbb{P}(\ell^\perp)$ of the union of $\mathcal{H}(x_{\gamma\cdot\ell}, x_\ell)$ for all $\gamma \in F$. This is a finite union of hyperplanes by Lemma 3.1.6, hence this intersection is non-empty.

Choose a line $w \subset \ell^\perp$ in this intersection. Let $u \in \partial\mathcal{H}^n$ be an isotropic line such that $w \subset u^\perp$. Since the limit set of Γ is all of $\partial\mathbb{H}^n$, there exist a sequence (γ_n) in Γ such that $(\gamma_n \cdot \ell)$ converges to u. The bisectors $\mathrm{Bis}(o, \gamma_n o)$ converge to $u^\perp$ by Lemma 3.1.6. In particular, these bisectors eventually meet the interior of the Dirichlet-Selberg domain. Then, since the bisectors have empty interior in $\mathbb{RP}^n$, the complements of $\mathcal{H}(o, \gamma_n \cdot o)$ eventually intersect the Dirichlet-Selberg domain, yielding a contradiction. □

Remark 3.1.7. It is not clear if such groups admit finite-sided Dirichlet-Selberg domains for other basepoints $o \in \mathcal{X}$.

3.2 Background on symmetric spaces and Anosov subgroups

In this section we introduce the notion of an *ω-undistorted* subgroup of G. First we review some important properties of the visual boundary of a symmetric space of non-compact type, and fix some notation. We then recall the relevant properties of Anosov subgroups.

3.2.1 The visual boundary of a symmetric space of non-compact type

Let G be a connected semisimple Lie group with finite center and let $\mathbb{X}$ be the associated symmetric space of non-compact type. The symmetric space $\mathbb{X}$ is a Hadamard manifold. Its *visual boundary*, denoted $\partial_{\text{vis}}\mathbb{X}$, is the set of asymptote classes of geodesic rays. The visual boundary of a symmetric space has the structure of a thick spherical building, see [KLP17, Eps86] for further discussion.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Let $\Sigma \subset \mathfrak{a}^*$ be the associated (restricted) root system, and Δ be a choice of simple roots. This choice defines a positive *(Euclidean) Weyl chamber* $\mathfrak{a}^+ := \{\mathrm{v} \in \mathfrak{a} \mid \forall \alpha \in \Delta, \, \alpha(\mathrm{v}) \geqslant 0\}$.

For $x, y \in \mathbb{X}$, there is an isometry $g \in G$ conjugating the transvection from x to y into $\exp(\mathfrak{a}^+)$. The corresponding element of $\mathfrak{a}^+$ is called the the vector-valued distance, and denoted by $\vec{d}(x, y)$.

The projecitivization $\mathbb{S}\mathfrak{a}^+$ is naturally identified with a subset of $\partial_{\text{vis}}\mathbb{X}$ called a *(spherical) Weyl chamber*. It is a fundamental domain for the natural action of G on $\partial_{\text{vis}}\mathbb{X}$. In particular, $\partial_{\text{vis}}\mathbb{X}$ is a union of Weyl chambers σ, and each is naturally identified with a *model Weyl chamber* σ_{mod}. Every element of the visual boundary has a *type* in the model Weyl chamber:

$$\tau \colon \partial_{\text{vis}}\mathbb{X} \to \sigma_{mod}.$$

Non-empty faces τ_{mod} of σ_{mod} are in one-to one correspondence with non-empty subsets Θ of Δ. To such a face/subset of simple roots one can associate a *flag manifold* $\mathcal{F}_\Theta = \mathrm{Flag}(\tau_{mod})$ defined as the set of faces $\tau \subset \partial_{\text{vis}}\mathbb{X}$ of type τ_{mod}. It can also be written as $\mathcal{F}_\Theta = G/P_\Theta$ where P_Θ is the standard parabolic subgroup associated to Θ. The *star* of a simplex τ in $\partial_{\text{vis}}\mathbb{X}$ is the union of chambers containing τ, and denoted $\mathrm{st}(\tau) \subset \partial_{\text{vis}}\mathbb{X}$.

We often fix a subset $\mathbb{C}$ of σ_{mod}; which can be for instance the limit cone of some discrete subgroup, see below.[1] The $\mathbb{C}$-*star* of τ is the subset of the star of τ with types in $\mathbb{C}$:

$$\mathrm{st}_{\mathbb{C}}(\tau) := \mathrm{st}(\tau) \cap \tau^{-1}(\mathbb{C}).$$

We further consider certain subsets of $\mathbb{X}$ which appear as cones on subsets of $\partial_{\text{vis}}\mathbb{X}$. For $x \in \mathbb{X}$ and $A \subset \partial_{\text{vis}}\mathbb{X}$, we let $\mathcal{V}(x, A)$ denote the union of points on geodesic rays from x to A. In particular, we will consider later the *Weyl cone* $\mathcal{V}(x, \mathrm{st}_{\mathbb{C}}(\tau))$ of a simplex τ in $\partial_{\text{vis}}\mathbb{X}$.

To $\omega \in \mathfrak{a}^*$, one can associate its orthogonal vector $\omega^\perp \in \mathfrak{a}$ for the Killing form. Up to the action of the Weyl group W, we may assume that $\omega^\perp \in \mathfrak{a}^+$. We let $\mathcal{F}_\omega$ denote the flag manifold

$$\mathcal{F}_\omega := \tau^{-1}(\omega^\perp) = G \cdot [c_{\omega^\perp}] \subset \partial_{\text{vis}}\mathbb{X} \tag{3.2}$$

[1] In [KLP17], $\mathbb{C}$ would be denoted Θ, but we reserve that notation for a collection of simple roots.

where $c_{\omega^\perp}$ is a geodesic ray determined by $\omega^\perp$. The flag manifold $\mathcal{F}_\omega$ is naturally identified with $\mathcal{F}_\Xi$ for the subset of simple roots $\Xi = \{\alpha \in \Delta \mid \alpha(\omega^\perp) \neq 0\}$.

Remark 3.2.1. In the following sections, we will consider Θ-Anosov subgroups which have limit maps with values in $\mathcal{F}_\Theta$ and domains of discontinuity in the flag manifold $\mathcal{F}_\omega = \mathcal{F}_\Xi$. We emphasize that Ξ is typically not equal to Θ in this setup.

3.2.2 The ω-undistorted condition.

Let Γ be a discrete subgroup of G. The *limit cone* $\mathcal{C}_\Gamma$ of Γ is given by:

$$\mathcal{C}_\Gamma := \bigcap_{n\in\mathbb{N}} \overline{\left\{ \frac{\vec{d}(o,\gamma\cdot o)}{d(o,\gamma\cdot o)} | \gamma \in \Gamma,\, d(o,\gamma\cdot o) \geqslant n \right\}} \subset \sigma_{mod}. \tag{3.3}$$

This definition does not depend of the base-point $o \in \mathbb{X}$. The limit cone is non-empty when Γ is unbounded and is compact in general. Note that for a fixed $o \in \mathbb{X}$, the map $\mu\colon G \to \mathfrak{a}^+$ given by $\mu(g) := \vec{d}(o, g\cdot o)$ is often called the *Cartan projection*.

Before introducing the ω-undistorted notion, we recall a similar condition that characterizes the Anosov property.

Definition 3.2.2 ([KLP17, Definition 5.17])**.** Let $\Theta \subset \Delta$ be a set of simple roots. A finitely generated subgroup Γ is Θ-*Anosov* if and only if for one (and hence any) word metric $|\cdot|$ on Γ, there exist $\epsilon, C > 0$ such that for all $\alpha \in \Theta$ and $\gamma \in \Gamma$:

$$\alpha\left(\vec{d}(o,\gamma\cdot o)\right) \geqslant \epsilon|\gamma| - C.$$

Equivalently, Γ is Θ-*Anosov* if it is quasi-isometrically embedded and the limit cone $\mathcal{C}_\Gamma$ avoids $\mathrm{Ker}(\alpha)$ for all $\alpha \in \Theta$.

We introduce a similar notion.

Definition 3.2.3. Let $\omega \in \mathfrak{a}^*$ be nonzero. We say that a finitely generated subgroup $\Gamma < G$ is ω-*undistorted* if for one (and hence any) word metric $|\cdot|$ on Γ, there exist $\epsilon, C > 0$ such that for all w in the Weyl group and $\gamma \in \Gamma$:

$$\left|\omega\left(w\cdot\vec{d}(o,\gamma\cdot o)\right)\right| \geqslant \epsilon|\gamma| - C. \tag{3.4}$$

Equivalently Γ is ω-undistorted if and only if it is quasi-isometrically embedded in G and if its limit cone $\mathcal{C}_\Gamma$ avoids $w\cdot\mathrm{Ker}(\omega)$ for all w in the Weyl group.

Remark 3.2.4. Note that if $G = \mathrm{SL}(n,\mathbb{R})$, the logarithm of each singular value σ_i of γ for $1 \leqslant i \leqslant n$ is equal to $\omega_1\left(w\cdot\vec{d}(o,\gamma\cdot o)\right)$ for some $w \in W$. It follows that an ω_1-undistorted subgroup of G is exactly a subgroup which is $|\log\sigma_i|$-*undistorted for all* i.

If ω is a simple root this exactly means that Γ is Anosov with respect to the set of simple roots $\Theta = \{\omega \circ w | w \in W\} \cap \Delta$. Recall that an elementary subgroup is a subgroup that is virtually cyclic.

Proposition 3.2.5. *Let Γ be a non-elementary ω-undistorted subgroup of G, and let σ_Γ be the connected component of $\sigma_{mod} \backslash \bigcup_{w \in W} w \cdot \mathrm{Ker}(\omega)$ containing $\mathcal{C}_\Gamma$. Let $\Theta(\sigma_\Gamma) \in \Delta$ be the set of simple roots whose associated walls do not intersect σ_Γ. The set $\Theta(\sigma_\Gamma)$ is nonempty and Γ is $\Theta(\sigma_\Gamma)$-Anosov.*

Proof. The limit cone $\mathcal{C}_\Gamma$ is connected, see Proposition 3.8.2. The fact that $\Theta(\sigma_\Gamma) \neq \varnothing$ is exactly [Dav23, Lemma 5.20]. Finally, since $\mathcal{C}_\Gamma \subset \sigma_\Gamma$, the limit cone $\mathcal{C}_\Gamma$ avoids the walls associated to the simple roots in $\Theta(\sigma_\Gamma)$, and Γ is quasi-isometrically embedded, so Γ is $\Theta(\sigma_\Gamma)$-Anosov. $\square$

Note that the connected component σ_Γ must be invariant by the opposition involution $\iota : \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$, since the limit cone is itself invariant. Hence for $\omega = \omega_1$ and $G = \mathrm{SL}(d, \mathbb{R})$:

Proposition 3.2.6. *If $d = 2n$, then any non-elementary ω_1-undistorted subgroup of $\mathrm{SL}(d, \mathbb{R})$ is n-Anosov. If d is odd, there exist no non-elementary ω_1-undistorted subgroups of $\mathrm{SL}(d, \mathbb{R})$.*

Proof. For $1 \leqslant k < d$, let c_k denote the set of tuples $(\lambda_1, \lambda_2, \cdots, \lambda_d)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0 > \lambda_{k+1} \geqslant \cdots \geqslant \lambda_d$. These are exactly the connected components of $\mathfrak{a}^+ \backslash \bigcup_{w \in W} w \cdot \mathrm{Ker}(\omega_1)$. The opposition involution maps c_k to c_{d-k}.

For d odd there are no invariant connected components. For $d = 2n$ there is only one invariant component, and this component avoids the wall $\lambda_n = \lambda_{n+1}$. Hence $\Theta(\sigma_\Gamma) = \{\lambda_n - \lambda_{n+1}\}$. $\square$

Remark 3.2.7. One could define and study other conditions for a subgroup by requiring for some fixed finite set of functionals $\omega \in \mathfrak{a}^*$ the inequality:

$$\left| \omega \left(\vec{d}(o, \gamma \cdot o) \right) \right| \geqslant \epsilon |\gamma| - C$$

In this framework one could view the Anosov condition as the special case where one considers a collection of simple roots. In the present paper, we are mainly interested in the case where that set is a Weyl group orbit.

3.2.3 Illustration of the ω-undistorted condition.

In this subsection we consider a few examples to illustrate the ω-undistorted condition, and its relation to the Anosov properties.

3.2.3.1 Let $G = \mathrm{PSL}(d, \mathbb{R})$.

Consider the following:

$$\omega_1 : \mathrm{Diag}(\sigma_1, \cdots, \sigma_d) \mapsto \sigma_1.$$

If $\Gamma < \mathrm{PSL}(d,\mathbb{R})$ is a subgroup $|\log \sigma_i|$-undistorted for all i, or ω_1-undistorted. Figure 3.2 illustrates the intersection of the model Weyl chamber σ_{mod} with the hyperplanes $w \cdot \mathrm{Ker}(\omega_1)$ as dotted lines. Since the limit cone $\mathcal{C}_\Gamma$ is connected and invariant by the opposition involution, it is contained in the gray region.

Now consider the following:

$$\omega_\Delta : \mathrm{Diag}(\sigma_1, \cdots, \sigma_d) \mapsto \sigma_1 - \sigma_d.$$

An ω_Δ-undistorted subroup $\Gamma < \mathrm{PSL}(d,\mathbb{R})$ is exactly a Δ-Anosov subgroup of $\mathrm{PSL}(d,\mathbb{R})$, also called *Borel Anosov*.

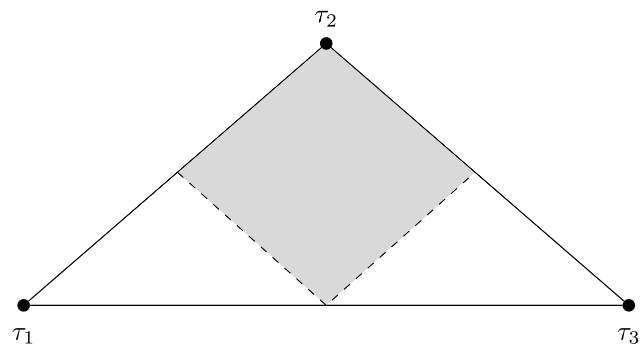


Figure 3.2: An illustration of ω_1 for $\mathrm{SL}(4,\mathbb{R})$.

3.2.3.2 Let $G = \mathrm{Sp}(6,\mathbb{R})$.

We consider representations of convex cocompact subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ that factor through a representation $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(6,\mathbb{R})$. Given a partition $\tau = \{\tau_1, \tau_2, \cdots \tau_k\}$ with repetition such that $\tau_1 + \cdots + \tau_k = 6$ and each odd integer appears an even number of times, we can define a representation $f_\tau\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(6,\mathbb{R})$ as the direct sum of irreducible representations $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(\tau_i,\mathbb{R})$. Let $j\colon \Gamma \to \mathrm{SL}(2,\mathbb{R})$ be a Fuchsian representation of the fundamental group of a closed surface. The representations $\rho_\tau = f_\tau \circ j$ are discrete and faithful for $\tau \neq \{1,1,1,1,1,1\}$ and their limit cone is a single point $\{\mathrm{v}_\tau\}$.

The positive Weyl chamber $\mathfrak{a}^+$ can be identified as the space of triples $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0$. The generators of the Weyl group are the involutions $\sigma_1 : (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_2, \lambda_1, \lambda_3)$, $\sigma_2 : (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_3, \lambda_2)$ and $\sigma_3 : (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_2, -\lambda_3)$.

Let $\omega_1 = \lambda_1$ denote the first fundamental weight, which is an element of $\mathfrak{a}^*$ in the same Weyl group orbit as half of the root $\alpha_3 = 2\lambda_3$. The second fundamental weight is $\omega_2 = \lambda_1 + \lambda_2$, which is in the same Weyl group orbit as the roots $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_2 = \lambda_2 - \lambda_3$. The third fundamental weight

$\omega_3 = \lambda_1 + \lambda_2 + \lambda_3$ is not in the Weyl group orbit of a simple root, even up to rescaling.

We illustrate some of these points in Figure 3.3, in the projective chart defined by $\lambda_1 = 1$. In this picture, the red line correspond to the union of $\mathrm{Ker}(w \cdot \omega_1)$ for $w \in W$, the blue lines show the same union for ω_2 and the gray line shows this union for ω_3.

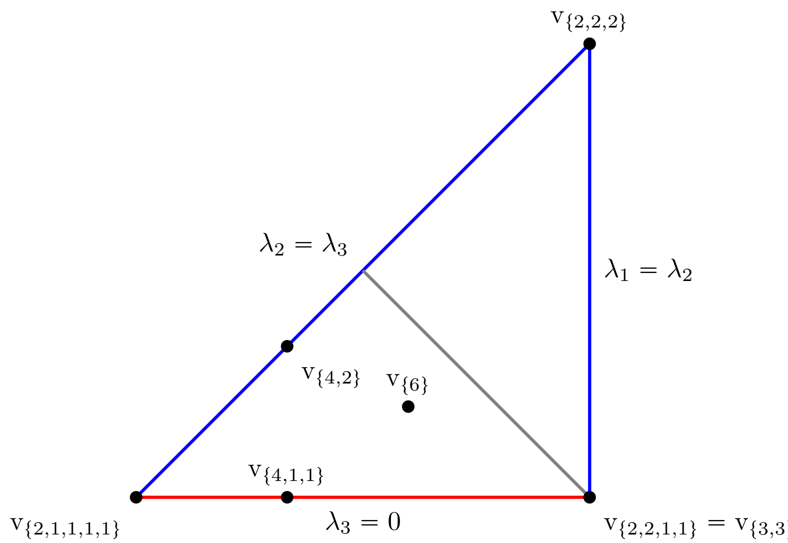


Figure 3.3: An illustration of the positive Weyl chamber $\mathbb{P}(\mathfrak{a}^+) = \sigma_{mod}$ for $\mathrm{Sp}(6, \mathbb{R})$.

In this picture, we see that the following are ω_1-undistorted, which is equivalent to being $\{\alpha_3\}$-Anosov:

$$\rho_{\{6\}}, \rho_{\{2,2,2\}}, \rho_{\{4,1,1\}}.$$

We also see that $\rho_{\{6\}}$ and $\rho_{\{4,1,1\}}$ are ω_2- An ω_Δ-undistorted, which is equivalent to being $\{\alpha_1, \alpha_2\}$-Anosov. Finally we see that teh representations $\rho_{\{6\}}$, $\rho_{\{4,2\}}$, $\rho_{\{2,1,1,1,1\}}$, $\rho_{\{4,1,1\}}$ and $\rho_{\{2,2,2\}}$ are ω_3-undistorted. There are two connected components of $\mathfrak{a}^+$ minus the gray line. The first one contains $\mathrm{v}_{\{2,2,2\}}$, and having a limit cone in this component implies being $\{\alpha_3\}$-Anosov. Representations whose limit cone lies in the other component are all $\{\alpha_1\}$-Anosov.

3.2.4 Boundary maps and the Morse property.

An important feature of Anosov subgroups is the existence of a boundary map, which can be characterized in the following way. Let Θ be a non-empty set of simple roots.

Theorem 3.2.8 ([KLP17, BPS19])**.** *Let Γ be Θ-Anosov subgroup of G. The group Γ is hyperbolic, with Gromov boundary $\partial\Gamma$. There exist a unique continuous Γ-equivariant map $\xi_\Theta : \partial\Gamma \to \mathcal{F}_\Theta$ such that for all $o \in \mathbb{X}$ and every geodesic ray (γ_n) in Γ converging to $\zeta \in \partial\Gamma$, every limit point of $(\gamma_n \cdot o)$ belongs to a Weyl chamber that contains $\xi_\Theta(\zeta)$, i.e. every limit point belongs to* $\mathrm{st}\,(\xi_\Theta(\zeta)) \subset \partial_{\mathrm{vis}}\mathbb{X}$.

The map $\xi_\Theta \colon \partial\Gamma \to \mathcal{F}_\Theta$ is called the *boundary map* of Γ.

Symmetric spaces of rank one satisfy the Morse Lemma: quasi-geodesics stay close to geodesics. That property fails in higher rank, but a suitable generalization holds: uniformly regular quasigeodesics stay close to Weyl cones.

Theorem 3.2.9 ([KLP17, KLP18b, BPS19])**.** *Let Γ be Θ-Anosov subgroup of G. Let $o \in \mathbb{X}$, and let us a fix a word metric $|\cdot|$ on Γ. There exist $D \geqslant 0$ such that if $\gamma \in \Gamma$ lies on a geodesic ray from $e \in \Gamma$ to $\zeta \in \partial\Gamma$ then the distance from $\gamma \cdot o$ to the Weyl cone $\mathcal{V}\,(o, \mathrm{st}(\xi_\Theta(\zeta)))$ is at most D.*

We observe that the orbit also stays close to Weyl cones on $\mathcal{C}$-stars. This has important consequences throughout the paper.

Lemma 3.2.10. *Let Γ be a Θ-Anosov subgroup of G. Let $\mathbb{C}$ be any compact neighborhood of the limit cone $\mathbb{C}_\Gamma$, let $o \in \mathbb{X}$ and fix some word metric $|\cdot|$ on Γ. There exists $D \geqslant 0$ such that:*

1. *If $(\gamma_n)_{n\in\mathbb{N}}$ is a geodesic ray in Γ converging to $\zeta \in \partial\Gamma$ with $\gamma_0 = e$, then for all $n \in \mathbb{N}$, the distance from $\gamma_n \cdot o$ to the Weyl cone $\mathcal{V}\,(o, \mathrm{st}_{\mathcal{C}}\,(\xi_\Theta(\zeta)))$ is at most D,*
2. *For all $\gamma \in \Gamma$, there exists $\zeta \in \partial\Gamma$ such that the distance from $\gamma \cdot o$ to the Weyl cone $\mathcal{V}\,(o, \mathrm{st}_{\mathcal{C}}\,(\xi_\Theta(\zeta)))$ is at most D.*

Proof. We first remark that since Γ is hyperbolic, there exist $D' \geqslant 0$ such that every $\gamma \in \Gamma$ is at distance at most D' from some element γ' that belongs to an infinite geodesic ray starting from $e \in \Gamma$. Hence, since Anosov subgroups are quasi-isometrically embedded, (2) follows from (1).

Now let γ lie on a geodesic ray (γ_n) in Γ with $\gamma_0 = e$. By Theorem 3.2.9 we know that there exist a point $x \in \mathcal{V}\,(o, \mathrm{st}\,(\xi_\Theta(\zeta)))$ for some $\zeta \in \partial\Gamma$ that is at distance at most D from $\gamma \cdot o$. Since $\mathcal{C}$ contains a neighborhood of $\mathcal{C}_\Gamma$, it contains $\vec{d}(o, \gamma \cdot o)$ for all γ large enough. Moreover since $d(o, \gamma \cdot o)$ goes to $+\infty$, $\mathcal{C}$ must contain $\vec{d}(o, x)$ if γ is large enough. Hence there exist D'' such that either $d(o, \gamma \cdot o) \leqslant D''$ or $x \in \mathcal{V}\,(o, \mathrm{st}_{\mathcal{C}}\,(\xi_\Theta(\zeta)))$. This concludes the proof. □

Example 3.2.11 (Taking a neighborhood of $\boldsymbol{\mathcal{C}_\Gamma}$ is necessary)**.** We discuss an example where Γ orbits fail to stay at uniform distance from Weyl cones on $\mathcal{C}_\Gamma$. Indeed, let γ be an isometry of $\mathcal{H}^2 \times \mathcal{H}^2$ which is hyperbolic on the first factor and unipotent on the second. Then $\Gamma = \langle\gamma\rangle$ is Anosov with respect to the first factor. Let $o = (o_1, o_2)$ be a basepoint in $\mathcal{H}^2 \times \mathcal{H}^2$. The fixed points of γ on

$\partial\mathcal{H}^2 \times \partial\mathcal{H}^2$ are $(\xi_1^\pm, \xi_2)$; let τ be the simplex in $\partial_{\mathrm{vis}}(\mathcal{H}^2 \times \mathcal{H}^2)$ corresponding to ξ_1^+. While the (forward) orbits of γ uniformly fellow travel the Weyl cone

$$\mathcal{V}(o, \mathrm{st}(\tau)) = \{(p, q) \in \mathcal{H}^2 \times \mathcal{H}^2 \mid o_1 p(+\infty) = \xi_1^+\},$$

they drift logarithmically away from

$$\mathcal{V}(o, \mathrm{st}_{\mathcal{C}_\Gamma}(\tau)) = \mathcal{V}(o, \tau) = \{(p, o_2) \in \mathcal{H}^2 \times \mathcal{H}^2 \mid o_1 p(+\infty) = \xi_1^+\}.$$

On the other hand, for any neighborhood $\mathcal{C}$ of $\mathcal{C}_\Gamma$, the orbits of γ uniformly fellow travel

$$\mathcal{V}(o, \mathrm{st}_{\mathcal{C}}(\tau)) = \{(p, q) \in \mathcal{H}^2 \times \mathcal{H}^2 \mid o_1 p(+\infty) = \xi_1^+, d(o_2, q) \leqslant C d(o_1, p)\}$$

where C is a constant depending on $\mathcal{C}$.

3.3 Finsler metrics and horofunction compactifications

In this section we review a class of G-invariant Finsler metrics on the symmetric space $\mathbb{X}$ as well as their horofunction compactifications, which were previously studied by Kapovich-Leeb [KL18]. In the sequel we will study Dirichlet domains for these Finsler metrics by considering their closure in the horofunction compactification, which are closely related to Satake compactifications, see [HSWW18].

3.3.1 A family of Finsler metrics on the symmetric space

To a non-zero element $\omega \in \mathfrak{a}^*$ one can associate a seminorm $\|\cdot\|_\omega$ on the model Cartan subalgebra for $\mathrm{v} \in \mathfrak{a}$ by:

$$\|\mathrm{v}\|_\omega = \max_{w \in W} \omega(w \cdot \mathrm{v}).$$

The seminorm only depends on the Weyl group orbit $W \cdot \omega$, and is symmetric if and only if $W \cdot \omega = W \cdot (-\omega)$.

When $W \cdot \omega$ spans $\mathfrak{a}^*$, $\|\cdot\|_\omega$ is moreover definite. This is always the case when G is simple. We will assume from now on that ω is chosen in such a way.

The seminorm on $\mathfrak{a}$ defines a G-invariant Finsler metric $\|\cdot\|_\omega$ on $\mathbb{X}$, that can be characterized for $\mathrm{v} \in T_o\mathbb{X}$ by:

$$\|\mathrm{v}\|_\omega = \max_{a \in \mathcal{F}_\omega} -\mathrm{d}b_{a,o}(\mathrm{v}).$$

Here for $o, x, y \in \mathbb{X}$ and $a \in \mathcal{F}_\omega$, $b_{a,o} : \mathbb{X} \to \mathbb{R}$ is the Busemann function associated to a and based at o and $\mathcal{F}_\omega$ denotes the G orbit of an ideal point dual to ω, see (3.2).

This defines a *Finsler distance* on $\mathbb{X}$, characterized for $x, y \in \mathbb{X}$ by:

$$d_\omega(x, y) = \|\vec{d}(x, y)\|_\omega = \max_{a \in \mathcal{F}_\omega} b_{a,y}(x).$$

Remark 3.3.1. Let $G = \mathrm{PSL}(2,\mathbb{R})^n$, and let $\omega\colon \mathfrak{a} \to \mathbb{R}$ be the linear form coming from the projection onto the first factor. For this functional $W \cdot \omega$ does not span $\mathfrak{a}^*$ and the seminorm $\|\cdot\|_\omega$ is not definite. Indeed, the degenerate pseudo-metric d_ω on $\left(\mathbb{H}^2\right)^n$ is the composition of projection onto the first coordinate with the distance in $\mathbb{H}^2$. In general, we only consider those functionals ω which lead to nondegenerate metrics, so this example is ruled out.

3.3.2 Horofunction compactification

We review the construction of a horofunction compactification for an asymmetric metric, see [Wal14, KL18, HSWW18] for further details. Let $\mathcal{Y}$ be the space of 1-Lipschitz functions $f : \mathbb{X} \to \mathbb{R}$ for a G-invariant Riemannian metric on $\mathbb{X}$, modding out the line of constant functions. This space is endowed with the compact open topology: a basis of neighborhoods of $[f] \in \mathcal{Y}$ is defined by the open sets of the form:

$$U_{K,\epsilon} = \{[g] \mid \forall x \in K,\, (g - f)(x) < \epsilon\},$$

for $K \subset \mathbb{X}$ compact and $\epsilon > 0$.

One can define a topological embedding $\iota : \mathbb{X} \to \mathcal{Y}$ by setting $\iota(x_0) : x \in \mathbb{X} \mapsto d_\omega(x, x_0)$ for $x_0 \in \mathbb{X}$. Since $\mathcal{Y}$ is compact Hausdorff, the closure of $\iota(\mathbb{X})$ in $\mathcal{Y}$ is compact Hausdorff and we denote it by $\partial_\omega \mathbb{X} = \overline{\iota(\mathbb{X})} \backslash \iota(\mathbb{X})$. The functions representing points in $\partial_\omega \mathbb{X}$ are called *horofunctions*.

3.3.3 Satake compactification

Let G be a semisimple real Lie group and V an irreducible real representation of G with finite kernel, and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced Lie algebra representation.

We define as previously the space S^2V of symmetric bilinear tensors $Q : V^* \to V$, and the subset $\mathcal{X} = \mathcal{X}(V) \subset \mathbb{P}(S^2V)$ of projectivizations of positive definite elements, see Section 3.1. The space $\mathcal{X}$ is the projective model for the symmetric space of $\mathrm{SL}(V)$. The symmetric space $\mathbb{X}$ associated to G can be identified with a unique totally geodesic submanifold of the symmetric space of $\mathrm{SL}(V)$, hence it can be seen as a subset $\mathbb{X} \subset \mathcal{X}$ [Kar53, Mos55]. Note that this subspace is not in general a linear subspace.

Definition 3.3.2. The *Satake compactification* of $\mathbb{X}$ associated to ρ is the closure of $\mathbb{X} \subset \mathcal{X}$ inside the compact space $\mathbb{P}(S^2V)$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. Given $\lambda \in \mathfrak{a}^*$, let $V_\lambda = \{x \mid \forall h \in \mathfrak{a},\, \rho(h) \cdot x = \lambda(h)x\}$. The *restricted weight system* associated to ρ is the set $\Phi_\rho \subset \mathfrak{a}^*$ of elements λ such that $V_\lambda \neq \{0\}$. We have the following *weight space decomposition*:

$$V = \bigoplus_{\lambda \in \Phi_\rho} V_\lambda.$$

The *highest weight* $\omega \in \Phi_\rho$ of the representation ρ is the unique element such that for any $\lambda \in \Phi_\rho$, $\omega - \lambda \geqslant 0$ on $\mathfrak{a}^+$.

Since $\rho\colon \mathfrak{g} \to \mathfrak{sl}(V)$ is injective, the kernel of ω cannot contain a simple factor of $\mathfrak{g}$, so the Finsler metric d_ω on $\mathbb{X}$ is nondegenerate. In this case the Satake compactification associated to ρ coincides with the horofunction compactification of $\mathbb{X}$ with respect to d_ω.

Theorem 3.3.3 ([HSWW18, Theorem 5.5])**.** *The Satake compactification of $\mathbb{X}$ associated to the representation ρ is G-equivariantly homeomorphic to the horofunction compactification $\mathbb{X} \cup \partial_\omega \mathbb{X}$.*

See also [HSWW18, Remark 5.6]. Note that [HSWW18, Theorem 5.5] also applies in the more general case of *generalized Satake compactifications* of reducible representations. In the present paper we restrict attention to the irreducible case, since the Finsler metrics we want to consider are defined by a single weight.

3.3.4 Description of horofunctions

Horofunctions for the polygonal Finsler metric d_ω can be constructed from Busemann functions associated to elements in $\mathcal{F}_\omega$.

Recall that a flag of ν of any type corresponds to a simplex in the visual boundary.

Definition 3.3.4 (Incidence)**.** We say that $\zeta \in \partial_{\mathrm{vis}}\mathbb{X}$ and a simplex $\nu \subset \partial_{\mathrm{vis}}\mathbb{X}$ are *incident*, denoted $\zeta \sim \nu$, if there exists a chamber $\sigma \subset \partial_{\mathrm{vis}}\mathbb{X}$ such that $\zeta \in \sigma$ and $\nu \subset \sigma$. We let $I^\omega_\nu \subset \mathcal{F}_\omega$ denote the set of $\zeta \in \mathcal{F}_\omega$ incident to ν.

Note that I^ω_ν can also be written as $\mathrm{st}(\nu) \cap \mathcal{F}_\omega$.

Proposition 3.3.5 ([KL18, Section 5])**.** *Every horofunction of $(\mathbb{X}, d_\omega)$ is of the form $b^\omega_{\nu,o}$ for some flag ν of any type and some point $o \in \mathbb{X}$:*

$$b^\omega_{\nu,o} \coloneqq \max\left\{ b_{\zeta,o} \mid \zeta \in I^\omega_\nu \right\}.$$

Example 3.3.6. Let $G = \mathrm{PSL}(d,\mathbb{R})$ and $\omega = \omega_1$, so that $\mathcal{F}_{\omega_1} \simeq \mathbb{RP}^{d-1}$. A flag τ of any type corresponds to a tuple $(E^{i_1}, E^{i_2}, \cdots, E^{i_k})$ of subspaces of $\mathbb{R}^d$ such that $E^{i_1} \subsetneq E^{i_2} \subsetneq \cdots \subsetneq E^{i_k}$ and $\dim(E^{i_\ell}) = i_\ell$ for all $1 \leqslant \ell \leqslant k$. A line $\ell \in \mathcal{F}_{\omega_1} \subset \partial_{\mathrm{vis}}\mathbb{X}$ satisfies $\ell \sim \tau$ if and only if $\ell \subset E^{i_1}$. Indeed this is equivalent to the existence of a full flag $(F^1, F^2, \cdots, F^{d-1})$ in $\mathbb{R}^d$ such that $F^1 = \ell$ and $F^{i_\ell} = E^{i_\ell}$ for all $1 \leqslant \ell \leqslant k$.

Let $G = \mathrm{PSL}(d,\mathbb{R})$ and $\omega = \omega_\Delta$, so that $\mathcal{F}_\omega \simeq \mathcal{F}_{1,d-1}$. A flag τ of any type as before and a pair $(\ell, H) \in \mathcal{F}_{\omega_\Delta} \subset \partial_{\mathrm{vis}}\mathbb{X}$ satisfies $(\ell, H) \sim \tau$ if and only if $\ell \subset E^{i_1}$ and $E^{i_k} \subset H$. Indeed this is equivalent to the existence of a full flag $(F^1, F^2, \cdots, F^{d-1})$ in $\mathbb{R}^d$ such that $F^1 = \ell$, $F^{d-1} = H$ and $F^{i_\ell} = E^{i_\ell}$ for all $1 \leqslant \ell \leqslant k$.

3.4 Domain of proper horofunctions

In this section we consider an ω-undistorted subgroup Γ of G and study two related domains of discontinuity: Ω^{ω}_{flag} in the flag manifold $\mathcal{F}_\omega$ and Ω^{ω}_{horo} in the horoboundary $\partial_\omega\mathbb{X}$. We show that when Γ is ω-undistorted, these are cocompact domains of proper discontinuity for Γ. In fact, the domains are constructed from a balanced metric thickening naturally associated to ω and the limit cone $\mathbb{C}_\Gamma$. The proper discontinuity and cocompactness of the domains can be deduced from [KLP18a, KL18], but we give a simpler proof in the present case. For ω-undistorted subgroups, the domains can be characterized as a space of horofunctions which are proper and bounded below on Γ-orbits, see Proposition 3.4.8.

3.4.1 Thickenings in flag manifolds and horoboundaries

Recall that we have fixed an $\omega \in \mathfrak{a}^*$, which defines a flag manifold $\mathcal{F}_\omega \subset \partial_{\mathrm{vis}}\mathbb{X}$ and a definite Finsler metric d_ω on $\mathbb{X}$.

Definition 3.4.1 ([KLM09, KL06])**.** The *asymptotic slope* of a convex Lipschitz function f is

$$\mathrm{slope}_f\colon \partial_{\mathrm{vis}}\mathbb{X} \to \mathbb{R}, \quad \mathrm{slope}_f(\eta) := \lim_{t\to\infty} \frac{f \circ c_\eta(t)}{t}.$$

This limit always exists for convex Lipschitz functions and is independent of the basepoint of the geodesic ray c_η.

For a Riemannian Busemann function associated to $\xi \in \partial_{\mathrm{vis}}\mathbb{X}$, the asymptotic slope is given by the Tits angle:

$$\mathrm{slope}_{b_{o,\xi}}(\eta) = -\cos\angle_{\mathrm{Tits}}(\xi,\eta).$$

Lemma 3.4.2. *The slope of a mixed Busemann function $b^{\omega}_{\nu,o}$ is given by*

$$\mathrm{slope}_{b^{\omega}_{\nu,o}}(\eta) = \max\left\{-\cos\angle_{\mathrm{Tits}}(\xi,\eta) \mid \xi \in I^{\omega}_{\nu}\right\}.$$

Here $I^{\omega}_{\nu} \subset \mathcal{F}_\omega$ refers to the set of $\xi \in \mathcal{F}_\omega$ incident to ν, see Definition 3.3.4.

Proof. Since $b^{\omega}_{\nu,o}$ and $b_{\xi,o}$ for $\xi \in \mathcal{F}_\omega$ are convex, we can replace the limits by a supremum in the definition of the slope. Therefore:

$$\mathrm{slope}_{b^{\omega}_{\nu,o}}(\eta) = \sup_{t\geqslant 0} \frac{b^{\omega}_{\nu,o} \circ c_{o,\eta}}{t} = \sup_{t\geqslant 0}\max_{\xi\in I^{\omega}_{\nu}} \frac{b_{o,\xi}\circ c_{o,\eta}}{t} = \max_{\xi\in I^{\omega}_{\nu}}\sup_{t\geqslant 0} \frac{b_{o,\xi}\circ c_{o,\eta}}{t},$$

$$\sup_{t\geqslant 0} \frac{b_{o,\xi}\circ c_{o,\eta}}{t} = \mathrm{slope}_{b^{\omega}_{\nu,o}}(\eta) = -\cos\angle_{\mathrm{Tits}}(\xi,\eta).$$

□

To a point $\xi \in \partial_{\rm vis}\mathbb{X}$ such that $\omega\left(w \cdot \tau(\xi)\right) \neq 0$ for all $w \in W$, one can associate its thickening in $\mathcal{F}_\omega$:

$$\mathrm{Th}^\omega(\xi) := \{a \mid \angle_{\rm Tits}(a,\xi) < \pi/2\} \subset \mathcal{F}_\omega.$$

Similarly we can define a thickening in the horoboundary:

$$\mathrm{Th}^\omega_{\rm horo}(\xi) := \{[h] \mid \mathrm{slope}_h(\xi) < 0)\} \subset \partial^\omega_{horo}\mathbb{X}.$$

Note that the intersection of $\mathrm{Th}^\omega_{\rm horo}(\xi)$ with $\mathcal{F}_\omega$ in $\partial_\omega\mathbb{X}$ coincides with $\mathrm{Th}^\omega(\xi)$.

Lemma 3.4.3. *Let $\xi_1, \xi_2 \in \sigma$ be two points in an ideal Weyl chamber whose types belong to the same connected component of $\sigma_{mod}\backslash\bigcup_{w\in W} w \cdot \mathrm{Ker}(\omega)$. The associated thickenings coincide:*

$$\mathrm{Th}^\omega(\xi_1) = \mathrm{Th}^\omega(\xi_2),$$

$$\mathrm{Th}^\omega_{\rm horo}(\xi_1) = \mathrm{Th}^\omega_{\rm horo}(\xi_2).$$

Proof. The fact that $\tau(\xi_1)$ and $\tau(\xi_2)$ lie in the same connected component of $\sigma_{mod}\backslash\bigcup_{w\in W} w \cdot \mathrm{Ker}(\omega)$ implies that the segment c between ξ_1 and ξ_2 in σ contains only elements whose types do not belong to $w \cdot \mathrm{Ker}(\omega)$ for any $w \in W$.

The Tits angle with a point $a \in \partial_{\rm vis}\mathbb{X}$ is a function on $\partial_{\rm vis}\mathbb{X}$ that is continuous on any ideal Weyl chamber: indeed given a Weyl chamber of $\partial_{\rm vis}\mathbb{X}$, there exist a flat containing it as well as a in its boundary. On this flat the Tits angle is just the standard Euclidean angle.

Now let $a \in \mathrm{Th}^\omega(\xi_1)$: on the segment c the Tits angle $\angle_{\rm Tits}(a,\xi)$ is never equal to $\frac{\pi}{2}$, and it varies continuously, so $a \in \mathrm{Th}^\omega(\xi_2)$, and vice versa. Hence $\mathrm{Th}^\omega(\xi_1) = \mathrm{Th}^\omega(\xi_2)$. The set $\mathrm{Th}^\omega_{\rm horo}(\xi_i)$ can be characterized as the set of mixed Busemann functions $b^\omega_{\nu,o}$ such that $I^\omega_\nu \subset \mathrm{Th}^\omega(\xi_i)$ for $i = 1, 2$. Hence also $\mathrm{Th}^\omega_{\rm horo}(\xi_1) = \mathrm{Th}^\omega_{\rm horo}(\xi_2)$. □

Let $\mathcal{C}$ be a subset of a connected component of $\sigma_{mod}\backslash\bigcup_{w\in W} w \cdot \mathrm{Ker}(\omega)$, and let $\Theta \subset \Delta$ be the set of roots which do not vanish on $\mathcal{C}$. Given a flag $\tau \in \mathcal{F}_\Theta$ we define therefore its thickenings:

$$\mathrm{Th}^\omega(\tau,\mathcal{C}) := \{\eta \mid \angle_{\rm Tits}(\eta,\zeta) < \pi/2\} \subset \mathcal{F}_\omega.$$

$$\mathrm{Th}^\omega_{\rm horo}(\tau,\mathcal{C}) := \{[h] \mid \mathrm{slope}_h(\zeta) < 0)\} \subset \partial^\omega_{\rm horo}\mathbb{X}.$$

For this definition we chose some $\zeta \in \mathrm{st}_\mathcal{C}(\tau) \in \partial_{\rm vis}\mathbb{X}$. The definition does not depend of this choice because of Lemma 3.4.3. In particular one has the following:

$$\mathrm{Th}^\omega(\tau,\mathbb{C}) = \{\eta \mid \forall\xi \in \mathrm{st}_\mathbb{C}(\tau),\, \angle_{Tits}(\xi,\eta) < \pi/2\} \subset \mathcal{F}_\omega$$

$$\mathrm{Th}^\omega_{\rm horo}(\tau,\mathbb{C}) = \left\{[b^\omega_{\nu,x}] \mid I^\omega_\nu \subset \mathrm{Th}^\omega(\tau,\mathbb{C})\right\} \subset \partial^\omega_{\rm horo}\mathbb{X}.$$

These thickenings are closely related to the *metric thickenings* considered by Kapovich-Leeb-Porti, see [KL18, Section 8.3] and Remark 3.4.4.

Note that the thickenings $\mathrm{Th}^{\omega}(\tau, \mathcal{C})$ and $\mathrm{Th}^{\omega}_{\mathrm{horo}}(\tau, \mathcal{C})$ depend only on τ and on the connected component of $\sigma_{mod} \backslash \bigcup_{w \in W} w \cdot \mathrm{Ker}(\omega)$ in which $\mathcal{C}$ lies. In practice we will apply this to the case when Γ is ω-undistorted and $\mathbb{C}$ is an auxiliary neighborhood of the limit cone $\mathbb{C}_\Gamma$ in the same connected component.

