Heidelberg University
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Master thesis

# Earthquakes in the hyperbolic plane

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supervised by

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# Declaration

I hereby declare that this thesis is the result of my own work and that I did not use any means or sources other than those indicated.

Heidelberg, September 11, 2017

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## Abstract

Thurston's earthquake theorem in the hyperbolic plane states that any orientation-preserving homeomorphism of the circle arises as the boundary homeomorphism of a so-called *earthquake map*. The goal of this thesis is to define and study these transformations and to give a detailed proof of the earthquake theorem, following Thurston's paper [Thu06]. We also define *relative hyperbolic structures* on the hyperbolic plane, discuss their relation with homeomorphisms of the circle and we deduce from the earthquake theorem that any two such structures can be related by an earthquake map.



## Zusammenfassung

Thurstons Theorem über Erdbeben in der hyperbolischen Ebene besagt, dass jeder orientierungs-erhaltende Homöomorphismus des Einheitskreises als Randhomöomorphismus einer sogenannten *Erdbeben-Abbildung* auftritt. Das Ziel dieser Arbeit ist es, Erdbeben-Abbildungen zu definieren, ihre Eigenschaften zu untersuchen und das Erdbeben-Theorem im Detail zu beweisen. Dabei folgen wir Thurstons Arbeit [Thu06]. Außerdem definieren wir *relative hyperbolische Strukturen* der hyperbolischen Ebene und untersuchen ihren Zusammenhang mit Homöomorphismen des Einheitskreises. Aus dem Erdbeben-Theorem schließen wir, dass sich zwei beliebige relative hyperbolische Strukturen durch ein Erdbeben miteinander in Verbindung setzen lassen.

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# Contents

# 1. Introduction

## 1.1. Motivation

In mathematics, surfaces are ubiquitous and their study, from different points of view, is an active area of research. Topologically, they are completely classified, but their geometry is not entirely understood, even in the simplest case of Riemannian metrics of constant sectional curvature. By the Gauss-Bonnet theorem, most surfaces, i.e. those with negative Euler characteristic, can only admit a metric of negative curvature. If the curvature is normalized to $-1$, the metric is called hyperbolic.
Given a topological surface $S$ of negative Euler characteristic, the hyperbolic metrics on it (up to an appropriate equivalence relation, see for instance [FM12] for a precise definition) are parametrized by a space called Teichmüller space and denoted by $T(S)$. If $S$ is the three-punctured sphere, $T(S)$ is trivial. For other surfaces, $T(S)$ is a large parameter space. For instance, if $S$ is a compact surface $S$ of genus $g \geq 2$, $T(S)$ has real dimension $6g-6$. All Teichmüller spaces can be embedded in a larger space $\mathcal{T}$, that for this reason is called *universal Teichmüller space*. One way to define the universal Teichmüller space is as quotient

$$\mathcal{T} := {}^{\mathrm{QS}(\mathbb{S}^1)}\big/_{\mathrm{M\ddot{o}b}(\mathbb{S}^1)}$$

where $\mathrm{QS}(\mathbb{S}^1)$ is the group of *quasisymmetric* maps of the circle, a subgroup of the space of orientation-preserving homeomorphisms of the circle, and $\mathrm{Möb}(\mathbb{S}^1)$ is the group of Möbiustransformations, seen as maps of $\mathbb{S}^1$. We refer to [Ser14] for the precise definitions and more on the universal Teichmüller space. By giving a unifying approach, universal Teichmüller space $\mathcal{T}$ helps us to better understand the classical Teichmüller spaces $T(S)$.
A basic question in Teichmüller theory is how to relate distinct points and how to move in Teichmüller space. One answer is that any two elements in $T(S)$ can be related by a transformation called earthquake map. Earthquake maps were first introduced by William P. Thurston in the 1970s. A basic example for an earthquake map is the following: Take a compact hyperbolic surface $S$ of genus $g \geq 2$ and choose a simple closed geodesic on the surface. Cut the surface along this geodesic and reglue with a twist. When the twist is $2\pi$, this transformation is called Dehn twist. Topologically, the surface is still the same - the genus did not change. But the resulting hyperbolic surface is different from the original one. A general earthquake map can be much more complicated than this example. Instead of one simple closed geodesic, we could cut along

a set of non-intersecting closed geodesics or along a complete biinfinite geodesic.
The result that for any two points in Teichmüller space, there is an earthquake map sending one to the other, is known as *Thurston's Earthquake Theorem* and has a high importance in the study of Teichmüller space. Thurston did not publish a proof of his theorem at first. It was not until 1983 when Stephen Kerckhoff published a proof in the appendix of his paper on the Nielsen Realization Problem ([Ker83]), where he made crucial use of the earthquake theorem. In 1986, Thurston published a different proof in his paper "Earthquakes in 2-dimensional hyperbolic geometry" ([Thu06]). In his words, this proof is "more elementary and more constructive" than the previous one. It is based on the fact that the universal cover of any hyperbolic surface is the hyperbolic plane. Thus, the theory of earthquakes is first developed in the setting of the hyperbolic plane, including a version of the earthquake theorem in this setup. He then shows how to deduce the earthquake theorem on surfaces from the hyperbolic plane version.

## 1.2. Structure of the thesis

The aim of this thesis is to survey the first part of Thurston's paper [Thu06] that deals with earthquakes on $\mathbb{H}^2$ and to understand the proofs in detail. If not stated otherwise, the results and proofs in this thesis are based on Thurston's paper [Thu06].
In Section 2 we start with a reminder on basics in hyperbolic geometry. Section 3 gives some background on circle maps. Section 4 then introduces relative hyperbolic structures that allow us to distinguish between different complete metrics on the hyperbolic plane $\mathbb{H}^2$. In Section 5, earthquakes on $\mathbb{H}^2$ are defined. The most important result in this section is that earthquakes, though usually not continuous, have a continuous extension to the boundary of $\mathbb{H}^2$ and that the resulting map on the boundary is an orientation-preserving homeomorphism. This leads to the question: For a given orientation-preserving homeomorphism $f$ on the boundary, is there an earthquake map having $f$ as extension? The answer is yes and moreover, the earthquake is essentially unique. This "Earthquake theorem in the hyperbolic plane" is the heart of the thesis and proven in Section 6. From this, it easily follows that any two relative hyperbolic structures are related by an earthquake map.

# 2. Preliminaries on two-dimensional hyperbolic geometry

## 2.1. The hyperbolic plane

In this section, we give a short introduction to the topology and geometry of the hyperbolic plane. The material presented is based on [Bea83, Ch. 3,4 and 7] and [Mar16, Ch. 2]. One way of defining the hyperbolic plane is as "the unique simply connected Riemannian manifold of dimension two having constant sectional curvature $-1$". The fact that such a manifold is indeed unique (up to isometry) is proven in [Mar16, Th. 3.1.2]. We denote the hyperbolic plane by $\mathbb{H}^2$. Topologically, it is homeomorphic to the Euclidean plane $\mathbb{E}^2$, but geometrically it is different. For instance, in $\mathbb{E}^2$, given any geodesic $\ell$ and a point $p$ that does not lie on $\ell$, there is a unique line $\ell'$ through $p$ that is parallel to $\ell$, i.e. $\ell$ and $\ell'$ do not intersect. This is not true in $\mathbb{H}^2$. In fact, in the same situation in the hyperbolic plane, there are infinitely many lines through $p$ that do not intersect $\ell$. Other aspects of the geometries are similar: both in $\mathbb{E}^2$ and $\mathbb{H}^2$ there exists a unique geodesic through any two distinct points.

An important role is played by "points at infinity". Informally, these are the directions of geodesics. Precisely, we have the following.