Remark 3.4.4. In the present paper we directly define thickenings as subsets of flag manifolds and horoboundaries. Kapovich-Leeb-Porti [KLP18a] define thickenings to be subsets of the Weyl group W and use such subsets to construct thickenings in flag manifolds and horoboundaries. When $\omega = \omega_1$ and $G = \mathrm{SL}(2n, \mathbb{R})$, we have $W = S_{2n}$ and the thickening is the subset of S_{2n} taking $1 \in \{1, \dots, 2n\}$ into $\{1, \dots n\}$. This thickening is balanced, left-invariant for the subgroup of W stabilizing the subset $\{1, \dots, n\}$ (equivalently, the vertex of σ_{mod} corresponding to $\mathrm{Gr}(n, 2n)$), and right-invariant for the subgroup of W stabilizing $\{1\}$ (equivalently, the vertex of σ_{mod} corresponding to $\mathrm{Gr}(1, 2n) = \mathbb{RP}^{2n-1}$). In general, the thickening is the metric thickening $\mathrm{Th}_{\zeta, \mathrm{w}, \pi/2}$ of [KLP18a] where w is the W-translate of the dual to ω in σ_{mod} and ζ is a point in the simplex τ_{mod} corresponding to $\Theta(\sigma_\Gamma)$, see Proposition 3.2.5.

3.4.2 Behaviour of horofunctions along geodesic rays

For a semi-simple Lie groups of real rank at least 2, for every geodesic in the symmetric space there exist a Busemann function that is constant on this geodesic. However if we restrict to some types of geodesics and if we consider only Busemann functions associated to points in $\mathcal{F}_\omega$ we can rule out this phenomenon. In this section we prove that the behavior of a Busemann function associated to a point $a \in \mathcal{F}_\omega$, and more generally the behavior of a horofunction in $\partial_\omega \mathbb{X}$ along the orbits of Γ are subject to a dichotomy.

Let $\mathcal{C} \subset \mathfrak{a}^+$ be a closed subset that avoids $w \cdot \mathrm{Ker}(\omega)$ for all $w \in W$. Let

$$C_{\mathcal{C},\omega} = \inf \left\{ \frac{|\omega(w \cdot \mathrm{v})|}{\|\mathrm{v}\| \|\omega\|} \mid \mathrm{v} \in \mathcal{C}, w \in W \right\},$$

which is a positive constant.

For $o \in \mathbb{X}$ and $\eta \in \partial_{\mathrm{vis}} \mathbb{X}$ we let $c_{o,\eta} \colon [0, \infty) \to \mathbb{X}$ denote the geodesic ray emanating from o asymptotic to η.

Lemma 3.4.5. *Let $[h] \in \partial_\omega \mathbb{X}$ be a horofunction, let $o \in \mathbb{X}$ be a basepoint and let $f \in \mathcal{F}_\Theta$. Exactly one of the following holds:*

(i) $[h] \in \mathrm{Th}^{\omega}_{\mathrm{horo}}(f, \mathcal{C})$ and for every $\eta \in \mathrm{st}_{\mathcal{C}}(f)$ the geodesic ray $c_{o,\eta}$ satisfies

$$h(c_{o,\eta}(t)) - h(o) \leqslant -C_{\mathcal{C},\omega} t.$$

(ii) $[h] \notin \mathrm{Th}^{\omega}_{\mathrm{horo}}(f, \mathcal{C})$ and for all $\epsilon > 0$ there exist $A > 0$ such that for every $\eta \in \mathrm{st}_{\mathcal{C}}(f)$ the geodesic ray $c_{o,\eta}$ satisfies

$$h(c_{o,\eta}(t)) - h(o) \geqslant (C_{\mathcal{C},\omega} - \epsilon) t - A.$$

We emphasize that A depends on $o, [h]$, and f.

Proof. By Proposition 3.3.5 we may write $h = b^{\omega}_{\nu,x}$ for some simplex ν, i.e. some flag of any type, and some $x \in \mathbb{X}$. Let $c = c_{o,\eta} \colon \mathbb{R}_{\geqslant 0} \to \mathbb{X}$ be a geodesic ray based at o, corresponding to a point $\eta \in \partial_{\mathrm{vis}}\mathbb{X}$.

Suppose that $[h] \in \mathrm{Th}^{\omega}_{\mathrm{horo}}(f, \mathcal{C})$ and $\eta \in \mathrm{st}_{\mathcal{C}}(f)$, i.e. $I^{\omega}_{\nu} \subset \mathrm{Th}^{\omega}(f, \mathcal{C})$ by the discussion after Lemma 3.4.3. This means that every $\xi \in \mathcal{F}_{\omega}$ incident to ν satisfies $\angle_{\mathrm{Tits}}(\xi, \eta) < \pi/2$. The slope of $b_{x,\xi}$ along η is then $-\cos \angle_{\mathrm{Tits}}(\xi, \eta)$, which is at most $-C_{\mathcal{C},\omega}$. So each $b_{x,\xi} \circ c_{\eta}$ is bounded above by $-C_{\mathcal{C},\omega} t$, and the same applies to their maximum, $h = b^{\omega}_{\nu,x}$.

Suppose now that $[h] \notin \mathrm{Th}^{\omega}_{\mathrm{horo}}(f, \mathcal{C})$, i.e. there exists some $\xi \in \mathcal{F}_{\omega}$ incident to ν and $\eta \in \mathrm{st}_{\mathcal{C}}(f)$ such that $\angle_{\mathrm{Tits}}(\xi, \eta) > \frac{\pi}{2}$. The asymptotic slope of the convex function $h \circ c_{o,\eta}$ is greater than or equal to $C_{\mathcal{C},\omega}$; in particular there exist $E > 0$ such that $h(c(E)) - h(c(0)) \geqslant (C_{\mathcal{C},\omega} - \epsilon)E$. The constant E can be chosen uniformly for all geodesic rays $c_{o,\eta}$ since h is continuous and the set of geodesic rays based at o with $\eta \in \mathrm{st}_{\mathcal{C}}(f)$ is compact. Since $h \circ c$ is convex, $h(c(t)) - h(o) \geqslant (C_{\mathcal{C},\omega} - \epsilon)t$ for $t > E$. Since h is 1-Lipshitz one has in particular $h(c(t)) - h(o) \geqslant (C_{\mathcal{C},\omega} - \epsilon)t - E$ for all $t \geqslant 0$, which concludes the proof. ☐

Example 3.4.6. For $G = \mathrm{SL}(d, \mathbb{R})$, the Busemann functions $b_{o,\zeta}$ with $\zeta = \mathbb{R}w \in \mathcal{F}_{\omega_1} = \mathbb{RP}^{d-1}$ are given by

$$b_{o,\zeta}(q) = \log\left(\frac{|w|_q}{|w|_o}\right).$$

A maximal flat containing o corresponds to a line decomposition which is orthogonal with respect to o. $\mathbb{R}w$ is in the boundary of this flat if and only if it is one of these lines. If it is, then the Busemann function along a ray in this flat is linear.

3.4.3 Characterization of the domain of discontinuity

In this section we consider a non-elementary ω-undistorted subgroup Γ of G. In particular Γ is Anosov for a set of roots $\Theta = \Theta(\sigma_\Gamma)$ by Proposition 3.2.5.

The limit map $\xi \colon \partial\Gamma \to \mathcal{F}_{\Theta}$ and the thickenings from the previous section provide the data to define domains in $\mathcal{F}_{\omega}$ and $\partial_{\omega}\mathbb{X}$, following [KLP18a, KL18].

Recall that $\Theta \subset \Delta$ is the set of simple roots determined by ω and $\mathbb{C}_\Gamma$ such that Γ is Θ-Anosov, and that $\mathcal{C} \subset \mathfrak{a}^+$ is a compact neighborhood of $\mathcal{C}_\Gamma$ that avoids $w \cdot \mathrm{Ker}(\omega)$ for all $w \in W$.

Definition 3.4.7. Let us define the following domains:

$$\Omega^{\omega}_{\mathrm{flag}} := \mathcal{F}_{\omega} \setminus \bigcup_{x \in \partial\Gamma} \mathrm{Th}^{\omega}(\xi_{\Theta}(x), \mathcal{C}),$$

$$\Omega^{\omega}_{\mathrm{horo}} := \partial^{\omega}_{\mathrm{horo}}\mathbb{X} \setminus \bigcup_{x \in \partial\Gamma} \mathrm{Th}^{\omega}_{\mathrm{horo}}(\xi_{\Theta}(x), \mathcal{C}).$$

They can be characterized as domains of Busemann (resp. horofunctions) that are proper bounded from below on any/every Γ-orbit.

Proposition 3.4.8. *A point $\eta \in \mathcal{F}_\omega$ belongs to $\Omega^\omega_{\mathrm{flag}}$ if and only if the associated Busemann function $b_{\eta,o}$ restricted to the Γ-orbit of o is bounded from below.*

An element $[h] \in \partial^\omega_{\mathrm{horo}}\mathbb{X}$ belongs to $\Omega^\omega_{\mathrm{horo}}$ if and only if h restricted to the Γ-orbit of o is bounded from below. In this case, the horofunction is proper on any Γ-orbit.

Proof. Since $\mathcal{F}_\omega$ includes into $\partial^\omega_{horo}\mathbb{X}$, the first statement follows from the second.

Let $[h] \in \partial^\omega_{\mathrm{horo}}\mathbb{X}$ be a horofunction. If $[h] \in \partial^\omega_{\mathrm{horo}}\mathbb{X}$ is not in $\Omega^\omega_{\mathrm{horo}}$, then it belongs to $\mathrm{Th}^\omega_{\mathrm{horo}}(\xi_\Theta(z), \mathcal{C})$ for some $z \in \partial\Gamma$. If (γ_n) is a geodesic ray in Γ converging to z, Lemma 3.2.10 implies that there exist a constant $D \geqslant 0$ such that for all $n \geqslant 0$, $\gamma_n \cdot o$ is at distance at most D from a point $x_n \in \mathcal{V}(o, \mathrm{st}_{\mathcal{C}}(\xi_\Theta(z)))$.

Lemma 3.4.5 implies that $h(x_n) - h(o) \leqslant -C_{\mathcal{C},\omega} d(o, x_n)$. Since h is 1-Lipshitz, this implies that $h(\gamma_n \cdot o)$ goes to $-\infty$, so h is unbounded from below.

To conclude the proof, we need to show that if h is unbounded from below or fails to be proper on a Γ-orbit, then h does not belong to Ω^ω_{horo}. In either case, there is a diverging sequence of elements $(\gamma_n)_{n\in\mathbb{N}}$ of Γ such that $h(\gamma_n \cdot o)$ is bounded from above by a constant $D \geqslant h(o)$. The sequence of geodesic segments $([o, \gamma_n \cdot o])$ converges up to subsequence to a geodesic ray $[o, \eta)$ with $\eta \in \partial_{\mathrm{vis}}\mathbb{X}$. Since Γ is Θ-Anosov with limit cone inside $\mathcal{C}$ one has $\eta \in \mathrm{st}_{\mathcal{C}}(\xi_\Theta(z))$ for some $z \in \partial\Gamma$ by Lemma 3.2.10.

Note that since h is convex, it is bounded from above by D on all the geodesic segments $([o, \gamma_n \cdot o])$ and hence also on the geodesic ray $[o, \eta)$. Lemma 3.4.5 therefore implies that $h \in \mathrm{Th}^\omega_{\mathrm{horo}}(\xi_\Theta(\zeta), \mathcal{C})$, so $[h]$ does not belong to $\Omega^\omega_{\mathrm{horo}}$. □

We show that the horofunctions belonging to $\mathbb{X} \cup \Omega^\omega_{\mathrm{horo}}$ are not only proper and bounded from below on Γ-orbits, but are moreover locally uniformly proper. More precisely, the constant A from Lemma 3.4.5 can be chosen to be uniform on a neighborhood of $[h]$.

Lemma 3.4.9. *Let $[h_0] \in \mathbb{X} \cup \Omega^\omega_{\mathrm{horo}}$, and let $o \in \mathbb{X}$. There exists a neighborhood $U \subset \mathbb{X} \cup \partial^\omega_{\mathrm{horo}}\mathbb{X}$ of $[h_0]$ and a constant $A > 0$ such that for $[h] \in U$ and $\gamma \in \Gamma$:*

$$h(\gamma \cdot o) - h(o) \geqslant C_{\mathcal{C},\omega} d(o, \gamma \cdot o) - A.$$

We check that this result still holds for elementary groups Γ in 3.4.4.

Proof. Let $\mathcal{C}'$ be a compact neighborhood of $\mathcal{C}_\Gamma$ that lies in the interior of $\mathcal{C}$. Note that $C_{\mathcal{C}',\omega} > C_{\mathcal{C},\omega}$. We consider the following subset of the visual boundary:

$$E = \bigcup_{\zeta \in \partial\Gamma} \mathrm{st}_{\mathcal{C}'}(\xi_\Theta(\zeta)).$$

Applying Lemma 3.4.5 for the subset $\mathcal{C}'$ implies that for every $\xi \in E$ there exist $t_0 > 0$ large enough that any point x on the geodesic ray $[o, \xi)$ at distance $t \geqslant t_0$ from o satisfies:

$$\frac{h_0(x) - h_0(o)}{t} > C_{\mathcal{C},\omega}.$$

Moreover the same property holds for every ξ' close enough to ξ and for $[h]$ close enough to $[h_0]$. Since E is compact, there exist a real number t_1 and a neighborhood U of $[h_0]$ such that for all $[h] \in U$ and $\xi \in E$ the point x on the geodesic ray $[o, \xi)$ at distance t_1 from 0 satisfies:

$$\frac{h(x) - h(o)}{t_1} > C_{\mathcal{C},\omega}.$$

Lemma 3.2.10 implies that for some $D > 0$, for every $\gamma \in \Gamma$, $\gamma \cdot o$ is at distance at most D from some point $y \in [o, \xi)$ for some $\xi \in E$. Let $[h] \in U$; the function $h - h(o)$ is convex on the geodesic ray $[o, \xi)$ and greater than $C_{\mathcal{C},\omega} t_1$ at the point $x \in [o, \xi)$ such that $d(o, x) = t_1$. Moreover it is 1-Lipshitz, which implies that for all $y \in [o, \xi)$, $h(y) - h(o) \geqslant d(o, y) C_{\mathcal{C},\omega} - t_1 C_{\mathcal{C},\omega} - t_1$. Using again the fact that h is 1-Lipshitz we get:

$$h(\gamma \cdot o) - h(o) \geqslant C_{\mathcal{C},\omega} d(o, \gamma \cdot o) - (t_1 C_{\mathcal{C},\omega} + t_1 + D).$$

□

3.4.4 Elementary subgroups.

In the previous subsection we required the group Γ to be non-elementary, i.e. not virtually cyclic. The fact that nonelementary ω-undistorted groups are Anosov, Proposition 3.2.5, relies on proving that the limit cone $\mathcal{C}_\Gamma$ is connected, which often fails for virtually cyclic groups. For instance the following subgroup of $\mathrm{SL}(3, \mathbb{R})$ has disconnected limit cone and is not Anosov, but is ω_1-undistorted:

$$\left\{ \begin{pmatrix} 4^n & & \\ & 2^{-n} & \\ & & 2^{-n} \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

In this subsection we adapt the previous results for elementary subgroups. Note that the only non-trivial case is the case of infinite virtually cyclic groups.

Let $\Gamma \subset G$ be an infinite virtually cyclic group that is quasi-isometrically embedded, and $\langle \gamma \rangle \subset \Gamma$ be a finite index infinite cyclic subgroup. The element $\gamma \in G$ admits a Jordan decomposition into $\gamma_t \gamma_e \gamma_u$ where γ_t is a transvection, γ_e is elliptic, and γ_u is unipotent, and the factors commute [Ebe96].

The element γ_t is nontrivial since Γ is quasi-isometrically embedded, so γ_t is the transvection corresponding to an oriented geodesic axis $c \colon \mathbb{R} \to \mathbb{X}$ parametrized with speed one with endpoints $\eta_\pm \in \partial_{\mathrm{vis}} \mathbb{X}$. Let m_g denote the translation length of $g \in G$.

Lemma 3.4.10. *Let $o \in \mathbb{X}$ and $\gamma \in G$ with γ_t non-trivial. There exists a constant C depending on o, γ such that: For all $n \in \mathbb{Z}$*

$$d(\gamma^n \cdot o, c(nm_\gamma)) \leqslant 2\log(n) + C.$$

Proof. Let $\gamma = \gamma_t \gamma_u \gamma_e$ be the Jordan decomposition. Let p be a fixed point of γ_e. Then

$$d(\gamma^n \cdot o, c(nm_\gamma)) \leqslant d(o,p) + d(\gamma_u^n \cdot p, p) + d(p, c(0)).$$

By the proof of [GGKW17a, Claim 2.28], there exists C' depending only on γ_u and p such that $d(\gamma_u^n \cdot p, p) \leqslant 2\log(n) + C'$. We may set $C = C' + d(o,p) + d(p, c(0))$ to see the desired result. □

Lemma 3.4.10 is weaker than the Morse Lemma, but it will be sufficient to generalize our results to cyclic groups.

Now let $\omega \in \mathfrak{a}^*$, and let us assume that for all $w \in W$, $\tau(\eta_+)$ and $\tau(\eta_-)$ do not belong to $\mathrm{Ker}(w \cdot \omega)$. This property is the equivalent to the ω-undistorted condition, since the limit cone $\mathcal{C}_\Gamma$ consist of the two points $\tau(\eta_+)$ and $\tau(\eta_-)$.

In the virtually cyclic case we adapt our definitions as follows:

$$\Omega^\omega_{\text{flag}} = \mathcal{F}_\omega \backslash \left\{ \xi \in \mathcal{F}_\omega \mid \angle_{\text{Tits}}(\xi, \eta_+) < \frac{\pi}{2} \text{ or } \angle_{\text{Tits}}(\xi, \eta_-) < \frac{\pi}{2} \right\},$$

$$\Omega^\omega_{\text{horo}} = \partial^\omega_{\text{horo}} \mathbb{X} \backslash \left\{ [h] \in \mathcal{F}_\omega \mid \text{slope}_h(\eta_+) < 0 \text{ or } \text{slope}_h(\eta_-) < 0 \right\}.$$

We now adapt the following results.

Proposition 3.4.11 (Analog of Proposition 3.4.8)**.** *An element $[h] \in \partial^\omega_{\text{horo}}\mathbb{X}$ belongs to $\Omega^\omega_{\text{horo}}$ if and only if h restricted to the Γ-orbit of o is bounded from below. In this case, the horofunction is proper on any Γ-orbit.*

Proof. Let $[h] \in \partial^\omega_{\text{horo}}\mathbb{X}$ be a horofunction. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a diverging sequence of elements of Γ such that $h(\gamma_n \cdot o)$ is bounded from above. In our case one can assume that $\gamma_n = \gamma^n$, or γ^{-n}. We consider the first case, as the other one is identical. By Lemma 3.4.10, $\gamma^n \cdot o$ is at logarithmic distance from $c(nm_g)$. Hence h grows t most logarithmically on c, but then Lemma 3.4.5 implies that h has non-positive slope, and hence negative slope on c, hence h does not belong to $\Omega^\omega_{\text{horo}}$.

The other part of the proof works as in Proposition 3.4.8. □

Lemma 3.4.12 (Analog of Lemma 3.4.9)**.** *Let $[h_0] \in \mathbb{X} \cup \Omega^\omega_{\text{horo}}$, and let $o \in \mathbb{X}$. There exists a neighborhood $U \subset \mathbb{X} \cup \partial^\omega_{\text{horo}}\mathbb{X}$ of $[h_0]$ and a constant $A > 0$ such that for $[h] \in U$ and $n \in \mathbb{Z}$:*

$$h(\gamma^n \cdot o) - h(o) \geqslant C_{\mathcal{C}_\Gamma, \omega} d(o, \gamma^n \cdot o) - A.$$

Proof. Here again we replace the Morse property by Lemma 3.4.10. Let $\mathcal{C}$ be a neighborhood of $\mathcal{C}_\Gamma$ avoiding $w \cdot \ker(\omega)$ for all $w \in W$. We obtain for some $D, E, t_0 > 0$:

$$h(\gamma^n \cdot o) - h(o) \geqslant C_{\mathcal{C}, \omega} d(o, \gamma^n \cdot o) - (t_0 C_{\mathcal{C}, \omega} + D\log(n) + E).$$

This is still enough to get the desired result for A large enough since $C_{\mathcal{C},\omega} > C_{\mathcal{C}_\Gamma,\omega}$ and $d(o, \gamma^n \cdot o)$ grows linearly in n. $\square$

Remark 3.4.13. Du proves in [Du23] that the group generated by

$$\gamma = \begin{pmatrix} \lambda & & \\ & 1 & \\ & & \lambda^{-1} \end{pmatrix}$$

has a finite-sided Dirichlet-Selberg domain at $o \in \mathcal{X}_3$ if and only if o is in the axis of γ, i.e. is diagonal. But even for such points $o \in \mathcal{X}$, the domain is not properly finite-sided. One can deduce from this that an elementary subgroup of $\mathrm{SL}(3, \mathbb{R})$ admits a properly finite-sided Dirichlet-Selberg domain if and only if it is ω_1-undistorted, in which case every Dirichlet-Selberg domain is properly finite-sided.

3.5 Dirichlet domains for Finsler metrics

In this section, we consider Dirichlet domains associated to Finsler metrics. Using the results of the previous section, we show in Theorem 3.5.2 that such Dirichlet domains with respect to d_ω are properly finite-sided for ω-undistorted subgroups. Moreover we deduce Theorem 1.3.15 in Corollary 3.5.4. In the rest of the section we demonstrate some partial converse results. In Theorem 3.5.7 we show that any discrete group admitting a properly finite-sided Dirichlet domain is quasi-isometrically embedded in $\mathbb{X}$. In Section 3.5.3 we show that the ω-undistorted condition is equivalent to the *disjoint half-space property*, see Definition 3.5.9.

Recall that we assume throughout the paper that $\omega \in \mathfrak{a}^*$ defines a definite Finsler metric d_ω on $\mathbb{X}$.

3.5.1 ω-undistorted implies properly finite-sided

For $x, y \in \mathbb{X}$, the *Finsler half-space* is

$$\mathcal{H}^\omega(x, y) := \{[h] \in \mathbb{X} \cup \partial^\omega_{horo}\mathbb{X} \mid h(x) \leqslant h(y)\}.$$

It is the closure in $\mathbb{X} \cup \partial^\omega_{horo}\mathbb{X}$ of the set of points $z \in \mathbb{X}$ satisfying $d_\omega(x, z) \leqslant d_\omega(y, z)$.

Let Γ be a discrete subgroup of G.

Definition 3.5.1. The *Dirichlet domain* associated to Γ based at o with respect to the Finsler distance d_ω is given by:

$$\mathcal{D}^\omega_\Gamma(o) := \{[h] \mid \forall \gamma \in \Gamma,\ h(o) \leqslant h(\gamma \cdot o)\} = \bigcap_{\gamma \in \Gamma \setminus \Gamma_o} \mathcal{H}^\omega(o, \gamma \cdot o) \subset \mathbb{X} \cup \partial^\omega_{horo}\mathbb{X}.$$

We call a Dirichlet domain $\mathcal{D}^\omega_\Gamma(o)$ *properly finite-sided* if there exists a neighborhood U of $\mathcal{D}^\omega_\Gamma(o)$ in $\mathbb{X} \cup \partial^\omega_{horo}\mathbb{X}$ and a finite set $F \subset \Gamma$ such that for all $\gamma \in \Gamma \backslash F$, $U \subset \mathcal{H}^\omega(o, \gamma \cdot o)$.

Theorem 3.5.2. *If Γ is ω-undistorted then for all $o \in \mathbb{X}$, the Dirichlet domain $\mathcal{D}^\omega_\Gamma(o)$ is properly finite-sided.*

Moreover for any $A > 0$ one can find a finite set $S \subset \Gamma$ and a neighborhood U of $\mathcal{D}^\omega_\Gamma(o)$ such that for all $[h] \in U$ and $\gamma \in \Gamma \backslash S$, $h(\gamma \cdot o) > h(o) + A$.

Proof. Every horofunction in $\mathcal{D}^\omega_\Gamma(o)$ is bounded from below on the Γ-orbit of o. Hence if Γ is ω-undistorted one has $\mathcal{D}^\omega_\Gamma(o) \subset \mathbb{X} \cup \Omega^\omega_{\mathrm{horo}}$ by Proposition 3.4.8 (or Proposition 3.4.11 in the elementary case). Let $K \subset \mathbb{X} \cup \Omega^\omega_{\mathrm{horo}}$ be a compact neighborhood of $\mathcal{D}^\omega_\Gamma(o)$. By considering an open cover of K, Lemma 3.4.9 (or Lemma 3.4.12) implies the existence of constants $B, C > 0$ such that for $[h] \in K$ and $\gamma \in \Gamma$:

$$h(\gamma \cdot o) - h(\gamma) \geqslant Cd(o, \gamma \cdot o) - B.$$

Hence for all $\gamma \in \Gamma$ such that $d(o, \gamma \cdot o) > \frac{A+B}{C}$, one has $h(\gamma \cdot o) - h(\gamma) > A$. In particular the half-space $\mathcal{H}^\omega(o, \gamma \cdot o)$ contains K for all but finitely many $\gamma \in \Gamma$, so $\mathcal{D}^\omega_\Gamma(o)$ is properly finite-sided. □

When $G = \mathrm{SL}(d, \mathbb{R})$, the Selberg invariant is close to the Riemannian metric for points that are close, but it also always stays at bounded distance from the Finsler distance d_{ω_1}. The identification $\mathbb{X} \simeq \mathcal{X}$ induces a continuous identification $\mathbb{X} \cup \partial^\omega_{horo}\mathbb{X} \simeq \mathcal{X} \cup \partial\mathcal{X}$, see Theorem 3.3.3.

Lemma 3.5.3. *Let $x_1, x_2 \in \mathbb{X} \simeq \mathcal{X}$:*

$$d_{\omega_1}(x_1, x_2) - \log(d) \leqslant \mathfrak{s}(x_1, x_2) \leqslant d_{\omega_1}(x_1, x_2).$$

Moreover for $[h] \in \mathbb{X} \cup \partial^\omega_{horo}\mathbb{X} \simeq \mathcal{X} \cup \partial\mathcal{X}$ and $o, x \in \mathbb{X} \simeq \mathcal{X}$:

$$h(x) - h(o) - \log(d) \leqslant \mathfrak{s}_o(x, [h]) \leqslant h(x) - h(o) + \log(d).$$

Proof. Let $\lambda_1, \cdots, \lambda_d$ be the eigenvalues of $x_1^{-1}x_2$. Applying Proposition 3.1.1 we see that:

$$d_{\omega_1}(x_1, x_2) = \log\left(\max_{1 \leqslant i \leqslant d} |\lambda_i|\right) \leqslant \log\left(\sum_{1 \leqslant i \leqslant d} |\lambda_i|\right) = \mathfrak{s}(x_1, x_2) + \log(d).$$

$$d_{\omega_1}(x_1, x_2) = \log\left(\max_{1 \leqslant i \leqslant d} |\lambda_i|\right) \geqslant \log\left(\frac{1}{d}\sum_{1 \leqslant i \leqslant d} |\lambda_i|\right) = \mathfrak{s}(x_1, x_2).$$

In particular one gets for $x, y, o \in \mathcal{X} \simeq \mathbb{X}$:

$$d_{\omega_1}(x, y) - d_{\omega_1}(o, y) - \log(d) \leqslant \mathfrak{s}_o(x, y) \leqslant d_{\omega_1}(x, y) - d_{\omega_1}(o, y) + \log(d).$$

By passing to the limit as y goes to $[h]$, we get the desired result. □

We may now deduce Theorem 1.3.15 from Theorem 3.5.2.

Corollary 3.5.4. *Let* $\Gamma < \mathrm{SL}(2n, \mathbb{R})$ *be an* ω_1*-undistorted subgroup. The Dirichlet-Selberg domain* $\mathcal{DS}_\Gamma(o)$ *is properly finite-sided.*

Proof. Let $o \in \mathbb{X}$ be any basepoint. Let U be a neighborhood of $\mathcal{D}^\omega_\Gamma(o)$ in $\mathbb{X} \cup \partial^\omega_{horo}\mathbb{X}$ provided by Theorem 3.5.2 for $A = \log(d)$. There exists a finite set $F \subset \Gamma$ such that for all $[h] \in U$ and $\gamma \in \Gamma \backslash F$, $h(\gamma \cdot o) - h(o) > A$. Hence $\mathfrak{s}_o(o, \gamma \cdot o) > 0$ by Lemma 3.5.3, so U is contained in each of the projective half-spaces $\mathcal{H}(o, \gamma \cdot o)$ with $\gamma \in \Gamma \backslash F$. Therefore $\mathcal{DS}_\Gamma(o)$ is properly finite-sided. □

We will give a second proof of Theorem 1.3.15 using Theorem 3.7.3 in Section 3.7

For ω-undistorted subgroups, we can also give a direct proof that the domains $\Omega^\omega_{\mathrm{flag}}$ and $\Omega^\omega_{\mathrm{horo}}$ are properly discontinuous and cocompact.

Proposition 3.5.5. *The action of an* ω*-undistorted subgroup* Γ *on* $\Omega^\omega_{\mathrm{flag}}$ *and* $\Omega^\omega_{\mathrm{horo}}$ *is properly discontinuous and cocompact.*

The idea is that these domain *coarsely fiber* over Γ, via the map that associates to a horofunction its minimum on the Γ-orbit of $o \in \mathbb{X}$.

Proof. To a horofunction $[h] \in \Omega^\omega_{\mathrm{horo}}$ we associate the non-empty finite *set of minima* $M_{[h]} \subset \Gamma$ of elements γ_0 such that $h(\gamma_0 \cdot o) = \min_{\gamma \in \Gamma} h(\gamma \cdot o)$. This set is well defined and finite since h is proper and bounded from below by Proposition 3.4.8. Moreover this association is equivariant, i.e. $M_{\gamma \cdot [h]} = \gamma M_{[h]}$ for all $\gamma \in \Gamma$.

Let $K \subset \Omega^\omega_{\mathrm{horo}}$ be a compact set. For each $[h_0] \in K$, by Lemma 3.4.9 there is a neighborhood U of $[h_0]$ and a finite set $M \subset \Gamma$ such that for all $[h] \in U$, $M_{[h]} \subset M$. Since K is compact, one can therefore find a finite set M_K such that for all $[h] \in K$, $M_{[h]} \subset M_K$. All but finitely many $\gamma \in \Gamma$ satisfy $\gamma \cdot M_K \cap M_K = \varnothing$. Therefore for all such $\gamma \in \Gamma$, $\gamma \cdot K \cap K = \varnothing$. Hence Γ acts properly on $\Omega^\omega_{\mathrm{horo}}$.

A horofunction $[h] \in \Omega^\omega_{\mathrm{horo}}$ belongs to the Dirichlet domain $\mathcal{D}^\omega_\Gamma(o)$ if (and only if) the neutral element $e \in \Gamma$ belongs to $M_{[h]}$. For every $[h'] \in \Omega^\omega_{\mathrm{horo}}$, there exists some $\gamma' \in M_{[h']}$, and one has $(\gamma')^{-1} \cdot [h'] \in \mathcal{D}^\omega_\Gamma(o)$. Therefore $\mathcal{D}^\omega_\Gamma(o) \cap \Omega^\omega_{\mathrm{horo}} = \mathcal{D}^\omega_\Gamma(o) \cap \partial^\omega_{\mathrm{horo}}\mathbb{X}$ is a fundamental domain for the action of Γ. Moreover it is closed in the metrizable compact $\partial^\omega_{\mathrm{horo}}\mathbb{X}$, hence it is compact. So this is a compact fundamental domain for the action of Γ on $\Omega^\omega_{\mathrm{horo}}$, and the action is cocompact. □

Remark 3.5.6. The fact that Ω^ω_{flag} is a cocompact domain of proper discontinuity is a special case of a result due to Kapovich-Leeb-Porti [KLP18a], and Guichard-Wienhard [GW12] for ω_1-undistorted subgroups of $\mathrm{PSL}(2n, \mathbb{R})$. The construction of Ω^ω_{horo} from a thickening is similar to a construction due to Kapovich-Leeb [KL18], where they consider the case when ω is dual to a regular point of σ_{mod}. They prove proper discontinuity and cocompactness when Γ is an arbitrary Anosov subgroup. Since their Finsler compactification is the maximal

Satake compactification, it dominates the compactification we consider here, and the proper discontinuity and cocompactness follows. In fact, it follows from Proposition 3.4.8 and their Theorem that the Dirichlet domains we consider are properly finite-sided for ω-undistorted subgroups, without the use of Lemma 3.4.9.

3.5.2 Properly finite-sided implies undistorted

We now prove the following necessary condition for a group to admit a properly finite-sided Dirichlet domain.

Theorem 3.5.7. *Let Γ be a discrete subgroup of G and suppose that a Dirichlet domain $\mathcal{D}^{\omega}_{\Gamma}(o)$ is properly finite-sided. The orbit map $\Gamma \to \mathbb{X}$ is a quasi-isometric embedding.*

This result is an adaptation of the Milnor-Schwarz Lemma, replacing the cocompactness of the action by the fact that the Dirichlet domain is *tame at infinity.*

Lemma 3.5.8. *Let Γ be a group acting by isometries on a geodesic metric space X. Suppose that there exists a subset $D \subset X$ such that:*

$$X = \bigcup_{\gamma \in \Gamma} \gamma \cdot D.$$

Suppose moreover that there exist a finite subset $F \subset \Gamma$ and $\epsilon > 0$ such that the ϵ-neighborhood D_ϵ of D satisfies for all $\Gamma \in \Gamma \backslash F$:

$$\gamma \cdot D \cap D_\epsilon = \varnothing.$$

For any $o \in X$ the orbit map $\gamma \in \Gamma \mapsto \gamma \cdot o \in X$ is a quasi-isometric embedding.

Proof. We consider the word metric on Γ defined for $\gamma \in \Gamma$ by:

$$|\gamma| = \min\{n | \gamma = s_1 s_2 \cdots s_n,\ s_i \in F\}.$$

Any other word metric with respect to a finite generating set is quasi-isometric to this one. Let $o \in X$. Let $A = \max_{s \in F} d(o, s \cdot o)$.

Let $\gamma \in \Gamma$ be any element. First note that $d(o, \gamma \cdot o) \leqslant A|\gamma|$. Now let $n = \lceil \frac{d(o,\gamma \cdot o)}{\epsilon} \rceil$. We consider a sequence $x_0, x_1, \cdots, x_n$ of points on a geodesic in X between o and $\gamma \cdot o$ with $x_0 = o, x_n = \gamma \cdot o$ and such that $d(x_i, x_{i+1}) \leqslant \epsilon$ for $0 \leqslant i < n$. Since $X = \bigcup_{\gamma \in \Gamma} \gamma \cdot D$, there exist for all $1 \leqslant i < n$ an element $\gamma_i \in \Gamma$ such that $x_i \in \gamma_i \cdot D$. We set $\gamma_0 = e$ and $\gamma_n = \gamma$.

By the definition of F we know that for all $1 \leqslant i < n$, $\gamma_i^{-1}\gamma_{i+1} \in F$. Indeed $x_{i+1} \in \gamma_{i+1} \cdot D \cap \gamma_i \cdot D_\epsilon$ and hence $\gamma_i^{-1} \cdot x_{i+1} \in \gamma_i^{-1}\gamma_{i+1} \cdot D \cap D_\epsilon$. Therefore γ can be written as the product of n elements of F, so $|\gamma| \leqslant n$. Hence:

$$\epsilon|\gamma| - \epsilon \leqslant d(o, \gamma \cdot o) \leqslant A|\gamma|.$$

This concludes the proof. $\square$

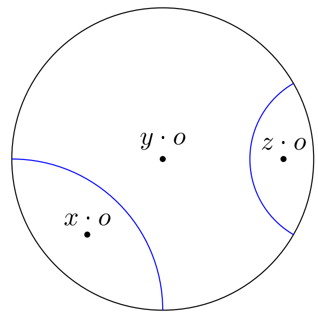


Figure 3.4: Illustration of the disjoint half-space property.

Proof of Theorem 3.5.7. Let U be an open neighborhood of $\mathcal{D}^\omega_\Gamma(o) \subset \mathbb{X} \cup \partial_\omega \mathbb{X}$ such that there exist only finitely many $\gamma \in \Gamma$ such that $\gamma \cdot \mathcal{D}^\omega_\Gamma(o) \cap U \neq \varnothing$. In order to apply Lemma 3.5.8, it suffices to prove that for some $\epsilon > 0$, the intersection $U \cap \mathbb{X}$ contains the ϵ-neighborhood of $\mathcal{D}^\omega_\Gamma(o) \cap \mathbb{X}$. Let U^c be the complement of U in $\mathbb{X} \cup \partial_\omega \mathbb{X}$, which is a compact set.

Suppose the contrary; then there exists a sequence (x_n) of points in $\mathcal{D}^\omega_\Gamma(o) \cap \mathbb{X}$ and a sequence (y_n) of points in $U^c \cap \mathbb{X}$ such that $d(x_n, y_n)$ converges to zero (note that we consider here the Riemannian metric). Up to taking a subsequence, one can assume that the sequences converge to x_∞ in $\mathcal{D}^\omega_\Gamma(o)$ and to y_∞ in U^c respectively. But since $d(x_n, y_n)$ converges to zero, the function $x \in \mathbb{X} \mapsto d_\omega(x, x_n) - d_\omega(x, y_n)$ also converges to zero, uniformly on $\mathbb{X}$. Therefore $x_\infty = y_\infty$, which is not possible since $\mathcal{D}^\omega_\Gamma(o) \subset U$. $\square$

3.5.3 Disjoint half-spaces and the ω-undistorted condition

In this section, we show that ω-undistorted subgroups can be characterized by having sufficiently disjoint half-spaces for the Finsler distance d_ω.

Definition 3.5.9. We say that a finitely generated subgroup Γ satisfies the *ω-disjoint half-space property* if for some $o \in \mathbb{X}$, some word metric on Γ and some integer $D > 0$, for all triples (x, y, z) in Γ that lie in this order on a geodesic such that $d_\Gamma(x, y) = d_\Gamma(y, z) = D$, the half-space $\mathcal{H}^\omega(x \cdot o, y \cdot o)$ is disjoint from $\mathcal{H}^\omega(z \cdot o, y \cdot o)$.

We say that Γ satisfies the *ω-flag disjoint half-space property* if for some $o \in \mathbb{X}$, some word metric on Γ and some integer $D > 0$, for all triples (x, y, z) in Γ that lie in this order on a geodesic such that $d_\Gamma(x, y) = d_\Gamma(y, z) = D$, the intersection of $\mathcal{F}_\omega$ with the half-space $\mathcal{H}^\omega(x \cdot o, y \cdot o)$ is disjoint from $\mathcal{H}^\omega(z \cdot o, y \cdot o)$.

Recall that the half-spaces are closed subsets of $\mathbb{X} \cup \partial_\omega \mathbb{X}$. A priori the flag property is weaker, but we see later that when ω is symmetric, the two are equivalent.

Remark 3.5.10. If the (flag) disjoint half-space property holds for all triples (x, id, z) in Γ with x, z of word length D then it holds for all triples in Γ. Hence the (flag) disjoint half-space property can be verified on a finite subset of triples in Γ.

When $G = \mathrm{SL}(d, \mathbb{R})$, and $\omega = \omega_1$, the (flag) disjoint half-space property can be rephrased in terms of Selberg bisectors and yields another characterization of subgroups $|\log \sigma_i|$-undistorted for all i. Indeed, the intersection of a Selberg bisector with $\mathcal{F}_{\omega_1} = \mathbb{RP}^{d-1}$ equals the intersection of the corresponding Finsler bisector with $\mathcal{F}_{\omega_1}$, and two Selberg bisectors are disjoint if and only if their intersection with $\mathcal{F}_{\omega_1}$ are disjoint. Note that the intersection of a Selberg bisector with $\mathbb{RP}^{d-1}$ is the zero set of a quadratic form.

Theorem 3.5.11. *Let Γ be an ω-undistorted subgroup. It satisfies the ω-disjoint half-space property and the $(-\omega)$-disjoint half-space property.*

Proof. Let $\mathcal{G}$ be the space of bi-infinite geodesic $\eta : \mathbb{Z} \to \Gamma$ such that $\eta(0) = e$. This is a compact set for the standard compact-open topology.

Let $\eta \in \mathcal{G}$. We denote by η^+ and η^- respectively the endpoints in $\partial\Gamma$ of the geodesic ray when n goes respectively to $+\infty$ and $-\infty$. The thickenings $\mathrm{Th}^{\omega}_{\mathrm{horo}}(\xi_\Theta(\eta^+), \mathcal{C}_\Gamma)$ and $\mathrm{Th}^{\omega}_{\mathrm{horo}}(\xi_\Theta(\eta^-), \mathcal{C}_\Gamma)$ are disjoint since the flags $\xi_\Theta(\eta^+)$ and $\xi_\Theta(\eta^-)$ are transverse.

Let $U \subset \mathbb{X} \cup \partial^{\omega}_{horo}\mathbb{X}$ be an open set containing $\mathrm{Th}^{\omega}_{\mathrm{horo}}(\xi_\Theta(\eta^+), \mathcal{C}_\Gamma)$ whose closure is disjoint from $\mathrm{Th}^{\omega}_{\mathrm{horo}}(\xi_\Theta(\eta^-), \mathcal{C}_\Gamma)$. Lemma 3.4.9 implies that all $[h]$ in the complement of U go to $+\infty$ locally uniformly along $\eta(n)$, and all $[h]$ in $\overline{U}$ go to $+\infty$ locally uniformly along $\eta(-n)$. If Γ is elementary we can apply Lemma 3.4.12 instead. Since $\overline{U}$ and U^c are compact, the local uniform behavior is global. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\mathcal{H}^{\omega}(\eta(n) \cdot o, o) \subset U$ and $\overline{U}$ is contained in the complement of $\mathcal{H}^{\omega}(\eta(-n) \cdot o, o)$.

We write $V_{\eta,m} \subset \mathcal{G}$ for the open and closed set of geodesics $\eta' : \mathbb{Z} \to \Gamma$ such that $\eta'_{|[-m,m]} = \eta_{|[-m,m]}$, for the compact open topology. The collection of neighborhoods $\{V_{\eta,n_0} \mid \eta \in \mathcal{G}\}$ covers $\mathcal{G}$, so it admits a finite subcover. Hence there exists $m_0 \in \mathbb{N}$ such that for all $\eta \in \mathcal{G}$, the half-spaces $\mathcal{H}^{\omega}(\eta(m_0) \cdot o, o)$ and $\mathcal{H}^{\omega}(\eta(-m_0) \cdot o, o)$ are disjoint. Therefore Γ satisfies the ω-disjoint half-space property. Since Γ is also $(-\omega)$-undistorted, Γ also satisfies the $(-\omega)$-disjoint half-space property. $\square$

The following result can be seen as a coarse analogue of Proposition 2.4.20, and has a similar proof.

Theorem 3.5.12. *Let Γ be a finitely generated subgroup of G. If Γ satisfies the ω-flag disjoint half-space property and the $(-\omega)$-flag disjoint half-space property, it is ω-undistorted.*

To prove this theorem, we first relate the disjoint half-space property to the convexity of Busemann functions. Let $\epsilon > 0$. We say that a sequence $(s_n)_{n\in\mathbb{Z}}$ is *ϵ-convex at critical points* if for all $n \in \mathbb{Z}$ such that $s_{n+1} - s_n \geqslant -\epsilon$ one has $s_{n+2} - s_{n+1} \geqslant \epsilon$. If for such a sequence one has $s_{-1} \leqslant s_0$, then for all $n \in \mathbb{N}$:

$$s_n \geqslant \epsilon n + s_0.$$

Lemma 3.5.13. *Suppose that for some $o \in \mathbb{X}$, Γ satisfies the flag disjoint half-space property. Then there exist $D \in \mathbb{N}$ and $\epsilon > 0$ such that for every geodesic sequence $(\gamma_n)_{n\in\mathbb{Z}}$ in Γ and every $[b_\xi] \in \mathcal{F}_\omega \subset \partial^\omega_{horo}\mathbb{X}$, the sequence $(b_\xi(\gamma_{Dn} \cdot o))$ is ϵ-convex at critical points.*

Proof. We assume that $\mathcal{D}^\omega_\Gamma(o)$ satisfies the flag disjoint half-space property and let D be the constant from the definition. Suppose to the contrary that there exist sequences (x_n), (y_n) and (z_n) in Γ and a sequence $[b_n] \in \mathcal{F}_\omega \subset \partial^\omega_{\mathrm{horo}}\mathbb{X}$ such that for all $n \in \mathbb{N}$, $d(x_n, y_n) = d(y_n, z_n) = D$ while also $\liminf b_n(y_n \cdot o) - b_n(x_n \cdot o) \geqslant 0$ and $\limsup b_n(z_n \cdot o) - b_n(y_n \cdot o) \leqslant 0$.