**Definition 2.1.** *The set $\partial\mathbb{H}^2$ of* points at infinity *in $\mathbb{H}^2$ is the set of all geodesic half-lines up to the equivalence relation*

$$\gamma_1 \sim \gamma_2 \quad \Leftrightarrow \quad \sup_{t\in[0,\infty)} \mathrm{d}(\gamma_1(t), \gamma_2(t)) < \infty \ , \tag{2.1}$$

*where* d *denotes the distance function induced by the Riemannian metric on $\mathbb{H}^2$. The set of points of infinity $\partial\mathbb{H}^2$ is sometimes called the* visual boundary *of $\mathbb{H}^2$. We set $\overline{\mathbb{H}^2} := \mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

For any two points $p, q \in \partial\mathbb{H}^2$, there is a unique geodesic $\gamma$ having $p$ and $q$ as endpoints in the sense that if we split $\gamma$ in two geodesic half-lines $\gamma_1$ and $\gamma_2$, then $[\gamma_1] = p$ and $[\gamma_2] = q$, where $[\cdot]$ denotes the equivalence class for the equivalence relation defined in (2.1).

In order to work with $\overline{\mathbb{H}^2}$, we need to have a topology on it. For $p \in \partial\mathbb{H}^2$, we define a neighbourhood system as follows: Let $\gamma$ be a geodesic half-line with $[\gamma] = p$. Let further $V$ be an open neighbourhood of $\gamma'(0)$ in $T_{\gamma(0)}\mathbb{H}^2$ and pick $r > 0$. Set

$$U(\gamma, V, r) := \{\alpha(t) \mid \alpha(0) = \gamma(0),\ \alpha'(0) \in V,\ t > 0\} \cup \{[\alpha] \mid \alpha(0) = \gamma(0),\ \alpha'(0) \in V\}$$

where $\alpha$ is a half-line in $\mathbb{H}^2$. Letting $\gamma, V$ and $r$ vary, we obtain a system of open neighbourhoods of $p$ in $\overline{\mathbb{H}^2}$. For any $p \in \mathbb{H}^2$, a neighbourhood system is given as usual by open balls of radius $r > 0$. Thus, we have a topology on $\overline{\mathbb{H}^2}$ and one can show that with this topology, $\overline{\mathbb{H}^2}$ is homeomorphic to the closed disk $\mathbb{D}^2$. Note that the intersection of the sets $U(\gamma, V, r)$ with $\mathbb{H}^2$ is open in $\mathbb{H}^2$, so the topology defined above restricts to the original topology on $\mathbb{H}^2$ (see [Mar16, Section 2.2]).
The easiest way to get used to hyperbolic geometry is through various models of the hyperbolic plane. The models all have their own particular advantage. We present and use three different models. In the following, $|\cdot|$ will denote the euclidean metric on $\mathbb{R}$ and $\|\cdot\|$ the euclidean metric on $\mathbb{R}^2$ (or $\mathbb{C}$).

### The half-space model $H^2$

Let $H^2 := \{x + iy \in \mathbb{C} \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-space in $\mathbb{C}$. We define a Riemannian metric on $H^2$ by

$$g_z^{H^2} = \frac{1}{\mathrm{Im}(z)^2}\, g_z^E,$$

where $g_z^E$ denotes the Euclidean metric on $H^2 \subseteq \mathbb{R}^2$ in the point $z$. Let $\mathrm{d}\colon H^2 \times H^2 \to \mathbb{R}_{\geq 0}$ be the distance function on $H^2$ induced by $g^{H^2}$. The geodesics in $H^2$ are Euclidean semi-circles and Euclidean lines orthogonal to the real line. In $H^2$, the visual boundary $\partial\mathbb{H}^2$ identifies with $\mathbb{R} \cup \{\infty\}$ and we have $\overline{H^2} = \{x + iy \in \mathbb{C} \mid y \geq 0\} \cup \{\infty\}$.

**Proposition 2.2.** *With the hyperbolic distance* $\mathrm{d}$ *in* $H^2$ *as above, it holds that*

*i)* $\mathrm{d}(ip, iq) = |\log\left(\frac{p}{q}\right)|$ *for* $p, q \in \mathbb{R}_{>0}$ *and*

*ii)* $\sinh\left(\frac{1}{2}\mathrm{d}(z, w)\right) = \frac{\|z-w\|}{2(\mathrm{Im}(z)\mathrm{Im}(w))^{\frac{1}{2}}}$ *for* $z, w \in H^2$.

*Proof.* See [Bea83, Th. 7.2.1.]. □

### The disk model $D^2$

Let $D^2 := \{z \in \mathbb{C} \mid \|z\| < 1\}$ be the open unit disk equipped with the Riemannian metric

$$g_z^{D^2} = \frac{4}{(1 - \|z\|^2)^2}\, g_z^E.$$

Let $\mathrm{d}\colon D^2 \times D^2 \to \mathbb{R}_{\geq 0}$ be the distance function on $D^2$ induced by $g^{D^2}$. We call both the distance functions on $H^2$ and on $D^2$ *hyperbolic distance* and will always write $\mathrm{d}(\cdot,\cdot)$. It will be clear from the context which model we are using.

**Proposition 2.3.** *For $r \in (0,1) \subseteq D^2$ we have*

$$\mathrm{d}(0,r) = \log\left(\frac{1+r}{1-r}\right) .$$

*Proof.* See [Bea83, Ch. 7.2]. □

The map

$$C\colon \mathbb{C} \to \mathbb{C}, \qquad C(z) = \frac{z-i}{z+i} ,$$

known as *Cayley map*, is an isometry from $H^2$ onto $D^2$. The geodesics in $D^2$ are Euclidean circles and Euclidean lines orthogonal to the boundary $\partial D^2 = \mathbb{S}^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$, which is the visual boundary in this model.

### The Klein model $K^2$

There is another way to model the hyperbolic plane on the open unit disk: the Klein model $K^2$. As a set, $K^2 = D^2$, but the distance function is different. In the Klein model, geodesics are Euclidean lines. There exists a map from $D^2$ to $K^2$ mapping a geodesics $\ell$ in the disk model to a Euclidean straight line segment $\ell^*$, i.e. a geodesic in $K^2$, such that $\ell$ and $\ell^*$ have the same endpoints on the boundary ([Bea83, Ch. 7.1.]). One downside of the Klein model is the fact that it is not conformal. However, it is useful for showing convexity properties as it transfers problems from hyperbolic to Euclidean geometry. Also in this model, $\partial\mathbb{H}^2$ is identified with the circle $\mathbb{S}^1$.

*Remark* 2.4. We use $\mathbb{H}^2$ to denote the hyperbolic plane in general when we do not specify which model we consider. When working with one of the models, we will use the notation $H^2$, $D^2$ and $K^2$ respectively. Further, we use $S^1_\infty$ to denote the visual boundary $\partial\mathbb{H}^2$, indicating that those are the points at infinity of $\mathbb{H}^2$ and that for $D^2$, it coincides with $\mathbb{S}^1$.

## 2.2. Isometries of the hyperbolic plane

Also this section is based on [Bea83, Ch. 3,4 and 7]. In this thesis, we only consider orientation-preserving isometries. In the half-plane model $H^2$, the orientation-preserving isometries are given by

$$\mathrm{Isom}^+(H^2) \cong \mathrm{PSL}(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = 1\} \big/ \{\pm I\} \, ,$$

where $I$ denotes the $2 \times 2-$identity matrix and the matrices act on $H^2$ by Möbius transformations

$$A(z) := \frac{az+b}{cz+d} \qquad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}), \ z \in H^2.$$

By setting $A(-\frac{d}{c}) = \infty$ and $A(\infty) = \frac{a}{c}$ for $c \neq 0$ and $A(\infty) = \infty$ for $c = 0$, this extends to an action on $\overline{H^2}$.
In the disk model $D^2$,

$$\mathrm{Isom}^+(D^2) \cong \mathrm{PSU}(1,1) = \left\{ \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \mid |a|^2 - |c|^2 = 1 \right\} \Big/ \{\pm I\} \, ,$$

where again the action is by Möbius transformations and extends naturally to $\overline{D^2} = \mathbb{D}^2$. We will always use the identification $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$. One easily computes that $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on oriented triples of $S^1_\infty$. In particular, for any two geodesics $\ell_1$ and $\ell_2$, there is an isometry $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ with $\varphi(\ell_1) = \ell_2$. $\varphi$ is not unique, but can be made unique by fixing a point $z \in \ell_1$ and its image points $\varphi(z) \in \ell_2$. The non-trivial isometries of $\mathbb{H}^2$ can be classified into three types.