Up to acting by Γ, one can assume that the sequence (y_n) is constant and equal to the identity element of Γ. Up to taking a subsequence one can assume that the sequences (x_n) and (z_n) are constant and equal to x and z respectively.

By the flag disjoint half-space property, $\mathcal{F}_\omega \cap \mathcal{H}^\omega(x \cdot o, o)$ is disjoint from

$$\mathcal{H}^\omega(z \cdot o, o) = \{[h] \mid h(z \cdot o) - h(o) \leqslant 0\}.$$

In particular, the function $[b] \mapsto b(o) - b(z \cdot o)$ is continuous and positive on the compact set $\mathcal{F}_\omega \cap \mathcal{H}^\omega(x \cdot o, o)$, so has a positive minimum η. Hence if any $[b]$ in $\mathcal{F}_\omega$ satisfies $b(o) - b(x \cdot o) \leqslant 0$, then $b(z \cdot o) - b(o) \leqslant -\eta$. This contradicts the assumptions on (x_n), (y_n) and (z_n). ☐

In order to handle the disjoint half-space property at two different basepoints, we need to improve the constant ϵ by coarsifying the sequence.

Lemma 3.5.14. *Suppose the sequence (x_n) is ϵ-convex at critical points. For any positive integer N, the sequence (x_{nN}) is $(N\epsilon)$-convex at critical points.*

Proof. If $x_{(n+1)N} - x_{nN} \geqslant -N\epsilon$ then there exists an integer $nN \leqslant k < (n+1)N$ such that $x_{k+1} - x_k \geqslant -\epsilon$. For an ϵ-convex sequence, if there exists k such that $x_{k+1} - x_k \geqslant -\epsilon$, then for all $k'' > k' > k$, it holds that $x_{k''} - x_{k'} \geqslant (k'' - k')\epsilon$. In particular, $x_{(n+2)N} - x_{(n+1)N} \geqslant N\epsilon$. ☐

Proof of Theorem 3.5.12. Let Γ be a finitely generated subgroup of G satisfying the ω-flag disjoint half-space property for the basepoint o and the $(-\omega)$-flag disjoint half-space property for another basepoint o'.

We first show that one can assume $o' = o$, up to replacing D by some $D' > 0$. By Lemma 3.5.13, there exists D such that for every $[b] \in \mathcal{F}_{-\omega}$, and geodesic (γ_n) the sequence $(b(\gamma_{nD} \cdot o'))$ is ϵ-convex at critical points. By Lemma 3.5.14, for any positive integer N, the sequence $(b(\gamma_{nDN} \cdot o'))$ is ϵN-convex at critical points. If $\epsilon N - 2d(o, o') \geqslant \epsilon$, then the sequence $(b(\gamma_{nDN} \cdot o))$ is also ϵ-convex at critical points, since Busemann functions are 1-Lipschitz. We set $D' = DN$.

Now let $\gamma \in \Gamma$ and let $e = \gamma_0, \gamma_1, \cdots, \gamma_N = \gamma$ be a geodesic sequence. We consider a maximal flat passing through o and $\gamma \cdot o$. In the visual boundary of this flat there exist $\xi_1 \in \mathcal{F}_\omega$ and $\xi_2 \in \mathcal{F}_{-\omega}$ such that the associated Busemann functions $[b_{\xi_1}]$ and $[b_{\xi_2}]$ satisfy:

$$\omega(\vec{d}(o, \gamma \cdot o) = b_{\xi_1}(\gamma \cdot o) - b_{\xi_1}(o),$$

$$-\omega(\vec{d}(o, \gamma \cdot o) = b_{\xi_2}(\gamma \cdot o) - b_{\xi_2}(o).$$

Note also that $x \mapsto b_{\xi_1}(x) - b_{\xi_1}(o) + b_{\xi_2}(x) - b_{\xi_2}(o)$ is nonnegative on $\mathbb{X}$, since it is a convex function that vanishes at o and whose gradient also vanishes at o. Hence up to exchanging ω and $-\omega$, one can assume that:

$$b_{\xi_1}(\gamma_{D'} \cdot o) - b_{\xi_1}(o) \geqslant 0.$$

Letting n be the integer part of $\frac{N}{D'}$, we have that

$$b_{\xi_1}(\gamma_{D'n} \cdot o) - b_{\xi_1}(o) \geqslant \epsilon(n-1).$$

Let $E > 0$ be the maximum distance between o and $\gamma_0 \cdot o$ for $\gamma_0 \in \Gamma$ at distance at most D' from the identity. Then:

$$\omega\left(\vec{d}(o, \gamma \cdot o)\right) = b_{\xi_1}(\gamma \cdot o) - b_{\xi_1}(o) \geqslant b_{\xi_1}(\gamma_{nD'} \cdot o) - b_{\xi_1}(o) - E \geqslant \epsilon n - E - \epsilon.$$

Hence Γ is ω-undistorted. $\square$

3.6 Locally symmetric spaces

In this section we consider any locally symmetric space $\mathbb{X}/\Gamma$ where Γ is an ω-undistorted subgroup of G. The Finsler distance d_ω on $\mathbb{X}$ descends to a natural metric on the quotient. We show that the horofunction compactification agrees with the quotient $(\mathbb{X} \cup \partial_{horo}^{\omega}\mathbb{X})/\Gamma$. As a consequence, we recover that such locally symmetric spaces are topologically tame, a special case of [GKW15, Theorem 1.4].

Recall that the symmetric space of $\mathrm{SL}(V)$ embeds in $\mathbb{P}(S^2V)$ as the space $\mathcal{X}$ of positive tensors. The embedding $G \subset \mathrm{SL}(V)$ induces a totally geodesic embedding between the corresponding symmetric spaces, $\mathbb{X} \subset \mathcal{X} \subset \mathbb{P}(S^2V)$.

Given any $x, y \in \mathbb{X} \subset \mathcal{X}$ one can define the *restricted Selberg invariant*. For this we chose representatives $X, Y : V^* \to V$ of x, y such that $\det(X^{-1}Y) = 1$, and we set:

$$\mathfrak{s}^V(x, y) = \log \mathrm{Tr}(X^{-1}Y).$$

This restricted Selberg invariant has the following formula:

Proposition 3.6.1. *Let $x, y \in \mathbb{X}$, the restricted Selberg invariant is equal to:*

$$\mathfrak{s}^V(x, y) = \log \sum_{\alpha \in \Xi} n(\alpha) e^{\alpha(\vec{d}(x,y))}.$$

In this expression, $\Xi \subset \mathfrak{a}^*$ is the weight system associated to the representation V, and for $\alpha \in \Xi$, $n(\alpha)$ is the dimension of the associated weight space.

Proof. The restricted Selberg invariant $\mathfrak{s}^V$ is the restriction to the totally geodesic symmetric space $\mathbb{X}$ of the Selberg invariant defined for the symmetric space of $\mathrm{SL}(V)$. The eigenvalues of the element of $\mathrm{SL}(V)$ corresponding to $\exp(\mathrm{v})$ for $\mathrm{v} \in \mathfrak{a}$ are equal to $e^{\alpha(\mathrm{v})}$ for $\alpha \in \Xi$. Hence this formula follows from 3.1.1. $\square$

This definition can be extended for any semi-positive $y \in \mathbb{P}(S^2V)$. If $y \in \mathbb{P}(S^2V)$ is semi-positive and if $o \in \mathbb{X}$ one can extend the Selberg invariant by taking representatives O, X, Y of o, x, y such that $\det(X^{-1}O) = 1$:

$$\mathfrak{s}_o^V(x, y) = \log \operatorname{Tr}(X^{-1}Y) - \log \operatorname{Tr}(O^{-1}Y).$$

Proposition 3.6.2. *The embedding $y \in \mathbb{X} \mapsto [\mathfrak{s}^V(\cdot, y)] \in \mathcal{Y}(\mathbb{X})$ induces a horofunction compactification of $\mathbb{X}$, which is naturally identified with the generalized Satake compactification $\overline{\mathbb{X}}$ of $\mathbb{X} \subset \mathcal{X}$.*

Proof. Indeed the map $y \in \overline{\mathbb{X}} \mapsto [\mathfrak{s}_o^V(\cdot, y)] \in \mathcal{Y}(\mathbb{X})$ is an embedding from a compact space with dense image, hence it is a homeomorphism onto the horofunction compactification of $\mathbb{X}$. $\square$

If $\Gamma \subset G$ is a discrete subgroup, one can define the Selberg invariant on $\mathbb{X}/\Gamma$ to be:

$$\mathfrak{s}^V(\Gamma \cdot x, \Gamma \cdot y) = \min_{\gamma \in \Gamma} \mathfrak{s}^V(x, \gamma \cdot y).$$

Note that this minimum is reached because the action of Γ on $\mathbb{X}$ is proper. This maps also defines an embedding $y \in \mathbb{X}/\Gamma \mapsto \mathfrak{s}^V(\cdot, y) \in \mathcal{Y}(\mathbb{X}/\Gamma)$, from which one can define a horofunction compactification of $\mathbb{X}/\Gamma$.

Theorem 3.6.3. *Let V be an irreducible representation of G with highest restricted weight ω. Let Γ be a torsion-free ω-undistorted subgroup of G. The horofunction compactification $\overline{\mathbb{X}/\Gamma}$ of $\mathbb{X}/\Gamma$ for the restricted Selberg invariant is naturally identified with $\left(\mathbb{X} \cup \Omega^{\omega}_{\text{horo}}\right)/\Gamma$.*

In other words the compactification of $\mathbb{X}/\Gamma$ is equal to the quotient by Γ of a domain of discontinuity in the compactification of $\mathbb{X}$.

Proof. Let $\widetilde{\phi} : \mathbb{X} \cup \Omega^{\omega}_{\text{horo}} \to \mathcal{Y}(\overline{\mathbb{X}/\Gamma})$ be the map that associates to a class of functions $[h : \mathbb{X} \to \mathbb{R}]$ the class of functions $[\min_{\gamma \in \Gamma} \gamma \cdot h]$, where $\gamma \cdot h(x) = h(\gamma^{-1} \cdot x)$. Since every horofunction in $\mathbb{X} \cup \Omega^{\omega}_{\text{horo}}$ is proper and bounded from below on one and hence any Γ-orbit, $\widetilde{\phi}$ is well defined. Moreover on every open set $U \subset \mathbb{X} \cup \Omega^{\omega}_{\text{horo}}$, there exist a finite set $S \subset \Gamma$ such that on U one has $\min_{\gamma \in \Gamma} \gamma \cdot h = \min_{\gamma \in S} \gamma \cdot h$, see Lemma 3.4.9. Hence $\widetilde{\phi}$ is continuous. Moreover the image of $\mathbb{X}$ by $\tilde{\phi}$ lies in $\mathbb{X}/\Gamma$, so the image of $\tilde{\phi}$ lies in $\overline{\mathbb{X}/\Gamma}$.

The map $\widetilde{\phi}$ is also Γ-invariant by definition, so it induces a map:

$$\phi : \left(\mathbb{X} \cup \Omega^{\omega}_{\text{horo}}\right)/\Gamma \to \overline{\mathbb{X}/\Gamma}.$$

The restriction of this map to $\mathbb{X}/\Gamma$ is the identity: indeed the restricted Selberg invariant between $\Gamma \cdot x$ and $\Gamma \cdot y$ is equal to $\min_{\gamma \in \Gamma} \mathfrak{s}(x, \gamma \cdot y)$. Since $\widetilde{\phi}$ is continuous, so is ϕ and because $(\mathbb{X} \cup \Omega)/\Gamma$ is compact, its image is compact. Hence ϕ is surjective.

It remains to show that ϕ is injective. Let $[h_1], [h_2] \in \mathbb{X} \cup \Omega^{\omega}_{\text{horo}}$ be such that $\widetilde{\phi}([h_1]) = \widetilde{\phi}([h_2])$. Since h_1 and h_2 are proper and bounded from below, given any compact set K there exist a finite set $S_K \subset \Gamma$ such that for $i = 1, 2$, $\min_{\gamma \in \Gamma} \gamma \cdot h_i = \min_{\gamma \in S_K} \gamma \cdot h_i$ on K.

This implies that K is covered by the closed sets $K_{\gamma_1,\gamma_2} = \{x \in K \mid \gamma_1 \cdot h_1(x) = \gamma_2 \cdot h_2(x)\}$ for $\gamma_1, \gamma_2 \in S_K$. If we take a compact set K with non-empty interior, then one of these sets K_{γ_1,γ_2} must have non-empty interior for some $\gamma_1, \gamma_2 \in S_K$. Hence $\gamma_1 \cdot h_1 = \gamma_2 \cdot h_2$ on an open set on $\mathbb{X}$. Note that h_1 and h_2 are analytic as the restriction of the log of a linear map to an analytic submanifold. This implies that $[h_1] = [\gamma_1^{-1}\gamma_2 \cdot h_2]$ and hence the two points must correspond to the same element in $\left(\mathbb{X} \cup \Omega^{\omega}_{\text{horo}}\right)/\Gamma$.

In conclusion ϕ is injective, so we have proven that it induces an homeomorphism. □

Remark 3.6.4. If we try to apply the same argument for the horofunction compatification of $\mathbb{X}/\Gamma$ using the Finsler distance, the proof of the injectivity does not immediately apply since horofunctions are not analytic. However the map ϕ is still well-defined and surjective.

A consequence of this result is that the locally symmetric space $\mathbb{X}/\Gamma$ is topologically tame. A manifold is *topologically tame* if it is the interior of a compact manifold with boundary.

Proposition 3.6.5 ([GGKW17b, Proposition 6.1])**.** *Let X be a real semi-algebraic set and Γ a torsion-free discrete group acting on X by real algebraic homeomorphisms. Suppose Γ acts properly discontinuously and cocompactly on some open subset Ω of X. Let U be a Γ-invariant real semi-algebraic subset of X contained in Ω. If U is a manifold and $\overline{U} \subset \Omega$, then U/Γ is topologically tame.*

We can apply this to semi-algebraic compactifications of $\mathbb{X}$. Note that the Tarski principle implies that semi-algebraic set are closed under projection. Hence any orbit of the algebraic action of an algebraic group on a finite dimensional vector space is semi-algebraic. For instance, given an irreducible representation V of the semi-simple group G with highest restricted weight ω, the totally geodesic embedding $\mathbb{X} \subset \mathcal{X}$ is algebraic. Note also that the closure of a semi-algebraic set is semi-algebraic.

In particular we get the following result for every ω that is the highest restricted weight of a representation:

Corollary 3.6.6. *Let Γ be a torsion-free ω-undistorted subgroup of G. The locally symmetric space $\mathbb{X}/\Gamma$ is topologically tame.*

This recovers a particular case of [GKW15, Theorem 1.4] since ω-undistorted representations are Anosov. Note that the set of $\omega \in \mathfrak{a}$ up to positive scalar and the action of the Weyl group that are the highest restricted wieght of a representation is dense. Therefore any ω-undistorted representation for $\omega \in \mathfrak{a}$ is ω'-undistorted for such an ω', and hence the Theorem applies.

3.7 Restriction of Selberg's construction.

In this section we consider a discrete subgroup $\Gamma \subset \mathrm{SL}(V)$ that is not necessarily $|\log \sigma_i|$-undistorted for all i, and we try to find a smaller domain of $\overline{\mathcal{X}}$ on which the Dirichlet-Selberg domain is properly finite-sided.

3.7.1 The general statement

We first introduce a general but technical statement that we will then apply to more specific situations.

Let G be a semisimple real Lie group and V a finite-dimensional linear real representation of G. Recall from Section 3.3.3 the associated restricted weight space decomposition:

$$V = \bigoplus_{\lambda \in \Phi} V_\lambda$$

with Φ denoting the set of restricted weights.

More precisely for every pair of transverse full flags $f, g \in \mathcal{F}_\Delta$ for G, one has an identification of the model maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ with another $\mathfrak{a}_{f,g} \subset \mathfrak{g}$. For each such choice we get a restricted weight decomposition:

$$V = \bigoplus_{\lambda \in \Phi} V_\lambda^{f,g}.$$

Given a closed subset $\mathcal{C} \subset \sigma_{\mathrm{mod}} = \mathbb{S}\mathfrak{a}^+$, and a subset $\Theta \subset \Delta$ of simple roots of G, we define the following subsets of the space of restricted weights:

$$\Phi^+_{\mathcal{C},\Theta} = \{\lambda \in \Phi \mid \lambda > 0 \text{ on } W_\Theta \cdot \mathcal{C}\},$$

$$\Phi^-_{\mathcal{C},\Theta} = \{\lambda \in \Phi \mid \lambda < 0 \text{ on } W_\Theta \cdot \mathcal{C}\},$$

$$\Phi^0_{\mathcal{C},\Theta} = \Phi \backslash \left(\Phi^+_{\mathcal{C},\Theta} \cup \Phi^-_{\mathcal{C},\Theta}\right).$$

Here W_Θ is the subgroup of the Weyl group generated by the involutions s_α associated to the simple roots $\alpha \in \Delta \backslash \Theta$ and $W_\Theta \cdot \mathcal{C} \subset \mathbb{S}\mathfrak{a}$.

Lemma 3.7.1. *Let us fix some $\mathcal{C} \subset \sigma_{mod}$ and $\Theta \subset \Delta$, and let $\xi \in \mathcal{F}_\Theta$. The following subspaces are independent of the choice of transverse flags $f, g \in \mathcal{F}_\Delta$ such that the simplex corresponding to ξ is included in the simplex corresponding to f:*

$$V_+^\xi \coloneqq \bigoplus_{\lambda \in \Phi^+_{\mathcal{C},\Theta}} V_\lambda^{f,g}, \quad V_\geqslant^\xi \coloneqq \bigoplus_{\lambda \in \Phi^+_{\mathcal{C},\Theta} \cup \Phi^0_{\mathcal{C},\Theta}} V_\lambda^{f,g}.$$

Proof. Note that $A = \Phi^+_{\mathcal{C},\Theta}$ or $\Phi^+_{\mathcal{C},\Theta} \cup \Phi^0_{\mathcal{C},\Theta}$ are ideals for the order relation of Φ: for all $\lambda \in A$, if $\lambda' \in \Phi$ satisfies $\lambda' - \lambda \geqslant 0$ on $\mathfrak{a}^+$ then $\lambda' \in A$.

If g' is an other full flag transverse to f, then $g' = u \cdot g$ for some unipotent element of G fixing f, which in turn is the exponential of an element $\mathrm{u} \in \mathfrak{g}$

that belongs to the sum $\mathfrak{u}_\Delta$ of all positive root spaces $\mathfrak{g}_\alpha^{f,g}$ for α a positive root. However if $\mathrm{u} \in \mathfrak{g}_\alpha^{f,g}$ and $\mathrm{v} \in V_\lambda^{f,g}$ in the root space associated to the root α satisfies $[\mathrm{u}, \mathrm{v}] \in V_{\lambda+\alpha}^{f,g}$. Since A is an ideal, $\lambda \in A$ implies that $\lambda + \alpha \in A$.

The fact that this is independent of the choice of f is due to the fact that A is W_Θ-invariant.

□

Let $\mathcal{F} \subset \mathbb{P}(V)$ be a compact Γ-invariant subset. We define $S^2\mathcal{F} \subset \mathbb{P}(S^2V)$ to be the corresponding set of rank one tensors. We call $\mathrm{Hull}(S^2\mathcal{F}) \subset \mathcal{X}(V) = \mathbb{P}(S^2V^{\geqslant 0})$ the convex hull of these points. We denote by $\mathrm{Hull}(S^2\mathcal{F})^*$ the open dual convex domain, i.e. the set of linear form that do not vanish on $S^2\mathcal{F}$. This space contains the space $\mathcal{X}(V)^*$ of projectivizations of positive definite bilinear forms on V. We note that there is a natural identification of $\mathcal{X}$ with $\mathcal{X}^*$ given by $[X : V^* \to V] \mapsto [X^{-1} : V \to V^*]$.

Given $[O] = o \in \mathrm{Hull}(S^2\mathcal{F})^*$, we define $\mathcal{DS}_\Gamma^{\mathcal{F}}(o)$ to be the set of elements $[X] \in \mathrm{Hull}(S^2\mathcal{F})$ such that for all $\gamma \in \Gamma$:

$$\mathrm{Tr}\left(X(O - \gamma \cdot O)\right) \geqslant 0.$$

Here we chose the signs of the representatives of X and O so that $\mathrm{Tr}(XO) > 0$. When $o \in \mathcal{X}^* \simeq \mathcal{X}$, this domain coincides with $\mathcal{DS}_\Gamma(o^{-1}) \cap \mathrm{Hull}(S^2\mathcal{F})$.

Definition 3.7.2. Let $[O] = o \in \mathrm{Hull}(S^2\mathcal{F})^*$. We say that $\mathcal{DS}_\Gamma^{\mathcal{F}}(o)$ is *properly finite-sided* in $\mathrm{Hull}(S^2\mathcal{F})$ if there exists a neighborhood U of $\mathcal{DS}_\Gamma^{\mathcal{F}}(o)$ in $\mathrm{Hull}(S^2\mathcal{F})$ and a finite set $F \subset \Gamma$ such that for all γ not in F, U is contained in the set

$$\mathcal{H}_{\mathcal{F}}(o, \gamma \cdot o) := \{[X] \in \mathrm{Hull}(S^2\mathcal{F}) \mid \mathrm{Tr}\left(X(O - \gamma \cdot O)\right) \geqslant 0\},$$

with representatives of X and O chosen so that $\mathrm{Tr}(XO) > 0$.

Theorem 3.7.3. *Let Γ be a Θ-Anosov subgroup of G, and let $\mathcal{C} = \mathcal{C}_\Gamma$. Let V be a representation of G, and let $\mathcal{F} \subset \mathbb{P}(V)$ be a Γ-invariant compact subset. Suppose that $\mathcal{F}$ is disjoint from $V_\geqslant^\xi \backslash V_+^\xi$ for all $\xi = \xi_\Theta(x)$ for $x \in \partial\Gamma$ (see Lemma 3.7.1). Then $\mathcal{DS}_\Gamma^{\mathcal{F}}(o)$ is properly finite-sided in* $\mathrm{Hull}(S^2\mathcal{F})$ *for all $o \in \mathrm{Hull}(S^2\mathcal{F})^*$.*

Remark 3.7.4. One can consider $\mathcal{F}_{\max}$ the largest subset of $\mathbb{P}(V)$ that avoids $V_\geqslant^\xi \backslash V_+^\xi$ for all $\xi = \xi_\Theta(x)$ for $x \in \partial\Gamma$. This subset is not always closed, so one can only apply our result to compact Γ-invariant subsets $\mathcal{F} \subset \mathcal{F}_{\max}$.

The proof of Theorem 3.7.3 will be done in Section 3.7.2.

3.7.2 The general argument.

Throughout Section 3.7.2 we assume that Γ and $\mathcal{F}$ satisfy the assumptions of Theorem 3.7.3. Namely, we assume that Γ is Θ-Anosov and take $\mathcal{F}$ to be a compact Γ-invariant subset of $\mathbb{P}(V)$ disjoint from $V^{\xi}_{\geqslant}\backslash V^{\xi}_{+}$ for all $\xi = \xi_{\Theta}(x)$ for $x \in \partial\Gamma$ with respect to $\mathcal{C}_{\Gamma}$.

For every closed subset $\mathcal{C} \subset \sigma_{\mathrm{mod}} = \mathbb{S}\mathfrak{a}$ we define $C_{\mathcal{C},\Phi}$ to be the infimum of $\frac{|\lambda(v)|}{\|v\|}$ for $v \in \mathfrak{a}$ such that $[v] \in \mathcal{C}$ and $\lambda \in \Phi^{+}_{\mathcal{C},\Theta} \cup \Phi^{-}_{\mathcal{C},\Theta}$. We will consider sufficiently small neighborhoods $\mathcal{C}$ of $\mathcal{C}_{\Gamma}$ so that the sets of weights $\Phi^{+}_{\mathcal{C},\Theta}$ etc. are unchanged.

We will first focus on the case when the basepoint o^{-1} belongs to a totally geodesic $\mathbb{X} \subset \mathcal{X}$ corresponding to the symmetric space of G, and then we will see that the result still holds for other basepoints.

Each line $\ell \in \overline{\mathcal{X}(V)}$ defines a function $h_{\ell} \colon \mathcal{X} \to \mathbb{R}$, up to an additive constant, by setting:

$$h_{[L]}([X]) \coloneqq \log\left(\frac{1}{d}\operatorname{Tr}(X^{-1}L)\right),$$

for representatives X satisfying $\det(X^{-1}O) = 1$ where O is a positive definite representative of a basepoint $o \in \mathcal{X}$. Here d is the dimension of V. Each such line is a convex combination of some rank 1 lines in S^2V; i.e. for each L there exists $v_i \in V, i \in I$ such that $L = \sum v_i \otimes v_i$. The corresponding functions are then related by

$$h_{[L]}(x) = \log\big(\frac{1}{d}\sum_{i\in I} e^{h_{v_i}(x)-h_{v_i}(o)}\big). \tag{3.5}$$

At rank 1 points, these functions are exactly the Busemann functions on $\mathcal{X}$ centered at the minimal flag manifold $\mathbb{P}(V)$. The rest of the projective boundary $\partial\mathcal{X}$ can be interpreted as a sort of horoboundary with respect to the Selberg invariant. Note that these horofunctions are hence equal to, for some functions f_i that are convex and 1-Lipshitz with respect to the Riemannian metric on $\mathcal{X}$:

$$h = \log\left(\frac{1}{d}\sum_{i=1}^{d} e^{f_i}\right)$$

Therefore these functions are also 1-Lipschitz and convex.

If we fix $\xi \in \mathcal{F}_{\Theta}$, and we take a line $\ell \in \mathbb{P}(S^2V)$ exactly one of the three possibilities occur:

(a) $\ell \in \operatorname{Hull}\left(S^2\mathbb{P}(V^{\xi}_{+})\right)$

(b) $\ell \in \operatorname{Hull}\left(S^2\mathbb{P}(V^{\xi}_{\geqslant})\right)$ and $\ell \notin \operatorname{Hull}\left(S^2\mathbb{P}(V^{\xi}_{+})\right)$,

(c) $\ell \notin \operatorname{Hull}\left(S^2\mathbb{P}(V^{\xi}_{\geqslant})\right)$.

The hypothesis that we put on $\mathcal{F}$ in the statement of Theorem 3.7.3 implies that case (b) never occurs for [v] $\in S^2\mathcal{F}$. The subset that will play the role of the thickening here will be

$$\operatorname{Th}(\xi) \coloneqq \operatorname{Hull}\left(S^2\mathbb{P}(V^{\xi}_{+})\right) \cap \operatorname{Hull}(S^2\mathcal{F}).$$

We obtain the following dichotomy:

Lemma 3.7.5. *Let $\ell \in \mathrm{Hull}(S^2\mathcal{F}) \subset \mathbb{P}(S^2V)$, with $\mathcal{F}$ as in Theorem 3.7.3. Let $o \in \mathbb{X}$ be a basepoint and let $\tau \in \mathcal{F}_\Theta$. Exactly one of the following holds:*

(i) $\ell \in Th(\tau)$ and for every $\eta \in \mathrm{st}_{\mathcal{C}}(\tau)$ the geodesic ray $c_{o,\eta}$ satisfies

$$h_\ell(c_{o,\eta}(t)) - h(o) \leqslant -C_{\mathcal{C},\omega}t.$$

(ii) $\ell \notin Th(\tau)$ and for all $\epsilon > 0$ there exist $A > 0$ such that for every $\eta \in \mathrm{st}_{\mathcal{C}}(\tau)$ the geodesic ray $c_{o,\eta}$ satisfies

$$h_\ell(c_{o,\eta}(t)) - h(o) \geqslant (C_{\mathcal{C},\omega} - \epsilon)t - A.$$

Proof. We first consider the case when $\ell \in S^2\mathcal{F}$. We want to compute the asymptotic slope for h_ℓ and $\eta \in \mathrm{st}_{\mathcal{C}}(\tau)$, as defined in Section 3.4.1. Let us fix two opposite full flags $f, g \in \mathcal{F}_\Delta$ such that such that η belongs to the ideal Weyl chamber associated to f; note that τ belongs to this chamber as well. Let ζ be the projection of η to $\sigma_{\mathrm{mod}} = \mathbb{S}\mathfrak{a}^+$. Let $\mathrm{v} \otimes \mathrm{v} \in \ell$ be non-zero, we can decompose v for some $\mathrm{v}_\lambda \in V_\lambda^{f,g}$ as:

$$\mathrm{v} = \sum_{\lambda \in \Phi} \mathrm{v}_\lambda.$$

The basepoint $o \in \mathbb{X} \subset \mathcal{X}$ determines a norm $\|\cdot\|$ on $\mathbb{R}^n$, and one has:

$$h_\ell(c_{o,\eta}(t)) - h(o) = \frac{1}{d}\log\left(\left\|\sum_{\lambda \in \Phi} e^{\lambda(\zeta)t}\mathrm{v}_\lambda\right\|\right) - \frac{1}{d}\log\left(\|\mathrm{v}\|\right).$$

Since $\zeta \in \mathcal{C}$, the behavior of this quantity depends on the same case distinction as before:

(a) if $\mathrm{v} \in \mathrm{Hull}\left(S^2\mathbb{P}(V_+^\xi)\right)$, for all $t \geqslant 0$, $h_\ell(c_{o,\eta}(t)) - h(o) \leqslant -C_{\mathcal{C},\omega}t$,

(b) if $\mathrm{v} \in \mathrm{Hull}\left(S^2\mathbb{P}(V_\geqslant^\xi)\right)$ but $\mathrm{v} \notin \mathrm{Hull}\left(S^2\mathbb{P}(V_+^\xi)\right)$, the situation is unclear,

(c) if $\mathrm{v} \notin \mathrm{Hull}\left(S^2\mathbb{P}(V_\geqslant^\xi)\right)$, for all $\epsilon > 0$ there exist $A > 0$ such that for all $t > 0$

$$h_\ell(c_{o,\eta}(t)) - h(o) \geqslant (C_{\mathcal{C},\omega} - \epsilon)t - A.$$

Here case (b) cannot occur by the hypothesis that was put on $\mathcal{F}$. Note that case (a) means exactly that $\ell \in \mathrm{Th}(\tau)$. Hence we got the desired result for $\ell \in \mathrm{Hull}(S^2\mathcal{F})$.

We now consider an arbitrary $\ell = [v] \in \mathrm{Hull}(S^2\mathcal{F})$. Then v can be written as a convex combination of extremal points of $\mathrm{Hull}(S^2\mathcal{F})$. The way we defined the associated function on $\mathbb{X}$ was taking the log of a linear expression, so the associated function can be written for some $p_i \in \mathcal{F}$, $\lambda_i > 0$ and $o_i \in \mathbb{X}$ for $i \in I$ as:

$$h_\ell(x) = \frac{1}{d}\log(\sum_{i\in I}\lambda_i e^{h_{p_i}(x)-h_{p_i}(o)}) = \frac{1}{d}\log(\sum_{i\in I} e^{h_{p_i}(x)-h_{p_i}(o_i)}).$$

If $\ell \in \mathrm{Th}(\tau)$, one can choose $(p_i)_{i\in I}$ such that for all $i \in I$ one has $p_i \in V_+^\tau$. Therefore for all $t \geqslant 0$, $b_{p_i,o}(c_{o,\eta}(t)) - b_{p_i,o}(o) \leqslant -C_{\mathcal{C},\Phi}t$. Hence $h_\ell(c_{o,\eta}(t)) - h(o) \leqslant -C_{\mathcal{C},\Phi}t$.

Suppose now that $\ell \notin \mathrm{Th}(\tau)$, then one can choose $(p_i)_{i\in I}$ such that $p_{i_0} \notin V_+^\tau$ for some $i_0 \in I$, and hence $p_{i_0} \notin V_\geqslant^\tau$. Therefore there exists $A > 0$ such that the geodesic ray $c_{o,\eta}$ satisfies $h_{p_{i_0}}(c_{o,\eta}(t)) - h_{p_{i_0}}(o) \geqslant (C_{\mathcal{C},\Phi} - \epsilon)t - A$. Hence:

$$h_\ell(c_{o,\eta}(t)) - h(o) \geqslant (C_{\mathcal{C},\Phi} - \epsilon)t - A + \log(\lambda_{i_0}).$$

□

Remark 3.7.6. A consequence of this argument is that the thickening can also be described as follows: let $p \in \mathrm{Hull}(S^2\mathcal{F})$, we consider the exposed face F_p of the compact $\mathrm{Hull}(S^2\mathcal{F})$ containing p, i.e. the intersection of $\mathrm{Hull}(S^2\mathcal{F})$ with all support hyperplanes passing through p. The point p belongs to $\mathrm{Th}(\tau)$ if and only if the extremal points of F_p are all in V_+^τ.

Definition 3.7.7. Let us define the following domain:

$$\Omega := \mathrm{Hull}(S^2\mathcal{F}) \backslash \bigcup_{x\in\partial\Gamma} \mathrm{Th}(\xi_\Theta(x)).$$

The following statements are the analog in this setting of Proposition 3.4.8, Lemma 3.4.9, Proposition 3.5.5 and Theorem 3.5.2 respectively. The exact same proofs apply, by replacing $\mathbb{X} \cup \partial^\omega_{\mathrm{horo}}\mathbb{X}$ by $\mathrm{Hull}(S^2\mathcal{F})$, $\mathrm{Th}^\omega_{\mathrm{horo}}$ by Th, $\mathcal{D}^\omega_\Gamma(o)$ by $\mathcal{DS}^{\mathcal{F}}_\Gamma(o)$ and $\Omega^\omega_{\mathrm{horo}} \cup \mathbb{X}$ by Ω.

Proposition 3.7.8 (Analog of Proposition 3.4.8)**.** *An element $\ell \in \mathrm{Hull}(S^2\mathcal{F})$ belongs to Ω if and only if h_ℓ restricted to the Γ-orbit of $o \in \mathbb{X}$ is bounded from below. In this case, h_ℓ is proper on any Γ-orbit. In particular $\mathcal{DS}^{\mathcal{F}}_\Gamma(o) \subset \Omega$ for all $o \in \mathbb{X}$.*

Lemma 3.7.9 (Analog of Lemma 3.4.9)**.** *Let $\ell_0 \in \Omega$, and let $o \in \mathbb{X}$. There exists a neighborhood $U \subset \Omega$ of ℓ_0 and a constant $A > 0$ such that for $\ell \in U$ and $\gamma \in \Gamma$:*

$$h_\ell(\gamma \cdot o) - h_\ell(o) \geqslant C_{\mathcal{C},\Phi} d(o, \gamma \cdot o) - A.$$

Theorem 3.7.10 (Analog of Theorem 3.5.2)**.** *For all $o \in \mathbb{X}$ and for any $A > 0$ one can find a finite set $S \subset \Gamma$ and a neighborhood U of $\mathcal{DS}^{\mathcal{F}}_\Gamma(o)$ such that for all $\ell \in U$ and $\gamma \in \Gamma \backslash S$, $h_\ell(\gamma \cdot o) > h_\ell(o) + A$.*

Note that in this theorem we use the fact that $\mathcal{DS}^{\mathcal{F}}_\Gamma(o)$ is compact as the intersection of closed spaces in a compact space. Hence we need here to have $\mathcal{F}$ closed.

We can now prove Theorem 3.7.3.

Proof of Theorem 3.7.3. Let $o \in \mathbb{X}$ and $o' \in \mathrm{Hull}(\mathcal{F})^*$, and fix two representatives O^{-1} and $O' \colon V \to V^*$ of o^{-1} and o'. The following quantity is well-defined and continuous in $x = [X]$ on the compact set $\mathrm{Hull}(\mathcal{F}) \subset \mathbb{P}(S^2V^{\geqslant 0})$, since $o^{-1}, o' \in \mathrm{Hull}(\mathcal{F})^*$:

$$\left|\log\left|\mathrm{Tr}(XO^{-1})\right| - \log\left|\mathrm{Tr}(XO')\right|\right|.$$

We denote by B the supremum of this quantity.

Now we apply Theorem 3.7.10 for $o \in \mathbb{X}$ and $A = 2B$, and we get that there exist a finite set $S \subset \Gamma$ and a neighborhood U of $\mathcal{DS}_\Gamma^{\mathcal{F}}(o)$ such that for all $\ell \in U$ and $\gamma \in \Gamma\backslash S$, $h_\ell(\gamma \cdot o) > h_\ell(o) + 2B$. This implies that the half-space $\mathcal{H}_{\mathcal{F}}(o', \gamma \cdot o')$ contains U for all but finitely many $\gamma \in \Gamma$, so $\mathcal{DS}_\Gamma^{\mathcal{F}}(o')$ is properly finite-sided. □

3.7.3 Dirichlet-Selberg domains.

In the remainder of Section 3.7 we present applications of Theorem 3.7.3. We first deduce Theorem 1.3.15 for a second time.

Corollary 3.7.11. *Let Γ be a subgroup of $\mathrm{SL}(2n, \mathbb{R})$ which is $|\log \sigma_i|$-undistorted for all i. Then for any $o \in \mathcal{X}$, the Dirichlet-Selberg domain $\mathcal{DS}_\Gamma(o)$ is properly finite-sided.*

Proof. As observed in Proposition 3.2.6, Γ is n-Anosov. We set $\mathcal{F} = \mathbb{P}(V)$, which is clearly compact and Γ-invariant. The condition of being $|\log \sigma_i|$-undistorted for all i guarantees that Φ^0 is empty. Therefore we may apply Theorem 3.7.3. Since $\overline{\mathcal{X}} = \mathrm{Hull}(S^2\mathcal{F})$, $\mathcal{DS}_\Gamma(o) = \mathcal{DS}_\Gamma^{\mathcal{F}}(o^{-1})$. □

3.7.4 Projective Anosov subgroups.

We give two applications of the previous theorem for projective Anosov representations. In these examples the set $\mathcal{F}$ will depend on the representation.

Theorem 3.7.12. *Let Γ be a projective Anosov subgroup of $\mathrm{SL}(d, \mathbb{R})$. Let Λ be the projective limit set, i.e., $\Lambda = \{\xi_\Gamma^1(x) | x \in \partial\Gamma\} \subset \mathbb{P}(\mathbb{R}^d)$. The domain $\mathcal{DS}_\Gamma^\Lambda(o)$ is properly finite-sided in $\mathrm{Hull}(S^2\Lambda) \subset \mathbb{P}(S^2\mathbb{R}^d)$ for all $o \in \mathrm{Hull}(S^2\Lambda)^*$.*

Proof. We apply Theorem 3.7.3. Given a partial flag $\xi = (\xi^1, \xi^{n-1})$ in $\mathbb{R}^n$ consisting of a line and a hyperplane, the corresponding set V_+^ξ is equal to ξ^1 and $V_\geqslant^\xi$ is equal to ξ^{n-1}. The transversality of the boundary map ξ_Γ implies that Λ satisfies the hypothesis of Theorem 3.7.3. □

If Γ is convex-cocompact in the sense of [DGK18] or [Zim21], we can choose $\mathcal{F}$ to be larger.

Theorem 3.7.13. *Let $\Gamma \subset \mathrm{SL}(d, \mathbb{R})$ be a projective Anosov subgroup that is convex cocompact, i.e. that preserves a properly convex domain Ω and acts cocompactly on a convex set $\mathcal{C} \subset \Omega$. Let $\Lambda = \{\xi_\Gamma^1(x) | x \in \partial\Gamma\}$. The domain $\mathcal{DS}_\Gamma^{\mathcal{C}\cup\Lambda}(o)$ is properly finite-sided in $\mathrm{Hull}\left(S^2\left(\mathcal{C} \cup \Lambda\right)\right) \subset \mathbb{P}(S^2\mathbb{R}^d)$ for all $o \in \mathrm{Hull}(S^2\Lambda)^*$.*

Note that following from [DGK18, Zim21], every projective Anosov subgroup that preserves a convex domain admits such a non-empty convex set $\mathcal{C}$.

Proof. We can again apply Theorem 3.7.3. As previously it suffices to verify that if $x \in \partial\Gamma$, then $\mathcal{C} \cup \Lambda$ does not intersect $\xi_\Gamma^{n-1}(x)\backslash\xi_\Gamma^1(x)$. Since Ω is a proper convex domain preserved by the projective Anosov subgroup Γ, the hyperplane $\xi_\Gamma^{n-1}(x)$ is disjoint from Ω for all $x \in \partial\Gamma$. Moreover since Γ acts cocompactly on $\mathcal{C} \subset \Omega$, one has $\overline{\mathcal{C}} = \mathcal{C} \cup \Lambda$. Complete details for the proofs of the previous two sentences can be found in [DGK18, Section 8]. □

3.7.5 ω-undistorted subgroups through a representation.

We consider a semisimple Lie group G and an irreducible finite dimensional representation V. We construct a subset $\mathcal{F}$ that satisfies the hypothesis of Theorem 3.7.3 for ω-undistorted subgroups.

Let $I \subset \Phi$ be an ideal and write $V_I^f = \bigoplus_{\lambda \in I} V_\lambda^f$ for $f \in \mathcal{F}_\Delta$. Set

$$\mathcal{F}_I \coloneqq \left\{ [v] \mid v \in V_I^f \backslash \{0\}, f \in \mathcal{F}_\Delta \right\} \subset \mathbb{P}(V).$$

Theorem 3.7.14. *Let $\Gamma \subset G$ be an ω-undistorted subgroup for all $\omega \in I$. The domain $\mathcal{DS}_\Gamma^{\mathcal{F}_I}(o)$ is properly finite-sided in $\mathrm{Hull}(S^2\mathcal{F}_I)$ for all $o \in \mathrm{Hull}(S^2\mathcal{F}_I)^*$.*

An example of an ideal can always be obtained by taking the highest restricted weight $\{\omega\} \subset \Phi$. In this case, for $G = \mathrm{SL}(d, \mathbb{R})$ with the standard representation on $\mathbb{R}^d$, we recover Theorem 1.3.15.

Proof. Note that $\mathcal{F}$ is closed. For each $\lambda \in I$, we get a set $\Theta_\lambda \subset \Delta$ such that Γ is Θ-Anosov. We fix Θ to be the union of all these sets.

Let $[v] \in \mathcal{F}$ and let $x \in \partial\Gamma$. For some full flag $f \in \mathcal{F}_\Delta$, one has $v \in V_I^f$. One can find an opposite full flag $g \in \mathcal{F}_\Delta$ such that the flat determined by f, g contains $\xi_\Gamma(x)$. For all $w \in W$, either $w \cdot \omega > 0$ on $W_\Theta \cdot \mathcal{C}$ or $w \cdot \omega < 0$ on $W_\Theta \cdot \mathcal{C}$. Hence v does not belong to $V_\geqslant^\xi \backslash V_+^\xi$, so $\mathcal{F}$ satisfies the hypothesis of Theorem 3.7.3. □

One can apply Proposition 3.6.5 to the quotient of this convex hull.

Corollary 3.7.15. *Let Γ be a torsion-free subgroup of G that is ω-undistorted for all $\omega \in I$. The quotient by Γ of the relative interior of $\mathrm{Hull}(S^2\mathcal{F}_I)$ is topologically tame.*

Indeed $S^2\mathcal{F}_I$ is the orbit of an algebraic set by an algebraic group and hence it is semi-algebraic. Moreover the convex hull in a finite dimensional vector space of a semi-algebraic set is also semi-algebraic: given a semi-algebraic set $A \subset \mathbb{R}^n$ we consider the subset $B \subset (\mathbb{R}^n)^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n$ of elements $((x_0, \dots, x_n), (\lambda_0, \dots, \lambda_n), x)$ such that $x_0, \dots, x_n \in A$, $\lambda_0, \dots, \lambda_n \geqslant 0$, $\lambda_0 + \cdots \lambda_n = 1$ and $x = \lambda_0 x_0 + \cdots + \lambda_n x_n$. By Carathéodory's theorem, projecting B to the last copy of $\mathbb{R}^n$ yields the convex hull.