**Definition 2.5.** *Let $\varphi \in \mathrm{Isom}^+(\mathbb{H}^2) \setminus \{id\}$. We say that*

*i) $\varphi$ is* parabolic *if it has exactly one fixed point on $S^1_\infty$ and no other fixed points.*

*ii) $\varphi$ is* hyperbolic *if it has exactly two fixed points on $S^1_\infty$ and no other fixed points.*

*iii) $\varphi$ is* elliptic *if it has no fixed points on $S^1_\infty$ and exactly one fixed point in $\mathbb{H}^2$.*

Indeed, these are the only cases that can occur: In the half-plane model $H^2$, consider first the case when $\infty$ is a fixed point of $\varphi$. Then $\varphi(z) = az + b$ for some $a, b \in \mathbb{R}$. For $a = 1$ this does not have a fixed point in $\mathbb{R}$. For $a \neq 0$, this has a unique fixed point in $\mathbb{R}$. In both cases, $\varphi$ has either one or two fixed points in $\mathbb{R} \cup \infty$. If $\infty$ is not a fixed point of $\varphi$, finding the fixed points $z$ of $\varphi$ is equivalent to finding the zeroes of a polynomial with real coefficients of degree at most two. If the polynomial is of degree one, then it has exactly one real zero, so $\varphi$ has exactly one fixed point. If the polynomial is of degree two, it has exactly two zeroes that are either both real or both non-real and conjugated to each other. In this case, exactly one of the fixed points of $\varphi$ lies in the upper half plane $H^2$.

**Lemma 2.6.** *Let $\varphi \in \mathrm{Isom}^+(\mathbb{H}^2) \setminus \{id\}$. Consider the model $H^2$. Then*

*i) $\varphi$ is* parabolic *if and only if it is conjugate to the map $z \mapsto z + 1$.*

*ii) $\varphi$ is* hyperbolic *if and only if it is conjugate to a map of the form $z \mapsto e^{\lambda} z$ for some $\lambda \neq 0$.*

*Proof.* See [Bea83, Ch. 4.3]. □

If $\varphi$ is hyperbolic, then it preserves the unique geodesic connecting its fixed points. Note that no other geodesic is preserved, since otherwise, also its endpoints on $S^1_\infty$ would be fixed, contradicting the fact that $\varphi$ has exactly two fixed points on $S^1_\infty$.

**Definition 2.7.** *Let $\varphi \in \mathrm{Isom}^+(\mathbb{H}^2)$ be hyperbolic. The unique geodesic preserved by $\varphi$ is called the* axis *of $\varphi$. The* translation distance *of $\varphi$ is*

$$\tau(\varphi) := \inf_{z \in \mathbb{H}^2} \mathrm{d}(z, \varphi(z)).$$

From the definition, it is immediate that $\tau(\varphi) = \tau(\varphi^{-1})$.

*Remark* 2.8. $\tau(\varphi)$ can also be defined for elliptic and parabolic isometries. In both cases $\tau(\varphi) = 0$, where in case that $\varphi$ is elliptic the infimum is attained at the fixed point in $\mathbb{H}^2$ and in case that $\varphi$ is parabolic the infimum is not attained.

**Lemma 2.9.** *Let $\varphi \in \mathrm{Isom}^+(\mathbb{H}^2)$ be hyperbolic with axis $\ell$ and $z \in \mathbb{H}^2$. Then*

$$\tau(\varphi) = \mathrm{d}(z, \varphi(z)) \quad \Leftrightarrow \quad z \in \ell.$$

*Proof.* In the half-plane model $H^2$, suppose first that $\varphi$ is hyperbolic a hyperbolic transformation with axis the imaginary axis, i.e. it is of the form $\varphi(z) = e^{\lambda} z$ with $\lambda \neq 0$. Then by Proposition 2.2

$$\sinh\left(\frac{1}{2}\,\mathrm{d}(z, \varphi(z))\right) = \frac{|1 - e^{\lambda}|\ \|z\|}{2e^{\frac{\lambda}{2}}\ \mathrm{Im}(z)}$$

and this attains its minimum if and only if $\mathrm{Re}(z) = 0$, i.e. if $z$ is on the axis. If $\varphi$ is a general hyperbolic transformation with axis $\ell$, then there is some $\psi \in \mathrm{Isom}^+(\mathbb{H}^2)$

mapping the imaginary axis to $\ell$ and $\psi \circ \varphi \circ \psi^{-1}$ is hyperbolic with axis the imaginary axis. Since $\psi^{-1}$ is bijective, it holds that

$$\inf_{z \in \mathbb{H}^2} \mathrm{d}(z, \varphi(z)) = \inf_{z \in \mathbb{H}^2} \mathrm{d}(\psi^{-1}(z), \varphi(\psi^{-1}(z))) = \inf_{z \in \mathbb{H}^2} \mathrm{d}(z, \psi \circ \varphi \circ \psi^{-1}(z)) \ , \tag{2.2}$$

so $\tau(\varphi) = \tau(\psi \circ \varphi \circ \psi^{-1})$ and the minimum is attained if and only if $\varphi^{-1}(z)$ lies on the imaginary axis, so exactly if $z$ lies on $\ell$. □

As seen in Lemma 2.6, a hyperbolic isometry $\psi$ is conjugated to a hyperbolic isometry with axis connecting 0 and $\infty$, so an isometry of the form $\varphi(z) = e^\lambda z$, for $z \in \mathbb{H}^2$, with $\lambda \neq 0$. If $\lambda < 0$, then all points on the axis are moved towards 0 and 0 is called the *attracting* fixed point of $\varphi$. If $\lambda > 0$, then all points on the axis are moved away from 0 and 0 is called *repelling* fixed point. If 0 is the attracting fixed point, then $\infty$ is repelling and vice versa. The fixed point of $\psi$ identified with 0 is called attracting (respectively, repelling) if 0 is the attracting (respectively, repelling) fixed point of $\varphi$.

**Proposition 2.10.** *Consider the half-plane model $H^2$. Let $\varphi \in \mathrm{Isom}^+(H^2)$ be hyperbolic with repelling fixed point $x^- \in \mathbb{R}$. Then*

$$\tau(\varphi) = \log(\varphi'(x^-)) \ . \tag{2.3}$$

*Proof.* We have seen in (2.2) that $\tau$ is invariant under conjugation. To show that also the right side of (2.3) is invariant, let $\psi \in \mathrm{Isom}^+(H^2)$ and $y^- := \psi(x^-)$ be the repelling fixed point of $\psi \circ \varphi \circ \psi^{-1}$ and $y^- \in \mathbb{R}$. Then

$$\begin{aligned}
(\psi \circ \varphi \circ \psi^{-1})'(y^-) &= \psi'(\varphi \circ \psi^{-1}(y^-)) \cdot \varphi'(\psi^{-1}(y^-)) \cdot (\psi^{-1})'(y^-) \\
&= \psi'(x^-) \cdot \varphi'(x^-) \cdot (\psi^{-1})'(\psi(x^-)) \\
&= \varphi'(x^-)
\end{aligned}$$

by the inverse function theorem. Hence also the right side is invariant under conjugation and it suffices to show (2.3) for $\varphi$ of the form $\varphi(z) = e^\lambda z$ with $\lambda > 0$. Then by Lemma 2.9 and Proposition 2.2 we have

$$\tau(\varphi) = \log(e^\lambda) = \lambda = \log(\varphi'(0)),$$

so indeed, (2.3) holds for $\varphi$ and by invariance under conjugation also for any hyperbolic transformation. □

Note that we cannot compute the right side of (2.3) when $\varphi$ has repelling fixed point $\infty$. However, in this case the attracting fixed point $x^+$ lies in $\mathbb{R}$ and we have

$$\tau(\varphi) = \tau(\varphi^{-1}) = \log((\varphi^{-1})'(x^+)) \ .$$

So also in this case, we can use Proposition 2.10 to calculate $\tau(\varphi)$.