Let Γ now be an ω_Δ-undistorted subgroup of $G = \mathrm{SL}(d,\mathbb{R})$, equivalently a Δ-Anosov subgroup, or sometimes called a Borel Anosov subgroup. Recall that ω_Δ is the highest root, i.e. the highest weight of the adjoint representation $V = \mathfrak{sl}(d,\mathbb{R})$. The highest weight space is given by the span of the unit matrix $E_{1,d}$. The set $\mathcal{F}_{\{\omega_\Delta\}}$ can be identified with the flag manifold $\mathcal{F}_{\omega_\Delta}$ which is the partial flag manifold of lines in hyperplanes in $\mathbb{R}^d$. The identification is given by $([v],[\alpha]) \mapsto [v \otimes \alpha] \in \mathbb{P}(\mathfrak{sl}(d,\mathbb{R}))$.

Therefore Theorem 3.7.14 implies that one can obtain a properly finite sided domain in the convex hull of $S^2\mathcal{F}_{\{\omega_\Delta\}} \subset \mathbb{P}(S^2\mathfrak{sl}(n,\mathbb{R}))$ for every Borel Anosov subgroup of $\mathrm{SL}(d,\mathbb{R})$.

However one can apply this theorem to a larger ideal, and for any semi-simple Lie group G with its adjoint representation $V = \mathfrak{g}$. We consider $I = \Sigma^+$ the set of positive roots. The set $\mathfrak{n} = \mathcal{F}_I \subset \mathbb{P}(\mathfrak{g})$ is the closed set of nilpotent elements in the Lie algebra $\mathfrak{g}$ of G.

Corollary 3.7.16. *If $\Gamma \subset G$ is Δ-Anosov, the Dirichlet-Selberg domain $\mathcal{DS}^{\mathfrak{n}}_\Gamma(o)$ is properly finite-sided in $\mathrm{Hull}(S^2\mathfrak{n}) \subset \mathbb{P}(S^2\mathfrak{g})$ for all $o \in \mathrm{Hull}(S^2\mathfrak{n})^*$.*

3.8 Connected limit cone.

We show that the limit cone of a discrete finitely generated subgroup of G that is not virtually cyclic is connected. When Γ is a Zariski dense subgroup of G, Benoist proved that its limit cone is convex [Ben97], hence connected.

To deal with the general case we use the fact that Γ acts topologically transitively on its space of ends. First we recall the definition of the space of ends and its topology.

Let Γ be a finitely generated group with finite symmetric generating system S, and let $C(\Gamma)$ be the associated Cayley graph. Let $K_n \supset C(\Gamma)$ be the ball of radius n around the identity $e \in \Gamma$ for the word metric associated to S. The *space of ends* $\mathcal{E}$ is the space of senquences $(U_i)_{i\in\mathbb{N}}$ such that for $i \in \mathbb{N}$, $U_i \subset U_{i+1}$ and U_i is a connected component of $C(\Gamma)\backslash K_i$, where two such sequences are considered equal if they are eventually equal.

The space $C(\Gamma) \cup \mathcal{E}$ is equipped with the topology generated by the open sets $\mathcal{V}^n_V = V \cup \{[(U_i)] \mid U_n \subset V\}$ for open sets $V \subset C(\Gamma)\backslash K_n$. This topology makes it a compact space.

Proposition 3.8.1. *Let Γ be a torsion free finitely generated group that is not virtually cyclic. Then the action of Γ on the space of ends admits a dense orbit.*

Proof. Stalling's theorem implies that such a group has either one end or infinitely many ends. If there are infinitely many ends the group can be written as a non-trivial free product of infinite groups $A * B$ [Löh17, Theorem 8.2.14].

We fix some symmetric generating sets S_A and S_B for respectively A and B. Fix $a \in S_A$ and $b \in S_b$ and consider the diverging sequence $x_{2n} = (ab)^n$ and

$x_{2n+1} = (ab)^n a$. This sequence is a path in the Cayley graph of $A * B$ that is diverging. Therefore it determines an end of $A * B$ and we show that the orbit of this end is dense.

Let V be an infinite connected component of $C(\Gamma)\backslash K_n$ for some $n \in \mathbb{N}$. We want to prove that for some $\gamma \in \Gamma$ the end associated with the sequence $(\gamma \cdot x_n)_{n\in\mathbb{N}}$ is included in V. Let $w = a_1 b_1 a_2 b_2 \cdots a_k b_k$ be a reduced expression for some element $v \in V$ where $a_i \in A$ and $b_i \in B$ are non-trivial, except maybe a_1 and b_k.

We consider $\gamma = w$ if b_k is not the trivial element and $\gamma = wb$ otherwise. Then the words representating γx_n obtained by concatenating the words defining γ and x_n are reduced. Therefore for all $n \in \mathbb{N}$, γx_n does not belong to K_n. This implies that the left action of γ on the end determined by (x_n) belongs to the open set determined by V. We have therefore proven that there exist a dense orbit in the boundary. $\square$

Proposition 3.8.2. *Let G be a connected semisimple Lie group with finite center and let $\Gamma < G$ be a finitely generated, discrete, and not virtually cyclic subgroup. Then its limit cone $\mathcal{C}_\Gamma$ is connected.*

Proof. Since G is connected and semisimple with finite center, the adjoint representation has finite kernel. The image of Γ under the adjoint representation admits a finite index torsion-free subgroup Γ' by Selberg's Lemma. The limit cone of Γ' is the same as the limit cone of Γ, so without loss of generality we may assume that Γ is torsion-free.

Let $n \in \mathbb{N}$. We say that two elements γ', γ'' in $\Gamma\backslash K_n$ are *K_n-connected* if one can construct a finite sequence (γ_i) of elements of $\Gamma\backslash K_n$ such that for all $1 \leqslant i < N, \gamma_{i+1} = a_i \gamma_i b_i$ for some $a_i, b_i \in S \cup \{e\}$, with $\gamma' = \gamma_0$ and $\gamma'' = \gamma_N$.

Let $[(U_i)]$ in $\mathcal{E}$. Using only right translations, i.e. $a_i = e$, any two points in U_n are K_n-connected, since by definition U_n is a connected component of $C(\Gamma)\backslash K_n$.

Using left translations, we see that for all ends $[(U_i)]$ and $\gamma \in \Gamma$, there is a point in U_n and a point in $\gamma \cdot U_n$ that are K_n-connected. Hence for any end $[(U_i)]$, any two points in the following union are K_n-connected:

$$X = \bigcup_{\gamma\in\Gamma} \gamma \cdot U_n.$$

Since Γ admits a dense orbit in $\mathcal{E}$, up to choosing a specific point $[(U_i)]$ one can assume that $X \cup \mathcal{E}$ is a neighborhood of $\mathcal{E}$ in $C(\Gamma) \cup \mathcal{E}$. Hence its complement is a closed subset of the Hausdorff space $C(\Gamma)$, hence it is compact. Therefore there exist a cobounded subset of $C(\Gamma)$ such that every pair of elements in this set is K_n-connected.

We define the distance d on $\mathbb{S}\mathfrak{a}$, as $d([\mathrm{x}], [\mathrm{y}]) = |\frac{\mathrm{x}}{|\mathrm{x}|} - \frac{\mathrm{y}}{|\mathrm{y}|}|$ for $\mathrm{x}, \mathrm{y} \in \mathfrak{a}$.

Assume that there exists a partition $A \cup B$ of $\mathcal{C}_\rho$ into two open and closed sets. These sets are compact and hence are at uniform distance $\epsilon > 0$. Let F

be a finite symmetric generating set for Γ. Let M be the maximum for $\gamma \in F$ of $d_{\mathbb{X}}(o, \rho(\gamma) \cdot o)$. For some integer $n \in \mathbb{N}$, the ball K_n in $C(\Gamma)$ contains all the elements $\gamma \in \Gamma$ such that either $d_{\mathbb{X}}(o, \rho(\gamma) \cdot o) \leqslant \frac{12M}{\epsilon}$ or:

$$d([\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o)], \mathcal{C}_\rho) \geqslant \frac{\epsilon}{3}.$$

Indeed since ρ is discrete and by the definition of $\mathcal{C}_\rho$, the set of such elements is finite.

Since $A, B \neq \varnothing$, the definition of the limit cone allows us to pick two large enough elements $\gamma', \gamma'' \in \Gamma \backslash K_n$ such that :

$$d(\left[\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma') \cdot o)\right], A) \leqslant \frac{\epsilon}{3},$$

$$d(\left[\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma'') \cdot o)\right], B) \leqslant \frac{\epsilon}{3}.$$

One can assume that γ', γ'' are K_n-connected by taking them large enough. Therefore there exist $\gamma \in \Gamma \backslash K_n$ and $a, b \in F$ such that:

$$d([\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o)], A) \leqslant \frac{\epsilon}{3},$$

$$d([\mathbf{d}_{\mathfrak{a}}(o, \rho(a\gamma b) \cdot o)], B) \leqslant \frac{\epsilon}{3}.$$

Using Lemma 2.1.4 one gets that :

$$|\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o) - \mathbf{d}_{\mathfrak{a}}(o, \rho(a\gamma b) \cdot o)| \leqslant |\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o) - \mathbf{d}_{\mathfrak{a}}(o, \rho(a\gamma) \cdot o)| + d_{\mathbb{X}}(o, \rho(b) \cdot o).$$

$$|\mathbf{d}_{\mathfrak{a}}(\rho(\gamma)^{-1} \cdot o, o) - \mathbf{d}_{\mathfrak{a}}(\rho(\gamma)^{-1}\rho(a)^{-1} \cdot o, o)| \leqslant d_{\mathbb{X}}(\rho(a)^{-1} \cdot o, o).$$

And therefore:

$$\left| \frac{\mathbf{d}_{\mathfrak{a}}(o, \rho(\gamma) \cdot o)}{d_{\mathbb{X}}(o, \rho(\gamma) \cdot o)} - \frac{\mathbf{d}_{\mathfrak{a}}(o, \rho(a\gamma b) \cdot o)}{d_{\mathbb{X}}(o, \rho(a\gamma b) \cdot o)} \right| < \frac{\epsilon}{3}.$$

This contradicts the fact that A and B are at distance ϵ, so $\mathcal{C}_\rho$ is connected. $\square$

Chapter 4

Fibration by pencils of quadrics

This chapter begins with a recall of some facts about maximal and Anosov representations in Section 4.1.

The main definitions of the chapter are introduced in Section 4.2 where fibrations of a projective convex set by projective subspaces are discussed, more precisely fibrations of the projective model for the symmetric space of $\mathrm{SL}(2n,\mathbb{R})$. In Section 4.3 we introduce the notion of fitting flows. Section 4.4 discusses how the existence of an equivariant continuous map with a fitting flow implies the Anosov property and a fibration of a domain of discontinuity in projective space.

In Section 4.5 we focus on representations into $\mathrm{Sp}(2n,\mathbb{R})$ and prove our main result, which is the characterization of maximal representations by the existence of a locally fitting map of maximal pencils of quadrics that admits a fitting flow. In section 4.6 we show how spacelike surfaces in $\mathbb{H}^{2,2}$ with a bound on their principal curvatures define a fitting immersion of pencils. In Section 4.7 we prove two independent propositions. Finally in Section 4.8 we briefly discuss a decomposition of the projective structure associated to a maximal representation into hexagons of quadric hypersurfaces.

4.1 Maximal and Anosov representations.

4.1.1 Maximal representations.

Let us fix a symplectic from ω on $\mathbb{R}^{2n}$, i.e. a non-degenerated bilinear antisymmetric pairing. A *symplectic basis* of $\mathbb{R}^{2n}$ is a basis $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ in which :

$$\omega = \sum_{i=1}^{n} x_i^* \wedge y_i^*.$$

We define $\mathrm{Gr}_n(\mathbb{R}^{2n})$ as the space of n-dimensional subspaces of $\mathbb{R}^{2n}$ A *Lagrangian* in $(\mathbb{R}^{2n}, \omega)$ is an element $\ell \in \mathrm{Gr}_n(\mathbb{R}^{2n})$ such that ω restricted to ℓ is equal to zero. We denote by $\mathcal{L}_n$ the space of Lagrangians in $(\mathbb{R}^{2n}, \omega)$. We say that two Lagrangians are transverse if their intersection is trivial.

Let $\mathrm{Sp}(2n,\mathbb{R})$ be the subgroup of elements in $\mathrm{SL}(2n,\mathbb{R})$ that preserves ω. This groups acts transitively on $\mathcal{L}_n$, as well as on the space of pairs of transverse Lagrangians. Given a triple (ℓ_1, ℓ_2, ℓ_3) of transverse Lagrangians, one can find a symplectic basis such that for some $(\epsilon_i) \in \{1, -1\}$:

$$\ell_1 = \langle x_1, x_2, \cdots x_n \rangle,$$

$$\ell_2 = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, \cdots x_n + \epsilon_n y_n \rangle,$$

$$\ell_3 = \langle y_1, y_2, \cdots y_n \rangle.$$

The sum of the (ϵ_i) is an invariant of the triple of flags that is called the *Maslov index* $M(\ell_1, \ell_2, \ell_3)$. The group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on the space of triples of transverse Lagrangians with a given Maslov index. We say that (ℓ_1, ℓ_2, ℓ_3) is *maximal* if the Maslov index of the triple in equal to n.

The Lie group $\mathrm{Sp}(2n,\mathbb{R})$ is of Hermitian type and tube type. Hence it admits a special class in its continuous cohomology group $[\tau] \in H^2_c(\mathrm{Sp}(2n,\mathbb{R}), \mathbb{Z})$. Let S_g be a closed oriented surface of genus $g \geqslant 2$. The fundamental class of S_g defines a cohomology class $[S_g] \in H^2(\pi_1(S_g), \mathbb{Z}) \simeq \mathbb{Z}$. Given a representation $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n,\mathbb{R})$ one can consider the pullback of this class $\rho^*[\tau] = T(\rho)[S_g]$. the integer $T(\rho)$ is called the *Toledo number* of ρ.

The Toledo number can take only finitely many values as the space of representations can only have finitely many connected components. More precisely:

Lemma 4.1.1 ([BIW11])**.** *Let $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n,\mathbb{R})$, the Toledo number satisfies :*

$$(-2g+2)n \leqslant T(\rho) \leqslant (2g-2)n.$$

Such a representation is called *maximal* if its Toledo number is equal to $(2g-2)n$.

4.1.2 Anosov representations.

Let Γ be a finitely generated group. Anosov representations are representations with some exponential gaps between singular values.

Fix a word metric $|\cdot|$ on Γ and a scalar product on $\mathbb{R}^{2n}$ allowing us to define the singular values $(\sigma_1(g) \geqslant \sigma_2(g) \geqslant \cdots \geqslant \sigma_{2n}(g)$ of $g \in \mathrm{SL}(2n,\mathbb{R})$ as the eigenvalues of $\sqrt{g^t g}$. The following definition is independent of these choices.

Definition 4.1.2 ([BPS19])**.** We say that a representation $\rho : \Gamma \to \mathrm{SL}(2n,\mathbb{R})$ is $\{n\}$-Anosov if there exist $A, B > 0$ such that for all $\gamma \in \Gamma$:

$$\frac{\sigma_n(\rho(\gamma))}{\sigma_{n+1}(\rho(\gamma))} \geqslant e^{A|\gamma|+B}.$$

If a group admits an Anosov representation, it must be Gromov hyperbolic [BPS19]. We denote by $\partial\Gamma$ its Gromov boundary. Anosov representations come with boundary maps.

Theorem 4.1.3. *Let $\rho : \Gamma \to \mathrm{SL}(2n,\mathbb{R})$ be $\{n\}$-Anosov. There exist a unique ρ-equivariant continuous map $\xi_\rho^n : \partial\Gamma \to \mathrm{Gr}_n(\mathbb{R}^{2n})$ such that:*

- *for all distinct $x, y \in \partial\Gamma$, $\xi_\rho^n(x) \oplus \xi_\rho^n(y) = \mathbb{R}^{2n}$ (*transverse*),*
- *for all $\gamma \in \Gamma$ that admit an attracting fixed point $\gamma^+ \in \partial\Gamma$, $\xi_\rho^n(\gamma^+)$ is the attracting fixed of the action of $\rho(\gamma)$ on $\mathrm{Gr}_n(\mathbb{R}^{2n})$ (*dynamic preserving*),*

If moreover $\rho(\Gamma) \subset \mathrm{Sp}(2n,\mathbb{R})$, then $\xi_\rho^n(x)$ is a Lagrangian for all $x \in \partial\Gamma$.

Maximal representations have been characterized in [BILW05], [BIW03]:

Theorem 4.1.4. *A representation $\rho : \pi_1(S_g) \to \mathrm{PSp}(2n,\mathbb{R})$ is maximal if and only if it is $\{n\}$-Anosov and for one and hence any positively oriented triple $(x, y, z) \in \partial\pi_1(S)$ the triple $(\xi_\rho^n(x), \xi_\rho^n(y), \xi_\rho^n(z))$ is a maximal triple of Lagrangians.*

One can also characterize maximal representations among $\{n\}$-Anosov representations by looking at the homotopy type of their boundary map. The fundamental group of the space of Lagrangians $\mathcal{L}_n$ is isomorphic to $\mathbb{Z}$ [Wig98] where a generator is :

$$\tau : \theta \in \mathbb{S}^1 \mapsto \langle \cos\left(\frac{\theta}{2}\right) x_1 + \sin\left(\frac{\theta}{2}\right) y_1, x_2, \cdots, x_n\rangle \in \mathcal{L}_n.$$

Theorem 4.1.5. *A representation $\rho : \pi_1(S_g) \to \mathrm{Sp}(2n,\mathbb{R})$ is maximal if and only if it is $\{n\}$-Anosov and the free homotopy type of the curve ξ_ρ^n is equal to $n[\tau]$.*

Proof. Let $\rho : \pi_1(S_g) \to \mathrm{PSp}(2n,\mathbb{R})$ be $\{n\}$-Anosov. Let (x, y, z) be a positively oriented triple in $\partial\pi_1(S_g)$. Up to changing the symplectic basis, we can assume that for some $(\epsilon_i) \in \{-1, 1\}$:

$$\xi_\rho^n(x) = \langle x_1, x_2, \cdots x_n\rangle,$$
$$\xi_\rho^n(y) = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, \cdots x_n + \epsilon_n y_n\rangle,$$
$$\xi_\rho^n(z) = \langle y_1, y_2, \cdots y_n\rangle.$$

Here the Maslov index of the triple $(\xi_\rho^n(x), \xi_\rho^n(y), \xi_\rho^n(z))$ is equal to the sum of the (ϵ_i).

Consider the following curve:

$$\tau_0 : \theta \mapsto \langle \cos\left(\frac{\theta}{2}\right) x_1 + \epsilon_1 \sin\left(\frac{\theta}{2}\right) y_1, \cdots, \cos\left(\frac{\theta}{2}\right) x_n + \epsilon_n \sin\left(\frac{\theta}{2}\right) y_n\rangle.$$

This loop is homotopic to the concatenation of the loops τ_i for $1 \leqslant i \leqslant n$:

$$\tau_i : \theta \mapsto \langle x_1, x_2, \cdots, \cos\left(\frac{\theta}{2}\right) x_i + \epsilon_i \sin\left(\frac{\theta}{2}\right) y_i, \cdots, x_n\rangle.$$

These loops are homotopic to τ or its inverse depending on the sign of ϵ_i. The homotopy type of τ_0 is hence equal to $(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)[\tau]$. Moreover the set of Lagrangians tranverse to a given Lagrangian is contractible, so one can homotope τ_0 on the intervals $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$ and $[\pi, 2\pi]$ to coincide with ξ_ρ^n. Hence the free homotopy class of ξ_ρ^n is equal to the one of τ_0, which is equal to $M(\xi_\rho^n(x), \xi_\rho^n(y), \xi_\rho^n(z))[\tau]$. We therefore deduce that ρ is maximal if and only if it is n-Anosov and $[\xi_\rho^n] = n[\tau]$. $\square$

4.2 An invariant convex domain and its fibrations

In this section we define globally fitting maps and fitting immersions, which are maps that parametrize fibrations of the projective model for the symmetric space of $\mathrm{SL}(2n, \mathbb{R})$ by projective subspaces of codimension d.

4.2.1 Pencils of quadrics

Let V be a finite even-dimensional vector space. Let S^2V be the space of symmetric bilinear tensors on V, which we interpret as maps $V^* \to V$. The dual space $\mathcal{Q} = S^2V^*$ is the space of symmetric bilinear forms on V, or the space of *quadrics* on V, that we interpret as maps $V \to V^*$.

We denote by $S^2V^{\geqslant 0}$ and $S^2V^{>0}$ respectively the space of semi-positive and positive symmetric tensors, i.e. elements $p \in S^2V$ such that p^{-1} is respectively a semi-positive and positive bilinear form. The Lie group $\mathrm{SL}(V)$ acts on S^2V, and preserves the properly convex set $\mathbb{P}(S^2V^{\geqslant 0})$. The convex domain $\mathbb{P}(S^2V^{>0})$ is a projective model for the symmetric space associated to $\mathrm{SL}(V)$.

The Grassmanian of d-dimensional linear subspaces of $\mathcal{Q}$ will be denoted by $\mathrm{Gr}_d(\mathcal{Q})$. An element of $\mathrm{Gr}_2(\mathcal{Q})$ is usually called a *pencil of quadrics* on V. We will here also call elements of $\mathrm{Gr}_d(\mathcal{Q})$ pencils of quadrics.

To an element $P \in \mathrm{Gr}_d(\mathcal{Q})$ one can associate its annihilator codimension d subspace $P^\circ \subset S^2V$. This dual space can be described as the space of symmetric tensors p on which one has $q(p) = \mathrm{Tr}(q \circ p) = 0$ for all $q \in P$. Note that the projectivization $\mathbb{P}(P^\circ)$ also has codimension d in $\mathbb{P}(S^2V)$.

This subspace does not necessarily intersect the convex $\mathbb{P}(S^2V^{>0})$.

Definition 4.2.1. We say that a pencil $P \in \mathrm{Gr}_d(\mathcal{Q})$ is *mixed* if P° contains a positive element, i.e. if $P^\circ \cap S^2V^{>0} \neq \{0\}$. We call the set of mixed pencils $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$.

Equivalently P is mixed if and only if it does not contain any semi-positive quadric $0 \neq q \in P$. Indeed the dual of the cone of positive elements $S^2V^{>0}$ is the cone of semi-positive bilinear forms in $\mathcal{Q}$.

4.2.2 Fitting pairs.

To a pencil $P \in \mathrm{Gr}_d(\mathcal{Q})$ we associate the codimension d subspace $\mathbb{P}(P^\circ \cap S^2V^{\geqslant 0})$ in the convex set $\mathbb{P}(S^2V^{\geqslant 0})$.

We say that two elements $P_1, P_2 \in \mathrm{Gr}_d(\mathcal{Q})$ form a *fitting pair* if the associated subspaces $\mathbb{P}(P_1^\circ \cap S^2V^{\geqslant 0})$ and $\mathbb{P}(P_2^\circ \cap S^2V^{\geqslant 0})$ are disjoint.

The structure of the convex set $\mathbb{P}(S^2V^{\geqslant 0})$ is involved, but the set of its extremal points $S^2\mathbb{P}(V)$ is the projectivization of the set of rank one tensors $S^2\mathbb{P}(V)$, which is in one-to-one correspondence with $\mathbb{P}(V)$. We show that the condition of being a fitting pair can be checked by looking only at $\mathbb{P}(V)$.

Given a symmetric bilinear form $q \in \mathcal{Q}$, we will write respectively $\{q = 0\}$, $\{q > 0\}$ and $\{q \geqslant 0\} \subset \mathbb{P}(V)$ the set of lines that are respectively null, positive and non-negative for q.

Proposition 4.2.2. *Let $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be continuous, the following are equivalent:*

(i) *(P_1, P_2) form a fitting pair,*

(ii) *there exist $q_1 \in P_1$ and $q_2 \in P_2$ such that $q_2 - q_1$ is positive,*

(iii) *there exist $q_1 \in P_1$ and $q_2 \in P_2$ such that $\{q_1 \geqslant 0\} \subset \{q_2 > 0\}$.*

In order to prove this, note that the convex set $\mathbb{P}(S^2V^{\geqslant 0})$ does not have any segment as a *facet*, i.e. the intersection of the convex with a supporting hyperplane. Hence it satisfies the following lemma.

The set $\{q = 0\}$ is identified via the identification $\mathbb{P}(V) \simeq S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V)$ with the intersection $\langle q \rangle^\circ \cap S^2\mathbb{P}(V)$.

Lemma 4.2.3. *For all linear hyperplane H in S^2V the extremal points of $\mathbb{P}(H \cap S^2V^{\geqslant 0})$ are also extremal points of $\mathbb{P}(S^2V^{\geqslant 0})$. In particular:*

$$\mathbb{P}(H \cap S^2V^{\geqslant 0}) = \mathrm{Hull}\left(H \cap S^2\mathbb{P}(V)\right).$$

Furthermore if $H = \langle q \rangle^\circ$ for $q \in \mathcal{Q}$, $H \cap S^2V = S^2\{q = 0\}$.

Proof. Let f be a facet of $\mathbb{P}(S^2V^{\geqslant 0})$. It is the intersection of this convex with a projective hyperplane corresponding to $[q] \in \mathbb{P}(S^2V^*) = \mathbb{P}(\mathcal{Q})$. The fact that this is a supporting hyperplanes implies that $q \in \mathcal{Q}$ is a semi-positive element. Let $W \subset V$ be the vector subspace of isotropic vectors for q. The corresponding facet is equal to $\mathbb{P}(S^2W^{\geqslant 0})$.

Hence facets of $\mathbb{P}(S^2V^{\geqslant 0})$ are of the form $\mathbb{P}(S^2W^{\geqslant 0})$ for $W \subset V$ a linear subspace. This has dimension 0 or at least 2, and therefore no facet is a segment.

Suppose that for some general projective hyperplane, some extremal point p of $\mathbb{P}(H \cap S^2V^{\geqslant 0})$ is not an extremal point of $\mathbb{P}(S^2V^{\geqslant 0})$, then it belongs to the interior of a facet of $\mathbb{P}(S^2V^{\geqslant 0})$, that has dimension at least 2 as stated previously. The intersection of this face with H contains therefore a segment, so p is not an extremal point of $H \cap \mathbb{P}(S^2V^{\geqslant 0})$. ☐

Proof of Proposition 4.2.2. Let us prove that (i) implies (ii). The set $P_1^\circ \cap P_2^\circ$ is disjoint from $\mathbb{P}(S^2V^{\geqslant 0})$ if and only if there exist an element in $P_1 + P_2$ that belongs to the dual of $\mathbb{P}(S^2V^{\geqslant 0})$, i.e. if there exist a positive bilinear form $q \in P_1 + P_2$. This form can be written as $q = q_1 - q_2$ with $q_1 \in P_1$ and $q_2 \in P_2$, therefore (i) implies (ii).

Moreover (ii) implies (iii). Indeed, if $q_2 - q_1$ is positive then $\{q_1 \geqslant 0\} \subset \{q_2 > 0\}$.

It only remains to show that (iii) implies (i). Lemma 4.2.3 implies that:

$$\mathbb{P}(\langle q_1\rangle^\circ \cap S^2V^{\geqslant 0}) = \mathrm{Hull}(S^2\{q_1 = 0\}).$$

Hence $q_2 \in P_2 \subset \mathcal{Q} = S^2V^*$ is positive on the cone $\langle q_1\rangle^\circ \cap S^2V^{\geqslant 0} \subset S^2V$ and therefore $\mathbb{P}(P_2^\circ)$ does not intersect $\mathbb{P}(\langle q_1\rangle^\circ \cap S^2V^{\geqslant 0})$. $\square$

4.2.3 Fitting directions.

The space $\mathrm{Gr}_d(\mathcal{Q})$ inherits the structure of a smooth manifold. A chart around a point $P \in \mathrm{Gr}_d(\mathcal{Q})$ can be constructed given a subspace Q such that $P \oplus Q = V$. We denote by $U_Q \subset \mathrm{Gr}_d(\mathcal{Q})$ the open set of elements transverse to Q. Every element of U_Q can be written uniquely as the graph $\{x + u(x) | x \in P\}$ for some linear map $u : P \to Q$. Hence U_Q can be identified with the vector space $\mathrm{Hom}(P, Q)$.

The tangent space $T_P \mathrm{Gr}_d(\mathcal{Q})$ can be naturally identified with $\mathrm{Hom}(P, \mathcal{Q}/P)$, so that for each chart U_Q containing P, the tangent space identifies with the tangent space in the chart via the identification $\mathrm{Hom}(P, Q) \simeq \mathrm{Hom}(P, \mathcal{Q}/P)$.

Let $\mathrm{v} \in T_P \mathrm{Gr}_d(\mathcal{Q})$, that we see as an element of $\mathrm{Hom}(P, \mathcal{Q}/P)$. One can interpret $\mathrm{Ker}(\mathrm{v})$ as the set of vectors in P which remains at first order in the dimension d linear subspace when the subspace moves in the direction v. More precisely the lines ℓ in the kernel are exactly the one such that for any Riemannian metric on $\mathbb{P}(V)$ and every curve γ in $\mathrm{Gr}_P(V)$ with $\gamma(0) = P$ and $\gamma'(0) = \mathrm{v}$ one has $d\left(\mathbb{P}\left(\gamma(t)\right), \ell\right) = o(t)$ at $t = 0$. Because of the identification $T_P \mathrm{Gr}_d(\mathcal{Q}) \simeq T_{P^\circ} \mathrm{Gr}_{N-d}(\mathcal{Q}^*)$ where $N = \dim(\mathcal{Q})$, to such an element v corresponds an element:

$$\mathrm{v}^\circ \in \mathrm{Hom}\left(P^\circ, S^2V/P^\circ\right).$$

We will call *fitting directions* in the Grassmanian $\mathrm{Gr}_d(\mathcal{Q})$ the tangent directions such that if $(P_t) \in \mathrm{Gr}_d(\mathcal{Q})$ moves in this direction, the corresponding codimension d subspaces $\mathbb{P}\left(P_t^\circ \cap S^2V^{\geqslant 0}\right)$ are disjoint from $\mathbb{P}\left(P_0^\circ \cap S^2V^{\geqslant 0}\right)$, at order one, see Proposition 4.2.5.

Definition 4.2.4. We say that a vector $\mathrm{v} \in T_P \mathrm{Gr}_d(\mathcal{Q})$ is *fitting* if one of the following equivalent statements holds:

- $\mathrm{Ker}(\mathrm{v}^\circ) \subset P^\circ$ intersects trivially $S^2V^{\geqslant 0}$,
- $\mathrm{Im}(\mathrm{v}) \subset \mathcal{Q}/P$ contains $[q]$ where q is a positive element.

Proof. We check that these two statements are indeed equivalent. We first show that $\mathrm{Ker}(\mathrm{v}^\circ) = \mathrm{Im}(\mathrm{v})^\circ$, where we use the natural identification $(\mathcal{Q}/P)^* \simeq P^\circ$.

We prove this by writing an equation that relates v and v°. By definition, $\mathrm{Tr}(qp) = 0$ for $q \in P$ and $p \in P^\circ$. Let us choose some representatives $\overline{\mathrm{v}(p)} \in S^2V$ and $\overline{\mathrm{v}^\circ(q)} \in \mathcal{Q}$ for $\mathrm{v}(p) \in S^2V/P^\circ$ and $\mathrm{v}^\circ(q) \in \mathcal{Q}/P$. Let us fix $q \in P$ and $p \in P^\circ$. If (P_t) is a smooth curve with $P_0 = P$ and with derivative v at $t = 0$, $p + t\overline{\mathrm{v}^\circ(p)} + o(t^2) \in P_t$ and $q + t\overline{\mathrm{v}(q)} + o(t^2) \in P_t^\circ$. Hence we get:

$$\mathrm{Tr}\left(q\overline{\mathrm{v}^\circ(p)} + \overline{\mathrm{v}(q)}p\right) = 0.$$

An element $p \in P^\circ$ satisfies $p \in \mathrm{Ker}(\mathrm{v}^\circ)$ if and only if $\mathrm{Tr}(\overline{\mathrm{v}(q)}p) = 0$ for all $q \in \mathcal{Q}$, hence if and only if the corresponding linear form on Q/P belongs to $\mathrm{Im}(\mathrm{v})^\circ$.

Now we prove the equivalence of the two definitions. If there exist $[q] \in \mathrm{Im}(\mathrm{v})$ with q positive, then for any $p \in \mathrm{Ker}(\mathrm{v}^\circ)$, $\mathrm{Tr}(pq) = 0$. Hence p is not a positive tensor. Therefore $\mathrm{Ker}(\mathrm{v}^\circ) \subset P^\circ$ intersects trivially $S^2V^{\geqslant 0}$.

Conversely if $\mathrm{Ker}(\mathrm{v}^\circ)$ intersects trivially $S^2V^{\geqslant 0}$, there exist $q \in \mathcal{Q}$ that do not vanish on $S^2V^{\geqslant 0}$, i.e. q is positive. The class $[q] \in \mathcal{Q}/P$ belongs to $\mathrm{Im}(\mathrm{v})$. ☐

Fitting directions are related to fitting pairs. More precisely:

Proposition 4.2.5. *A vector $\mathrm{v} \in T_P\,\mathrm{Gr}_d(\mathcal{Q})$ is fitting if and only if for every $\mathcal{C}^1$ curve $\gamma : [0,1] \to \mathrm{Gr}_d(\mathcal{Q})$ with $\gamma_0 = P$ and $\gamma_0' = \mathrm{v}$, (γ_t, γ_0) is a fitting pair for all $t > 0$ small enough.*

Moreover in this case for any Riemannian metric on $\mathbb{P}(S^2V)$ there exists an $\epsilon > 0$ such that for any $t > 0$ small enough the Riemannian distance between $\mathbb{P}(\gamma_0^\circ \cap S^2V^{\geqslant 0})$ and $\mathbb{P}(\gamma_t^\circ \cap S^2V^{\geqslant 0})$ is greater that ϵt.

Proof. Let us fix a complement H of P° in S^2V. Let $N = \dim(\mathcal{Q}) = \frac{n(n+1)}{2}$. We identify a neighborhood of $P^\circ \subset \mathrm{Gr}_{N-d}(\mathcal{Q})$ with $\mathrm{Hom}(P^\circ, H)$. If there exist a non-zero element $p \in \mathrm{Ker}(\mathrm{v}^\circ) \cap S^2V^{\geqslant 0}$, one can consider the curve where γ_t corresponds to

$$q \in P^\circ \mapsto t\mathrm{v}(q).$$

The non-zero element $p \in S^2V^{\geqslant 0}$ belongs to $P^\circ = \gamma_0^\circ$ and γ_t° hence the pair (γ_0, γ_t) is not fitting for any $t > 0$.

In general the element corresponding to γ_t is equal for t close to 0 to the element of $\mathrm{Gr}_{N-d}(\mathcal{Q})$ which is the graph of the map $P^\circ \to H$:

$$q \mapsto t\mathrm{v}(q) + o(t).$$

The pair (γ_o, γ_t) is fitting if and only if $\mathrm{Ker}(\mathrm{v}) \cap S^2V^{\geqslant 0} = \{0\}$, hence if v is fitting.

In this case the distance between $\mathbb{P}(\gamma_0^\circ \cap S^2V^{\geqslant 0})$ and $\mathbb{P}(\gamma_t^\circ \cap S^2V^{\geqslant 0})$ grows at least linearly in t for t close to 0. ☐

Fitting vectors can be thought of analogs of spacelike vectors in a pseudo-Riemannian manifold of signature (d, N) for some $N > 0$. For instance the set of spacelike vectors is the union of a family of cones parametrized by $\mathbb{S}^{d-1}$, and so is the set of fitting vectors, as shown in the following proposition. This analogy is also emphasized by Remark 4.2.8 and Theorem 4.6.1.

For a vector space W we write $\mathbb{S}W = (W \backslash \{0\})/\mathbb{R}_{>0}$

Proposition 4.2.6. *The set of fitting vectors in $T_P \operatorname{Gr}_d(\mathcal{Q})$ is equal to the union:*

$$\bigcup_{[q] \in \mathbb{S}\mathcal{E}_P} C_{[q]}.$$

Here $C_{[q]}$ is the convex open cone of elements $\mathrm{v} \in T_P \operatorname{Gr}_d(\mathcal{Q})$ such that there exist a positive element in the class $\mathrm{v}(q) \in \mathcal{Q}/P$.

Recall that we use the identification $\mathrm{v} \in T_P \operatorname{Gr}_d(\mathcal{Q}) \simeq \operatorname{Hom}(P, \mathcal{Q}/P)$.

Proof. As in the third item of Proposition 4.2.11 a vector is well-fitting if and only if it belongs to $C_{[q]}$ for some $[q] \in \mathbb{S}\mathcal{E}_P$. We just check that the sets $C_{[q]}$ are indeed open convex cones.

Let us fix a complement H of P in $\mathcal{Q}$ to identify $T_P \operatorname{Gr}_d(\mathcal{Q})$ with $\operatorname{Hom}(P, H)$. If $\mathrm{v}_1, \mathrm{v}_2$ lie in $C_{[q]}$ and if $\lambda, \mu \in \mathbb{R}_{>0}$, then for some $q_1, q_2 \in P$ one has $\mathrm{v}_1(q) + q_1$ and $\mathrm{v}_2(q) + q_2$ positive. Therefore $(\lambda \mathrm{v}_1 + \mu \mathrm{v}_2)(q) + \lambda q_1 + \mu q_2$ is positive so $\lambda \mathrm{v}_1 + \mu \mathrm{v}_2$ belongs to $C_{[q]}$.

□

4.2.4 The space of geodesics in $\mathbb{H}^3$.

In this section we fix $V = \mathbb{C}^2$ and restrict ourselves to pencils of Hermitian quadrics. The result and notations of this Section are only used again in Remark 4.2.12 and Section 4.4.3, but they serve as an illustration of the previously introduced notions of fitting pairs and fitting directions.

Let $\mathcal{H} \subset \mathcal{Q}$ be the subset of Hermitian bilinear forms on $V = \mathbb{C}^2$. A Hermitian form $q \in \mathcal{H}$ that is not semi-positive or semi-negative is of Hermitian signature $(1, 1)$, and hence its zero set in $\mathbb{CP}^1$ is a circle.

Let $S^2V = S_h^2V \oplus S_a^2V$ be the eigenspace decomposition for the operator $J \otimes J$ where J is the complex conjugation. Here S_h^2V is the 4-dimensional eigenspace associated to 1 and $S_a^2V = \mathcal{H}^\circ$, the 6-dimensional space associated to -1. The intersection of $\mathbb{P}(S_h^2V)$ with the space of positive tensors is the projective Klein model for $\mathbb{H}^3$.

The annihilator of a pencil $P \in \operatorname{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ is equal to $P^\circ = \mathcal{H}^\circ \oplus H$ where H is a plane in S_h^2V, which in turns corresponds to a geodesic in $\mathbb{H}^3$ in the projective Klein model.

A pencil of quadrics $P \in \operatorname{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ vanishes completely on two points in $\mathbb{CP}^1$, as in Figure 4.1 where the zero set of three elements of the pencil are

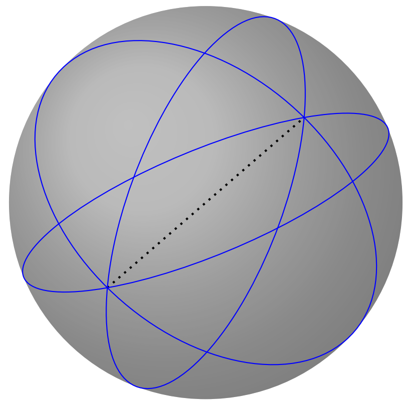


Figure 4.1: Three Hermitian quadrics in a pencil and the corresponding geodesic in $\mathbb{H}^3$.

depicted. The space of mixed pencils in $\mathcal{H}$ can be identified with the space $\mathcal{G}$ of unoriented geodesics in $\mathbb{H}^3$.

Two pencils in $\mathcal{H}$ form a fitting pair if and only if we can find a circle in each of the pencils that are disjoint. This is possible if and only if the two corresponding geodesics in $\mathbb{H}^3$ are disjoint, with disjoint endpoints. A fitting pair of pencils is illustrated in Figure 4.2.

The space $\mathcal{G}$ of unoriented geodesics in $\mathbb{H}^3$ admits a pseudo-Riemannian metric of signature $(2,2)$: we now compare the notion of fitting vectors with the notion of spacelike vectors. The tangent space at a geodesic with endpoints $(x,y) \in \mathbb{CP}^1$ can be identified with $T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1$. The choice of a point in the geodesic provides an identification $\phi : T_x\mathbb{CP}^1 \to T_y\mathbb{CP}^1$ and a metric q_0 on $T_y\mathbb{CP}^1$. Taking a different point in the geodesic means replacing ϕ by $\lambda\phi$ and q_0 by $\lambda^{-1}q_0$ for some $\lambda \in R_{>0}$.

Hence we can consider the pseudo Riemannian metric that is invariant by the action of the isometry group of $\mathbb{H}^3$:

$$q : (T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1)^2 \to \mathbb{R}$$

$$(v_1,w_1),(v_2,w_2) \mapsto q_0(\phi(v_1),w_2) + q_0(\phi(v_2),w_1).$$

For this metric a vector $(v,w) \in T_x\mathbb{CP}^1 \times T_y\mathbb{CP}^1$ is spacelike if and only if $q_0(\phi(v),w) > 0$.

Proposition 4.2.7. *A pair of geodesics is a fitting pair if and only if the corresponding geodesics are disjoint.*

A tangent vector (v,w) to $\mathcal{G} \simeq \mathrm{Gr}_2(\mathcal{H})$ is fitting if and only if $\phi(v)$ and w are not positively anti-colinear, i.e there are no $\lambda,\mu \in \mathbb{R}^{\geqslant 0}$ such that $\lambda\phi(v) = -\mu w$.

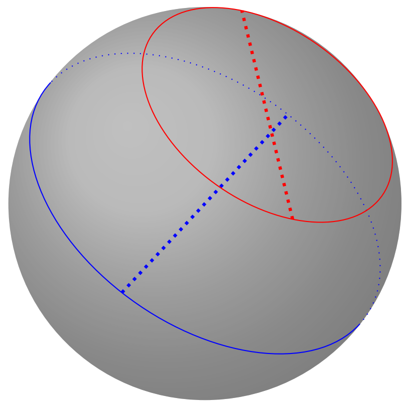


Figure 4.2: Two disjoint geodesics in $\mathbb{H}^3$ and disjoint circles in $\mathbb{CP}^1$ between their endpoints.

Remark 4.2.8. In particular spacelike vectors are fitting, but not all fitting vectors are spacelike.

Proof. Let $P_1, P_2 \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$. One has $\mathbb{P}(S^2_a V) \cap \mathbb{P}(S^2 V^{\geqslant 0}) = \varnothing$. Hence the subsets $\mathbb{P}(P_1^\circ \cap S^2 V^{\geqslant 0})$ and $\mathbb{P}(P_2^\circ \cap S^2 V^{\geqslant 0})$ for $P_1, P_2 \in \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ are disjoint if and only if the corresponding geodesics are disjoint.

Let (v, w) be a tangent vector to $\gamma \in \mathcal{G}$ and let γ_t be a curve in $\mathcal{G}$ with this derivative at $t = 0$. If $\phi(v)$ and w are not non-positively colinear, the distance between $\gamma = \gamma_0$ and γ_t is greater that ϵt for some $\epsilon > 0$ and t small enough. Indeed there exist $z \in T_x\mathbb{CP}^1$ such that $q_0(\phi(z), \phi(x)), q_0(\phi(z), w) \geqslant 0$, and the totally geodesic disk in $\mathbb{H}^3$ through x, y normal to z at x contains γ_0 while being at distance ϵt to γ_t.