### The topology of $\mathrm{Isom}^+(\mathbb{H}^2)$

Before introducing a topology on $\mathrm{Isom}^+(\mathbb{H}^2)$, we introduce a metric on $\hat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$. Let $\pi\colon \hat{\mathbb{R}}^2 \to \mathbb{S}^2$ be the stereographic projection given by

$$\pi(x_1, x_2) = \left( \frac{2x_1}{\|x\|^2+1}, \frac{2x_2}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1} \right) \qquad \text{with } x = (x_1, x_2) \in \mathbb{R}^2$$

and $\pi(\infty) = (0, 0, 1)$. This a bijective map of $\hat{\mathbb{R}}^2$ onto $\mathbb{S}^2$ and we can transfer the Euclidean metric from $\mathbb{S}^2$ to a metric on $\hat{\mathbb{R}}^2$. This is called the *chordal metric* on $\hat{\mathbb{R}}^2$, defined by

$$\mathrm{d}_c(x, y) = \|\pi(x) - \pi(y)\| \qquad \text{for } x, y \in \hat{\mathbb{R}}^2. \tag{2.4}$$

The metric $\mathrm{d}_c$ restricted to $\mathbb{R}^2$ induces the same topology as the Euclidean metric. Hence, a function on $\mathbb{R}^2$ is continuous with respect to $\mathrm{d}_c$ if and only if it is continuous with respect to the Euclidean metric (see [Bea83, Ch. 3.1]). The restriction of $\mathrm{d}_c$ to $\overline{H^2}$ gives a metric on $\overline{H^2}$.

**Definition 2.11.** *The* topology of uniform convergence $\mathcal{T}^*$ *on* $\mathrm{Isom}^+(H^2) = \mathrm{PSL}(2, \mathbb{R})$ *is the topology induced by the metric*

$$D(\varphi, \psi) := \sup\{ \ \mathrm{d}_c(\varphi(z), \psi(z)) \mid z \in H^2\}.$$

The fact that $D$ is positive definit, symmetric and satisfies the triangle inequality immediately follows from the fact that $\mathrm{d}_c$ is a metric. As $\mathrm{d}_c$ is continuous on the compact set $\overline{H^2} \times \overline{H^2}$, it is bounded, so $D$ is finite and hence a metric. Further, $\varphi_n$ converges to $\varphi$ in this metric if and only if $\varphi_n$ converges to $\varphi$ uniformly on $\overline{H^2}$.
The group $\mathrm{SL}(2, \mathbb{R})$ has a norm induced by the Euclidean norm on $\mathbb{R}^4$, namely for a

matrix in $\mathrm{SL}(2,\mathbb{R})$ we have

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)^{\frac{1}{2}} \ .$$

This norm gives a topology on $\mathrm{SL}(2,\mathbb{R})$ and we can endow $\mathrm{PSL}(2,\mathbb{R})$ with the quotient topology, which we denote with $\mathcal{T}$.

**Theorem 2.12.** *The topology $\mathcal{T}^*$ of uniform convergence on* $\mathrm{Isom}^+(H^2) \cong \mathrm{PSL}(2,\mathbb{R})$ *and the topology $\mathcal{T}$ coincide.*

*Proof.* See [Bea83, Ch. 4.5]. □

**Lemma 2.13.** *Fix $x_0 \in \mathbb{H}^2$. Then the map $p_0\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2$ given by $p_0(\varphi) = \varphi(x_0)$ is continuous and surjective.*

*Proof.* Surjectivity immediately follows from the fact that $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$. For continuity, we have

$$D(\varphi,\psi) < \varepsilon \ \Rightarrow \ \mathrm{d}_c(\varphi(x_0),\psi(x_0)) < \varepsilon \ .$$

Since the standard topology on $\mathbb{H}^2$ and the topology induced by the chordal metric agree, it follows that $p_0$ is continuous. □

*Remark* 2.14. If not stated otherwise, all function spaces will be endowed with the topology of uniform convergence of functions with respect to a certain metric. When working with functions on $\overline{\mathbb{H}^2}$, this will be the chordal metric (as in Definition 2.11). For functions on $\mathbb{R}$ and $\mathbb{S}^1$, we use the euclidean metric.

# 3. Circle maps

We now introduce some background on circle maps which will play an important role in our study of earthquakes.

### Cyclic orders and order preserving maps

The following definition and more properties of cyclic orders can be found in [Qui89].

**Definition 3.1.** *A family $F$ of triples of points of a set $X$ is a* cyclic order *if the following axioms are satisfied:*

*i) Cyclicity:* $(a,b,c) \in F \Rightarrow (b,c,a) \in F$ *and* $(c,a,b) \in F$.

*ii) Antisymmetry:* $(a,b,c) \in F \Rightarrow (b,a,c) \notin F$.

*iii) Transitivity:* $(a,b,c) \in F$ *and* $(c,d,a) \in F \Rightarrow (b,c,d) \in F$ *and* $(d,a,b) \in F$.

*We call* $(a,b,c) \in F$ *a* cyclically ordered triple *or just* ordered. *More generally, we say that an n-tuple* $(a_0,\ldots,a_{n-1})$ *is ordered if every subtriple* $(a_i,a_j,a_k)$ *is ordered for* $i<j<k \in \{0,\ldots,n-1\}$.

We fix an orientation of the circle $\mathbb{S}^1$. With *counterclockwise* (respectively, *clockwise*) we denote the positive (respectively, negative) direction with respect to this orientation. We define a set $C$ of triples of points of $\mathbb{S}^1$ by

$$(x,y,z) \in C \quad \Leftrightarrow \quad \text{When moving from } x \text{ counterclockwise, one first reaches } y, \text{ then } z\ ,$$

where $x,y,z \in \mathbb{S}^1$ are pairwise distinct. It can be easily verified that $C$ is a cyclic order on $\mathbb{S}^1$.

**Definition 3.2.** *A map* $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ *is* order preserving *or* preserves the cyclic order *if* $(x,y,z) \in C$ *implies* $(f(x),f(y),f(z)) \in C$.

**Definition 3.3.** *For distinct points* $x,y \in \mathbb{S}^1$*, we define the open interval between* $x$ *and* $y$ *by*

$$((x,y)) := \{z \in \mathbb{S}^1 \mid (x,z,y) \in C\}$$

*and the closed interval by* $[[x,y]] := ((x,y)) \cup \{x,y\}$.

## Lifts of circle maps

We introduce some basics on lifts of circle maps, based on [HK03]. The universal cover of $\mathbb{S}^1$ is $\mathbb{R}$ with covering map $\pi\colon \mathbb{R} \to \mathbb{S}^1$ given by $\pi(x) := e^{ix}$. In particular, $\pi$ is surjective and $2\pi$-periodic.

**Proposition 3.4.** *If $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is continuous, then there exists a continuous map $F\colon \mathbb{R} \to \mathbb{R}$, called a* lift *of $f$, such that*

$$f \circ \pi = \pi \circ F \ .$$

*$F$ is unique up to an additive constant of the form $2\pi k$, $k \in \mathbb{Z}$ and*

$$\frac{1}{2\pi}\left(F(x+2\pi) - F(x)\right)$$

*is an integer independent on $x \in \mathbb{R}$ and on the lift $F$. If $f$ is a homeomorphism, then $|\frac{1}{2\pi}\left(F(x+2\pi) - F(x)\right)| = 1$.*

*Proof.* See [HK03, Prop.4.3.1]. $\square$

**Definition 3.5.** *A homeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is called* orientation-preserving *if any of its lifts $F$ is increasing.*

Note that to show that $f$ is orientation-preserving it suffices to show that one specific lift $F$ is increasing, since for any other lift $\tilde{F}$ of $F$ it holds that $\tilde{F}(x) = F(x) + 2\pi k$ for some $k \in \mathbb{Z}$. The fact that $f$ is orientation-preserving is equivalent to

$$\frac{1}{2\pi}\left(F(x+2\pi) - F(x)\right) = 1 \ , \quad \text{i.e. } F(x+2\pi) = F(x) + 2\pi \tag{3.1}$$

for all $x \in \mathbb{R}$. We denote the set of all orientation-preserving homeomorphisms of the circle by $\mathrm{Homeo}^+(\mathbb{S}^1)$.

**Lemma 3.6.** *A homeomorphism $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is orientation-preserving if and only if it preserves the cyclic order.*

*Proof.* Let $f$ be an orientation-preserving homeomorphism, $F$ its lift and $(x_1, x_2, x_3) \in C$. Let $\varphi_i \in \mathbb{R}$ such that $x_i = e^{i\varphi_i}$ for $i = 1, 2, 3$ and chosen such that $\varphi_1 < \varphi_2 < \varphi_3$ and $|\varphi_3 - \varphi_1| < 2\pi$. As $F$ is a monotonically increasing homeomorphism, we have $F(\varphi_1) < F(\varphi_2) < F(\varphi_3)$ and $|F(\varphi_3) - F(\varphi_1)| < 2\pi$. It follows that

$$(f(x_1), f(x_2), f(x_3)) = (e^{iF(\varphi_1)}, e^{iF(\varphi_2)}, e^{iF(\varphi_3)}) \in C \ .$$

Vice versa, let $f$ be an order preserving homeomorphism and let $F$ be a lift of $f$. Since $F$ is a homeomorphism, we have for all $x \in \mathbb{R}$ either $F(x+2\pi) = F(x) + 2\pi$ or $F(x+2\pi) = F(x) - 2\pi$, so it suffices to show that $F$ is monotonically increasing on $[0, 2\pi)$. Let $\varphi_1, \varphi_3 \in [0, 2\pi)$ with $\varphi_1 < \varphi_3$ and let $\varphi_2 \in (\varphi_1, \varphi_2)$. Then the triple $(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$ is ordered, so also $(f(e^{i\varphi_1}), f(e^{i\varphi_2}), f(e^{i\varphi_3})) = (\pi \circ F(\varphi_1), \pi \circ F(\varphi_2), \pi \circ F(\varphi_3))$ is ordered. Since $F$ is a homeomorphism, we have $|F(\varphi_3) - F(\varphi_1)| < 2\pi$ and it follows that $F(\varphi_1) < F(\varphi_2) < F(\varphi_3)$, so $f$ is orientation-preserving. □

A circle homeomorphism and its lift are closely related in the sense that a lift inherits some properties from the homeomorphism. This can be seen in the following two lemmata.

**Lemma 3.7.** *Suppose $f \in \mathrm{Homeo}^+(\mathbb{S}^1)$ has a fixed point. Then there exists a unique lift of $f$ having fixed points.*

*Proof.* Let $p \in \mathbb{S}^1$ be a fixed point of $f$ and let $x \in \mathbb{R}$ with $\pi(x) = p$. Let $F \colon \mathbb{R} \to \mathbb{R}$ be any lift of $f$. Then

$$e^{ix} = p = f(p) = f \circ \pi(x) = \pi \circ F(x) = e^{iF(x)},$$

so $F(x) = x + 2\pi k$ for some fixed $k \in \mathbb{Z}$. Set $\tilde{F}(y) := F(y) - 2\pi k$ for $y \in \mathbb{R}$. Then $\tilde{F}$ is a lift of $f$ satisfying $\tilde{F}(x) = F(x) - 2\pi k = x$. Further, any other lift $G$ of $f$ is of the form $G(y) = \tilde{F}(y) + 2\pi m$ for some $m \in \mathbb{Z}$, $m \neq 0$. If now $G$ has a fixed point $y$, then, by (3.1), it also has a fixed point in $[0, 2\pi)$. Thus, we can assume that $y$ and also $x$ lie in $[0, 2\pi)$. Then

$$y = G(y) = \tilde{F}(y) + 2\pi m \ . \tag{3.2}$$

If $y \in [x, 2\pi)$, then it follows with $\tilde{F}(x) = x$ that $\tilde{F}(y) \in [x, x + 2\pi)$ and (3.2) can only hold for $m = 0$, so $G = \tilde{F}$. An analogous argument works for $y \in [0, x]$. In total, $\tilde{F}$ is the only lift of $f$ with fixed points. The fixed points of $\tilde{F}$ are precisely the preimages of fixed points of $f$. □

**Lemma 3.8.** *Let $f \in \mathrm{Homeo}^+(\mathbb{S}^1)$ and $F \colon \mathbb{R} \to \mathbb{R}$ be a lift of $f$. Then there exists a lift $G$ of $f^{-1}$ such that $G = F^{-1}$.*

*Proof.* Let $p, q \in \mathbb{S}^1$ such that $f(p) = q$ and $x, y \in \mathbb{R}$ such that $\pi(x) = p$, $\pi(y) = q$ and $F(x) = y$. Let $G$ be the lift of $f^{-1}$ satisfying $G(y) = x$. Then

$$\pi \circ (F \circ G) = f \circ (\pi \circ G) = f \circ f^{-1} \circ \pi = \pi,$$

i.e. $F \circ G(t) = t + 2\pi k$ for all $t \in \mathbb{R}$ for a fixed $k \in \mathbb{Z}$. But $F \circ G(y) = F(x) = y$, so $k = 0$ and $F \circ G = id$. In the same way, one shows $G \circ F = id$, so $G = F^{-1}$. □

### Homeomorphisms of the disk

In the sequel, it will be useful to be able to extend homeomorphisms of the circle to homeomorphisms of the closed disk $\mathbb{D}^2$. Given a homeomorphism $f$ of $\mathbb{S}^1$, one such extension is given by setting $\tilde{f}(0) = 0$ and for $x \in \mathbb{D}^2 \setminus \{0\}$

$$\tilde{f}(x) = \|x\| \ \cdot f\left(\frac{x}{\|x\|}\right) \ . \tag{3.3}$$

There are various other ways of extending $f$ to a homeomorphism of $\mathbb{D}^2$. For us, it is sufficient to know that an extension exists; uniqueness will not be a concern for us.
We will also use the *Alexander Lemma* - a result that gives a sufficient condition for a homeomorphism of the unit disk to be isotopic to the identity.