Hence the distance between the subsets $\mathbb{P}(P_0^\circ \cap S^2 V^{\geqslant 0})$ and $\mathbb{P}(P_t^\circ \cap S^2 V^{\geqslant 0})$ is also greater than $\epsilon' t$ for some $\epsilon' > 0$ and all t small enough. Therefore by Proposition 4.2.5 this direction is fitting.

Conversely if If $\phi(v)$ and w are non-positively colinear, then there is such a curve γ_t such that the corresponding geodesics all have a common point. Therefore by Proposition 4.2.5 this direction is not fitting.

□

4.2.5 Fibration of a convex set and globally fitting maps.

We consider continuous and smooth fibrations of the $\mathrm{SL}(V)$-invariant convex set $\mathbb{P}(S^2 V^{\geqslant 0})$ by projective codimension d subspaces. Let M be a connected manifold of dimension d. We are interested in continuous injective maps, or smooth immersions $u : M \to \mathrm{Gr}_d(\mathcal{Q})$.

We write $u^\circ(x) = (u(x))^\circ$. The map u determines a collection of projective subsets of codimension d:

$$\left(\mathbb{P}\left(u^\circ(x) \cap S^2V^{\geqslant 0}\right)\right)_{x\in M}. \tag{4.1}$$

If the image of u contains only mixed elements, then all the submanifolds in (4.1) are non-empty.

Definition 4.2.9. We call a continuous map $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ a *globally fitting map* if the subsets in the collection (4.1) are disjoint.

A continuous map $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ is a *locally fitting map* if for all $x \in M$ there is a neighborhood $U \subset M$ of x such that $u_{|U}$ is a globally fitting map.

Since $\dim(M) = d$, the invariance of domain implies that the sets (4.1) for a globally fitting map form a fibration for all $x \in M$ of a neighborhood in $\mathbb{P}(S^2V^{\geqslant 0})$ of $\mathbb{P}\left(u^\circ(x) \cap S^2V^{\geqslant 0}\right)$.

We now consider immersions from a manifold M of dimension d whose tangent directions are all fitting.

Definition 4.2.10. A smooth immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ is a *fitting immersion* if $\mathrm{d}u(v)$ is fitting for all $v \in TM$.

Because of proposition 4.2.5, fitting immersions are locally fitting maps.

The following proposition is the infinitesimal equivalent of Proposition 4.2.2. We write the statement in a way to emphasize this analogy.

Consider the tautological rank d vector bundle $p : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$, which admits a tautological projection $\pi : \mathcal{E} \to \mathcal{Q}$.

Proposition 4.2.11. *Given an immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$, let $x_0 \in M$. The following are equivalent:*

(i) the manifolds $\left(\mathbb{P}(u^\circ(x))\right)_{x\in M}$ define locally a smooth fibration of an open neighborhood of $\mathbb{P}\left(u^\circ(x_0) \cap S^2V^{\geqslant 0}\right)$,

(ii) for all $v \in T_{x_0}M$, $\mathrm{d}u(v)$ is fitting,

(iii) for all $v \in T_{x_0}M$ there exist $\mathrm{w} \in T\mathcal{E}$ such that $\mathrm{d}p(\mathrm{w}) = \mathrm{d}u(v)$ and $\mathrm{d}\pi(\mathrm{w}) \in T\mathcal{Q}$ is positive.

Note that that an element in $T\mathcal{Q}$ is a pair $(q, \dot{q})$ and we say that it is positive if the tangent vector $\dot{q} \in \mathcal{Q}$ is positive.

Proof. Note that since M has dimension d and $u^\circ(x_0)$ has codimension d, the statement (i) is equivalent to having for any Riemannian distance d_R on $\mathbb{P}(S^2V)$ and d_M on M, for some $\epsilon > 0$ when x is close to x_0:

$$d_R\left(\mathbb{P}\left(u^\circ(x) \cap S^2V^{\geqslant 0}\right), \mathbb{P}\left(u^\circ(x_0) \cap S^2V^{\geqslant 0}\right)\right) \geqslant \epsilon d_M(x, x_0).$$

Proposition 4.2.5 shows that the fitting condition is equivalent to having this distance growing linearly, hence the statements (i) and (ii) are equivalent.

A pair $(P, q_0) \in \mathrm{Gr}_d(\mathcal{Q}) \times \mathcal{Q}$ correspond to an element in $\mathcal{E}$ if and only if $p \in P$. A pair $(\mathrm{v}, \dot{q}) \in T_P \mathrm{Gr}_2(\mathcal{Q}) \times T_q \mathcal{Q}$ can be written as $(\mathrm{d}p(\mathrm{w}), \mathrm{d}\pi(\mathrm{w}))$ for some $\mathrm{w} \in T\mathcal{E}$ if and only if $\dot{q}$ belongs to the class defined by $\mathrm{v}(q_0)$.

Let us show this last claim. Let H be a complement of P in $\mathcal{Q}$ let $u : [0,1] \to \mathrm{Hom}(P, H)$ and $q : [0,1] \to \mathcal{Q}$ be smooth curves with derivative v and $\dot{q}$ at $q = 0$ and such that $q(t)$ belongs the the graph of $u(t)$ for $t \in [0,1]$. For some smooth curve $\tilde{q} : [0,1] \to P$ with $\tilde{q}(0) = q_0$ and all $t \in [0,1]$:

$$q(t) = \tilde{q}(t) + u(t)(\tilde{q}(t)).$$

Differentiating this at $t = 0$ and we get exactly $\mathrm{v}(q_0) + \tilde{q}'(0) = \dot{q}$, so $\dot{q}$ belongs to the class defined by $\mathrm{v}(q_0)$, since $\tilde{q}'(0) \in P$. Reciprocally if this holds, one can construct such a curve $\tilde{q}$, so the pair corresponds to an element of $T\mathcal{E}$.

We conclude that (iii) is equivalent to the second characterization of fitting vectors in Definition 4.2.4: one can find such a positive lift w if and only if one can find a class in $\mathrm{Im}(\mathrm{v})$ that contains a positive element.

$\square$

In Proposition 4.7.1 we show how to construct some examples of fitting maps from a totally geodesic immersion in the symmetric space.

Remark 4.2.12. If S is a surface in $\mathbb{H}^3$ with principal curvature in $(-1, 1)$, then the set of normal geodesics forms spacelike surface, for the pseudo-Riemannian structure on the space of geodesics described in Section 4.2.4. The corresponding map $\mathcal{G}u : S \to \mathcal{G}$ is called the *Gauss map*. Nearly Fuchsian representations are representations of a closed surface group Γ_g admitting an equivariant surface with principal curvature in $(-1, 1)$. They are a priori a larger class of representations than almost Fuchsian representations, for which the equivariant surface with principal curvature in $(-1, 1)$ is required to be minimal.

The space $\mathcal{G}$ also admits a special $\mathrm{SL}_2(\mathbb{C})$-invariant symplectic structure. An immersion in $\mathcal{G}$ is locally the Gauss map of an immersion with principal curvature in $(-1, 1)$ if and only if it is spacelike and Lagrangian for this symplectic structure [ES22]. Therefore if the fitting immersion is not Lagrangian, it does not come as the Gauss map of a surface in $\mathbb{H}^3$. Hence there could be representations admitting fitting immersions that are not nearly Fuchsian.

Remark 4.2.13. The definition of a fitting immersion and the previous two propositions can be generalized to the more general setup when S^2V is replaced by a vector space W and $S^2V^{\geqslant 0}$ is replaced by a closed proper convex cone C in W. In this setup positive quadrics should be replaced by elements in the dual cone of C in W^*.

4.3 Fitting flows.

In this section we define the notion of a fitting flow, and study the consequence of the existence of such a flow. We show next that such flows always exist for fitting immersions. In this section let us fix a map $u : M \to \mathrm{Gr}_d(\mathcal{Q})$.

4.3.1 Definition and application of fitting flows.

The pullback $u^*\mathcal{E}$ of the tautological bundle $p : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$ defines a rank d vector bundle over M. We define the sphere bundle $\mathbb{S}u^*\mathcal{E}$ as the quotient of the vector bundle $u^*\mathcal{E}$ by the action of positive scalars.

Recall that M has dimension d. We consider flows on $\mathbb{S}u^*\mathcal{E}$ so that some form of contraction occurs along the flow lines. We denote also by p the bundle maps $u^*\mathcal{E} \to M$, $\mathbb{S}u^*\mathcal{E} \to M$, with a slight abuse of notations.

The fiber at $P \in \mathrm{Gr}_d(\mathcal{Q})$ of the bundle $\mathcal{E}$ is identified with the vector subspace $P \subset \mathcal{Q}$. Since all the fibers are naturally identified with subsets of $\mathcal{Q}$, there is a natural projection $\pi : \mathcal{E} \to \mathcal{Q}$. We still denote by $\pi : u^*\mathcal{E} \to \mathcal{Q}$ the corresponding projection with a slight abuse of notations.

Definition 4.3.1. A *fitting flow* for a continuous map $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ is a continuous flow $(\Phi_t)_{t\in\mathbb{R}}$ on $\mathbb{S}u^*\mathcal{E}$ such that one can choose a representative q' of $[q'] = \Phi_t(q)$ such that $\pi(q') - \pi(q) \in \mathcal{Q}$ is positive.

Note that the last condition is equivalent to asking that $\{\pi(q) \geqslant 0\} \subset \{\pi(q') > 0\}$ in $\mathbb{RP}^{2n-1}$. Along such flows, the associated quadric hypersurfaces are nested into one another. In particular if u admits a fitting flow it is locally a fitting map.

Lemma 4.3.2. *Let $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a continuous map that admits a fitting flow. The projection to $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ of the flow lines of the fitting flow are embedded.*

Proof. Assume by contradiction that for some $t_0 > 0$ and $q \in \mathbb{S}u^*\mathcal{E}$ one has $u(x) = p(q) = p(\Phi_{t_0}(q))$. The fact that the flow is fitting implies that for some $\lambda > 0$, $\lambda\pi(\Phi_{t_0}(q)) - \pi(q) \in \mathcal{Q}$ is positive. This positive quadric belongs to $u(x)$, contradicting the fact that $u(x) \in \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$.

□

Some fitting flows can be constructed by taking a geodesic flow on M for some Riemannian metric and identifying $u^*\mathcal{E}$ with the tangent bundle to M. In general the projections of the flow lines of a fitting flow satisfy the following topological property, which is clearly satisfied for geodesic flows.

Lemma 4.3.3. *Let $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a continuous map equipped with a fitting flow Φ in a neighborhood of $x \in M$. For t small enough the sphere $S_t : [q] \in \mathbb{S}u^*\mathcal{E}_{|x} \mapsto p \circ \Phi_t([q]) \in M$ is homotopic to a generator of the homology of $U\backslash\{x\}$ for any open neighborhood U of x in M that is diffeomorphic to $\mathbb{R}^d$.*

The proof relies on the fact the dimension of M is equal to d, and hence the manifolds $\mathbb{P}(u^\circ(x) \cap S^2V^{\geqslant 0})$ locally define a fibration of $\mathbb{P}(S^2V^{\geqslant 0})$.

Proof. Let $P = u(x)$. Let $p_0 \in P^\circ \subset S^2V$ be a positive tensor, which exists since the pencil P is assumed to be in $\mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$. Let $P' \in \mathrm{Gr}_d(S^2V)$ be a supplement of P°. Since u is continuous for all y close enough to x the vector subspace

$u^\circ(y)$ is transverse to P', therefore there exist a unique vector $\phi(y) \in P'$ such that:

$$p_0 + \phi(y) \in (p_0 + P') \cap u^\circ(y) \subset S^2V^{>0}.$$

This defines a continuous map ϕ from a neighborhood U of $x \in M$ to P'.

Let $[q] \in \mathbb{S}u^*\mathcal{E}_x$. For all t such that $p \circ \Phi_t(q) \in U$, the linear form $\pi(\Phi_t(q)) \in \mathcal{Q} = S^2V^*$ vanishes on $\phi(p \circ \Phi_t(q)) \in S^2V$ since this point belongs to $u^\circ(p \circ \Phi_t(q))$. Moreover $\pi(\Phi_t(q)) - \pi(q)$ is a positive bilinear form since Φ is a fitting flow.

In particular $\pi(q) \in \mathcal{Q} = S^2V^*$ is always negative on $\phi(p \circ \Phi_t(q)) \in S^2V$. Hence for t small enough $[\phi \circ S_t] : \mathbb{S}u^*\mathcal{E}_x \to \mathbb{S}P'$ has the same degree as $[\pi] : \mathbb{S}u^*\mathcal{E}_x \to \mathbb{S}P'^* \simeq \mathbb{S}P$ which associates to $[q] \in \mathbb{S}u^*\mathcal{E}_x$ the class $[\pi(q)] \in \mathbb{S}P$. The map $[\pi]$ is a diffeomorphism, so in particular $1 = |\deg([\pi])| = |\deg(\phi)\deg(S_t)|$. Hence S_t is a generator of the homotopy group of $U\backslash\{x\}$.

□

Remark 4.3.4. In particular for an immersion $u : M \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ that admits a fitting flow, choosing continuously an orientation of the pencils $u(x)$ for $x \in M$ is equivalent to choosing an orientation of M.

When $M = \widetilde{N}$ where N is compact and $\Gamma = \pi_1(M)$, the quotient $\mathbb{S}u^*\mathcal{E}/\rho(\Gamma)$ is compact. Hence any Riemannian metric on $\mathbb{S}u^*\mathcal{E}/\rho(\Gamma)$ is quasi-isometric to Γ via any orbit map. When a fitting flow exists, the flow lines project to quasi-geodesics in $\widetilde{N}$.

Proposition 4.3.5. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits a ρ-equivariant fitting flow Φ. There exist $C, D > 0$ such that the projection to $\widetilde{N}$ of the flow lines of Φ are (C, D)-quasi geodesics. Moreover for every $(x, y) \in \widetilde{N}^2$ there exist a flow line whose projection to $\widetilde{N}$ starts at x and ends at y.*

Remark 4.3.6. Note that since there exist a flow line between any pair of points in $\widetilde{N}$, the map u is necessarily a globally fitting map.

In this proof we will use the *Hilbert distance* d_H on the properly convex domain $\mathbb{P}(S^2V^{>0})$. It is defined which is defined using the cross ratio as $d_H([p_1], [p_2]) = \log(\mathrm{cr}([p^-], [p_1], [p_2], [p^+]))$ where $[p_1], [p_2] \in \mathbb{P}(S^2V^{>0})$ and $[p^-]$, $[p^+]$ are the intersection of the projective line through $[p_1]$ and $[p_2]$ with the boundary of the domain $\mathbb{P}(S^2V^{>0})$. If the closure in $\mathbb{P}(S^2V)$ of two sets $A, B \subset \mathbb{P}(S^2V^{>0})$ are disjoint, then the two sets are at positive distance for the Hilbert distance. Indeed for every Riemannian metric $d_{\mathbb{S}}$ on the compact manifold $\mathbb{P}(S^2V)$, the Hilbert distance between any two points in $\mathbb{P}(S^2V^{>0})$ is bounded from below by some uniform multiple of $d_{\mathbb{S}}$.

Proof. For all $[q] \in \mathbb{S}u^*\mathcal{E}$ we consider the convex $\mathbb{P}(\langle \pi(q) \rangle^\circ \cap S^2V^{>0})$.

We choose a continuous and ρ-equivariant map $s : \mathbb{S}u^*\mathcal{E} \to \mathbb{P}(S^2V^{>0})$ with the property that $s[q] \in \mathbb{P}(\langle \pi(q) \rangle^\circ \cap S^2V^{>0})$ for all $[q] \in \mathbb{S}u^*\mathcal{E}$.

We fix a Γ-invariant Riemannian metric g on $\mathbb{S}u^*\mathcal{E}$ with associated distance d_g. We set C_1 to be the supremum of $d_H(s([q_1]), s([q_2]))$ for all $[q_1],[q_2] \in \mathbb{S}u^*\mathcal{E}$ such that $d_g([q_1],[q_2]) \leqslant 1$, which exists since Γ acts cocompactly on $\widetilde{N}$. The following inequality follows for all $[q_1],[q_2] \in \mathbb{S}u^*\mathcal{E}$ from the triangular inequality:

$$d_H(s([q_1]), s([q_2])) \leqslant C_1 d_g([q_1],[q_2]) + C_1.$$

Indeed for some integer $n \leqslant d_g([q_1],[q_2]) + 1$ one can find elements $x_0 = [q_1], x_1, \cdots, x_n = [q_2]$ in $\mathbb{S}u^*\mathcal{E}$ such that for all $1 \leqslant i \leqslant n$ one has $d_g(x_{i-1}, x_i) \leqslant 1$. The triangular inequality for the Hilbert distance implies that:

$$d_H(s([q_1]), s([q_2])) \leqslant nC_1.$$

We now set C_2 to be the supremum of $d_g(s([q_1]), s(\Phi_t([q])))$ for all $[q_1] \in \mathbb{S}u^*\mathcal{E}$ and $0 \leqslant t \leqslant 1$. Similarly we get the following inequality for all $[q] \in \mathbb{S}u^*\mathcal{E}$ and $t \geqslant 0$:

$$d_g(s([q]), s(\Phi_t([q]))) \leqslant C_2 t + C_2.$$

Let K be a compact fundamental domain for the action of Γ on $\mathbb{S}u^*\mathcal{E}$ and let ϵ be the infimum of the Hilbert distance for any $[q] \in K$ between $\mathbb{P}\left(\langle\pi(q)\rangle^\circ \cap S^2V^{>0}\right)$ and $\mathbb{P}\left(\langle\pi(\Phi_1(q))\rangle^\circ \cap S^2V^{>0}\right)$. Since the flow is fitting, the closures of these two sets in $\mathbb{P}(S^2V)$ are disjoint for any $[q]$, and hence their Hilbert distance is positive. Since K is compact, the infimum ϵ is also positive.

The Hilbert distance between $s(q)$ and $s(\Phi_t(q))$ for $t > 0$ and $q \in \mathbb{S}u^*\mathcal{E}$ is greater than $\epsilon(t-1)$. Indeed for all integer $0 \leqslant n \leqslant t$ the projective segment between $s(q)$ and $s(\Phi_t(q))$, which is a geodesic for the Hilbert distance, intersects $\mathbb{P}(\langle\pi(\Phi_n(q))\rangle^\circ \cap S^2V^{>0}$ in exactly one point x_n. Moreover the Hilbert distance between x_n and x_{n+1} for $0 \leqslant n \leqslant t-1$ is at least ϵ.

Putting all of these inequalities together we get that for all $t \geqslant 0$ and $[q] \in \mathbb{S}u^*\mathcal{E}$:

$$\frac{\epsilon}{C_1}(t-1) - 1 \leqslant d_g(s([q]), s(\Phi_t([q]))) \leqslant C_2(t+1)$$

Hence the flow lines are quasi-isometric embeddings.

We now check that flow lines exist between any pair of points. Let $x \in M$. Given $t \in \mathbb{R}$ we consider the d-sphere $S_t : q \in \mathbb{S}u^*\mathcal{E}_{|x} \mapsto \Phi_t(q) \in u^*\mathcal{E}$. Suppose that some $y \in M$, avoids the sphere S_t for all $t > 0$. Consider a curve η between x and y. The homological intersection between this segment and the spheres S_t in $M \backslash \{x, y\}$ is constant, and is equal to zero for t large enough since the spheres S_t are then uniformly far from x. However for t small enough, the homotopy class of S_t is the one of any small sphere encircling x by Lemma 4.3.3. This leads to a contradiction since such a sphere will have homological intersection equal to 1 or -1 with the curve η. Hence there exists a flow line joining any pair of points.

$\square$

As a consequence, we get the following.

Corollary 4.3.7. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{\mathrm{mix}}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits an equivariant fitting flow, then it is an embedding. In particular it is a globally fitting map. Moreover $\rho : \Gamma \to G$ is a quasi-isometric embedding.*

Proof. Since there are flow lines between any pair of points in $\widetilde{N}$, for every $x \neq y \in M$ one can find $q \in u(x)$ and $q' \in u(y)$ such that $q - q'$ is positive. Since the pencils $u(x)$ and $u(y)$ are mixed, one cannot have $q - q' \in u(x) = u(y)$ so $u(x) \neq u(y)$. Furthermore the pair $(u(x), u(y))$ is a fitting pair for all $x \neq y \in M$, hence u is a globally fitting map by Proposition 4.2.2.

For any Γ-invariant Riemannian metric g on $\mathbb{S}\mathcal{E}$, the map s from the proof of Proposition 4.3.5 is a quasi-isometric embedding. Indeed let δ be the maximum of $d_g(x, x')$ or $d_H(s(x), s(x'))$ for any x, x' in the same fiber of $p : \mathbb{S}\mathcal{E} \to \widetilde{N}$. For every $x, y \in \mathbb{S}\mathcal{E}$ one can find x', y' in the same fibers respectively as x, y and in the same flow line for Φ. Hence for the constants C, D from Proposition 4.3.5:

$$\frac{1}{C} d_g(x, y) - D - (\frac{2}{C} + 2)\delta \leqslant d_H(s(x), s(y)) \leqslant C d_g(x, y) + D + (2C + 2)\delta.$$

Hence ρ is a quasi-isometric embedding. $\square$

4.3.2 Existence of fitting flows

We now prove that the existence of a fitting flow is guaranteed on compacts for fitting immersions. It is not clear if it is the case in general for locally fitting maps.

Proposition 4.3.8. *Let M be a manifold of dimension d.*

(i) A fitting immersion $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ admits a fitting flow.

(ii) An equivariant fitting immersion $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ for a representation $\rho : \Gamma \to \mathrm{SL}(V)$, and a proper action of Γ on M admits a ρ-equivariant fitting flow.

Recall that $p : \mathcal{E} \to \mathrm{Gr}_d(\mathcal{Q})$ is the bundle map and $\pi : \mathcal{E} \to \mathcal{Q}$ is the tautological map.

In order to construct the fitting flow we construct the vector field W on $\mathbb{S}u^*\mathcal{E}$ that generates the flow. This first step of the proof uses crucially the hypothesis that $\dim(M) = d$.

Lemma 4.3.9. *Let $u : M \to \mathrm{Gr}_d(\mathcal{Q})$ be a fitting immersion with $\dim(M) = d$. For every $x \in M$ and any $q \in u(x)$, there exists a lift $\mathrm{w} \in T_{(u(x),q)} u^*\mathcal{E}$ such that $\mathrm{d}\pi(\mathrm{w}) \in T\mathcal{Q}$ is positive.*

We say that an element $(q, \dot{q}) \in T\mathcal{Q}$ is positive if $\dot{q} \in \mathcal{Q}$ is positive. Given any quadric in the pencil $u(x)$, we want to find an infinitesimal direction in which to move this quadric as well as the pencil containing it inside the image of u so that the derivative of the quadrics is positive.

This lemma is an inverse of point (iii) of Proposition 4.2.11: here we fix an element $q \in u(x)$ whereas before we were fixing a $v \in TM$. In order to construct this inverse we will use the fact that a continuous odd map between spheres of equal dimensions is surjective.

Proof. Let us fix $x \in M$. We construct a continuous map :

$$\phi : T_xM\backslash\{0\} \to u^*\mathcal{E}_x\backslash\{0\}.$$

We require that this map satisfies $\lambda\phi(v) = \phi(\lambda v)$ for all $v \in T_xM\backslash\{0\}$ and $\lambda \in \mathbb{R}$. Note that in particular ϕ defines an odd map:

$$\overline{\phi} : \mathbb{S}T_xM \to \mathbb{S}u^*\mathcal{E}_x.$$

We furthermore construct a lift:

$$\psi : T_xM\backslash\{0\} \to Tu^*\mathcal{E}.$$

In other words we assume that $\psi(v) \in T_{\phi(v)}(u^*\mathcal{E})$ for $v \in T_xM\backslash\{0\}$. We require that $\psi(\lambda v) = \lambda(\mathrm{d}m_\lambda)(\psi(v))$ for all $v \in T_xM\backslash\{0\}$ and $\lambda \in \mathbb{R}$ where $m_\lambda : u^*\mathcal{E} \to u^*\mathcal{E}$ is the multiplication by λ. We require $\mathrm{d}\pi(\psi(v)) \in T\mathcal{Q}$ to be positive for all $v \in T_xM\backslash\{0\}$.

Finally we will make this construction so that in addition $\mathrm{d}p\,(\psi(v)) = v$ for $v \in T_xM\backslash\{0\}$, but this property will be only used during the construction.

The following diagram illustrates the situation.

$$\begin{tikzcd}
Tu^*\mathcal{E} \arrow[r, "\mathrm{d}\pi"] \arrow[d, "\mathrm{d}p"'] \arrow[dr] \arrow[loop left, "\mathrm{d}m_\lambda"] & T\mathcal{Q} \arrow[dr] & \\
TM \arrow[u, dashed, bend right, "\psi"'] \arrow[r, dashed, "\phi"] \arrow[dr] & u^*\mathcal{E} \arrow[r, "\pi"] \arrow[d, "p"'] \arrow[loop right, "m_\lambda"] & \mathcal{Q} \\
& M &
\end{tikzcd}$$

If we can construct such continuous maps, the fact that $\overline{\phi}$ is an odd map between two spheres of the same dimension implies that it is homotopically non-trivial and therefore surjective. In particular for all $[q] \in \mathbb{S}\mathcal{E}_{u(x)} \simeq \mathbb{S}u(x)$ there exist a $v \in T_xM$ such that $\phi(v) = q$. The element $\mathrm{w} = \psi(v)$ then satisfies the required conditions, so this finishes the proof.

Now let us construct the maps ϕ and ψ. We first show that for all $v \in T_xM\backslash\{0\}$ we can define $\phi(v)$ and $\psi(v)$, and then we explain how to glue these maps together to get a continuous map.

Since u is a fitting immersion, the point (iii) of Proposition 4.2.11 implies that given $v_0 \in T_xM$ one can find $\phi(v_0) \in u^*\mathcal{E}_x$ an $\psi(v_0) \in T_{\phi(v_0)}u^*\mathcal{E}$ such that $\mathrm{d}\pi(\psi(v_0))$ is positive and $\mathrm{d}p\left(\psi(v_0)\right) = v_0$.

Note that the condition that $\mathrm{d}\pi(\psi(v))$ is positive is an open condition and the condition that $\mathrm{d}p\left(\psi(v)\right) = v$ requires that ψ is a section of an affine subbundle. Hence for every $v \in T_xM\backslash\{0\}$ we can find a small neighborhood $\mathcal{S}$ in a sphere in T_xM containing v_0 on which we can define ϕ and a lift ψ such that $\mathrm{d}\pi(\psi(v))$ is positive and $\mathrm{d}p\left(\psi(v)\right) = v$ for all $v \in \mathcal{S}$. We take $\mathcal{S}$ small enough so that it does not contain any antipodal pair of points.

We define U to be the set of non-zero elements λw for all $\lambda \in \mathbb{R}$ and $w \in \mathcal{S}$, and we extend ϕ and ψ to U in a homogeneous way. We define ϕ on U so that $\phi_i(\lambda v) = \lambda\phi_i(v)$ for all $\lambda \in \mathbb{R}$ non-zero and $v \in \mathcal{S}$. We set $\psi(\lambda v) = \lambda(\mathrm{d}m_\lambda)\psi_i(v)$ for $\lambda \in \mathbb{R}$, where m_λ is the multiplication by λ on $u^*\mathcal{E}_x$. Note that $\mathrm{d}\pi\left(\psi_i(\lambda v)\right) = \lambda^2\mathrm{d}\pi\left(\psi_i(v)\right)$ is positive and $\mathrm{d}p\left(\psi_i(\lambda v)\right) = \lambda v$ for all $\lambda v \in U$. Indeed $\pi \circ m_\lambda = \lambda\pi$ so $\mathrm{d}\pi \circ \mathrm{d}m_\lambda = \lambda\mathrm{d}\pi$ and $p \circ m_\lambda = p$ so $\mathrm{d}p \circ \mathrm{d}m_\lambda = \mathrm{d}p$.

We therefore can construct an open cover $\{U_i\}_{i\in I}$ of T_xM and continuous maps $\phi_i : U_i \to \mathbb{S}u^*\mathcal{E} \subset u^*\mathcal{E}$, and lifts $\psi_i : U_i \to Tu^*\mathcal{E}$ such that $\mathrm{d}\pi(\psi_i(v))$ is positive and $\mathrm{d}p\left(\psi_i(v)\right) = v$ for any $i \in I$ and $v \in U_i$. The U_i can be assumed invariant by scalar multiplication, and ϕ and ψ satisfy the aforementioned homogeneity conditions.

We now glue these maps together. Let $\chi_i : U_i \to [0,1]$ for $i \in I$ be a family of functions that forms a locally finite partition of the unit. We define :

$$\phi : v \in \mathbb{S}T_xM \mapsto \sum_{i\in I}\chi_i(v)\phi_i(v) \in \left(u^*\mathcal{E}\right)_x.$$

Let us check that $\phi(v)$ is always non-zero. This is where we use that $\mathrm{d}p\left(\psi_i(v)\right) = v$ for $i \in I$, and we also use that the fitting immersion u is defining a smooth fibration of the cone $S^2V^{>0}$. Let $\gamma : \mathbb{R} \to M$ be a curve such that $\gamma(0) = x$ and $\gamma'(0) = v$. For all $t \in \mathbb{R}$ the intersection $u^\circ(\gamma(t)) \cap S^2V^{>0}$ is a non-empty convex set, so we can construct a section $s : t \in \mathbb{R} \mapsto S^2V^{>0}$ of the fibration, i.e. such that for all $t \in \mathbb{R}$, $s(t) \in u^\circ(\gamma(t))$.

Let us fix $i \in I$. Let $q : \mathbb{R} \to \mathbb{S}u^*\mathcal{E}$ be such that $q'(0) = \psi(v)$, which implies that $p \circ q(t)$ is equal to $\gamma(t)$ at the first order around $t = 0$ since $\gamma'(0) = \mathrm{d}p\left(\psi(v)\right) = v$. Since $s(t) \in u^\circ(\gamma(t))$, one has $\pi(q(t)) \cdot s(t) = 0$ at the first order around $t = 0$. Taking the first derivative at $t = 0$ of this equation we get:

$$\pi(\phi_i(v)) \cdot s'(0) + \mathrm{d}\pi(\psi_i(v)) \cdot s(0) = 0.$$

Since $\mathrm{d}\pi(\psi_i(v))$ is positive and $s(0)$ is a positive tensor:

$$\mathrm{d}\pi(\psi_i(v)) \cdot s(0) > 0.$$

Hence for all $i \in I$, $\pi(\phi_i(v)) \cdot s'(0) < 0$ and therefore $\pi(\phi(v)) \cdot s'(0) < 0$. In particular $\phi(v)$ does not vanish.

In order to glue the ψ_i we need to be careful since the vectors $\psi_i(v)$ do not belong to the same fiber of the tangent bundle $Tu^*\mathcal{E}$. Let Σ be the following map:

$$\Sigma : (q_i)_{i\in I} \in (u^*\mathcal{E})_x^I \mapsto \sum_{i\in I} q_i \in (u^*\mathcal{E})_x .$$

Given $v \in \mathbb{S}T_x M$ we set:

$$\psi(v) = \mathrm{d}\Sigma\left((\chi_i(v)\psi_i(v))_{i\in I}\right).$$

This combination still satisfies that $\mathrm{d}\pi(\psi)$ is positive, indeed:

$$\mathrm{d}\pi(\psi(v)) = \sum_{i\in I} \chi_i(v)\mathrm{d}\pi(\psi_i(v)).$$

Note that we also get the following:

$$\mathrm{d}p(\psi(v)) = \sum_{i\in I} \chi_i(v)\mathrm{d}p(\psi_i(v)) = \left(\sum_{i\in I} \chi_i(v)\right) v = v.$$

This concludes the construction of ϕ and ψ, and hence this concludes the proof.

$\square$

To prove Proposition 4.3.8 we use the directions w from Lemma 4.3.9 and we glue these vectors into a vector field using a partition of the unit. We construct the vector field $W : \mathbb{S}u^*\mathcal{E} \to T\mathbb{S}u^*\mathcal{E}$ in a similar manner as $\psi : \mathbb{S}TM \to T\mathbb{S}u^*\mathcal{E}$. Morally "$W = \psi \circ \phi^{-1}$", but the map ϕ constructed previously is not a priori bijective.

Proof of Proposition 4.3.8. We construct a continuous vector field W over $u^*\mathcal{E}$, except the zero section, such that for all $x \in M$ and non zero $q \in u(x)$, $W_{\lambda q} = \mathrm{d}(m_\lambda)W_q$ for $\lambda \in \mathbb{R}^{>0}$, where m_λ is the multiplication by λ on $\mathcal{E}$. Such a vector field defines a vector field $\overline{W}$ on $\mathbb{S}u^*\mathcal{E}$. We require moreover that $\mathrm{d}\pi(W)$ is always positive.

Given any non-zero $q_0 \in u^*\mathcal{E}$, Lemma 4.3.9 provides the existence of some $\mathrm{w} \in T_{(u(x),q)}\mathcal{E}$ such that $\mathrm{d}p(\mathrm{w}) \in \mathrm{d}u(T_x M)$ and $\mathrm{d}\pi(\mathrm{w})$ is positive. The first properties implies that w defines a vector in $T_{x,q_0}u^*\mathcal{E}$. For each such q_0 one can find a neighborhood U of q in $u^*\mathcal{E}$ that is invariant by the $\mathbb{R}^{>0}$-action, and on which one can define a map W satisfiying the required properties.

Using a partition of the unit as in Lemma 4.3.9 we construct the desired vector field on $u^*\mathcal{E}$. This vector field generates a flow that is a fitting flow.

Finally note that the collection U_i as well as the partition of the unity can be chosen to be ρ-equivariant, so that the vector field W is also ρ-equivariant, and hence also the fitting flow Φ.

$\square$

4.4 The Anosov property and fibrations.

In this section we show that the existence of an equivariant map of pencils that admits a fitting flow implies that the representation is Anosov. Moreover we describe the domain that is fibered in $\mathbb{RP}^{2n-1}$. Finally we apply this to show that some quasi-Fuchsian representations do not admit equivariant maps that admit a fitting flow.

4.4.1 The Anosov property

In order to show that a uniform contraction is taking place along the flow lines of the fitting flow, we define a way to measure the distance between two quadric hypersurfaces nested into one another. The characterization of Anosov representations that we use is similar to the characterization in terms of inclusion of multicones from [BPS19].

Let $\mathbb{S}\mathcal{Q}^{\mathrm{mix}}$ be the set of quadrics that are not semi-positive or semi-negative up to a positive scalar.

Definition 4.4.1. Let $q_1, q_2 \in \mathbb{S}\mathcal{Q}^{\mathrm{mix}}$ be such that for some choice of representatives, the difference $q_2 - q_1$ is positive. We define the *cross ratio distance* $\mathrm{cr}([q_2],[q_1])$ between $[q_2]$ and $[q_1]$ as the minimum of $[\ell_1, \ell_2, \ell_2', \ell_1'] \in [1, \infty]$ for every quadruple $\ell_1, \ell_2, \ell_2', \ell_1' \in \mathbb{P}(V)$ of points in this order on a projective line L such that the zeros of q_1 on L with multiplicity are ℓ_1 and ℓ_1' and the zeros of q_2 on L are ℓ_2 and ℓ_2'.

If we fix an affine chart of L so that $\ell_1, \ell_2, \ell_2', \ell_1'$ correspond to the real numbers $x_2' < x_1' \leqslant x_1 < x_2$, the cross ratio $[\ell_1, \ell_2, \ell_2', \ell_1']$ is defined as:

$$[\ell_1, \ell_2, \ell_2', \ell_1'] = \frac{x_2 - x_1'}{x_2 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}.$$

Note that this quantity is greater than 1, and is infinite if and only if $\ell_1 = \ell_1'$ or $\ell_2 = \ell_2'$. However if q_1 and q_2 are not mixed, there exist a projective line L between a point on which q_2 is positive and a point on which q_1 is negative. Therefore on this line $\ell_1 \neq \ell_1'$ and $\ell_2 \neq \ell_2'$, so $\mathrm{cr}([q_2],[q_1])$ is finite.

The logarithm of this quantity satisfies a triangular inequality.

Proposition 4.4.2. *Let $[q_1], [q_2], [q_3] \in \mathbb{S}\mathcal{Q}^{\mathrm{mix}}$ be such that $q_3 - q_2$ and $q_2 - q_1$ are positive. Then :*

$$cr([q_3],[q_1]) \geqslant cr([q_3],[q_2])\, cr([q_2],[q_1]).$$

We illustrate this proposition and its proof in Figure 4.4.1. This figure illustrates 3 quadrics of signature $(1, 2)$ in $\mathbb{RP}^2$. In these pictures the quadrics are positive on the inside of the ellipse they define.

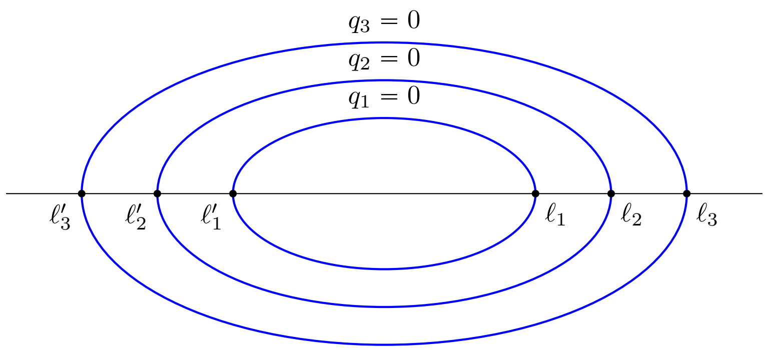


Figure 4.3: Illustration of Proposition 4.4.2

Proof. Pick any projective line that crosses the zeroes of the quadric q_1 and q_3. Let $(\ell_1, \ell_2, \ell_3, \ell_3', \ell_2', \ell_1')$ be the intersections of L with the zeroes of q_1, q_2 and q_3 respectively, counted with multiplicity and cyclically ordered. We fix an affine chart for L such that this tuple corresponds to the tuple $x_3' < x_2' < x_1' \leqslant x_1 < x_2 < x_3$ of real numbers. This yields the following :

$$[\ell_1, \ell_3, \ell_3', \ell_1'] = \frac{x_3 - x_1'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_1'}.$$

$$[\ell_1, \ell_3, \ell_3', \ell_1'] = \left(\frac{x_3 - x_1'}{x_3 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}\right) \times \left(\frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_2'}\right).$$

Moreover one has:

$$\mathrm{cr}([q_2], [q_1]) \leqslant [\ell_1, \ell_2, \ell_2', \ell_1'] = \frac{x_2 - x_1'}{x_2 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'} \leqslant \frac{x_3 - x_1'}{x_3 - x_2'} \times \frac{x_1 - x_2'}{x_1 - x_1'}.$$

$$\mathrm{cr}([q_3], [q_2]) \leqslant [\ell_2, \ell_3, \ell_3', \ell_2'] = \frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_3 - x_2'}{x_3 - x_2'} \leqslant \frac{x_3 - x_2'}{x_3 - x_3'} \times \frac{x_1 - x_3'}{x_1 - x_2'}.$$

Hence one has $[\ell_1, \ell_3, \ell_3', \ell_1'] \geqslant \mathrm{cr}([q_2], [q_1])\mathrm{cr}([q_3], [q_2])$ for every such projective line L.

Therefore $\mathrm{cr}([q_3], [q_1]) \geqslant \mathrm{cr}([q_3], [q_2])\mathrm{cr}([q_2], [q_1])$. $\square$

A sequence of quadrics such that the cross ratio distance between the first and last quadric goes to $+\infty$ satisfies that the intersection of all half-spaces determined by the quadrics is a projective subspace.

Proposition 4.4.3. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of quadrics such that $q_{n+1} - q_n$ is positive for all $n \in \mathbb{N}$ and cr$([q_n], [q_0])$ goes to $+\infty$. Then $\bigcap_{n\in\mathbb{N}}\{q_n \leqslant 0\} \subset \mathbb{P}(V)$ is a projective subspace.*

This proposition is proven in [BG09]. We write a version of the argument here for the sake of completeness. We illustrate this proposition with quadrics of signature $(1, 2)$ in $\mathbb{RP}^2$ in Figure 4.4.1.

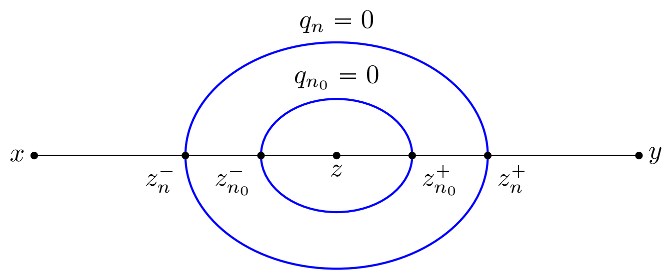


Figure 4.4: Illustration of Proposition 4.4.3

Proof. The intersection $I = \bigcap_{n\in\mathbb{N}}\{q_n \leqslant 0\}$ is a compact non-empty subset. Let $x \neq y \in I$ and let L be the projective line from x to y. Suppose that there exist $z \in L$ such that $z \notin I$. Without any loss of generality one can assume that the open interval $S \subset L$ bounded by x, y and containing z does not intersect I. Indeed one can otherwise replace x, y by the points on $L \cap I$ closest to z on both sides.

Since $z \notin I$ there exist $n_0 \in \mathbb{N}$ such that q_{n_0} is positive on z. For $n \geqslant n_0$, let z_1, z_2 be the two intersections of L with the zeroes of q_n so that the points (x, z_n^-, z, z_n^+, y) are cyclically ordered. The sequences (z_n^-) and (z_n^+) are monotonic in S and must converge to x and y since $S \cap I = \varnothing$. The value of $\mathrm{cr}([q_{n_0}], [q_n])$ is bounded from above by the cross ratio $[z_{n_0}^+, z_n^+, z_n^-, z_{n_0}^-]$, which is turns converges to the cross ratio $[z_{n_0}^+, x, y, z_{n_0}^-] < \infty$ when n goes to $+\infty$. This contradicts the fact that $\mathrm{cr}([q_n], [q_0])$ goes to $+\infty$. Hence for every pair of points in I, the associated projective line is contained in I. In particular I is a vector subspace. $\square$

We now apply these results to prove that representations that admit an equivariant fitting immersion, and more generally a fitting map with a fitting flow, are Anosov.

Theorem 4.4.4. *Let $u : \widetilde{N} \to \mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ be a continuous ρ-equivariant map that admits an equivariant fitting flow. The representation ρ is $\{n\}$-Anosov, and for any $q \in \mathbb{S}u^*\mathcal{E}$, the limit map at the limit $\zeta \in \partial\Gamma$ of the the flow line $(\Phi_t(q))$ can be characterized as:*

$$\mathbb{P}\left(\xi_\rho^n(\zeta)\right) = \bigcap_{t\geqslant 0}\{\pi(\Phi_t(q)) \leqslant 0\}.$$

Note that the flow lines of an equivariant fitting flow are quasi-geodesics by Proposition 4.3.5, hence the limit point ζ is well-defined.