**Lemma 3.9.** *Any homeomorphism $h\colon \mathbb{D}^2 \to \mathbb{D}^2$ that is the identity on the boundary $\mathbb{S}^1$ is isotopic to the identity, where the isotopy is relative to the boundary $\mathbb{S}^1$.*

*Proof.* We reproduce the proof from [FM12, Lemma 2.1]. Let $h\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a homeomorphism with $h|_{\mathbb{S}^1} = id$. We define

$$H\colon \mathbb{D}^2 \times [0,1] \to \mathbb{D}^2, \quad H(x,t) = \begin{cases} (1-t)h\left(\frac{x}{1-t}\right) & \text{for } 0 \le \|x\| < 1-t, \\ x & \text{for } 1-t \le \|x\| \le 1 \ . \end{cases}$$

Then $H$ is an isotopy from $h$ to $id$ relative to the boundary: $H$ is continuous as $h$ is continuous and $h|_{\mathbb{S}^1} = id$. Further, $H(\cdot,t)|_{\mathbb{S}^1} = id$ for all $t \in [0,1]$. It remains to show that for a fixed $t \in (0,1)$, $H(\cdot,t)$ is bijective. For surjectivity, let $y \in \mathbb{D}^2$. If $\|y\| \ge 1-t$, then $H(y,t) = y$. If $\|y\| < 1-t$, then by surjectivity of $h$ there is some $\tilde{x}$ in the interior of $\mathbb{D}^2$ such that $h(\tilde{x}) = \frac{y}{1-t}$. Since $\|(1-t)\tilde{x}\| < 1-t$, we have $H((1-t)\tilde{x},t) = (1-t)h\left(\frac{(1-t)\tilde{x}}{1-t}\right) = y$, so $H(\cdot,t)$ is surjective. For injectivity, suppose that $H(x,t) = H(y,t)$ for $x, y \in \mathbb{D}^2$. From injectivity of $h$ it follows that the only case that this can happen is for $\|x\| < 1-t$ and $\|y\| \ge 1-t$. But then

$$\left\| h\left(\frac{x}{1-t}\right) \right\| = \left\| \frac{y}{1-t} \right\| \ge 1$$

a contradiction. So all $H(\cdot,t)$ for $t \in [0,1]$ are homeomorphisms of $\mathbb{D}^2$. □

# 4. Relative hyperbolic structures

On $\mathbb{H}^2$, all complete hyperbolic metrics are isometric (see [Mar16, Th. 3.1.2]). However, if we want to distinguish between different hyperbolic metrics, we can do so by defining *relative hyperbolic structures.* They take into account what happens on the boundary $\partial\mathbb{H}^2$.

**Definition 4.1.** *A* relative hyperbolic structure *on* $(\mathbb{H}^2, S^1_\infty)$ *is an equivalence class of complete hyperbolic metrics on* $\mathbb{H}^2$*, where two metrics* $h$ *and* $h'$ *are equivalent if*

*i) there exists exists an isometry* $\phi\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h')$ *that is isotopic to the identity and*

*ii)* $\phi$ *extends continuously to the identity at the boundary* $S^1_\infty$*.*

*A relative hyperbolic structure* $[h]$ *on* $(\mathbb{H}^2, S^1_\infty)$ *is called* continuous*, if there exists an isometry* $f\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h_{st})$ *from* $h$ *to the standard hyperbolic metric* $h_{st}$ *which extends continuously to a map* $\overline{f}\colon \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$ *that is a homeomorphism on the boundary.*

Being a continuous relative hyperbolic structure is well-defined, i.e. independent on the representative: Let $h$ and $h'$ be two representatives of a hyperbolic structure $[h]$ and let $\phi$ be the isometry between them as in Definition 4.1. If there exists an isometry $f\colon (\mathbb{H}^2, h') \to (\mathbb{H}^2, h_{st})$ that extends continuously to be a homeomorphism at the boundary, then also $f\circ\phi\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h_{st})$ is an isometry that extends continuously to a homeomorphism at the boundary since $\phi$ is an isometry and $\phi|_{S^1_\infty} = id_{S^1_\infty}$. This also implies that if one isometry $f_h\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h_{st})$ extends to a homeomorphism on the boundary, then any such isometry $\tilde{f}_h$ does, since $\tilde{f}_h \circ f_h^{-1}$ has to lie in $\mathrm{PSL}(2,\mathbb{R})$, so $\tilde{f}_h = \varphi \circ f_h$ for some $\varphi \in \mathrm{PSL}(2,\mathbb{R})$. The claim follows as $\varphi$ extends to a homeomorphism on the boundary.
The fact that $\phi$ is isotopic to the identity is redundant here, since by the Alexander trick (Lemma 3.9), any self-homeomorphism of the unit disk that is the identity on $\mathbb{S}^1$ is isotopic to the identity. Therefore, to show that two metrics are equivalent, it suffices to

boundary. The property that $\phi$ is isotopic to the identity is needed in the more general case when considering relative hyperbolic structures on arbitrary complete hyperbolic surfaces with boundary. However, in this thesis we restrict ourselves to relative hyperbolic structures on $(\mathbb{H}^2, S^1_\infty)$.
If we just consider complete hyperbolic metrics on $\mathbb{H}^2$, then all of them are isometric (see

[Mar16, Th. 3.1.2]). When looking at relative hyperbolic structures in contrast, we can distinguish between them by looking at what happens on the boundary. In particular, we can classify continuous relative hyperbolic structures by homeomorphisms of the boundary $\mathbb{S}^1$.

**Theorem 4.2.** *There is a one-to-one correspondence between the set of continuous relative hyperbolic structures and the set of right cosets*

$$\mathrm{PSL}(2,\mathbb{R})\backslash^{\mathrm{Homeo}^+(\mathbb{S}^1)}$$

*where* $\mathrm{Homeo}^+(\mathbb{S}^1)$ *denotes the orientation-preserving homeomorphisms of* $\mathbb{S}^1$*.*

*Proof.* We construct a map

$$\mathcal{B}\colon \{\text{continuous relative hyperbolic structures}\} \to \mathrm{PSL}(2,\mathbb{R})\backslash^{\mathrm{Homeo}^+(\mathbb{S}^1)}$$

and show that it is a bijection. Let $[h]$ be a continuous relative hyperbolic structure and $h$ a metric representing it. Since all hyperbolic metrics on $\mathbb{H}^2$ are isometric (see [Mar16, Th. 3.1.2]), there exists a homeomorphism

$$f_h\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h_{st})$$

that is an orientation-preserving isometry from $h$ to the standard hyperbolic metric $h_{st}$. Since $[h]$ is continuous, $f_h$ extends continuously to a homeomorphism on the boundary. To simplify notation, we will denote the extension by $f_h$ as well. We set $\mathcal{B}([h]) = [f_h|_{S^1_\infty}]$. First, we have to show that this assignment is independent on $f_h$. Let $\hat{f}_h$ be another such homeomorphism. Then

$$f_h \circ \hat{f}_h^{-1}\colon (\mathbb{H}^2, h_{st}) \to (\mathbb{H}^2, h_{st}) \tag{4.1}$$

is an isometry of the hyperbolic plane, so $f_h = \varphi \circ \hat{f}_h$ for some $\varphi \in \mathrm{PSL}(2,\mathbb{R})$. In particular, $[f_h|_{S^1_\infty}]$ is independent on the choice of the isometry $f_h$. To show that it is also independent on the choice of the representative, let $h'$ be another representative of $[h]$, $\phi\colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h')$ an isometry with $\phi|_{S^1_\infty} = id_{S^1_\infty}$ and let $f_{h'}$ be any isometry sending $h'$ to $h_{st}$. Consider the diagram

$$\begin{tikzcd} (\mathbb{H}^2, h) \arrow[rr, "\phi"] \arrow[dr, "f_h"'] & & (\mathbb{H}^2, h') \arrow[dl, "f_{h'}"] \\ & (\mathbb{H}^2, h_{st}) & \end{tikzcd}$$

Since all maps are isometries, also $f_{h'} \circ \phi \colon (\mathbb{H}^2, h) \to (\mathbb{H}^2, h_{st})$ is an isometry. As seen before it follows that there is some $\varphi \in \mathrm{PSL}(2, \mathbb{R})$ such that

$$f_{h'} \circ \phi = \varphi \circ f_h \ .$$

If we now look at what happens at infinity, using the fact that $\phi|_{S^1_\infty} = id_{S^1_\infty}$, we get

$$f_{h'}|_{S^1_\infty} = (f_h \circ \phi)|_{S^1_\infty} = (\varphi \circ f_h)|_{S^1_\infty},$$

so $f_{h'}|_{S^1_\infty}$ and $f_h|_{S^1_\infty}$ are in the same right coset. This proves that the map $\mathcal{B} \colon [h] \mapsto [f_h|_{S^1_\infty}]$ is well-defined. It remains to show bijectivity. Let $[h]$ and $[h']$ be two continuous relative hyperbolic structures with $\mathcal{B}([h]) = \mathcal{B}([h'])$. With the same notation as above, we have

$$[f_h|_{S^1_\infty}] = [f_{h'}|_{S^1_\infty}],$$

so $f_{h'}|_{S^1_\infty} = \varphi \circ f_h|_{S^1_\infty}$ for some $\varphi \in \mathrm{PSL}(2, \mathbb{R})$. Consider the diagram

$$\begin{tikzcd} (\mathbb{H}^2, h) \arrow[r, "f_h"] \arrow[d, "\psi"'] & (\mathbb{H}^2, h_{st}) \arrow[d, "\varphi"] \\ (\mathbb{H}^2, h') \arrow[r, "f_{h'}"] & (\mathbb{H}^2, h_{st}) \end{tikzcd}$$

Then $\psi := f_{h'}^{-1} \circ \varphi \circ f_h$ is an isometry from $h$ to $h'$ and $\psi|_{S^1_\infty} = f_{h'}|_{S^1_\infty}^{-1} \circ (\varphi \circ f_h)|_{S^1_\infty} = id_{S^1_\infty}$, so $[h] = [h']$, i.e. $\mathcal{B}$ is injective. For surjectivity, let $f$ be an orientation-preserving homeomorphism of $\mathbb{S}^1$. Extend it to a homeomorphism $F$ of the disk in an arbitrary way (see (Section 3, equation 3.3). We can define a new metric $h_F$ on $\mathbb{H}^2$ by pulling back the standard metric $h_{st}$ using $F$, i.e.

$$d_{h_F}(x, y) = d_{st}(F(x), F(y)) \qquad \forall x, y \in \mathbb{H}^2,$$

where $d_{st}$ denotes the standard hyperbolic metric. Then $F \colon (\mathbb{H}^2, h_F) \to (\mathbb{H}^2, h_{st})$ is an isometry by construction, so $\mathcal{B}([h_F]) = [F|_{S^1_\infty}] = [f]$. Hence $\mathcal{B}$ is also surjective. In total,

$\mathcal{B}$ gives a one-to-one correspondence between the set of continuous relative hyperbolic structures and $\mathrm{PSL}(2,\mathbb{R})\backslash\mathrm{Homeo}^+(\mathbb{S}^1)$. $\square$

For a continuous relative hyperbolic structure, the most important thing is what happens on the boundary. This is why in the proof of surjectivity above, we can choose an arbitrary extension $F$ of $f$ to the hyperbolic plane. Indeed, the relative hyperbolic structure $[h_F]$ is independent on the choice of the extension $F$: Let $F_1,\ F_2$ be two homeomorphisms of the disk satisfying $F_1|_{S^1_\infty} = F_2|_{S^1_\infty} = f$ and $h_1,\ h_2$ the corresponding hyperbolic metrics. With $\phi := F_2^{-1} \circ F_1$ we have the following diagram:

$$\begin{tikzcd}
(\mathbb{H}^2, h_1) \arrow[rr, "\phi"] \arrow[dr, "F_1"'] & & (\mathbb{H}^2, h_2) \arrow[dl, "F_2"] \\
& (\mathbb{H}^2, h_{st}) &
\end{tikzcd}$$

So $\phi$ is an isometry from $(\mathbb{H}^2, h_1)$ to $(\mathbb{H}^2, h_2)$ satisfying $\phi|_{S^1_\infty} = (F_2^{-1} \circ F_1)|_{S^1_\infty} = id_{S^1_\infty}$, as $F_1$ and $F_2$ agree on the boundary. Thus indeed, $[h_1] = [h_2]$.

*Remark* 4.3. Theorem 4.2 gives a connection between the set of all continuous relative hyperbolic structures and universal Teichmüller space $\mathcal{T}$ introduced in Section 1.1: As the quasisymmetric maps of the circle are a subset of all orientation-preserving homeomorphisms, we can see the universal Teichüller space $\mathcal{T}$ as a subset of the set of all continuous relative hyperbolic structures.

# 5. Earthquakes in the hyperbolic plane

In geology, roughly speaking, earthquakes arise as a consequence of tectonic plate shifts when two plates move against each other. Earthquakes in the hyperbolic plane can be described similarly: If we cut the hyperbolic plane along a geodesic, we obtain two half-planes. Using an isometry, we can "move" or "shift" one of the planes against the other and then reglue the two half-planes. Such a map is called an elementary earthquake. To make precise in what direction we shift, we have to fix an orientation of $\mathbb{H}^2$. We can not only shift along one single geodesic but we could as well use a finite or even infinite set of disjoint geodesics, where with *geodesic* we always mean a complete geodesic $\gamma\colon \mathbb{R} \to \mathbb{H}^2$ homeomorphic to the real line where the endpoints $\lim_{t\to+\infty}\gamma(t)$ and $\lim_{t\to-\infty}\gamma(t)$ are in $\partial\mathbb{H}^2$. To give the definition of an earthquake, we first need to specify the geodesics along which we shear. They are given by a so-called *lamination*. The material presented in this section is based on [Thu06] and [Hu12].

## 5.1. Definition and basic properties

**Definition 5.1.** *A* geodesic lamination $\lambda$ *of the hyperbolic plane is a collection of geodesics that foliate a closed subset $L$. This means that the union of the geodesics equals $L$ and every $p \in L$ lies in exactly one geodesic. The closed set $L$ is the* locus *of $\lambda$, the geodesics are* leaves*. The components of $\mathbb{H}^2 \setminus L$ are* gaps*. The leaves together with the gaps are the* strata *of the lamination $\lambda$. We say that $\lambda$ is a* finite lamination *if the collection of geodesics is finite. Otherwise, $\lambda$ is* infinite.

**Definition 5.2.** *Let $\lambda$ be a geodesic lamination of the hyperbolic plane. A* $\lambda$-left earthquake *is a bijective map $E\colon \mathbb{H}^2 \to \mathbb{H}^2$ such that*

*i) for any stratum $A$ of $\lambda$, the restriction $E|_A$ of $E$ to $A$ agrees with the restriction of an isometry of the hyperbolic plane which we denote by $(E|A)$ and*

*ii) for any two strata $A \neq B$ of $\lambda$, the* comparison isometry

$$\operatorname{cmp}(A, B) = (E|A)^{-1} \circ (E|B)\colon \mathbb{H}^2 \to \mathbb{H}^2$$

*is a hyperbolic transformation whose axis $\ell$ weakly separates $A$ and $B$ and which translates to the left as viewed from $A$.*

*The fact that $\ell$ weakly separates $A$ and $B$ means that any path connecting some points $a \in A$ and $b \in B$ intersects $\ell$.*

When $A$ is a geodesic contained in the closure of $B$, then we permit $\operatorname{cmp}(A, B)$ to be trivial. Further, shifting to the left as viewed from $A$ has to be understood as follows: Consider the connected component of $\mathbb{H}^2 \setminus \ell$ containing $A$. The orientation of $\mathbb{H}^2$ induces an orientation on this component. In particular this gives an orientation on the translation axis $\ell$. Now shifting to the left means shifting in the positive direction with respect to this orientation.
Note that in general, the leaves of a lamination $\lambda$ are not fixed by a $\lambda$-left earthquake map. For a leaf $\ell$, $E|_\ell$ agrees with an isometry and can map $\ell$ to another geodesic of the hyperbolic plane. It can be convenient to keep track of what happens to the lamination $\lambda$ when we apply a $\lambda$-left earthquake. That is why sometimes when talking about a left earthquake, we include this data.