Proof. The flat V-bundle over $\mathbb{S}u^*\mathcal{E}$ associated with ρ admits a continuous split-

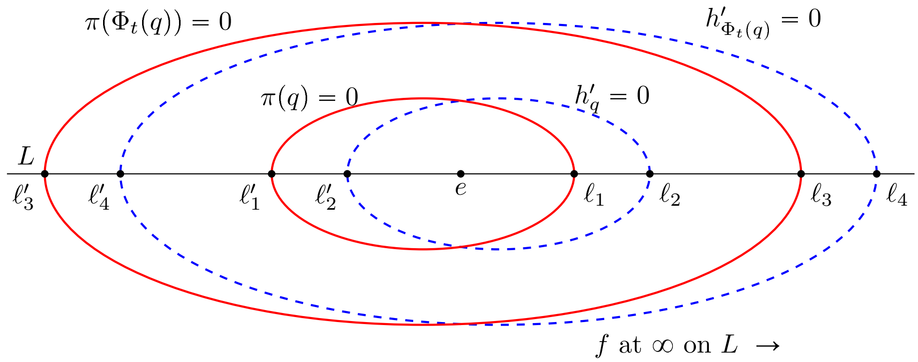


Figure 4.5: Illustration of the proof of Theorem 4.4.4.

ting $E \oplus F$ where for $q \in \mathbb{S}u^*\mathcal{E}$:

$$\mathbb{P}(E_q) = \bigcap_{t \geqslant 0} \{\pi(\Phi_{-t}(q)) \geqslant 0\},$$

$$\mathbb{P}(F_q) = \bigcap_{t \geqslant 0} \{\pi(\Phi_t(q)) \leqslant 0\}.$$

This defines transverse vector subspaces by Proposition 4.4.3 since $\mathrm{cr}(\Phi_t(q), q)$ and $\mathrm{cr}(-\Phi_{-t}(q), -q)$ go to $+\infty$ when t goes to $+\infty$. Moreover the quadrics in the pencils in the image of u are of signature (n, n), one must have $\dim(E_q) = \dim(F_q) = n$ so this splitting is well-defined. This splitting is preserved by Φ.

We now construct a metric h on this flat V-bundle over $\mathbb{S}u^*\mathcal{E}$. Given $q \in \mathbb{S}u^*\mathcal{E}$ we define the symmetric bilinear form h_q on $V = E_q \oplus F_q$ so that this sum is orthogonal and h_q is equal to $\pi(q) \in \mathcal{Q}$ on E_q and $-\pi(q) \in \mathcal{Q}$ on F_q. Note that by definition of E_q and F_q, h_q is positive.

We also introduce an auxiliary symmetric bilinear form h' of signature (n, n) on this flat V-bundle over $\mathbb{S}u^*\mathcal{E}$ so that the sum $E_q \oplus F_q$ is orthogonal and h'_q is equal to $\pi(q) \in \mathcal{Q}$ on E_q and on F_q.

Our first step is to compare the quadric h'_q with $\pi(q)$. Let L be a projective line intersecting $\mathbb{P}(E_q)$ at some e and $\mathbb{P}(F_q)$ at some f. In Figure 4.4.1, we illustrate some $\mathbb{RP}^2 \subset \mathbb{RP}^{2n-1}$ containing the projective line L. Let ℓ_1, ℓ'_1 be the zeroes of $\pi(q)$ on L and ℓ_2, ℓ'_2 be the zeroes of h'_q on L, so that ℓ_1, ℓ_2 lie in the same connected component of $L \backslash \{e, f\}$. Since N is compact, there exist a maximum $\delta < \infty$ for all $q \in \mathbb{S}u^*\mathcal{E}$ and all such projective line L of the following quantity:

$$|\log([e, \ell_1, \ell_2, f])|.$$

Now we turn our attention to the contraction properties of Φ. Let $t > 0$ be a real number and let $q \in \mathbb{S}u^*\mathcal{E}$. Let $v \in E_q$ and $w \in F_q$. We are interested in the following ratio :

$$R = \frac{h_{\Phi_t(q)}(v)h_q(w)}{h_{\Phi_t(q)}(w)h_q(v)}.$$

Let e, f be the lines generated by v, w and L be the projective line joining them. Let ℓ_1, ℓ_1' be the zeroes of $\pi(q)$ on L, ℓ_2, ℓ_2' the zeroes of h_q' on L, ℓ_3, ℓ_3' be the zeroes of $\pi(\Phi_t(q))$ on L and finally ℓ_4, ℓ_4' the zeroes of $h_{\Phi_t(q)}$ on L. We assume that $\ell_1, \ell_2, \ell_3, \ell_4$ all lie on the same component of $L \backslash \{e, f\}$.

The cross ratio $[e, \ell_2, \ell_4, f]$ is equal to $R^{\frac{1}{2}}$. Indeed ℓ_2 is generated by $h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w$ and ℓ_4 is generated by $h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w$, up to changing ℓ_i by ℓ_i' for $1 \leqslant i \leqslant 4$. Hence:

$$[e, \ell_2, \ell_4, f] = \frac{\left(h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w\right) \wedge v}{\left(h_{\Phi_t(q)}^{\frac{1}{2}}(w)v + h_{\Phi_t(q)}^{\frac{1}{2}}(v)w\right) \wedge w} \times \frac{\left(h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w\right) \wedge w}{\left(h_q^{\frac{1}{2}}(w)v + h_q^{\frac{1}{2}}(v)w\right) \wedge v} = R^{\frac{1}{2}}.$$

However due to our comparison of $\pi(q)$ and h_q' one has:

$$e^{-\delta} \leqslant [e, \ell_1, \ell_2, f], [e, \ell_3, \ell_4, f] \leqslant e^{\delta}.$$

Therefore $[e, \ell_2, \ell_4, f]/[e, \ell_1, \ell_3, f] \geqslant e^{-2\delta}$. Hence $R^{\frac{1}{2}} > e^{-2\delta}[e, \ell_1, \ell_3, f]$. This last cross ratio is larger than:

$$[\ell_1', \ell_1, \ell_3, \ell_3'] \geqslant \operatorname{cr}\left(\pi(\Phi_t(q)), \pi(q), \right).$$

Since Φ is a fitting flow and since N is compact, there exist $\alpha > 0$ such that $\operatorname{cr}\left(\pi(\Phi_1(q)), \pi(q), \right) \geqslant e^{\alpha}$ for all $q \in \mathbb{S}u^*\mathcal{E}$. Hence by the triangular inequality from Proposition 4.4.2, for all $t > 0$:

$$\operatorname{cr}\left(\pi(\Phi_t(q)), \pi(q), \right) \geqslant e^{\alpha(t-1)}.$$

Hence we get the following estimate:

$$\frac{h_{\Phi_t(q)}(v)h_q(w)}{h_{\Phi_t(q)}(w)h_q(v)} \geqslant e^{2\alpha t - 4\delta - 2\alpha}.$$

This implies that the splitting $V = E_q \oplus F_q$ is $\{n\}$-contracting in the sense of [BPS19] with respect to the flow Φ for the metric h. Moreover Γ acts cocompactly on $\mathbb{S}u^*\mathcal{E}$ and every geodesic in Γ is at uniform distance from a flow line of Φ. The domination of this splitting implies an exponential gap for the singular values [BPS19, Theorem 2.2], which implies that ρ is $\{n\}$-Anosov. The vector subspace $\xi_\rho^n(\zeta)$ is the contracted subspace F_q.

□

4.4.2 Fibered domain of discontinuity

Such an equivariant map into the space of pencils that admits a fitting flow induces a fibration of the Guichard-Wienhard domain of discontinuity.

Proposition 4.4.5. *Let N be a compact manifold of dimension d with fundamental group Γ. Let $\rho : \Gamma \to \mathrm{SL}(V)$ and $u : \widetilde{N} \to \mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ be a ρ-equivariant continuous map that admits an equivariant fitting flow Φ on $\mathbb{S}u^*\mathcal{E}$. The union of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for $x \in \widetilde{N}$ cover all of $\mathbb{P}(S^2V^{>0})$, and the closure of this union intersects the space of rank one points $\mathbb{P}(V) \simeq S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V^{\geqslant 0})$ exactly at the domain of discontinuity for $\{n\}$-Anosov representations considered by Guichard-Wienhard [GW12]:*

$$\Omega = \mathbb{P}(V) \setminus \bigcup_{\zeta \in \partial\Gamma} \mathbb{P}(\xi_\rho^n(\zeta)). \tag{4.2}$$

The intersection of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ with the set of rank one points for $x \in \widetilde{N}$ defines a fibration over $\widetilde{N}$ of Ω.

In this argument we will use the Hilbert distance on $\mathbb{P}(S^2V^{>0})$ already introduced for the proof of Proposition 4.3.5. For a subset $A \subset \mathbb{P}(V)$ we write $S^2A \subset \mathbb{P}(S^2V)$ the corresponding set of rank one lines.

Proving first that $\mathbb{P}(S^2V^{>0})$ is fully covered first helps us proving that the Guichard -Wienhard domain is also fully covered.

Proof. Let us fist prove that all of $\mathbb{P}(S^2V^{>0})$ is covered by the union of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for $x \in \widetilde{N}$. We fix a Riemanian metric on N that defines a Riemannian metric g on $\widetilde{N}$, with associated distance d_g. Since u is globally fitting, see Remark 4.3.6, for all $x \in \widetilde{N}$ there exist a neighborhood U of $\mathbb{P}(u^\circ(x) \cap S^2V^{\geqslant 0})$ in $\mathbb{P}(S^2V^{\geqslant 0})$ that is covered by the manifolds $\mathbb{P}(u^\circ(y) \cap S^2V^{\geqslant 0})$ for y in the ball for d_g of radius 1 centered at 0. This neighborhood contains an ϵ-neighborhood of $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ for the Hilbert metric on $\mathbb{P}(S^2V^{>0})$ for some $\epsilon > 0$. Since N is compact, this $\epsilon > 0$ can be chosen independently of x.

Now let us fix some $x_0 \in \widetilde{N}$ and some $[p_0] \in \mathbb{P}(u^\circ(x_0) \cap S^2V^{>0})$. Given any $[p'] \in \mathbb{P}(S^2V^{>0})$ one can find a finite sequence $p_0, p_1, \cdots, p_k = p'$ in $S^2V^{>0}$ so that the Hilbert distance between $[p_i]$ and $[p_{i+1}]$ is less than ϵ for all $0 \leqslant i < k$. By induction, and since the Riemannian metric d_g is complete, one can construct for all $1 \leqslant i \leqslant k$ a point $x_i \in \widetilde{N}$ such that $[p_i] \in \mathbb{P}(u^\circ(x_i) \cap S^2V^{>0})$ and $d_H(x_{i-1}, x_i) \leqslant 1$. Therefore the manifolds $\mathbb{P}(u^\circ(x) \cap S^2V^{>0})$ cover all of $\mathbb{P}(S^2V^{>0})$.

Now let us consider the fibered domain in projective space. Consider a rank one line $[p] \in S^2\mathbb{P}(\xi_\rho^n(x))$ for some $\zeta \in \partial\Gamma$. Suppose that p belongs to $u^\circ(\zeta)$ for some $x \in \widetilde{N}$. There exist a flow line $(\Phi_t([q]))_{t \geqslant 0}$ starting at $p([q]) = x$ and converging to $\zeta \in \partial\Gamma \simeq \partial\widetilde{N}$ by Proposition 4.3.5. Theorem 4.4.4 implies that $\pi(q) \in \mathcal{Q}$ must be negative on $\xi_\rho^2(\zeta)$ and hence p cannot belong to $u^\circ(x)$.

Conversely fix a point $x \in \widetilde{N}$ and take any rank one point $[p] \in S^2\Omega$ in the Guichard-Wienhard domain of discontinuity. There exist a sequence (x_n) such that $[p]$ belongs to the limit of $\mathbb{P}(u^\circ(x_n) \cap S^2V^{\geqslant 0})$, since these manifolds cover $\mathbb{P}(S^2V^{>0})$. We consider some $[q_n] \in \mathbb{S}\mathcal{E}_x$ such that $\Phi_t([q_n]) \in \mathbb{S}\mathcal{E}_{x_n}$ for some $t_n > 0$, which exist by Proposition 4.3.5. If t_n diverges when n varies, then the set $\mathbb{P}(u^\circ(x_n) \cap S^2\mathbb{P}(V))$ becomes arbitrarily close to $S^2\mathbb{P}(\xi^2_\rho(\zeta_n))$ where ζ_n is the limit when t goes to $+\infty$ of $\Phi_t([q_n])$. This would contradict the fact that $[p] \in S^2\Omega$, as in this case $[p] \in \mathbb{P}(\xi^n_\rho(\zeta))$ where ζ is a limit point of (ζ_n). Hence the sequence (x_n) is bounded and therefore converges up to subsequence to some $x_\infty \in \widetilde{N}$, and $[p] \in \mathbb{P}(u^\circ(x_\infty))$.

$\square$

4.4.3 A quasi-Fuchsian representation with no fitting immersions.

Having a representation that is Anosov is not sufficient to ensure that there exist an equivariant fitting immersion. We show that there are quasi-Fuchsian representations that admit no such immersions of Hermitian pencils of quadrics. We use here the notations from Section 4.2.4.

Theorem 4.4.6. *There exist a quasi-Fuchsian representation $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ for some genus g large enough that admits no ρ-equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H}) = \mathcal{G}$.*

Moreover it also admits no continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H}) = \mathcal{G}$ that admits a ρ-equivariant fitting flow.

Here $V = \mathbb{C}^2$ and if $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C}) \subset \mathrm{Sp}(4,\mathbb{R})$, the Guichard-Wienhard domain of discontinuity corresponds to the pullback in $\mathbb{RP}^3$ of the complement in $\mathbb{CP}^1$ of the limit set of ρ. Since Γ_g is a surface group, this domain in $\mathbb{CP}^1$ is the union of two topological disks, and hence for each $x \in \widetilde{S_g}$ the geodesic corresponding to $u(x)$ has one endpoint in each of these discs.

An other ingredient of the proof of Theorem 4.4.6 is the following.

Proposition 4.4.7 ([HW15, Corollary 3.5])**.** *Given any $\mathcal{C}^1$ embedded circle γ in $\mathbb{CP}^1$, and any $\epsilon > 0$, there exist a quasi-Fuchsian representation $\rho : \Gamma_g \to$ $\mathrm{SL}(2,\mathbb{C})$ for some genus g large enough whose limit set has Haussdorf distance at most ϵ with γ.*

Proof of Theorem 4.4.6. We consider the Jordan curve γ from Figure 4.6. Let x, y, z be as in the figure. We consider a quasi-Fuchsian representation of Γ_g for a genus large enough such that its limit set Λ contains x, z and is close enough to γ using Proposition 4.4.7. More precisely let s_x and s_z be the open arcs of the circle of $\mathbb{CP}^1$ passing through x, y, z respectively between x, y and y, z and let I be the interior of the Jordan curve Λ. We require that the union U_x of all the connected component of $I \backslash s_z$ whose closure contain x is disjoint from the union U_z of all the connected component of $I \backslash s_x$ whose closure contain z. These two disjoint sets are illustrated for the curve γ as the two gray regions.

Let $\rho : \Gamma_g \to \mathrm{SL}(2,\mathbb{C})$ then be such a quasi-Fuchsian representation. Suppose that it admits an equivariant continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{H})$ with an equivariant fitting flow Φ on $\mathbb{S}u^*\mathcal{E}$. By Proposition 4.3.5 there exist a flow line $(\Phi_t(q))_{t\in\mathbb{R}}$ such that its projection $\gamma : \mathbb{R} \to \widetilde{S_g}$ is a quasi-geodesic between the points of $\partial\Gamma$ corresponding to x and z in the limit set.

For all $t \in \mathbb{R}$, let $x_t \in \mathbb{CP}^1$ be the endpoint of the geodesic corresponding to $u(\gamma(t))$ that belongs to the interior of the Jordan curve Λ. When t goes to $+\infty$, x_t converges to z and it converges to x for t going to $-\infty$. Moreover x_t always belong to the circle determined by $\Phi_t(q)$.

There exist a $t_0 \in \mathbb{R}$ such that y belongs to the circle $\Phi_{t_0}(q)$. Note that this great circle splits $\mathbb{CP}^1$ in two parts, one containing x and s_x and one containing z and s_z. In particular $(x_t)_{t \geqslant t_0}$ must lie in U_z and $(x_t)_{t \leqslant t_0}$ must lie in U_x, leading to a contradiction. Hence no such map u can exist.

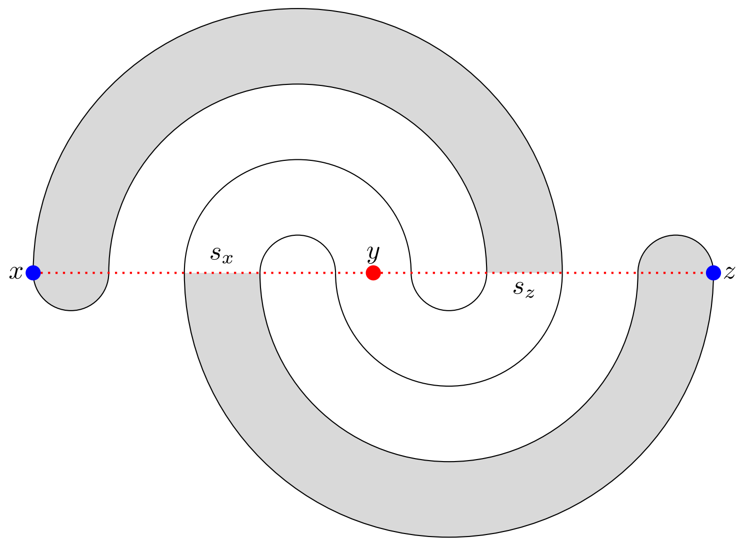


Figure 4.6: A Jordan curve in $\mathbb{CP}^1$

□

4.5 Fitting maps and maximal representations.

Let us consider representations $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbb{R})$. We prove our main result, which is the characterization of maximal representations in terms of maps of pencils. The first part introduces ω-regular pencils, as well as a connected component of the space of ω-regular pencils. We then state the characterization, and then present the construction of a map of pencils with a fitting flow for any maximal representation. This construction relies to a map from the space of pairs of Lagragians to the space of quadrics. We briefly discuss how one can also use this map to decompose the projective structure into polygons of quadric hypersurfaces in Section 4.8.

Throughout this section we set $d = 2$, and consider the case when $N = S_g$ is a surface.

4.5.1 Definition of maximal pencils.

We say that a quadric q in $\left(\mathbb{R}^{2n},\omega\right)$ is *ω-regular* if it is positive on some Lagrangian ℓ_1 and negative on some Lagrangian ℓ_2. We call $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ the space of ω-regular pencils. These pencils have in particular the property that the corresponding subsets of $\mathbb{P}(S^2V)$ intersect transversely the symmetric space of $\mathrm{Sp}(2n,\mathbb{R})$, see Lemma 4.7.4.

Remark 4.5.1. If a locally fitting map $u:\widetilde{S_g}\to \mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$ admits a fitting flow which is equivariant with respect to a representation $\rho:\Gamma\to\mathrm{Sp}(2n,\mathbb{R})$, the image of u must lie in $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ as ρ is $\{n\}$-Anosov and its limit map takes values in the space of Lagrangians.

There are non-maximal representations admitting equivariant fitting immersions, for instance almost-Fuchsian representations in $\mathrm{SL}(2,\mathbb{C})\subset\mathrm{Sp}(4,\mathbb{R})$. In order to obtain the maximality property, we need to restrict ourselves to the correct union of connected component of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$.

Let $P\in\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ be a pencil, and fix an orientation for P. Recall that $\mathcal{L}_n$ in the space of Lagrangians in $\mathbb{R}^{2n}$. We construct a "boundary map" for an ω-regular pencil of quadrics, defined up to homotopy. Before defining this map, note the following:

Lemma 4.5.2. *Let $q\in\mathcal{Q}$ be an ω-regular element. The set of Lagrangians ℓ such that q is positive on ℓ is an open ball.*

Proof. There exist some $\ell_+\in\mathcal{L}_n$ on which q is positive. Moreover there exist some $\ell_-\in\mathcal{L}_n$ on which q is negative.

Every Lagrangian ℓ on which q is positive must be transverse to ℓ_-, hence it can be written as the graph $\{x+u(x)\mid x\in\ell_+\}$ of some linear map $u:\ell_+\to\ell_-$, and one has for all $v\in\ell^+$:

$$q(v,v)+q(u(v),u(v))+2q(v,u(v))=q(v+u(v),v+u(v))>0.$$

Since $q(v,v)>0$ and $q(u(v),u(v))<0$, for all $0<\lambda<1$:

$$q(v+\lambda u(v),v+\lambda u(v))=q(v,v)+\lambda q(u(v),u(v))+2\lambda q(v,u(v))>0.$$

We can identify the elements of $\mathcal{L}_n$ transverse to ℓ_- as the vector subspace of the space of maps $u:\ell_+\to\ell_-$. We just proved that in this chart the set of elements on which q is positive is open and star-shaped, hence it is a open ball. □

We now define the "boundary map" of the pencil.

Proposition 4.5.3. *Let $P\in\mathrm{Gr}_2^{\omega}(\mathcal{Q})$. There exist a continuous map $\xi_P:\mathbb{S}P\to\mathcal{L}_n$ such that for all $[q]\in\mathbb{S}P$, $q>0$ on $\xi_P([q])$. Moreover any two such maps are homotopic, so the free homotopy type $[\xi_P]$ is well defined.*

Proof. A map ξ_P is exactly a section of the bundle $\{([q],\ell) \mid q_{|\ell} > 0\} \to \mathbb{S}P$, which is a fiber bundle whose fibers are open balls. Such sections always exist and are unique up to homotopy.

$\square$

We say that a pencil is *maximal* for some orientation if $[\xi_P] = n[\tau]$, where $[\tau]$ is the generator of $\pi_1(\mathcal{L}_n)$ introduced in Section 4.1.2. We denote by $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ the space of pencils that are maximal for some orientation. This is a union of connected components of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ as the homotopy type $[\xi_P]$ is locally invariant for $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$.

Remark 4.5.4. The previous discussion allow us to distinguish several connected components of the open subspace $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$ by looking at the homotopy type of the boundary map ξ_P.

As a recal we have the following inclusions:

$$\mathrm{Gr}_2^{\max}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\omega}(\mathcal{Q}) \subset \mathrm{Gr}_2^{(n,n)}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\mathrm{mix}}(\mathcal{Q}) \subset \mathrm{Gr}_2(\mathcal{Q}).$$

All these inclusions are open, and the inclusion $Gr_2^{\max}(\mathcal{Q}) \subset \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ is a union of connected components.

4.5.2 Statement of the characterization.

We obtain the following characterization of maximal representations in terms of the existence of locally fitting maps that admit a fitting flow.

Theorem 4.5.5. *A representation $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R})$ admits a ρ-equivariant locally fitting map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits a ρ-invariant fitting flow if and only if it is maximal for some orientation of S_g.*

In this case the orientation of S_g for which ρ is maximal is induced by the orientation of the maximal pencils $u(x)$ for $x \in \widetilde{S_g}$ and Lemma 4.3.3.

In particular if a representation ρ admits an equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ then it is maximal because of Proposition 4.3.8. This theorem leaves the following question open:

Question 4.5.6. Given a maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R})$, is there always an equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$?

We show in Section 4.6 that this is true for $\mathrm{Sp}(4,\mathbb{R})$, and in this case there exist a fitting immersion whose image lies in a single special $\mathrm{Sp}(4,\mathbb{R})$-orbit of $\mathrm{Gr}_2^{\max}(\mathcal{Q})$.

The following Lemma shows one direction of Theorem 4.5.5, the other direction is proven by Lemma 4.5.13.

Lemma 4.5.7. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R})$ be a representation that admits an equivariant continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits an equivariant fitting flow. Then ρ is maximal for some orientation of S_g.*

Proof. We first apply Theorem 4.4.4 which shows that ρ is $\{n\}$-Anosov. Let $x \in \widetilde{S_g}$, we prove that the homotopy type of the boundary curve ξ^n_ρ is the same as the homotopy type $[\xi_{u(x)}]$ from Proposition 4.5.3. We consider the map ζ_∞ that associates to $[q] \in \mathbb{S}u^*\mathcal{E}_x$ the limit of $\Phi_t([q])$ in $\partial\Gamma$. This map is homotopic in $\widetilde{S_g} \cup \partial\Gamma\backslash\{x\}$ to the map ζ_1 that associates the projection $p(\Phi_1)$. Hence by Lemma 4.3.3, ζ_1 defines a diffeomorphism of degree 1 between the circle $\mathbb{S}u(x)$ with its maximal orientation and the boundary $\partial\Gamma$ for the induced orientation.

The map $\xi^n_\rho \circ \zeta_\infty$ has the homotopy type $[\xi_{u(x)}]$ associated to $u(x)$: it defines a boundary map as in Proposition 4.5.3. The fact that ρ is maximal is then a consequence of the characterization of maximal representations from Theorem 4.1.5. Indeed $u(x) \in \mathrm{Gr}_2^{\max}(\mathcal{Q})$ implies that $[\xi_{u(x)}] = n[\tau]$, and we already know that the degree of ζ_∞ is equal to the degree of ζ_1 which is equal to 1.

□

4.5.3 Construction of a fitting flow

In this section we study special quadrics in $\mathbb{R}^{2n}$ associated to pairs of transverse Lagrangians. These objects will allow us to construct fitting continuous embeddings of pencils.

Definition 4.5.8. Let ℓ_1, ℓ_2 be two transverse Lagrangians in $\mathbb{R}^{2n}$. We define q_{ℓ_1,ℓ_2} to be the symmetric bilinear form on $\mathbb{R}^{2n}$ such that if π_1, π_2 are the projections on ℓ_1, ℓ_2 associated to the direct sum $\ell_1 \oplus \ell_2 = V$:

$$q_{\ell_1,\ell_2}(v, v) = \omega(\pi_1(v), \pi_2(v)).$$

Note that $q_{\ell_2,\ell_1} = -q_{\ell_1,\ell_2}$.

Remark 4.5.9. In particular q_{ℓ_1,ℓ_2} is characterized by the fact that ℓ_1 and ℓ_2 are isotropic and for all $v \in \ell_1, w \in \ell_2$:

$$q_{\ell_1,\ell_2}(v, w) = \omega(v, w).$$

Maximal triples of Lagrangians can be characterized as follows:

Lemma 4.5.10. *A triple of Lagrangians (ℓ_1, ℓ_2, ℓ_3) is maximal if and only if q_{ℓ_1,ℓ_3} is positive on ℓ_2.*

Proof. Let us write ℓ_1, ℓ_2, ℓ_3 as in Section 4.1.1 for some symplectic basis :

$$\begin{aligned} \ell_1 &= \langle x_1, x_2, \cdots x_n\rangle, \\ \ell_2 &= \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, \cdots x_n + \epsilon_n y_n\rangle, \\ \ell_3 &= \langle y_1, y_2, \cdots y_n\rangle. \end{aligned}$$

The form q_{ℓ_1,ℓ_3} can be written in this basis as:

$$q_{\ell_1,\ell_3} = \frac{1}{2}\sum_{i=1}^n x_i^* \otimes y_i^* + y_i^* \otimes x_i^*.$$

This form is positive on ℓ_2 if and only if all of the ϵ_i are positive, and hence if the triple is maximal. □

These quadrics have also the following remarkable properties for maximal quadruples of Lagrangians.

Lemma 4.5.11. *Let $(\ell_1, \ell_2, \ell_3, \ell_4)$ be a maximal quadruple of Lagrangians, i.e. such that each cyclic oriented subtriple is maximal. The bilinear form $q_{\ell_4,\ell_3} - q_{\ell_1,\ell_2}$ is positive. In particular the zero set of these quadrics define two disjoint quadric hypersurfaces in $\mathbb{P}(\mathbb{R}^{2n})$.*

Note that a triple of Lagrangians (ℓ_1, ℓ_2, ℓ_3) is maximal if and only if corresponding linear map $u \in \mathrm{Hom}(\ell_1, \ell_3)$ whose graph is equal to ℓ_2 is such that $\omega(\cdot, u(\cdot))$ is positive on ℓ_1.

Proof. Let us prove the first part of the statement. Since (ℓ_4, ℓ_1, ℓ_3) is a maximal triple of Lagrangians, ℓ_1 can be written as the graph of some linear map $u_1 : \ell_4 \to \ell_3$ such that $\omega(\cdot, u_1(\cdot))$ is a positive bilinear form on ℓ_4. Similarly since (ℓ_4, ℓ_2, ℓ_3) is a maximal triple of Lagrangians, ℓ_2 can be written as the graph of some linear map $u_2 : \ell_4 \to \ell_3$ such that $\omega(\cdot, u_2(\cdot))$ is a positive bilinear form on ℓ_4.

Let $v \in \mathbb{R}^{2n}$, it can be decomposed uniquely as $v = v_1 + v_2$ with $v_1 \in \ell_1$ and $v_2 \in \ell_2$. Moreover there exist some unique $x, y \in \ell_4$ such that $v_1 = x + u_1(x)$ and $v_2 = y + u_2(y)$. The vector v decomposes therefore as $v = x + y + u_1(x) + u_2(y)$. One computes that :

$$\begin{aligned} q_{\ell_4,\ell_3}(v,v) - q_{\ell_1,\ell_2}(v,v) &= \omega\left(x + y, u_1(x) + u_2(y)\right) - \omega\left(x + u_1(x), y + u_2(y)\right), \\ &= \omega\left(x, u_1(x)\right) + \omega\left(y, u_2(y)\right) + 2\omega\left(y, u_1(x)\right). \end{aligned}$$

Finally the fact that (ℓ_1, ℓ_2, ℓ_3) forms a maximal triple implies that the bilinear form $\omega\left(\cdot, (u_2 - u_1)(\cdot)\right)$ is positive. Therefore the previous expression, for $y \neq 0$, is strictly greater than $\omega\left(x, u_1(x)\right) + \omega\left(y, u_1(y)\right) + 2\omega\left(y, u_1(x)\right)$, which is non-negative since $\omega\left(\cdot, u_1(\cdot)\right)$ is positive. In the case when $y = 0$, this last inequality is strict for $x \neq 0$. Otherwise the previous inequality $\omega\left(y, (u_2 - u_1)(y)\right) > 0$ is strict. Therefore for $v \neq 0$, $q_{\ell_4,\ell_3}(v,v) - q_{\ell_1,\ell_2}(v,v) > 0$. □

We now state an infinitesimal version of Lemma 4.5.11 that we will use in Section 4.6.

Lemma 4.5.12. *Let $\ell^+, \ell^- : [0,1] \to \mathcal{L}_n$ be smooth and such that $\ell_0^+ = \ell^+(0)$ and $\ell_0^- = \ell^-(0)$ are transverse and the linear maps $\dot{u}^+ \in \mathrm{Hom}(\ell_0^+, \ell_0^-)$ and $\dot{u}^- \in \mathrm{Hom}(\ell_0^+, \ell_0^+)$ corresponding to $(\ell^+)'(0)$ and $(\ell^-)'(0)$ are such that $\omega(\cdot, \dot{u}^+(\cdot))$ and $\omega(\dot{u}^-(\cdot), \cdot)$ are positive respectively on ℓ_0^+ and ℓ_0^-. The derivative at $t = 0$ of $q_{\ell^+(t),\ell^-(t)}$ is positive.*

Proof. The proof is similar to the previous one. Let $u^+(t) \in \mathrm{Hom}(\ell_0^+, \ell_0^-)$ and $u^-(t) \in \mathrm{Hom}(\ell_0^-, \ell_0^+)$ be the linear maps whose graph is equal to $\ell^+(t)$ and $\ell^-(t)$ respectively. Let $v \in \mathbb{R}^{2n}$. The derivative of the evaluation of $q_t = q_{\ell^+(t),\ell^-(t)}$ to v can be written in term of the derivative $\dot{v}_t^+$, $\dot{v}_t^-$ of the vectors $v_t^+ \in \ell_+(t)$, $v_t^- \in \ell_-(t)$ such that $v = v_t^+ + v_t^-$:

$$\dot{q}_0(v,v) = \omega(v_0^+, \dot{v}_0^-) + \omega(\dot{v}_0^+, v_0^-).$$

One has $\dot{v}_0^+ = -\dot{v}_0^-$ since $v = v_t^+ + v_t^-$ does not vary with t. The fact that $v_t^+ \in \ell_+(t)$ for all t implies that $\dot{v}_0^+$ can be written $w^+ + \dot{u}^+(v_0^+)$ with $w^+ \in \ell_0^+$. Similarly $\dot{v}_0^-$ can be written $w^- + \dot{u}^-(v_0^+)$ with $w^- \in \ell_0^-$. Note that $\omega(v_0^+, w^+) = 0$ as they belong to the same Lagrangian ℓ_0^+. Similarly $\omega(v_0^-, w^-) = 0$ and hence:

$$\dot{q}_0 = 2\omega(v_0^+, \dot{u}^+(v_0^+)) + 2\omega(\dot{u}^-(v_0^-), v_0^-).$$

This is positive by our assumption.

□

We now construct an equivariant continuous map that admits a well-fitting flow.

Proposition 4.5.13. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation. There exist a continuous map $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ that admits a fitting flow.*

We prove this lemma using some averaging argument, where the basic building blocks are the quadrics associated to pairs of Lagrangians in the limit curve. This construction is not unique, as we start by fixing a hyperbolic metric.

Proof. Fix a hyperbolic metric on S_g. Since S_g is oriented it admits an associated complex structure J. For $v \in T_x^1\widetilde{S_g}$ write $\ell_v = \xi_\rho^n(\zeta_v)$ where $\zeta_v \in \partial\Gamma$ is the limit point of the geodesic with initial derivative v. We furthermore define:

$$q_v^\circ = q_{\ell_{Jv}, \ell_{-Jv}}.$$

These quadrics for a fixed $x \in \widetilde{S_g}$ do not in general define a pencil of quadrics. We therefore define the following quadric associated to $v \in T_x\widetilde{S_g}$:

$$q_v = \int_{w \in T_x^1\widetilde{S}} \langle v, w \rangle q_w^\circ \mathrm{d}\lambda.$$

Here we take the integral for the measure λ on $T_x^1\widetilde{S}$ induced by the hyperbolic metric. For each $x \in \widetilde{S}$, we consider the pencil $u(x) = \{q_v | v \in T\widetilde{S_g}\} \in \mathrm{Gr}_2(\mathcal{Q})$ which is well defined since q_v depends linearly on $v \in T_xS$.

First we check that these pencils are in $\mathrm{Gr}_2^{(n,n)}(\mathcal{Q})$, by proving that they are actually ω-regular.

Let $v \in T_x^1\widetilde{S_g}$ be non-zero for some $x \in \widetilde{X}$. For all $w \in T_x\widetilde{S_g}$ if $\langle w, v \rangle > 0$, the triple $(\zeta_{-Jw}, \zeta_v, \zeta_{Jw})$ is positively oriented and hence $(\ell_{-Jw}, \ell_v, \ell_{Jw})$ is maximal and hence q_w° is negative on ℓ_v. If $\langle w, v \rangle < 0$, the triple $(\ell_{Jw}, \ell_v, \ell_{-Jw})$ is maximal and hence q_w° is positive on ℓ_v. Hence q_v is negative on ℓ_v, and by a similar argument q_v is positive on ℓ_{-v}, which are Lagrangians. In particular q_v is ω-regular for all $v \in T_x^1\widetilde{S_g}$, and so $u(x)$ is ω-regular.

We consider the geodesic flow on $u^{\mathbb{S}}\mathcal{E} \simeq T^1\widetilde{S_g}$, and we prove that this flow is fitting. Let $t > 0$, $x \in \widetilde{S_g}$ and $v \in T_x^1\widetilde{S_g}$. Let (y, v') be the image of (x, v) by

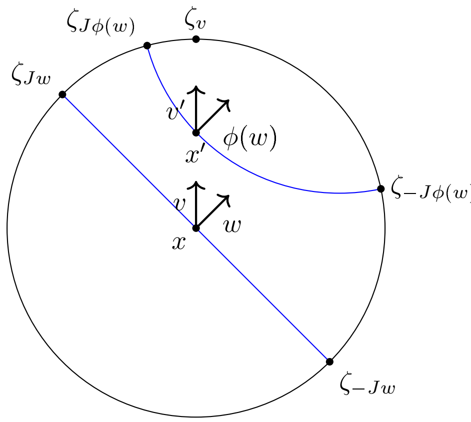


Figure 4.7: Proof of Lemma 4.5.13.

the geodesic flow at time t, and let $\phi : T^1_x\widetilde{S_g} \to T^1_y\widetilde{S_g}$ be the identification given by the parallel transport along the geodesic between x and y.

Let $w \in T^1_x\widetilde{S}$ be such that $\langle w, v\rangle > 0$. The following quadruple is positive $(\zeta_{-Jw}, \zeta_{-J\phi(w)}, \zeta_{J\phi(w)}, \zeta_{Jw})$ due to the negative curvature of the metric we put on $\widetilde{S_g}$, see Figure 4.7. Hence the corresponding quadruple of Lagrangians is maximal. Therefore $q^\circ_{\phi(w)} - q^\circ_w$ is positive by Lemma 4.5.11. When $\langle w, v\rangle < 0$, the following quadruple is positive $(\zeta_{-J\phi(w)}, \zeta_{-Jw}, \zeta_{Jw}, \zeta_{J\phi(w)})$ and therefore $q^\circ_{\phi(w)} - q^\circ_w$ is negative. Hence :

$$q_{v'} - q_v = \int_{w\in T^1_x\widetilde{S}} \langle v, w\rangle \left(q^\circ_{\phi(w)} - q^\circ_w\right) \mathrm{d}\lambda > 0.$$

We therefore have proven that the geodesic flow for the fixed hyperbolic metric is a fitting flow on $T^1\widetilde{S_g} \simeq \mathbb{S}u^{\mathcal{E}}$.

Finally as in the proof of Lemma 4.5.7 the homotopy type of ξ^n_ρ is equal to the homotopy type of $\xi_{u(x)}$ for all $x \in \widetilde{S_g}$. Hence these pencils are in $\mathrm{Gr}_2^{\max}(\mathcal{Q})$. $\square$

Remark 4.5.14. These pencils always lie in the same connected component of $\mathrm{Gr}_2^\omega(\mathcal{Q})$. Indeed on can construct such a pencil $u(x) \in \mathrm{Gr}_2^\omega(\mathcal{Q})$ given any maximal continuous map ξ from $\partial\widetilde{S_g}$ into the space of Lagrangians. The space of such maximal continuous maps being path connected, any two such pencils can be joined by a path in $\mathrm{Gr}_2^\omega(\mathcal{Q})$. It is not clear if $\mathrm{Gr}_2^{\max}(\mathcal{Q})$ only contains this connected component.

4.6 Fitting immersions and spacelike immersions for $\mathbf{PSp}(4,\mathbb{R})$

In this section we explain how the data of a maximal immersion into the Pseudo-Riemannian space $\mathbb{H}^{2,2}$ with principal curvature in $(-1,1)$ induces a fitting immersion. Combining this with a result from Collier-Tholozan-Toulisse we show that our characterization of maximal representations can be improved in $\mathrm{Sp}(4,\mathbb{R})$.

Note first that one has the following exceptional isomorphism:

$$\mathrm{PSp}(4,\mathbb{R}) \simeq \mathrm{SO}_o(2,3).$$

This isomorphism comes form the fact that $\mathrm{PSp}(4,\mathbb{R})$ preserves a subspace of dimension 5 of $\Lambda^2\mathbb{R}^4$, as well as a symmetric bilinear form of signature $(2,3)$ on this subspace. Hence $\mathrm{PSp}(4,\mathbb{R})$ acts naturally on the pseudo-Riemannian symmetric space with constant negative sectional curvature $\mathbb{H}^{2,2}$, which consists of vectors of norm -1 in $\mathbb{R}^{2,3}$. The space of Lagrangians $\mathcal{L}$ in $\mathbb{R}^4$ is naturally identified with the space of isotropic lines $\partial\mathbb{H}^{2,2}$ in $\mathbb{R}^{2,3}$.

To a pointed totally geodesic spacelike plane (p,P) in $\mathbb{H}^{2,2}$ one can associate an element in a special G-orbit of $\mathrm{Gr}_2^{\max}(\mathcal{Q})$. For every geodesic in this plane passing though the base point, we consider the endpoints $\ell_1,\ell_2 \in \mathcal{L} \simeq \partial\mathbb{H}^{2,2}$, and the space generated by all such quadrics q_{ℓ_1,ℓ_2} forms a plane which is a an element of $\mathrm{Gr}_2^{\omega}(\mathcal{Q})$. Indeed for some symplectic basis (x_1,x_2,y_1,y_2) the Lagrangians corresponding to the boundary of the spacelike plane P are for $\theta \in [0,2\pi]$:

$$\ell(\theta) = \langle \cos\left(\frac{\theta}{2}\right) x_1 + \sin\left(\frac{\theta}{2}\right) y_1, \cos\left(\frac{\theta}{2}\right) x_2 + \sin\left(\frac{\theta}{2}\right) y_2 \rangle.$$

The corresponding quadric $q_{\ell(0),\ell(\pi)}$ for $\theta = 0$ in the basis (x_1,y_1,x_2,y_2) is equal to:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let R_θ be the following rotation matrix:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The corresponding quadric $q_{\ell(\theta),\ell(\theta+\pi)}$ in the basis (x_1,y_1,x_2,y_2) is equal to:

$$\begin{pmatrix} R_{\frac{\theta}{2}} & 0 \\ 0 & R_{\frac{\theta}{2}} \end{pmatrix} q_{\ell(0),\ell(\pi)} \begin{pmatrix} R_{-\frac{\theta}{2}} & 0 \\ 0 & R_{-\frac{\theta}{2}} \end{pmatrix}.$$

Hence $q_{\ell(\theta),\ell(\theta+\pi)}$ is equal to:

$$\begin{pmatrix} \sin(\theta) & \cos(\theta) & 0 & 0 \\ \cos(\theta) & -\sin(\theta) & 0 & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix}.$$

These quadrics span a plane in $\mathcal{Q}$ when θ varies, and these quadrics are ω-regular so this plane is in $\mathrm{Gr}_2^\omega(\mathcal{Q})$.

We define the Gauss map $\mathcal{G}u : S \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ of a spacelike immersion $u : S \to \mathbb{H}^{2,2}$ the map that associates to $x \in S$ the pencils corresponding to the pointed totally geodesic spacelike plane $(u(x), P)$ where $T_{u(x)}P = \mathrm{d}u(T_xS)$.

Theorem 4.6.1. *Let $u : S \to \mathbb{H}^{2,2}$ be a spacelike immersion such that for all $v \in TS$, $\|I\!I_u(v,v)\| < \|v\|$. The Gauss map $\mathcal{G}u : S \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ is a fitting immersion.*

Let S be a spacelike surface in $\mathbb{H}^{2,n}$. Let $\gamma : [0,1] \to S$ be a geodesic for the induced metric on S parametrized with unit speed an let $V : [0,1] \to TS$ be the unit orthogonal vector field to γ' in S along γ. We denote by $V^+, V^- : [0,1] \to \partial\mathbb{H}^{2,2}$ the endpoints of the geodesic rays starting respectively at V and $-V$. Up to changing the sign of V one can assume that (V^+, γ^+, V^-) is a maximal triple where γ^+ is the endpoint of the geodesic ray srarting at γ'.

Lemma 4.6.2. *Suppose that $\|I\!I_u(V,\gamma')\| \leqslant 1$, then the curves $V^+, V^- : [0,1] \to \partial\mathbb{H}^{2,n}$ are spacelike.*

Note that in this lemma $I\!I_u(V,\gamma')$ is timelike, so its norm is the timelike norm that we see as a positive number.

Proof. We fix a orthogonal basis (e_i) of $\mathbb{R}^{2,n}$ such that e_1 and e_2 have norm 1 and e_i for $i \geqslant 3$ have norm -1. Without any loss of generality we suppose that $\gamma(0) = e_3$, $\gamma'(0) = e_1$ and $V(0) = e_2$. Let d be the flat connection on $\mathbb{R}^{2,n}$ and let ∇ be the Levi-civita connection on $u(S)$ for the induced spacelike metric. For t close to 0, $V(t) = V(0) + tdV(0)$ and $dV = \nabla V + I\!I_{\mathbb{H}^{2,n}}(V,\gamma') + I\!I_u(V,\gamma')$. Here the second fundamental form of $\mathbb{H}^{2,n}$ inside $\mathbb{R}^{2,n+1}$ is equal to $I\!I_{\mathbb{H}^{2,n}}(v_1,v_2) = \langle v_1, v_2\rangle v_0$ for $v_1, v_2 \in T_{v_0}\mathbb{H}^{2,n}$. Note also that since γ is a geodesic and V a orthogonal unit vector field along γ, $\nabla V = 0$.

Hence $V(t) = V(0) + tI\!I_u(V,\gamma') + o(t)$. Since V has norm 1, we can write a representative of the isotropic line $V^+(t)$ as $v^+(t) = V(t) + \gamma(t)$. Therefore:

$$v^+(t) = v^+(0) + t\left(I\!I_u(V,\gamma') + e_2\right) + o(t).$$

This curve is spacelike since e_2 is spacelike of norm 1 and the (timelike) norm of $I\!I_u(V,\gamma')$ is strictly less than 1. The same holds for V^-. $\square$

Proof of Theorem 4.6.1. The Gauss map $\mathcal{G}u : S \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$ comes with an identification between $u^*\mathcal{E}$ and TS. We consider the geodesic flow Φ on $\mathbb{S}TS$ for the metric induced by u. Lemma 4.6.2 implies that the Lagrangians ℓ^+, ℓ^- in $\mathbb{R}^4$ corresponding to the isotropic lines V^+ and V^- in $\mathbb{R}^{2,3}$ satisfy the hypothesis of the second part of Lemma 4.5.12. Hence the derivative of the associated quadrics along this flow is positive so Φ is a fitting flow. Since u admits a fitting flow it is in particular a well-fitting immersion.

□

The existence of a maximal spacelike immersion was proven by Collier-Tholozan-Toulisse and a bound of its second fundamental form is a consequence of a result from Cheng.

Theorem 4.6.3 ([CTT19],[Che93]). *Every maximal representation $\rho : \Gamma_g \to \mathrm{SO}_o(2,3)$ admits a unique ρ-equivariant maximal spacelike immersion $u : \widetilde{S_g} \to \mathbb{H}^{2,2}$. Moreover it is an embedding and for all $v \in TS$, $\|I\!I_u(v,v)\| < \|v\|$.*

The bound on the second fundamental from is a consequence of a maximal principle, see [LT22, Corollary 5.2]. Note that this reference it is written that the square norm of $I\!I_u$ is at most 2, but since u is maximal it implies that the principal values are at most equal to 1.

Putting together Theorem 4.6.1, Theorem 4.6.3 and Theorem 4.5.5 we obtain:

Corollary 4.6.4. *Every maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(4,\mathbb{R})$ admits a ρ-equivariant fitting immersion $u : \widetilde{S_g} \to \mathrm{Gr}_2^{\max}(\mathcal{Q})$. This characterizes representations which are maximal for some orientation of S_g.*

4.7 Geometry of the symmetric space.

In this section we prove Proposition 4.7.1 and Lemma 4.7.4 which are two facts independent from the main results of the chapter. We show how to construct fitting immersions of pencils using totally geodesic surfaces in the symmetric space $\mathbb{P}(S^2V^{>0})$. Then we prove that the codimension d submanifolds corresponding to pencils in $\mathrm{Gr}_d^{\omega}(\mathcal{Q})$ intersect transversely the symmetric space of $\mathrm{Sp}(2n,\mathbb{R})$ embedded in $\mathbb{P}(S^2V^{>0})$.

Given an immersion $h : M \to \mathbb{P}(S^2V^{>0})$ from a manifold M of dimension d we define its *Gauss map* $\mathcal{G}h : M \to \mathrm{Gr}_d(\mathcal{Q})$, that associates to $x \in M$ the pencil P associated with the codimension d projective subspace of $\mathrm{Gr}_d(\mathcal{Q})$ orthogonal to $u(h)$ at $h(x)$, for the $\mathrm{SL}(V)$-invariant Riemannian metric on $\mathbb{P}(S^2V)$.

The invariant Riemannian metric of the symmetric space associated to $\mathrm{SL}(V)$ can be described by a natural identification between $\mathbb{P}(S^2V^{>0})$ and its dual cone $\mathbb{P}(S^2(V^*)^{>0})$. We therefore reformulate the definition of $\mathcal{G}h$ as follows.

We first identify $\mathbb{P}(S^2V^{>0})$ with $\mathbb{P}(S^2(V^*)^{>0})$ via the map $[X] \mapsto [X^{-1}]$. Note as once again we view elements of S^2V and $S^2(V^*) = \mathcal{Q}$ respectively as maps $V^* \to V$ and $V \to V^*$. An immersion $h : M \to \mathbb{P}(S^2V^{>0})$ hence defines a dual immersion $h^* : M \to \mathbb{P}(S^2(V^*)^{>0})$. Fixing a volume form on V and V^* allows us to lift this map to a map $\overline{h}^*$ into the space of elements in $S^2(V^*)^{>0}$ whose corresponding map $V \to V^*$ has determinant 1. We define $\mathcal{G}h(x) = \mathrm{d}\overline{h}^*(T_xM) \subset S^2V^* = \mathcal{Q}$.

Proposition 4.7.1. *Let $h : M \to \mathbb{P}(S^2V^{>0})$ be a totally geodesic immersion. Suppose that the image of the Gauss map $\mathcal{G}h : M \to \mathrm{Gr}_d(\mathcal{Q})$ contains only regular pencils, i.e. pencils containing only non-degenerate quadrics. The immersion $\mathcal{G}h$ is then a fitting immersion. If h is complete it is a globally fitting map.*

Remark 4.7.2. The fibration of a domain of $\mathbb{P}(V)$ induced by the fitting map in this proposition is a particular case of Theorem 2.6.3 from Chapter 2.

Note that if $d \geqslant 2$, the signature (a, b) of the quadrics of a regular pencil must satisfy $a = b$ since $\mathbb{S}^{d-1}$ is connected. Hence a regular pencil for $d \geqslant 2$ is just an element of $\mathrm{Gr}_d^{(n,n)}(\mathcal{Q})$ where $\dim(V) = 2n$.

As a corollary on can construct fitting immersions for some representations that factor through $\mathrm{SL}(2,\mathbb{R})$. Indeed if one has a representation $\iota : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(V)$ there exist a ι-equivariant totally geodesic map $h : \mathbb{H}^2 \to \mathrm{SL}(V)$, see Section 2.7.1.

Proof. Let $\gamma : \mathbb{R} \to M$ be a geodesic for the metric induced by h. We write the representative of $h(\gamma(o))$ with determinant 1 in a basis $(e_i)_{i\in I}$ such that for some $\lambda^i \in \mathbb{R}$, for all $t \in \mathbb{R}$:

$$\overline{h}(\gamma(t)) = \sum_{i\in I} e^{t\lambda_i} e_i \otimes e_i.$$

The dual immersion can be written as:

$$\overline{h}^*(\gamma(t)) = \sum_{i\in I} e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

The element $q_t \in \mathcal{Q}$ corresponding to $(\overline{h}^* \circ \gamma)'(t)$ is the symmetric bilinear form:

$$q_t = \sum_{i\in I} -\lambda_i e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

The derivative of (q_t) at $t = 0$ equals:

$$\sum_{i\in I} \lambda_i^2 e^{-t\lambda_i} e_i^* \otimes e_i^*.$$

This is a positive bilinear form if and only if all the λ_i are non-zero, which is the case if and only if the bilinear forms q_t are non-degenerate, i.e. if the image of $\mathcal{G}h$ contains only regular pencils. In this case, the positivity of the derivative of (q_t) implies that the geodesic flow on $\mathbb{S}TM$ induces a fitting flow on $\mathcal{G}h^*\mathbb{S}\mathcal{E}$, so $\mathcal{G}h$ is a fitting immersion.

If moreover h is complete, it is ρ-equivariant for the discrete and faithful action of some closed surface group Γ. Hence Corollary 4.3.7 implies that h is a globally fitting map. $\square$

Let us fix a symplectic form ω on $V = \mathbb{R}^{2n}$. Let $\mathbb{X}_{\mathrm{Sp}}$ be the subset of $\mathbb{P}(S^2V^{>0})$ consisting of tensors $[q^{-1}]$ that are compatible with ω, i.e. such that for some complex structure J on $\mathbb{R}^{2n}$, $q + i\omega$ is a hermitian metric on V. Recall that $q : V \to V^*$ is a bilinear form, and $q^{-1} : V^* \to V$ is a tensor.

The space $\mathbb{X}_{\mathrm{Sp}}$ is a copy of the symmetric space associated to $\mathrm{Sp}(2n, \mathbb{R})$, which is a totally geodesic subspace of the symmetric space associated to the Lie group $\mathrm{SL}(2n, \mathbb{R})$ whose model is $\mathbb{P}(S^2V^{>0})$. However it is not a projective subset : the closure of the projective convex hull of $\mathbb{X}_{\mathrm{Sp}}$ in $\mathbb{P}(S^2V^{>0})$ is equal to $\mathbb{P}(S^2V^{\geqslant 0})$ since it contains all the extremal points of $\mathbb{P}(S^2V^{\geqslant 0})$, i.e. the rank one elements $S^2\mathbb{P}(\mathbb{R}^{2n})$.

The intersection of $\mathbb{X}_{\mathrm{Sp}}$ with a general linear subspace is not necessarily transverse. However it is the case for some special subspaces.

Definition 4.7.3. We say that an element $q \in \mathcal{Q}$ is *ω-regular* if for some Lagrangians ℓ^+, ℓ^- the bilinear form q is positive on ℓ^+ and negative on ℓ^-.

We denote by $\mathrm{Gr}_d^{\omega}(\mathcal{Q})$ the set of pencils whose non-zero elements are ω-regular.

In particular an ω-regular pencil q has signature (n, n).

Lemma 4.7.4. *Let $P \in \mathrm{Gr}_2^{\omega}(\mathcal{Q})$ be an ω-regular pencil, i.e. such that all its non-zero elements are ω-regular. The space $\mathbb{P}(P^\circ)$ intersects transversely the manifold $\mathbb{X}_{Sp}$ in a codimension 2 submanifold.*

Proof. Let $q \in \mathcal{Q}$ be ω-regular, and let $x \in \mathbb{X}_{\mathrm{Sp}} \cap \mathbb{P}(\langle q \rangle^\circ)$ be an intersection point. Up to acting by $\mathrm{Sp}(2n, \mathbb{R})$, one can assume that $x = [X^{-1}] \in \mathbb{P}(S^2V)$ where X is the bilinear form whose associated matrix in some symplectic basis is:

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

The annihilator of the tangent space to $\mathbb{X}_{\mathrm{Sp}}$ at this point can be identified with the space of following symmetric matrices where A is symmetric and B is antisymmetric:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Suppose that the intersection is not transverse, i.e. that q can be written in this form. Let ℓ^+ and ℓ^- be Lagrangians on which q is respectively positive and negative. Since the maximal compact $U(n)$ acts transitively on the space of Lagrangians one can assume that $\ell^+ = \langle x_1, \cdots, x_n \rangle$. Let $\ell^- = \langle x_1, \cdots, x_n \rangle$.

The Lagrangian ℓ^- is transverse to ℓ^+ so for some symmetric matrix U, one can write this Lagrangian as the image of :

$$\begin{pmatrix} U \\ I_n \end{pmatrix}.$$

The fact that q is positive on ℓ^+ implies that A is positive. The fact that q is negative on ℓ^- implies that the following is negative:

$$UAU + A - BU + UB.$$

But A and UAU are both positive and the bracket $[B, U]$ has trace zero so it cannot be negative. Hence the intersection must be transverse. $\square$

Proposition 4.7.5. *A tangent vector in $T\mathbb{X}_{Sp} \subset T\mathbb{P}(S^2(V^*)^{>0})$ is ω-regular if and only if the corresponding element of $\mathcal{Q}$ is non-degenerate.*

Proof. Given $\mathrm{v} \in T\mathbb{X}_{\mathrm{Sp}}$ one can write the corresponding element $q \in \mathcal{Q}$ for some symplectic basis $(x_1, \cdots x_n, y_1, \cdots y_n)$ as:

$$\sum_{i=1}^{n} \lambda_i (x_i^* \otimes x_i^* - y_i^* \otimes y_i^*).$$

This bilinear form is regular if and only if all the λ_i are non-zero. If this is the case that it is positive on the Lagrangians $\langle x_1, \cdots, x_n \rangle$ and negative on the Lagrangian $\langle y_1, \cdots, y_n \rangle$. $\square$

4.8 Polygons of quadric hypersurfaces

We briefly discuss in this section how the quadrics associated to pairs of Lagrangians defined previously can be used in an other way to decompose the geometric structures associated to maximal representations into hexagons of quadric hypersurfaces. This decomposition is a generalisation of the work Burelle-Treib for Schotty subgroups [BT17].

Let us cut the surface S_g by closed curves that intersect transversely so that no three curves intersect at a single point and all the complementary regions are hexagons.

Once such a topological decomposition is fixed, one can put a hyperbolic metric S_g so that the curves are geodesics, by making all complementary regions regular right-angled hexagons. Conversely by gluing right-angled hyperbolic hexagons one can construct such a decomposition. This defines a tiling of $\widetilde{S_g}$ into hexagons indexed by a set I.

Each hexagon $h \in I$ is described by its cyclically ordered oriented sides $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$. Note that non-adjacent sides are parallel for any hyperbolic metric on S_g.

Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a maximal representation, and let ξ_ρ^n be the associated limit map. Let $h = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$ be the positively ordered

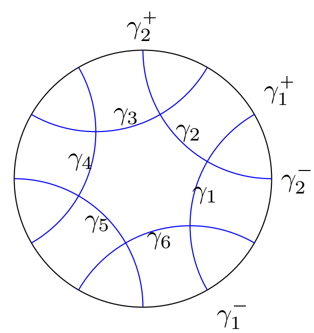


Figure 4.8: An hexagon in $\widetilde{S_g}$ and its oriented sides.

sides of an hexagon in the decomposition of $\widetilde{S_g}$ that we fixed. To each of these oriented geodesic γ with ordered endpoints (γ^+, γ^-) we associate the quadric $q_\gamma = q_{\ell^+,\ell^-}$ where $\ell^+ = \xi_\rho^n(\gamma^+)$, $\ell^- = \xi_\rho^n(\gamma^-)$. The intersection of the sets $\{q_{\gamma_i} \geqslant 0\}$ defines a compact set that we call a *hexagon of quadric hypersurfaces*.

Proposition 4.8.1. *The hexagons of quadric hypersurfaces associated to all of the hexagons $h \in I$ define a tiling of the Guichard-Wienhard domain of disontinuity* (4.2).

Proof. We first see that the hexagons of quadric hypersurfaces have disjoint interior. Indeed, given two distinct hexagons $h_1, h_2 \in I$, there exist a side γ_1 of h_1 that separates h_1 and h_2. Every side of h_2 is either parallel (or equal) to γ_1, or is orthogonal to γ_1. In the first case there are one or two sides γ_2 that separate γ_1 and h_2. In the second case let γ_2' be the side of h_2 intersecting γ_1, one of its two adjacent sides γ_2 of h_2 separates h_2 from γ_1.

The fact that γ_1 and γ_2 are parallel or equal with opposite orientations and the fact that ξ_ρ^n is maximal implies that $\{q_{\gamma_1} \geqslant 0\}$ is disjoint from $\{q_{\gamma_2} \geqslant 0\}$, see Lemma 4.5.11. Hence h_1 and h_2 have disjoint interior.

We now show that every hexagon in I of quadric hypersurfaces lies in the Guichard-Wienhard domain of discontinuity. Let us fix $\zeta \in \partial\Gamma_g$. For every hexagon $h \in I$ there is a side γ of h separating h from ζ. The maximality of the boundary map ξ_ρ^n implies that $\mathbb{P}\left(\xi_\rho^n(\zeta)\right)$ is disjoint from $\{q_\gamma \geqslant 0\}$ by Lemma lem:Triples of Lagrangians quadric. Hence the union of all the hexagons of quadric hypersurfaces do not intersect $\mathbb{P}\left(\xi_\rho^n(\zeta)\right)$ for any ζ.

Finally note that $\rho(\Gamma_g)$ preserves and acts cocompactly on the union of all the hexagons of quadric hypersurfaces as there are finitely such hexagons and they are compact. Moreover this union is open: indeed if x is a point in projective space in the boundary of the hexagon of quadrics associated to $h \in I$, there is exactly one or two adjacent sides γ of h such that q_γ vanishes on x. In

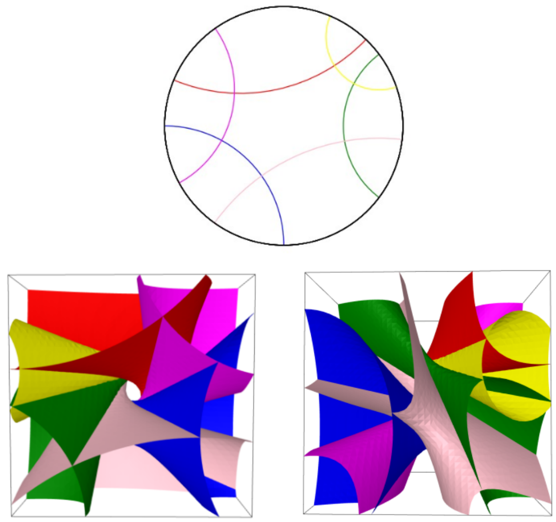


Figure 4.9: A hexagon in $\widetilde{S_g}$ and two views on the corresponding hexagon of quadric hypersurfaces in $\mathbb{RP}^3$.

this case x belongs to the interior of the union of the respectively two or four hexagon of quadrics corresponding to h_1 and its neighbors for the sides γ such that q_γ vanishes on x

The Guichard-Wienhard cocompact domain of discontinuity contains this non-empty cocompact domain of discontinuity, Therefore this this domain must be equal to a union of connected components of the Guichard-Wienhard domain. For $n \geqslant 3$ this domain is connected [GW12]. For $n = 2$ the domain has two connected components, but so does also the hexagons of quadric hypersurfaces as illustrated in Figure 4.9. Hence the hexagons of quadric hypersurfaces form a tiling of the Guichard-Wienhard domain of discontinuity. $\square$

Figure 4.9 illustrates a hyperbolic hexagon and the corresponding hexagon of quadric hypersurfaces in the case $n = 2$ for a maximal representation constructed as the composition of a Fuchsian representation and the diagonal embedding of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{Sp}(2n, \mathbb{R})$. The hexagon is the complement of the inside of the six ruled hyperboloids in the picture.

This hexagon of quadric hypersurfaces is a fiber bundle over an hyperbolic hexagon with fiber the union of two cricles. It is disconnected and the two views in Figure 4.9 put an emphasis on the two connected components.

In this Figure, Lemma 4.5.11 is illustrated by the fact that the quadric hypersurfaces corresponding to non-intersecting geodesics in $\mathbb{H}^2$ do not intersect in $\mathbb{RP}^3$.

Remark 4.8.2. To a hexagon $h \in I$ with sides $(\gamma_i)_{1 \leqslant i \leqslant 6}$ we can associate a subset of the convex set $\mathbb{P}(S^2V^{>0})$ and in the symmetric space $\mathbb{X}_{\mathrm{Sp}} \subset \mathbb{P}(S^2V^{>0})$ associated wit $\mathrm{Sp}(4, \mathbb{R})$ as the set of classes $[p]$ such that $p \in S^2V^{>0}$ and $q_{\gamma_i} \cdot p = \mathrm{Tr}(q_{\gamma_i} \circ p) \geqslant 0$. The hexagon of quadric hypersurfaces arises as the intersection of the closure of this set of the set of rank one points $\mathbb{P}(V) \simeq S^2\mathbb{P}(V) \subset \mathbb{P}(S^2V)$. One can similarly show that these subsets for $h \in I$ define a tiling of $\mathbb{P}(S^2V^{>0})$, which restricts to a tiling of $\mathbb{X}_{\mathrm{Sp}}$.

Chapter 5

Maximal and Borel Anosov representations

In this chapter we study maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ that satisfy additional Anosov properties.

We first recall in Section 5.1 the definition of Anosov representations in $\mathrm{Sp}(2n, \mathbb{R})$ and fix some notations. In Section 5.2 we describe a special chart of the space of Lagrangians. In Section 5.3 we discuss the link between hyperconvexity and smooth properties of the boundary maps.

In Section 5.4 we characterize maximal representations among $\{n-1, n\}$-Anosov representations by the hyperconvexity property H_n. In Section 5.5 and 5.6 we prove that maximal and Borel Anosov representations are Hitchin.

This chapter is an adaptation of the article [Dav24].

5.1 Anosov representations.

Let Γ_g denote the fundamental group of a closed orientable surface of genus $g \geqslant 2$. This is an hyperbolic group in the sense of Gromov, and we will denote by $\partial\Gamma_g$ its Gromov boundary, which is a topological circle.

Let $N \geqslant 2$ be an integer. Let us fix some Euclidean structure on $\mathbb{R}^N$, and for every element $M \in \mathrm{SL}(N, \mathbb{R})$ denote by $\sigma_1(M) \geqslant \sigma_2(M) \geqslant \cdots \geqslant \sigma_N(M)$ the singular values of M in non-decreasing order. Given $\gamma \in \Gamma_g$ we will denote by $|\gamma|_w$ the word length of γ with respect to some fixed finite generating set of Γ_g.

The following definition is not the original one, but a characterization due to Kapovich-Leeb-Porti [KLP17]:

Definition 5.1.1 ([BPS19, Section 4])**.** A representation ρ of Γ_g into $\mathrm{SL}(N, \mathbb{R})$ is Θ-Anosov with $\Theta \subset \{1, \cdots, N\}$ if there exists some constants $C, \alpha > 0$ such that for all $\gamma \in \Gamma_g$ and $k \in \Theta$:

$$\frac{\sigma_{k+1}(\rho(\gamma))}{\sigma_k(\rho(\gamma))} \leqslant Ce^{-\alpha|\gamma|_w}.$$

If a representation is Anosov with respect to $\Delta = \{1, \cdots, N\}$, then it is called *Borel Anosov*.

Remark 5.1.2. If a representation is Θ-Anosov then it is automatically Θ'-Anosov for all $\Theta' \subset \Theta$.

For a general semi-simple Lie group G, the Anosov property depends on a subset of the set of simple roots, or equivalently of a conjugacy class of parabolic subgroups. Here we identified the set Δ of simple roots of the simple Lie group $\mathrm{SL}(N, \mathbb{R})$ with the set $\{1, \cdots, N-1\}$.

Boundary maps are important objects naturally associated to an Anosov representation.

Theorem 5.1.3 ([GW12],[BPS19]). *Let $\rho : \Gamma_g \to \mathrm{SL}(N, \mathbb{R})$ be a $\{k\}$-Anosov representation. Let $Gr(k, N)$ be the Grassmannian of k-dimensional subspaces in $\mathbb{R}^N$. There exists a unique continuous ρ-equivariant map $\xi^k_\rho : \partial\Gamma_g \to Gr(k, N)$ that is* dynamic preserving*, i.e for all element $\gamma \in \Gamma_g$ if γ^+ is the unique attracting fixed point of γ in $\partial\Gamma_g$ then $\xi^k_\rho(\gamma^+)$ is the unique attracting fixed point of $\rho(\gamma)$ in $Gr(k, N)$.*

The property of being dynamic preserving determines ξ^k_ρ, since the set of attracting fixed point of elements of Γ_g is dense in $\partial\Gamma_g$.

Notation 5.1.4. For any $\{k\}$-Anosov representation and any $x \in \partial\Gamma_g$ we will write $x^k_\rho := \xi^k_\rho(x)$ as in [PSW21] to make expressions involving boundary maps lighter. We will still keep the notation ξ^k_ρ to denote the boundary map itself. As a convention $x^0_\rho = \{0\}$ and $x^N_\rho = \mathbb{R}^N$ for any $x \in \partial\Gamma_g$.

Boundary maps satisfy additional properties: they are *transverse* and *compatible.*

Proposition 5.1.5 ([GW12],[BPS19]). *Let $\rho : \Gamma_g \to \mathrm{SL}(N, \mathbb{R})$ be a $\{k\}$-Anosov representation. The representation ρ is also $\{N-k\}$-Anosov, and for every pair $x, y \in \partial\Gamma_g$ of distinct points, x^k_ρ and y^{N-k}_ρ are transverse* (transversality)*. If ρ is $\{k, \ell\}$-Anosov with $k \leqslant \ell$ then $x^k_\rho \subset x^\ell_\rho$ for all $x \in \partial\Gamma_g$* (compatibility)*.*

As a consequence the image of boundary maps at two different point are in general position.

Corollary 5.1.6. *Let $k, \ell \geqslant 1$. Let $x, y \in \partial\Gamma_g$ be distinct points :*

$$\dim\left(x^k_\rho \cap y^\ell_\rho\right) = \max(k + \ell - N, 0).$$

Let us assume now that $N = 2n$ is even and let us fix a symplectic form ω on $\mathbb{R}^{2n}$. Consider the subgroup $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$ consisting of elements which preserve ω: representations into $\mathrm{Sp}(2n, \mathbb{R})$ can be seen as particular examples of representations into $\mathrm{SL}(2n, \mathbb{R})$. The boundary maps of Anosov representations whose images lie in $\mathrm{Sp}(2n, \mathbb{R})$ have some additional properties.

Lemma 5.1.7. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{k\}$-Anosov representation. For any $x \in \partial\Gamma_g$, $\left(x_\rho^k\right)^\perp = x_\rho^{2n-k}$, where $\left(x_\rho^k\right)^\perp$ is the orthogonal of x_ρ^k with respect to ω. In particular if $k \leqslant n$ the space x_ρ^k is isotropic.*

Proof. The orthogonality condition holds for a closed Γ_g-equivariant subset of $\partial\Gamma_g$. Since the action of Γ_g is minimal on $\partial\Gamma_g$ ([KB02] Proposition 4.2), it is sufficient to check it for a single point. Let x be the attracting fixed point of an element $\gamma \in \Gamma_g$, so x_ρ^k is the unique attracting fixed k-dimensional subspace of $\rho(\gamma)$ and x_ρ^{n-k} the unique attracting fixed $(2n-k)$-dimensional subspace of $\rho(\gamma)$.

Since $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$, it maps any subspace $V^\perp$ for $V \subset \mathbb{R}^{2n}$ to $(\gamma \cdot V)^\perp$. Hence $\left(x_\rho^k\right)^\perp$ is an attracting fixed point for the action of γ on the space of $(2n-k)$-dimensional subspaces. Therefore $\left(x_\rho^k\right)^\perp = x_\rho^{2n-k}$. If $k \leqslant n$, then $x_\rho^{2n-k} \subset \left(x_\rho^k\right)^\perp$ and hence x_ρ^k is isotropic.

□

5.2 Charts of the space of Lagrangians and maximality.

Recall that we fixed a symplectic structure ω on $\mathbb{R}^{2n}$. Let $\mathcal{L}_n$ be the space of *Lagrangians* in $\mathbb{R}^{2n}$, i.e. the space of n-dimensional subspaces of $\mathbb{R}^{2n}$ on which ω vanishes. Let $P, Q \in \mathcal{L}_n$ be two transverse Lagrangians, i.e. with trivial intersection.

Definition 5.2.1. A linear map u between P and Q is *symmetric* (with respect to ω) if for all $\mathrm{v}, \mathrm{w} \in P$:

$$\omega\left(\mathrm{v}, u(\mathrm{w})\right) = \omega\left(\mathrm{w}, u(\mathrm{v})\right).$$

The space of symmetric linear maps u from P to Q will be denoted by $\mathrm{Sym}_{P,Q}$.

For $Q \in \mathcal{L}_n$ let U_Q be the set of Lagrangians transverse to Q. The open sets $(U_Q)_{Q\in\mathcal{L}_n}$ form an open covering of $\mathcal{L}_n$. Given a Lagrangian P transverse to the Lagrangian Q, we get an identification of U_Q with the vector space $\mathrm{Sym}_{P,Q}$. This provides a family of linear charts of $\mathcal{L}_n$.

Proposition 5.2.2. *The graph of an element $u \in \mathrm{Sym}_{P,Q}$ is an element of U_Q. This defines an identification of $\mathrm{Sym}_{P,Q}$ with $U_Q \subset \mathcal{L}_n$.*

Proof. Recall that the graph of a linear map $u : P \to Q$ is the vector subspace of elements $\mathrm{v} + u(\mathrm{v})$ for $\mathrm{v} \in P$. It is a Lagrangian if and only if for all $\mathrm{v}, \mathrm{w} \in P$:

$$\omega\left(\mathrm{v} + u(\mathrm{v}), \mathrm{w} + u(\mathrm{w})\right) = 0.$$

Since P, Q are Lagrangians this is equivalent to having for all $\mathrm{v}, \mathrm{w} \in P$:

$$\omega\left(\mathrm{v}, u(\mathrm{w})\right) - \omega\left(\mathrm{w}, u(\mathrm{v})\right) = 0.$$

Hence the graph of u is a Lagrangian if and only if $u \in \mathrm{Sym}_{P,Q}$.

$\square$

Notation 5.2.3. Let P, Q be transverse Lagrangians and R be a Lagrangian transverse to Q, i.e. in U_Q. We denote by $u^R_{P,Q}$ the corresponding element in $\mathrm{Sym}_{P,Q}$.

Bilinear symmetric forms can be degenerate: they can have singular spaces. For any vector space V let $\mathcal{Q}(V)$ be the space of symmetric bilinear forms on V.

Definition 5.2.4. A subspace U of a vector space V is *singular* for a symmetric bilinear form q in $\mathcal{Q}(V)$ if for all $\mathrm{v} \in V, \mathrm{w} \in U$, on has $q(\mathrm{v}, \mathrm{w}) = 0$.

Let P, Q be two transverse Lagrangians in $\mathcal{L}_n$.

Proposition 5.2.5. *An element $u \in \mathrm{Sym}_{P,Q}$ determines a symmetric bilinear form $q \in \mathcal{Q}(P)$ defined for $\mathrm{v}, \mathrm{w} \in P$ as:*

$$q(\mathrm{v}, \mathrm{w}) = \omega\left(\mathrm{v}, u(\mathrm{w})\right).$$

This defines an identification of $\mathrm{Sym}_{P,Q}$ and $\mathcal{Q}(P)$. Moreover $\mathrm{Ker}(u)$ is singular for q.

This identification also defines linear charts $U_Q \simeq \mathcal{Q}(P)$.

Definition 5.2.6. For $R \in U_Q$, define $q^R_{P,Q} \in \mathcal{Q}(P)$ as the following symmetric bilinear form on P:

$$q^R_{P,Q}(\mathrm{v}, \mathrm{w}) = \omega\left(\mathrm{v}, u^R_{P,Q}(\mathrm{w})\right).$$

An invariant that classifies orbit of triples of pairwise transverse Lagrangians up to the action of $\mathrm{Sp}(2n, \mathbb{R})$ is called the Maslov index [BILW05]. We will be only interested by triples with maximal Maslov index, so we will only define the notion of maximal triples of Lagrangians. For a vector space V, let $\mathcal{Q}^+(V)$ denote the open cone of scalar products in the space of symmetric bilinear forms $\mathcal{Q}(V)$.

Definition 5.2.7. Let (P, R, Q) be three pairwise transverse Lagrangians in $\mathbb{R}^{2n}$. This triple is called *maximal* if the symmetric bilinear form $q^R_{P,Q}$ is in $\mathcal{Q}(P)^+$, i.e. is a scalar product.

A triple (P, R, Q) is maximal in this sense if and only if its Maslov index is maximal, i.e. if its Maslov index is equal to n (see for instance [BP17, Lemma 2.10]).

Remark 5.2.8. The signature of $q^R_{P,Q}$ is locally constant on the space of triples of pairwise transverse Lagrangians in $\mathbb{R}^{2n}$. Hence the space of maximal triples of Lagrangians (P, R, Q) forms a connected component of this space.

Let us fix an orientation of $\partial\Gamma_g$, i.e. a connected component of the space of distinct triples in $\partial\Gamma_g$ that we will call *positive triples*. The *Toledo invariant* of a representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ of a surface group Γ_g is an integer T_ρ that depends only on the connected component of $\mathrm{Hom}(\Gamma_g, \mathrm{Sp}(2n, \mathbb{R}))$ in which ρ lies. This invariant satisfies $|T_\rho| \leqslant n(2g-2)$. A representation has *maximal* Toledo invariant when $T_\rho = n(2g-2)$ [BILW05]. The following characterization will be taken as a definition for the rest of the paper.

Definition 5.2.9. Given an orientation of $\partial\Gamma_g$, we say that a representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ is *maximal* if it is $\{n\}$-Anosov and for every positive triple of distinct points x, y, z in $\partial\Gamma_g$, the triple $(x_\rho^n, y_\rho^n, z_\rho^n)$ is maximal, in the sense of Definition 5.2.7.

Maximal representations in this sense are exactly representations with maximal Toledo invariant: any representation ρ with maximal Toledo invariant is $\{n\}$-Anosov ([BILW05], Theorem 6.1), and its boundary map sends positive triples to maximal triples ([BILW05], Theorem 7.6). Conversely any representation that admits a continuous equivariant map from $\partial\Gamma_g$ to $\mathcal{L}_n$ which sends positive triples to maximal triples has maximal Toledo invariant ([BIW03], Theorem 8) and in particular is $\{n\}$-Anosov.

An example of the boundary curve $\xi_{\rho_0}^2$ of a maximal representation $\rho_0 : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ is given Figure 5.1. The boundary curve which is represented is a part of the Veronese curve, which is the boundary curve of a 4-Fuchsian representation ρ_0, i.e. the composition of a fuchsian representation and the irreducible representation $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(4, \mathbb{R})$. The triple (x, y, z) for this picture is a positive triple in $\partial\Gamma_g$. In the picture the point $z_{\rho_0}^2$ is "at infinity".

5.3 Differentiability properties of the boundary maps.

The k-th boundary map of an Anosov representation ρ of a surface group Γ_g has smooth image if ρ satisfies the hyperconvexity property H_k, which we now define. Recall that we use Notation 5.1.4 for the boundary maps of an Anosov representation.

Definition 5.3.1. Let $N \geqslant 2$ and $1 \leqslant k \leqslant N-1$ be integers. Let $\rho : \Gamma_g \to \mathrm{SL}(N, \mathbb{R})$ be a $\{k-1, k, k+1\}$-Anosov representation. We say that ρ satisfies property H_k if for all triples of distinct points $x, y, z \in \partial\Gamma_g$, the following sum is direct :

$$\left(x_\rho^k \cap z_\rho^{N+1-k}\right) + \left(y_\rho^k \cap z_\rho^{N+1-k}\right) + z_\rho^{N-1-k}. \tag{5.1}$$

If ρ satisfies property H_k for all $1 \leqslant k \leqslant N-1$, we say that ρ satisfies property H.

These properties can be also written as follows.

Lemma 5.3.2. *For a triple of distinct points $x, y, z \in \partial\Gamma_g$, the sum* (5.1) *defining property H_k is direct if and only if the following sum is direct:*

$$x_\rho^k + \left(y_\rho^k \cap z_\rho^{N+1-k}\right) + z_\rho^{N-1-k}. \tag{5.2}$$

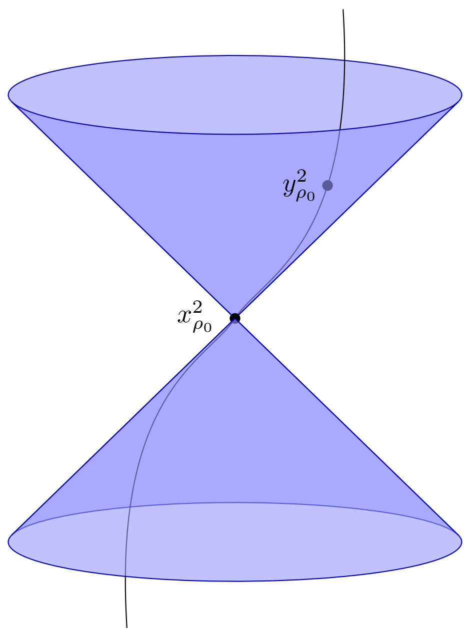


Figure 5.1: The Veronese curve $\xi^2_{\rho_0}(\partial\Gamma_g)$ in the chart $\mathcal{Q}(x^2_{\rho_0}) \simeq U_{z^2_{\rho_0}} \subset \mathcal{L}_2$ with the cone $\mathcal{Q}^+(x^2_{\rho_0})$.

Proof. The transversality of the boundary maps stated in Proposition 5.1.5 implies that the sum $\left(y_\rho^k \cap z_\rho^{N+1-k}\right) \oplus z_\rho^{N-1-k}$ is necessarily direct. If a vector in x_ρ^k belongs to this sum, it also belongs to $x_\rho^k \cap z_\rho^{N+1-k}$. Hence if (5.1) is direct then (5.2) is direct. The converse is immediate since $x_\rho^k \cap z_\rho^{N+1-k} \subset x_\rho^k$. $\square$

For a $\{k-1, k, k+1\}$-Anosov representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$, some of these properties are equivalent.

Proposition 5.3.3. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{k-1, k, k+1\}$-Anosov representation. It satisfies property H_k if and only if it satisfies property H_{2n-k}.*

Proof. Let $x, y, z \in \partial\Gamma_g$ be distinct points. Let us assume that the sum (5.2) is direct. Hence :

$$x_\rho^k \cap \left(\left(y_\rho^k \cap z_\rho^{2n+1-k}\right) \oplus z_\rho^{2n-1-k}\right) = \{0\}.$$

By considering the orthogonal of this set with respect to the bilinear form ω, and because of Lemma 5.1.7, one has:

$$x_\rho^{2n-k} + \left(\left(y_\rho^{2n-k} \oplus z_\rho^{k-1}\right) \cap z_\rho^{k+1}\right) = \mathbb{R}^{2n}.$$

Since $z_\rho^{k-1} \subset z_\rho^{k+1}$, then $\left(y_\rho^{2n-k} \oplus z_\rho^{k-1}\right) \cap z_\rho^{k+1} = \left(y_\rho^{2n-k} \cap z_\rho^{k+1}\right) \oplus z_\rho^{k-1}$. The following sum is equal to $\mathbb{R}^{2n}$ and the sum of the dimensions on the summands is equal to $2n$, so it is direct:

$$x_\rho^{2n-k} + \left(\left(y_\rho^{2n-k} \cap z_\rho^{k+1}\right) \oplus z_\rho^{k-1}\right) = \mathbb{R}^{2n}.$$

This means that this sum is direct for all distinct x, y, z, and hence property H_{2n-k} is satisfied. The converse implication is immediate by setting $k' = 2n - k$. $\square$

For any $x, y, z \in \partial\Gamma_g$ distinct and any $\{n, n-1\}$-Anosov representation $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$, the following subspace is a hyperplane in x_ρ^n:

$$(y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n.$$

Indeed $y_\rho^{n-1} \oplus z_\rho^n$ is an hyperplane of $\mathbb{R}^{2n}$ that cannot contain x_ρ^n since $x_\rho^n \oplus z_\rho^n = \mathbb{R}^{2n}$. This hyperplane can be seen as the image of y_ρ^{n-1} by the linear projection onto x_ρ^n in $\mathbb{R}^{2n}$ associated with the direct sum $\mathbb{R}^{2n} = x_\rho^n \oplus z_\rho^n$.

The transversality of boundary maps and property H_n imply the following transversality properties. These properties will be used in the case $n = 2$ to prove Lemma 5.5.4 and Theorem 5.5.5.

Lemma 5.3.4. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{n-1, n\}$-Anosov representation. Let $x, y, z \in \partial\Gamma_g$ be three distinct points. Then :*

(i) x_ρ^{n-1} *and* $z_\rho^{n+1} \cap x_\rho^n$ *are transverse;*

(ii) x_ρ^{n-1} *and* $y_\rho^{n+1} \cap x_\rho^n$ *are transverse;*

(iii) $(y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$ *and* $z_\rho^{n+1} \cap x_\rho^n$ *are transverse;*

(iv) *if moreover* ρ *satisfies property* H_n*, then* $(y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$ *and* $y_\rho^{n+1} \cap x_\rho^n$ *are transverse.*

Proof. The transversality of the boundary maps between x and z implies that x_ρ^{n-1} and z_ρ^{n+1} have trivial intersection so x_ρ^{n-1} and $z_\rho^{n+1} \cap x_\rho^{n-1}$ intersect trivially. The same argument shows that, x_ρ^{n-1} and $y_\rho^{n+1} \cap x_\rho^n$ are disjoint.

The transversality of the boundary maps between y and z implies that y_ρ^{n-1} and z_ρ^{n+1} have trivial intersection. In particular let $\mathrm{v} \in (y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$ and $\mathrm{w} \in z_\rho^n$ be such that $\mathrm{v} + \mathrm{w} \in y_\rho^{n-1}$. Suppose that moreover $\mathrm{v} \in z_\rho^{n+1}$. Then $\mathrm{v} + \mathrm{w} \in y_\rho^{n-1} \cap z_\rho^{n+1}$ since $z_\rho^n \subset z_\rho^{n+1}$. Hence $\mathrm{v} + \mathrm{w} = 0$, so $\mathrm{v} \in x_\rho^n \cap z_\rho^n$. Therefore $\mathrm{v} = 0$. As a conclusion $(y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$ and $z_\rho^{n+1} \cap x_\rho^n$ are disjoint.

Finally property H_n implies that if we replace (x, y, z) by (z, x, y) in (5.2), the sum is direct, and hence $x_\rho^n \cap y_\rho^{n+1}$ intersects trivially $z_\rho^n \oplus y_\rho^{n-1}$. Therefore $(y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$ and $y_\rho^{n+1} \cap x_\rho^n$ are disjoint. $\square$

The main tool that we are going to use in Sections 5.4 and 5.5 is the following result from Pozzetti, Sambarino and Wienhard [PSW21].

Theorem 5.3.5 ([PSW21], Theorem 4.2)**.** *Let* $\rho : \Gamma_g \to \mathrm{SL}(N, \mathbb{R})$ *be a* $\{k-1, k, k+1\}$*-Anosov representation. If* ρ *satisfies property* H_k *then the map* $\xi_\rho^k : x \mapsto x_\rho^k$ *has* $\mathcal{C}^1$ *image, i.e.* $\xi_\rho^k(\partial\Gamma_g) \subset Gr(k, N)$ *is a* 1*-dimensional* $\mathcal{C}^1$ *submanifold.*

At the point x_ρ^k *this* 1*-dimensional submanifold of* $Gr(k, N)$ *is tangent to the curve consisting of spaces containing* x_ρ^{k-1} *and contained in* x_ρ^{k+1}*.*

We will be interested in the regularity of the boundary curve ξ_ρ^n, whose image lies in the space of Lagrangians $\mathcal{L}_n$ when $\rho(\Gamma_g) \subset \mathrm{Sp}(2n, \mathbb{R})$. Once an $\{n\}$-Anosov representation ρ has been fixed, given 3 points $x, y, z \in \partial\Gamma_g$ with $x, y \neq z$ we will write for simplicity :

$$\mathrm{Sym}_{x,z} := \mathrm{Sym}_{x_\rho^n, z_\rho^n} \;\; , \;\; u_{x,z}^y := u_{x_\rho^n, z_\rho^n}^{y_\rho^n} \in \mathrm{Sym}_{x,y} \;\; , \;\; q_{x,z}^y = q_{x_\rho^n, z_\rho^n}^{y_\rho^n} \in \mathcal{Q}(x_\rho^n).$$

Let us rephrase Theorem 5.3.5 in the charts $U_Q \simeq \mathrm{Sym}_{x,z}$ of the space of Lagrangians $\mathcal{L}_n$.

Lemma 5.3.6. *Let* $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ *be an* $\{n-1, n\}$*-Anosov representation that satisfies property* H_n*. Let* $x, z \in \partial\Gamma_g$ *be distinct points. For* $y \neq z$*, the tangent space at* $u_{x,z}^y$ *to the image of the map:*

$$w \in \partial\Gamma_g \backslash \{z\} \mapsto u_{x,z}^w$$

is the affine line of $\mathrm{Sym}_{x,z}$ *passing through* $u_{x,z}^y$ *and directed by the vector line of elements* $\dot{u} \in \mathrm{Sym}_{x,y}$ *such that one of the following equivalent statements holds:*

(i) $\left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n \subset \mathrm{Ker}(\dot{u})$,

(ii) $\mathrm{Im}(\dot{u}) \subset y_\rho^{n+1} \cap z_\rho^n$.

In particular such an element $\dot{u} \in \mathrm{Sym}_{x,z}$ must have rank 1.

Proof. Because of Theorem 5.3.5, the image of the boundary map ξ_ρ^n is $\mathcal{C}^1$, and tangent at y_ρ^n to the one dimensional submanifold ℓ of $\mathcal{L}_n$ consisting of Lagrangians P satisfying the condition :

$$y_\rho^{n-1} \subset P \subset y_\rho^{n+1}.$$

Since y_ρ^{n-1} is orthogonal to y_ρ^{n+1} with respect to ω, and since P is a Lagrangian, this is equivalent to $y_\rho^{n-1} \subset P$ which is equivalent to $P \subset y_\rho^{n+1}$.

An element $u' \in \mathrm{Sym}_{x,z}$ corresponds to a Lagrangian P satisfying $y_\rho^{n-1} \subset P$ if and only if for all $\mathrm{v} \in x_\rho^n$ such that $\mathrm{v} + u_{x,z}^y(\mathrm{v}) \in y_\rho^{n-1}$ one has $\mathrm{w} + u'(\mathrm{w}) = \mathrm{v} + u_{x,z}^y(\mathrm{v})$ for some $\mathrm{w} \in x_\rho^n$ that must be equal to v since $\mathrm{v} - \mathrm{w} \in x_\rho^n \cap z_\rho^n$. But $\mathrm{v} + u_{x,z}^y(\mathrm{v}) \in y_\rho^{n-1}$ if and only if $\mathrm{v} \in (y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$. Hence $y_\rho^{n-1} \subset P$ if and only if :

$$\left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n \subset \mathrm{Ker}(u' - u_{x,z}^y).$$

Similarly an element $u' \in \mathrm{Sym}_{x,z}$ corresponds to a Lagrangian P satisfying $P \subset y^{n+1}$ if and only if for all $\mathrm{v} \in x_\rho^n$, $\mathrm{v} + u'(\mathrm{v}) \in y_\rho^{n+1}$. However $\mathrm{v} + u_{x,z}^y(\mathrm{v}) \in y_\rho^n \subset y_\rho^{n+1}$. Hence $P \subset y_\rho^{n+1}$ if and only if for all $\mathrm{v} \in x_\rho^n$, $u'(\mathrm{v}) - u_{x,z}^y(\mathrm{v}) \in y_\rho^{n+1}$, or in other words:

$$\mathrm{Im}(u' - u_{x,z}^y) \subset y_\rho^{n+1} \cap z_\rho^n.$$

Therefore the image ℓ' of the submanifold ℓ in the chart $\mathrm{Sym}_{P,Q}$ is the affine line directed by symmetric endomorphisms $\dot{u}$ satisfying $\left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n \subset \mathrm{Ker}(\dot{u})$, or equivalently $\mathrm{Im}(\dot{u}) \subset y_\rho^{n+1} \cap z_\rho^n$. Such a non-zero element must have rank 1.

□

Theorem 5.3.5 can be also rephrased in the chart $U_Q \simeq \mathcal{Q}(x_\rho^n)$ of $\mathcal{L}_n$. Recall that singular subspaces for a symmetic bilinear form were defined in Definition 5.2.4.

Lemma 5.3.7. *Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{n-1, n\}$-Anosov representation that satisfies property H_n. Let $x, z \in \partial\Gamma_g$ be distinct points. For $y \neq z$, the tangent space at $q_{x,z}^y$ to the image of the map*

$$w \in \partial\Gamma_g \backslash \{z\} \mapsto q_{x,z}^w$$

is the affine line of $\mathcal{Q}(x_\rho^n)$ passing through $q_{x,z}^y$ and directed by the vector line of elements $\dot{q} \in \mathrm{Sym}_{x,z}$ such that the hyperplane $\left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n$ is singular for $\dot{q}$. In particular such an element $\dot{q} \in \mathcal{Q}(x_\rho^n)$ must have signature $(1,0)$ *or* $(0,1)$.

Proof. Let ℓ' be the affine line in $\mathrm{Sym}_{x,z}$ defined in the proof of Lemma 5.3.6 part (i). The affine line $\tilde{\ell}$ in $\mathcal{Q}(x_\rho^n)$ corresponding to ℓ' via the linear identification $\mathrm{Sym}_{x,z} \simeq \mathcal{Q}(x_\rho^n)$ is directed by the elements $\dot{q} \in \mathcal{Q}(x_\rho^n)$ such that for some $\dot{u} \in \mathrm{Sym}_{P,Q}$ satisfying (i), and for all $\mathrm{v} \in x_\rho^n$ and $\mathrm{w} \in (y_\rho^{n-1} \oplus z_\rho^n) \cap x_\rho^n$, one has $\dot{q}(\mathrm{v}, \mathrm{w}) = \omega(\mathrm{v}, \dot{u}(\mathrm{w})) = 0$.

In other words $\tilde{\ell}$ is directed by the non-zero elements $\dot{q} \in \mathcal{Q}(x_\rho^n)$ such that $\dot{q}(\mathrm{v}, \mathrm{w}) = 0$ for all $\mathrm{v} \in z_\rho^n$ and $\mathrm{w} \in \left(y_\rho^{n-1} \oplus x_\rho^n\right) \cap z_\rho^n$, i.e. such that $\left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n$ is singular for $\dot{q}$.

Since $\dot{q}$ is non-zero but admits a singular hyperplane, its signature is equal to $(1, 0)$ or $(0, 1)$ □

A first application of this result is the following lemma, which will be used in the proof of Lemma 5.5.1.

Lemma 5.3.8. *Let ρ be a $\{n-1, n\}$-Anosov representation that satisfies property H_n. Let $z \in \partial\Gamma_g$. The map that associates to $y \in \partial\Gamma_g \backslash \{z\}$ the hyperplane $y_\rho^{n-1} \oplus z^n \subset \mathbb{R}^{2n}$ is constant on no open interval.*

Proof. Let $x \in \partial\Gamma_g$ be any point distinct from z. Let ψ be the map that associates to an element $y \in \partial\Gamma_g \backslash \{z\}$ the following hyperplane of x_ρ^n:

$$\psi(y) = \left(y_\rho^{n-1} \oplus z_\rho^n\right) \cap x_\rho^n.$$

If this map was constant on some open interval I, Lemma 5.3.6 would imply that the image by $y \mapsto u_{x,z}^y \in \mathrm{Sym}_{x,z}$ restricted to I has a tangent direction which is always a rank one symmetric element with constant kernel $\psi(x)$.

However in this case $u_{x,z}^y$ would be the integral of some elements $\dot{u} \in \mathrm{Sym}_{x,z}$ whose kernel always contains $\psi(x)$. In particular $u_{x,z}^y$ would have rank at most 1. However this would imply that this element has a kernel, and hence y_ρ^n has a non-trivial intersection with x_ρ^n. This would contradict the transversality of the boundary maps (Proposition 5.1.5).

Hence the map ψ cannot be constant on any open interval. □

5.4 Relation between maximality and property H_n.

Our goal will be to prove that a $\{n-1, n\}$-Anosov representation ρ is maximal if and only if it satisfies property H_n. In order to prove property H_n implies maximality we will use the smoothness of the n-th boundary curve and the following simple geometric fact.

A *closed cone* of a vector space is a closed subset that is stable by addition and multiplication by positive scalars.

Lemma 5.4.1. *Let V be a real vector space and S be a closed cone in V, Let $\eta : \mathbb{R} \to V$ be a $\mathcal{C}^1$ curve such that for all $t \in \mathbb{R}$, $\eta'(t) \in S$ and $\eta(0) \in S$. Then, for all $t \geqslant 0$, $\eta(t) \in S$.*

In other words, if the derivative of a curve stays in a closed cone and if the curve is in the closed cone initially, then the curve stays in this closed cone. We will use this fact again to prove Lemma 5.5.4.

Proof. Let $t \geqslant 0$ be a real number, the we can write $\eta(t)$ as :

$$\eta(t) = \eta(0) + \int_0^t \eta'(s)\mathrm{d}s.$$

Hence $\eta(t)$ can be approximated by finite sums of elements in S, which are also in S since S is a cone. Moreover S is closed so $\eta(t) \in S$. $\square$

Now we prove the following characterization of maximal representations that are $\{n-1, n\}$-Anosov.

Theorem 5.4.2. *Let $1 \leqslant k \leqslant n$. Let $\rho : \Gamma_g \to \mathrm{Sp}(2n, \mathbb{R})$ be a $\{n-1, n\}$-Anosov representation. The representation ρ satisfies property H_n if and only if it is maximal for some orientation of $\partial\Gamma_g$.*

Proof. Suppose first that ρ is maximal for some orientation of Γ_g. Let (x, y, z) be a positive triple of distinct points in $\partial\Gamma_g$. Suppose that the sum (5.1) is not direct, i.e. that there is a vector h belonging to the intersection:

$$\left(\left(x_\rho^n \cap z_\rho^{n+1}\right) \oplus z_\rho^{n-1}\right) \cap \left(y_\rho^n \cap z_\rho^{n+1}\right).$$

Note that in this expression, $\left(x_\rho^n \cap z_\rho^{n+1}\right) \oplus z_\rho^{n-1}$ is direct since $x_\rho^n \cap z_\rho^{n-1} = \{0\}$. In particular $\mathrm{h} = \mathrm{v} + \mathrm{w}$ for some $\mathrm{v} \in x_\rho^n \cap z_\rho^{n+1}$, $\mathrm{w} \in z_\rho^{n-1}$. Moreover $\mathrm{h} \in y_\rho^n$ so $u_{x,z}^y(\mathrm{v}) = \mathrm{w}$ with $u_{x,z}^y \in \mathrm{Sym}_{x,z}$ the element corresponding to y_ρ^n.

Lemma 5.1.7 implies that z_ρ^{n+1} and z_ρ^{n-1} are orthogonal with respect to ω, so $\omega(\mathrm{v}, \mathrm{w}) = 0$. Thus the symmetric bilinear form $q_{x,z}^y$ associated to $u_{x,z}^y$ satisfies $q_{x,z}^y(\mathrm{v}, \mathrm{v}) = \omega\left(\mathrm{v}, u_{x,z}^y(\mathrm{v})\right) = 0$. However $q_{x,z}^y$ is positive since ρ is maximal and (x, y, z) is positive. Hence $\mathrm{v} = \mathrm{w} = 0$ and the desired sum of spaces is direct. This means that the sum is direct for all positive triples (x, y, z), but since (5.1) stays invariant when x and y are exchanged, this sum is direct for all triples. Therefore if ρ is maximal, then property H_n holds.

Conversely, let us suppose that ρ satisfies H_n. Let $x, z \in \partial\Gamma_g$ be distinct points. Lemma 5.3.7 implies that there exists a parametrization $\phi : \mathbb{R} \to \partial\Gamma_g \backslash \{z\}$ which is a homeomorphism such that $f : \mathbb{R} \to \mathcal{Q}(x_\rho^n)$, $t \mapsto q_{x,z}^{\phi(t)}$ is a $\mathcal{C}^1$ embedding.

The derivative of f at all times is non-zero and has signature $(1, 0)$ or $(0, 1)$. Up to considering $t \mapsto \phi(-t)$, we can assume that at some point the derivative of f has signature $(1, 0)$, and hence it has signature $(1, 0)$ for all points $t \in \mathbb{R}$.

Let us assume that $\phi(0) = x$. The derivative of $t \mapsto q_{x,z}^{\phi(t)}$ has signature $(1, 0)$, hence it belongs to the closed cone $\overline{\mathcal{Q}^+(x_\rho^n)}$ of semi-positive elements. Moreover $q_{x,z}^x = 0$ is also in this closed cone. Hence by Lemma 5.4.1 the image of $\mathbb{R}_{\geqslant 0}$ consists only of semi-positive elements.

For $t > 0$, the transversality of boundary maps implies that:

$$x_\rho^n \cap \xi_\rho^n \circ \phi(t) = \{0\}.$$

Hence $q^{\phi(t)}_{x,z}$ is a non-degenerate symmetric bilinear form. Since it belongs to $\overline{\mathcal{Q}^+(x^n_\rho)}$, it is positive. Therefore the triple of Lagrangians $(x^n_\rho, \xi^n_\rho \circ \phi(t), z^n_\rho)$ is maximal for all $t > 0$.

Hence for at least one triple of distinct points $x, y, z \in \partial\Gamma_g$, the triple $(x^n_\rho, y^n_\rho, z^n_\rho)$ is maximal. Because of Remark 5.2.8, this holds for all triple of distinct points (x', y', z') ordered as (x, y, z) in $\partial\Gamma_g$. Therefore for the orientation of $\partial\Gamma_g$ such that (x, y, z) is positive, the representation ρ is maximal. ☐

5.5 From property H_2 to H_1 for $\mathrm{Sp}(4, \mathbb{R})$.

We proved in Section 5.4 that any maximal and $\{n-1, n\}$-Anosov representation satisfies property H_n. This means that we can use Theorem 5.3.5 to get more information on the boundary curve. In this section we prove that if additionally $n = 2$, such a representation must also satisfy property H_1.

For a triple (x, y, z) of distinct points in the circle $\partial\Gamma_g$, let $(x, y)_z$ and $[x, y]_z$ be respectively the open and closed arc in the circle $\partial\Gamma_g$ between x and y not containing z.

Before we prove the key result of this section in Lemma 5.5.4, we need to find a positive triple of points in $\partial\Gamma_g$ that satisfies the following lemma. Given a triple (x, w, z) in $\partial\Gamma_g$ such that $x, w \neq z$ we define:

$$\psi(w) = \left(w^1_\rho \oplus z^2_\rho\right) \cap x^2_\rho \in \mathbb{P}(x^2_\rho).$$

Lemma 5.5.1. *Let $\rho : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ be a $\{1, 2\}$-Anosov representation that is maximal for some orientation of $\partial\Gamma_g$. There exists a positive triple (x, y, z) in $\partial\Gamma_g$ such that $\psi(x) \neq \psi(y)$ and for all $w \in (x, y)_z$ and $\psi(w) \neq \psi(x), \psi(y)$.*

Proof. Let $z \in \partial\Gamma_g$ be any point. Let ψ_0 be the map that associates to an element $w \in \partial\Gamma_g \backslash \{z\}$ the hyperplane $w^1_\rho \oplus z^2_\rho \subset \mathbb{R}^4$. Theorem 5.4.2 implies that ρ satisfies property H_n, and because of Lemma 5.3.8 the map ψ_0 is not constant.

In particular we can find some distinct $x_0, y_0 \in \partial\Gamma_g \backslash \{z\}$ such that $\psi_0(x_0) \neq \psi_0(y_0)$. Let $x \in [x_0, y_0]_z$ be the unique point such that $\psi_0(x) = \psi_0(x_0)$ and for all $w \in (x, y_0)_z$, $\psi_0(w) \neq \psi_0(x)$. Then define similarly $y \in [x, y_0]_z$ as the unique point such that $\psi_0(y) = \psi_0(y_0)$ and for all $w \in (x, y)_z$, $\psi_0(w) \neq \psi_0(y)$.

Hence $\psi_0(x) \neq \psi_0(y)$, and for all $w \in (x, y)_z$, $\psi_0(w) \neq \psi_0(x), \psi_0(y)$. Up to exchanging x and y we can assume that (x, y, z) is a positive triple.

Now that we fixed a triple (x, y, z), the map ψ is defined. For all $w \neq z$, $\psi_0(w) = \psi(w) \oplus z^2_\rho$. Hence $\psi(x) \neq \psi(y)$ and for all $w \in (x, y)_z$, $\psi(w) \neq \psi(x), \psi(y)$. The triple (x, y, z) satisfies the desired condition.

☐

Let P, Q be two Lagrangians in $\mathbb{R}^4$. The space $\mathbb{P}(\mathcal{Q}^+(P))$ of positive symmetric bilinear forms on P up to a scalar is a projective model of the hyperbolic plane $\mathbb{H}^2$. There is a natural identification $\iota : \mathbb{P}(P) \to \mathbb{P}(\partial\mathcal{Q}^+(P))$. To a line

$\ell \in \mathbb{P}(P)$ we can associate the line $\iota(\ell)$ of symmetric bilinear elements $q \in \mathcal{Q}(P)$ for which ℓ is singular (see Definition 5.2.4).

Recall that we use Notation 5.1.4 for the boundary maps of an Anosov representation. Given a $\{1,2\}$-Anosov representation ρ that satisfies property H_2, the fact that $y^1_\rho \subset y^2_\rho \subset y^3_\rho$ implies the following result.

Lemma 5.5.2. *Let $\rho : \Gamma_g \to \mathrm{Sp}(4,\mathbb{R})$ be a $\{1,2\}$-Anosov representation that satisfies property H_2. Let $(x,y,z) \in \partial\Gamma_g$ be distinct points. The point $\left[q^y_{x,z}\right] \in \mathbb{P}(\mathcal{Q}^+(x^2_\rho))$ lies in the projective line between the two elements of $\mathbb{P}(\partial\mathcal{Q}^+(x^2_\rho))$:*

$$\iota(y^3_\rho \cap x^2_\rho)\ ,\ \iota\left((y^1_\rho \oplus z^2_\rho) \cap x^2_\rho\right).$$

This projective line is illustrated as a dotted line in Figure 5.2. Through the identification $\mathbb{P}(\mathcal{Q}^+(x^2_\rho)) \cong \mathbb{H}^2$, this line corresponds to a geodesic.

Proof. Let $\mathrm{v} \in y^3_\rho \cap x^2_\rho$ be a non-zero vector. One has $u^y_{x,z}(\mathrm{v}) + \mathrm{v} \in y^2_\rho \subset y^3_\rho$ and hence $u^y_{x,z}(\mathrm{v}) \in y^3_\rho \cap z^2_\rho$.

Let $u_0 \in \mathrm{Sym}_{x,y}$ be such that $\mathrm{Ker}(u_0) = (y^1_\rho \oplus z^2_\rho) \cap x^2_\rho$. Since u_0 is symmetric, $\mathrm{Im}(u_0)$ is orthogonal with respect to ω to $\mathrm{Ker}(u_0)$. Since y^1_ρ and y^3_ρ are orthogonal with respect to ω, then $\mathrm{Im}(u_0) = y^3_\rho \cap z^2_\rho$. Hence $u_0(\mathrm{v}) \in y^3_\rho \cap z^2_\rho$, therefore $u_0(\mathrm{v})$ and $u^y_{x,z}(\mathrm{v})$ are collinear.

By the part (iv) of Lemma 5.3.4 and since ρ satisfies property H_2, $y^3_\rho \cap x^2_\rho \cap \mathrm{Ker}(u_0) = \{0\}$, and hence $u_0(\mathrm{v}) \neq 0$. Therefore, for some $\lambda \in \mathbb{R}$, $u_1(\mathrm{v}) = 0$ with $u_1 = u^y_{x,z} - \lambda u_0$. In particular $q^y_{x,z} = q_1 + \lambda q_0$ where $q_0, q_1 \in \mathcal{Q}(P)$ are such that $(y^1_\rho \oplus z^2_\rho) \cap x^2_\rho$ is singular for q_0 and $y^3_\rho \cap x^2_\rho$ is singular for q_1. Hence $q_1 \in \iota(y^3_\rho \cap x^2_\rho)$ and $q_2 \in \iota\left((y^1_\rho \oplus z^2_\rho) \cap x^2_\rho\right)$, which concludes the proof. □

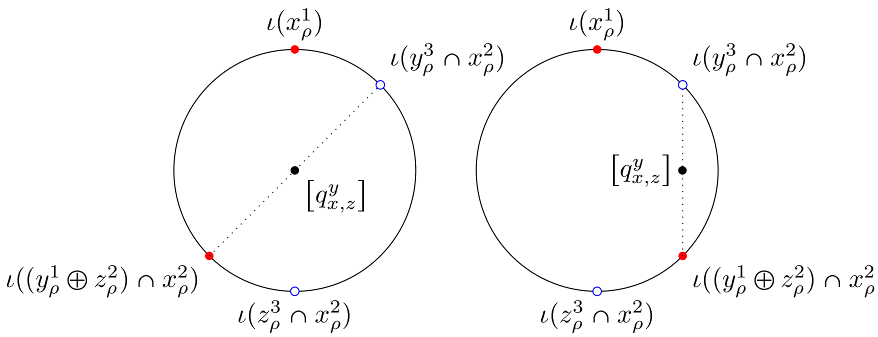


Figure 5.2: Two possible configurations of the image by ι of the points in (5.4) in $\mathbb{P}(\mathcal{Q}(x^2_\rho))$.

In order to state Lemma 5.5.4, we need to define a notion of *cyclically oriented* quadruple on a topological circle.

Definition 5.5.3. Let V be a vector space of dimension 2. A quadruple (a,b,c,d) of points in $\mathbb{P}(V)$ is *cyclically ordered* if b and d are in different components of $\mathbb{P}(V)\backslash\{a,c\}$, or equivalently if the following cross ratio is negative:

$$\mathrm{cr}(a,b;c,d) := \frac{\overline{a}\wedge\overline{b}}{\overline{c}\wedge\overline{b}} \times \frac{\overline{c}\wedge\overline{d}}{\overline{a}\wedge\overline{d}}. \tag{5.3}$$

Here $\overline{a},\overline{b},\overline{c},\overline{d}$ are any non-zero vectors representing the lines a,b,c,d.

The key argument of the proof of Theorem 5.5.5 is the Lemma 5.5.4. We will use the geometric fact from Lemma 5.4.1 that a curve whose derivative lies in a cone must remain in that cone.

Lemma 5.5.4. *Let $\rho:\Gamma_g\to\mathrm{Sp}(4,\mathbb{R})$ be a $\{1,2\}$-Anosov representation that satisfies property H_2. There exists some triple of distinct points $x,y,z\in\partial\Gamma_g$ such that the quadruple*

$$(\ z^3_\rho\cap x^2_\rho,\ \ (y^1_\rho\oplus z^2_\rho)\cap x^2_\rho,\ \ y^3_\rho\cap x^2_\rho,\ \ x^1_\rho\ ) \tag{5.4}$$

is cyclically ordered in $\mathbb{P}(x^2_\rho)$.

Figure 5.2 illustrates this Lemma. The depicted filled points are distinct from the unfilled ones because of Lemma 5.3.4. The 4 points depicted are cyclically ordered as in (5.4) on the right picture, but not on the left.

Proof. Our goal is to find $x,y,z\in\partial\Gamma_g$ such that the 4 points in (5.4) are cyclically ordered, i.e. are not ordered as in the left part of Figure 5.2. Because of Theorem 5.4.2, we can choose an orientation of $\partial\Gamma_g$ such that ρ is maximal. Let (x,y,z) be a positive triple in $\partial\Gamma_g$ that satisfies the properties from Lemma 5.5.1. Let $\psi(w)=\left(w^1_\rho\oplus z^2_\rho\right)\cap x^2_\rho$ for $w\neq z$ as defined in Lemma 5.5.1. Note that $\psi(x)=x^1_\rho$.

Let us assume that the four points in (5.4) are not cyclically ordered for the triple (x,y,z). This means that $y^3_\rho\cap x^2_\rho$ and $z^3_\rho\cap x^2_\rho$ are in the same connected component of $\mathbb{P}(x^2_\rho)\backslash\{\psi(x),\psi(y)\}$.

The linear plane in $\mathcal{Q}(x^2_\rho)$ passing through $\iota(\psi(x))$ and $\iota(\psi(y))$ cuts the closure $\overline{\mathcal{Q}^+(x^2_\rho)}$ of the cone of scalar products into two closed cones. Let C be the closure of the one whose projectivization satisfies $\iota(y^3_\rho\cap x^2_\rho),\iota(z^3_\rho\cap x^2_\rho)\notin\mathbb{P}(C)$.

The convex set $\mathbb{P}(C)$ is illustrated as the colored region in Figure 5.3. One has $\iota(\psi(x)),\iota(\psi(y))\in\partial\mathbb{P}(C)$. Moreover Lemma 5.5.2 implies that $[q^y_{x,z}]$ lies in the segment between $\iota(\psi(y))$ and $\iota(y^3_\rho\cap x^2_\rho)$. As a consequence $[q^y_{x,z}]\notin\mathbb{P}(C)$.

Because of Lemma 5.3.7, there exists a parametrization $\phi:\mathbb{R}\to\partial\Gamma_g\backslash\{z\}$, such that $\phi(0)=x,\phi(1)=y$ and $\xi^2_\rho\circ\phi$ is a $\mathcal{C}^1$ embedding. Let $\dot q(t_0)$ be the derivative at $t=t_0$ of the map:

$$t\mapsto q^{\phi(t)}_{x,z}.$$

Since (x,y,z) is positive, for any $t_0\in\mathbb{R}$, $\dot q(t_0)$ is an element of $\overline{\mathcal{Q}^+(x^2_\rho)}$, whose projectivization is $\iota(\psi(\phi(t_0)))$.

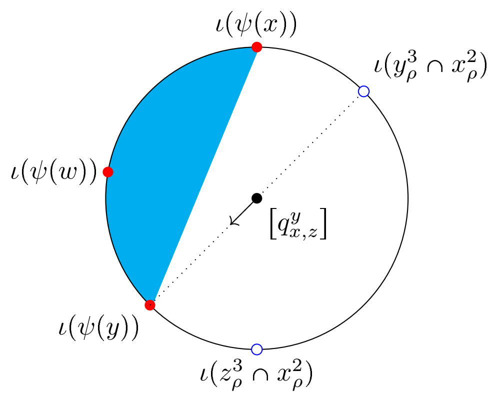


Figure 5.3: $\mathbb{P}(\mathcal{Q}(x^2_\rho))$ and the convex $\mathbb{P}(C)$ from the proof of Lemma 5.5.4.

The map $g : t \mapsto \iota(\psi(\phi(t)))$ is continuous from $[0,1]$ to the circle $\mathbb{P}(x^2_\rho)$. Because (x,y,z) was chosen as in Lemma 5.5.2, one has $g(0) \neq g(1)$ and for $t \in (0,1)$, $g(t) \neq g(0), g(1)$. Hence $g([0,1])$ is equal to one of the two arcs joining $g(0)$ and $g(1)$. Because of Lemma 5.3.4, $\iota(z^3_\rho \cap x^2_\rho)$ is not in the image of this map.

Therefore $g([0,1]) = \iota(\psi([x,y]_z))$ is equal to the closed arc in $\iota(\mathbb{P}(x^2_\rho))$ between $\iota(\psi(x))$ and $\iota(\psi(y))$ not containing $\iota(z^3_\rho \cap x^2_\rho)$. In particular $\iota(\psi(w)) \in \mathbb{P}(C)$ for $w \in [x,y]_z$. Hence for all $t \in [0,1]$, $\dot{q}(t) \in C$.

Moreover $0 = q^x_{x,z} = q^{\phi(0)}_{x,z} \in C$. Hence, since C is a closed cone, by Lemma 5.4.1 $q^{\phi(t)}_{x,z} \in C$ for all $t \in [0,1]$. But this would imply that $q^y_{x,z} \in C$. We proved already that $[q^y_{x,z}] \notin \mathbb{P}(C)$, so this is a contradiction.

Hence the four points (5.4) are cyclically ordered for this choice of a triple (x,y,z).

$\square$

Theorem 5.5.5. *Let $\rho : \Gamma_g \to \mathrm{Sp}(4,\mathbb{R})$ be a $\{1,2\}$-Anosov representation that satisfies property H_2. The representation ρ must satisfy property H_1.*

Proof. Let ρ be a $\{1,2\}$-Anosov representation that satisfies property H_2. By Theorem 5.4.2, ρ is maximal. Because of Lemma 5.5.4, there exist a triple of distinct points (x,y,z) in $\partial\Gamma_g$ such that the quadruple (5.4) is cyclically ordered. This is for instance the case in the right part of Figure 5.2.

Let (u,v,w) be a triple of distinct points on $\partial\Gamma_g$ that is oriented as (x,y,z). These two triples are joined by a continuous path in the space of disjoint triples in $\partial\Gamma_g$. Along this path, the cross ratio of the points (5.4) is defined and cannot vanish, because of Lemma 5.3.4. Hence the cross ratio of these points stays negative. In particular for every triple of distinct points (u,v,w) that are oriented in the circle $\partial\Gamma_g$ as (x,y,z), the following 4 points are cyclically ordered:

$$(\ w^3_\rho \cap u^2_\rho, \ \ (v^1_\rho \oplus w^2_\rho) \cap u^2_\rho, \ \ v^3_\rho \cap u^2_\rho, \ \ u^1_\rho \ )$$

In particular $w^3_\rho \cap u^2_\rho \neq v^3_\rho \cap u^2_\rho$, therefore the following sum is direct :

$$w^3_\rho \cap u^2_\rho + v^3_\rho \cap u^2_\rho + u^0_\rho.$$

Since this expression is invariant if one exchanges v and w, this holds for all triple (u,v,w) of distinct points in $\partial\Gamma_g$. Therefore property H_3 holds for ρ. Finally, because of Proposition 5.3.3, property H_1 holds for ρ.

□

We end this section by presenting a Proposition that describes the behavior of $y \mapsto [q^y_{x,z}]$. This proposition is not used in the proof of the main theorem but it helps to understand Figure 5.2.

Proposition 5.5.6. *Let $\rho : \Gamma_g \to \mathrm{Sp}(4,\mathbb{R})$ be a $\{1,2\}$-Anosov representation that satisfies property H_2. Let $x, z \in \partial\Gamma_g$ be two distinct points.*

(i) The limit in $\mathbb{P}(\mathcal{Q}(x^2_\rho))$ of $\left[q^y_{x,z}\right]$ is $\iota(x^1_\rho) \in \partial\mathbb{P}(\mathcal{Q}^+(x^2_\rho))$ when y converges to x.

(ii) If moreover ρ satisfies property H_1, the limit in $\mathbb{P}(\mathcal{Q}(x^2_\rho))$ of $\left[q^y_{x,z}\right]$ when y converges to z is $\iota(z^3_\rho \cap x^2_\rho) \in \partial\mathbb{P}(\mathcal{Q}^+(x^2_\rho))$.

Because of Theorem 5.5.5, it is actually not necessary to require property H_1 for part (ii).

Proof. Because of Lemma 5.3.7, there is a parametrization $\phi : \mathbb{R} \to \partial\Gamma_g \backslash \{z\}$ such that $t \mapsto q^{\phi(t)}_{x,z}$ is a $\mathcal{C}^1$ embedding. Let us assume that $\phi(0) = x$ and let $\dot{q}(t_0)$ be the derivative of $t \mapsto q^{\phi(t)}_{x,z}$ at $t = t_0$. Since $q^x_{x,z} = 0$, one can write for t close to 0:

$$q^{\phi(t)}_{x,z} = t\dot{q}(0) + t\epsilon(t),$$

with $\epsilon(t) \to 0$ when $t \to 0$.

This implies that the limit of $\mathbb{P}(q^{\phi(t)}_{x,z})$ when t goes to 0 is equal to $\mathbb{P}(\dot{q}) = \iota\left((x^1_\rho \oplus z^2_\rho) \cap x^2_\rho\right) = \iota(x^1_\rho)$, which proves (i).

Let's now prove (ii). A representation ρ satisfies property H_1, if and only if it is $(1,2,3)$-hyperconvex in the sense of ([PSW21] Definition 6.1). Hence ([PSW21], Theorem 7.1) implies that the hyperplane $y^1_\rho \oplus z^2_\rho$ converges to z^3_ρ when y converges to z.

Therefore $[\dot{q}(t)] \in \mathbb{P}(\mathcal{Q}(x^2_\rho))$ converges to $\iota(z^3_\rho \cap x^2_\rho)$. Hence for any small closed cone C in $\mathcal{Q}^+(x^2_\rho)$ containing $\iota(z^3_\rho \cap x^2_\rho)$ in its interior, there is an affine cone C_0 directed by C such that for any t big enough $q^{\phi(t)}_{x,z} \in C_0$.

Since $q^y_{x,z}$ diverges when $y \to z$, then any subsequence of $[q^y_{x,z}]$ converges to a point in $\mathbb{P}(C)$. Hence for $t \to +\infty$, $[q^y_{x,z}]$ converges to $\iota(z^3_\rho \cap x^2_\rho)$. When $t \to -\infty$, a similar argument holds.

□

5.6 Hyperconvex representations.

Let $N \geqslant 2$ be an integer. Labourie introduced the notion of an hyperconvex representation.

Definition 5.6.1. A Borel Anosov representation ρ is hyperconvex if for all distinct $x_1, \cdots, x_N \in \partial\Gamma_g$, the N lines $(x_1)^1_\rho, (x_2)^1_\rho, \cdots, (x_N)^1_\rho$ span the whole vector space $\mathbb{R}^N$.

A Borel Anosov representation ρ is $\{a, b, c\}$-hyperconvex if for all $x, z, y \in \partial\Gamma_g$ distinct, then the following sum is direct :

$$x^a_\rho + y^b_\rho + z^c_\rho.$$

If ρ is $\{a, b, c\}$-hyperconvex for all $1 \leqslant a \leqslant b \leqslant c$ such that $a + b + c \leqslant 2n$, then we say that it is 3-hyperconvex.

Remark 5.6.2. A representation is $\{a, b, c\}$-hyperconvex in this sense if and only if it is $(a, b, N - c)$-hyperconvex in the sense of [PSW21] .

The following theorem of Labourie [Lab06] will enable us to show hyperconvexity using property H for any maximal and Borel Anosov representation in $\mathrm{Sp}(4, \mathbb{R})$.

Theorem 5.6.3 ([Lab06, Lemma 7.1])**.** *Every Borel Anosov representation that satisfies property H and that is 3-hyperconvex is hyperconvex.*

Then the following theorem of Guichard [Gui08] will enable us to show that hyperconvex representation are Hitchin. This theorem is one part of the characterization of Hitchin representations by the hyperconvexity condition. The other part was proved by Labourie [Lab06].

Theorem 5.6.4 ([Gui08, Theorem 1])**.** *Any Borel Anosov and hyperconvex representation $\rho : \Gamma_g \to \mathrm{SL}(N, \mathbb{R})$ is Hitchin.*

Finally we can prove our main Theorem.

Theorem 5.6.5. *Every representation $\rho : \Gamma_g \to \mathrm{Sp}(4, \mathbb{R})$ that is maximal and Borel Anosov is Hitchin.*

Proof. Because of Theorem 5.4.2 and Theorem 5.5.5, the representation ρ must satisfy property H_2, H_1, and therefore H_3 by Proposition 5.3.3. Hence ρ satisfies property H.

When $n = 2$, property H_1 is equivalent to $\{1, 1, 2\}$-hyperconvexity for a representation ρ. Moreover if $a + b + c = 4$ with $a, b, c \geqslant 1$ then $\{a, b, c\}$ is equal to $\{1, 1, 2\}$. Therefore the representation ρ is 3-hyperconvex. By Theorem 5.6.3, the representation ρ is hyperconvex, and by Theorem 5.6.4 it is Hitchin.

□

Bibliography

[ADL21] Daniele Alessandrini, Colin Davalo, and Qiongling Li, *Projective structures with (quasi-)hitchin holonomy*, 2021.

[Ale19] Daniele Alessandrini, *Higgs bundles and geometric structures on manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl. **15** (2019), Paper 039, 32.

[AMTW23] Daniele Alessandrini, Sara Maloni, Nicolas Tholozan, and Anna Wienhard, *Fiber bundles associated with anosov representations.*, 2023.

[BCGP+21] Steven B. Bradlow, Brian Collier, Oscar Garcia-Prada, Peter Gothen, and André Oliveira, *A general cayley correspondence and higher teichmüller spaces*, 2021.

[Ben97] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. **7** (1997), no. 1, 1–47.

[BG09] Jairo Bochi and Nicolas Gourmelon, *Some characterizations of domination*, Math. Z. **263** (2009), no. 1, 221–231.

[BGPG12] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen, *Deformations of maximal representations in* $\mathrm{Sp}(4,\mathbb{R})$, Q. J. Math. **63** (2012), no. 4, 795–843.

[BILW05] Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590.

[BIW03] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Surface group representations with maximal Toledo invariant*, Comptes Rendus Mathematique **336** (2003), no. 5, 387–390.

[BIW11] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Higher teichmüller spaces: from sl(2,r) to other lie groups*, 2011, 1004.2894.

[BP17] Marc Burger and Maria Beatrice Pozzetti, *Maximal representations, non-Archimedean Siegel spaces, and buildings*, Geom. Topol. **21** (2017), no. 6, 3539–3599.

[BP21] Jonas Beyrer and Beatrice Pozzetti, *Positive surface group representations in po(p,q)*, 2021.

[BPS19] Jairo Bochi, Rafael Potrie, and Andrés Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3343–3414.

[BT17] Jean-Philippe Burelle and Nicolaus Treib, *Schottky groups and maximal representations*, Geometriae Dedicata **195** (2017), no. 1, 215–239.

[CG05] Suhyoung Choi and William M. Goldman, *The deformation spaces of convex* $\mathbb{RP}^2$*-structures on 2-orbifolds*, Amer. J. Math. **127** (2005), no. 5, 1019–1102.

[Che93] Qing Cheng, *Space-like surfaces in an anti-de sitter space*, Colloquium Mathematicum **66** (1993), 201–208.

[CTT19] Brian Collier, Nicolas Tholozan, and Jérémy Toulisse, *The geometry of maximal representations of surface groups into* $\mathrm{SO}_0(2,n)$, Duke Math. J. **168** (2019), no. 15, 2873–2949.

[Dav23] Colin Davalo, *Nearly geodesic immersions and domains of discontinuity*, 2023, 2303.11260.

[Dav24] ———, *Maximal and borel anosov representations into sp(4,r)*, Advances in Mathematics **442** (2024), 109578.

[DGK18] Jeffrey Danciger, François Guéritaud, and Fanny Kassel, *Convex cocompact actions in real projective geometry*, 2018, 1704.08711.

[DR24] Colin Davalo and Max Riestenberg, *Finite-sided dirichlet domains for anosov representations*, 2024.

[Du23] Yukun Du, *Geometry of selberg's bisectors in the symmetric space* $sl(n,\mathbb{R})/so(n,\mathbb{R})$, 2023, 2302.00643.

[Ebe96] Patrick B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[Eps86] Charles L. Epstein, *The hyperbolic Gauss map and quasiconformal reflections*, J. Reine Angew. Math. **372** (1986), 96–135.

[ES22] Christian El Emam and Andrea Seppi, *On the gauss map of equivariant immersions in hyperbolic space*, 2022, 2008.07390.

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211.

[FMS21] C. Fevola, Y. Mandelshtam, and B. Sturmfels, *Pencils of quadrics: old and new*, Matematiche (Catania) **76** (2021), no. 2, 319–335.

[Gal05] Światosław R. Gal, *On normal subgroups of Coxeter groups generated by standard parabolic subgroups*, Geom. Dedicata **115** (2005), 65–78.

[GGKW17a] François Guéritaud, Olivier Guichard, Fanny Kassel, and Anna Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), no. 1, 485–584.

[GGKW17b] ———, *Compactification of certain Clifford-Klein forms of reductive homogeneous spaces*, Michigan Math. J. **66** (2017), no. 1, 49–84.

[GKW15] Olivier Guichard, Fanny Kassel, and Anna Wienhard, *Tameness of riemannian locally symmetric spaces arising from anosov representations*, 2015, 1508.04759.

[GLW21] Olivier Guichard, François Labourie, and Anna Wienhard, *Positivity and representations of surface groups*, 2021, 2106.14584.

[Gol80] William Goldman, *Discontinuous groups and the euler class.*

[Gol22] William M. Goldman, *Geometric structures on manifolds*, Graduate Studies in Mathematics, vol. 227, American Mathematical Society, Providence, RI, [2022] ©2022.

[Gro81] M. Gromov, *Hyperbolic manifolds, groups and actions*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213.

[Gui08] Olivier Guichard, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, J. Differential Geom. **80** (2008), no. 3, 391–431.

[GW10] Olivier Guichard and Anna Wienhard, *Topological invariants of Anosov representations*, J. Topol. **3** (2010), no. 3, 578–642.

[GW12] ———, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), no. 2, 357–438.

[GW22] ———, *Generalizing Lusztig's total positivity*, preprint, August 2022.

[Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[Hit92] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473.

[HSWW18] Thomas Haettel, Anna-Sofie Schilling, Cormac Walsh, and Anna Wienhard, *Horofunction compactifications of symmetric spaces*, 2018, 1705.05026.

[HW15] Zheng Huang and Biao Wang, *Counting minimal surfaces in quasi-fuchsian three-manifolds*, Transactions of the American Mathematical Society **367** (2015), no. 9, 6063–6083.

[Kap23] Michael Kapovich, *Geometric algorithms for discreteness and faithfulness*, Computational aspects of discrete subgroups of lie groups, Contemp. Math., vol. 783, Amer. Math. Soc., [Providence], RI, [2023] ©2023, pp. 87–112.

[Kar53] F. I. Karpelevič, *Surfaces of transitivity of a semisimple subgroup of the group of motions of a symmetric space*, Doklady Akad. Nauk SSSR (N.S.) **93** (1953), 401–404.

[KB02] Ilya Kapovich and Nadia Benakli, *Boundaries of hyperbolic groups*, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39–93.

[KL06] Bruce Kleiner and Bernhard Leeb, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. **163** (2006), no. 3, 657–676.

[KL18] Michael Kapovich and Bernhard Leeb, *Finsler bordifications of symmetric and certain locally symmetric spaces*, Geometry and Topology **22** (2018), no. 5, 2533–2646.

[Kle93] Felix Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Mathematische Annalen **43** (1893), no. 1, 63–100.

[KLM09] Michael Kapovich, Bernhard Leeb, and John Millson, *Convex functions on symmetric spaces, side lengths of polygons, and the stability inequalities for weighted configurations at infinity*, J. Differential Geom **81** (2009), 297–354.

[KLP17] Michael Kapovich, Bernhard Leeb, and Joan Porti, *Anosov subgroups: dynamical and geometric characterizations*, European Journal of Mathematics **3** (2017), no. 4, 808–898.

[KLP18a] ———, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geom. Topol. **22** (2018), no. 1, 157–234.

[KLP18b] ———, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol. **22** (2018), no. 7, 3827–3923.

[Lab06] François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

[Lee08] Jaejeong Lee, *Convex fundamental domains for properly convex real projective structures*, 2008.

[Löh17] C. Löh, *Geometric group theory: An introduction*, Universitext, Springer International Publishing, 2017.

[LT22] François Labourie and Jérémy Toulisse, *Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces*, 2022, 2010.05704.

[Mar09] Ludovic Marquis, *Les pavages en géométrie projective de dimension 2 et 3*, Theses, Université Paris Sud - Paris XI, May 2009.

[Mil58] John Milnor, *On the existence of a connection with curvature zero*, Commentarii Mathematici Helvetici **32** (1958), no. 1, 215–223.

[Mos55] G. D. Mostow, *Some new decomposition theorems for semi-simple groups*, Mem. Amer. Math. Soc. **14** (1955), 31–54.

[OV90] A. L. Onishchik and È. B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990, Translated from the Russian and with a preface by D. A. Leites.

[Pla95] Pierre Planche, *Géométrie de finsler sur les espaces symétriques*, PhD thesis (1995).

[PSW21] Maria Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard, *Conformality for a robust class of non-conformal attractors*, J. Reine Angew. Math. **774** (2021), 1–51.

[Rat19] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, vol. 149, Springer, Cham, [2019] ©2019, Third edition [of 1299730].

[Rie21] J. Maxwell Riestenberg, *A quantified local-to-global principle for morse quasigeodesics*.

[Wal14] Cormac Walsh, *The horoboundary and isometry group of Thurston's Lipschitz metric*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 327–353.

[Wie18] Anna Wienhard, *An invitation to higher Teichmüller theory*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039.

[Wig98] Mark Wiggerman, *The fundamental group of a real flag manifold*, Indag. Math. (N.S.) **9** (1998), no. 1, 141–153.

[Woo71] John W. Wood, *Bundles with totally disconnected structure group*, Comment. Math. Helv. **46** (1971), 257–273.

[Zim21] Andrew Zimmer, *Projective Anosov representations, convex cocompact actions, and rigidity*, J. Differential Geom. **119** (2021), no. 3, 513–586.