**Definition 5.3.** *A* left earthquake *of the hyperbolic plane consists of two copies of* $\mathbb{H}^2$*, the* source *and the* target *of the earthquake map, together with laminations* $\lambda_s$ *and* $\lambda_t$ *and a* $\lambda_s$*-left earthquake sending the strata of* $\lambda_s$ *to the strata of* $\lambda_t$*.*

To shorten notation, we just talk about a *left earthquake* without mentioning the underlying lamination explicitly. Until now, we have only been talking about left earthquakes. Right earthquakes are defined analogous; we simply replace “left” by “right” in the definitions given above.

**Lemma 5.4.** *The inverse of a left earthquake is a right earthquake.*

*Proof.* Let $E$ be a left earthquake map with source lamination $\lambda_s$ and target lamination $\lambda_t$. For any stratum $A_t$ of $\lambda_t$, there is some stratum $A_s$ of $\lambda_s$ with $A_t = E(A_s)$. Let $F$ be the inverse of $E$. We have

$$F|_{A_t} = (E^{-1})|_{E(A_s)} = (E|_{A_s})^{-1}.$$

Since $E|_{A_s}$ agrees with the isometry $(E|A_s)$, also $F|_{A_t}$ agrees with an isometry, namely with $(F|A_t) = (E|A_s)^{-1}$. For the comparison isometry of two strata $A_t, B_t$, we have

$$\operatorname{cmp}_F(A_t, B_t) = (F|A_t)^{-1} \circ (F|B_t) = (E|A_s) \circ (E|B_s)^{-1}$$

where $\operatorname{cmp}_F$ denotes the comparison isometry with respect to $F$ and cmp always denotes the comparison isometry for $E$. We know that $\operatorname{cmp}(A_s, B_s) = (E|A_s)^{-1} \circ (E|B_s)$ is a hyperbolic isometry with axis $\ell$ weakly separating $A_s$ and $B_s$. Let $x_1, x_2$ be the endpoints of the translation axis $\ell$ and set $y_i := (E|A_s)(x_i) = (E|B_s)(x_i)$ for $i = 1, 2$. The last equality holds since $\mathrm{cmp}(A_s, B_s)$ fixes the points $x_i$. It then follows that for $i = 1, 2$

$$\mathrm{cmp}_F(A_t, B_t)(y_i) = (E|A_s) \circ (E|B_s)^{-1}(y_i) = (E|A_s)(x_i) = y_i \ ,$$

so $\mathrm{cmp}_F(A_t, B_t)$ fixes $y_1$ and $y_2$, so is hyperbolic with axis $\ell' := (E|A_s)(\ell) = (E|B_s)(\ell)$. Both $(E|A_s)$ and $(E|B_s)$ are orientation-preserving, so $A_t = (E|A_s)(A_s)$ and $(E|B_s)(A_s)$ lie on the same side of $\ell'$ and $(E|A_s)(B_s)$ and $B_t = (E|B_s)(B_s)$ lie on the other side of $\ell'$. Hence, $\ell'$ weakly separates $A_t$ and $B_t$. It remains to check that $\mathrm{cmp}_F(A_t, B_t)$ translates to the right as viewed from $A_t$. Let $x_1$ be the repelling fixed point of $\mathrm{cmp}(A_s, B_s)$. Since $(E|A_s)$ is orientation-preserving, shifting to the left as viewed from $A_t = (E|A_s)(A_s)$ is shifting from $y_1$ to $y_2$. We have to show that $y_1$ is attracting for $\mathrm{cmp}_F(A_t, B_t)$. Let $w \in \mathbb{S}^1$ be a point with $y_1 \neq w \neq y_2$. Then $z := (E|B_s)^{-1}(w)$ is not equal to $x_1 = (E|B_s)^{-1}(y_1)$ and

$$\mathrm{cmp}_F(A_t, B_t)(w) = (E|A_s) \circ (E|B_s)^{-1}((E|B_s)(z)) = (E|A_s)(z).$$

Since $x_1$ is repelling for $\mathrm{cmp}(A_s, B_s)$, $z$ is closer to $x_1$ than $(E|A_s)^{-1} \circ (E|B_s)(z)$. By applying $(E|A_s)$, we see that $(E|A_s)(z)$ is closer to $(E|A_s)(x_1 = y_1)$ than $(E|B_s)(z) = w$, i.e. $\mathrm{cmp}_F(A_t, B_t)(w)$ is closer to $y_1$ than $w$. This shows that $y_1$ is attracting, so $\mathrm{cmp}(A_t, B_t)$ shifts from $y_2$ to $y_1$, i.e. to the right as viewed from $A_t$. $\square$

In the following, an earthquake will always be a left earthquake unless specified otherwise. We now prove a technical lemma on compositions of hyperbolic isometries and give sufficient conditions for a map to be an earthquake. For that, we need a version of the Brouwer fixed point theorem.

**Theorem 5.5.** *Let $C \neq \emptyset$ be a compact convex subset of a finite dimensional normed space $X$ and let $f\colon C \to C$ be continuous. Then $f$ has a fixed point.*

*Proof.* See [Wer07, Satz IV.7.15]. $\square$

**Lemma 5.6.** *Let $S$ and $T$ be hyperbolic transformations with non-intersecting axes $\ell_S, \ell_T$, translating in the same direction. Then also $S \circ T$ is hyperbolic with axis weakly separating $\ell_S$ and $\ell_T$ and translating in the same direction as $S$ and $T$. Further, the translation distances satisfy*

$$\tau(S \circ T) \geq \tau(S) + \tau(T). \tag{5.1}$$

*Proof.* The proof here follows [Thu06, Prop. III.1.2.4] and [Hu12, Lemma 4.2]. Both $S$ and $T$ have two fixed points on $S^1_\infty$ that we denote by $s^+, s^-$ and $t^+, t^-$, where $+$ denotes

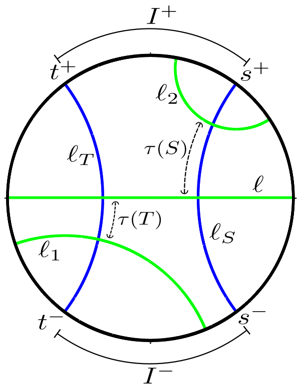



Figure 5.1: The composition of two hyperbolic transformations $S$ and $T$ with disjoint axes that translate in the same direction is again hyperbolic with axis weakly separating the axes of $S$ and $T$. Further, the translation distance for $S \circ T$ is bounded below by the sum of the translation distances for $S$ and $T$.

the attracting and $-$ the repelling fixed point. We assume that the tuple $(s^+, t^+, t^-, s^-)$ is ordered. We can do so since the axis of $S$ and $T$ do not intersect. Let $I^+ := ((s^+, t^+))$ and $I^- := ((t^-, s^-))$ (see Figure (5.1)). We have $T(I^+) \subseteq I^+$ and $S(I^+) \subseteq I^+$, so in total $S \circ T(I^+) \subseteq I^+$. $I^+$ is a compact convex subset of the normed space $S^1_\infty$, identified with $\mathbb{S}^1$, and $S \circ T$ is continuous and maps $I^+$ into itself. It follows by the Brouwer fixed point theorem 5.5 that $S \circ T$ has a fixed point $x^+ \in I^+$. Analogously, $(S \circ T)^{-1}$ has a fixed point $x^- \in I^-$. As isometry of $\mathbb{H}^2$ with at least two fixed points on $S^1_\infty$, $S \circ T$ is hyperbolic with axis $\ell^*$ connecting $x^+$ and $x^-$, so separating $\ell_S$ and $\ell_T$. Since $S \circ T(I^+) \subseteq I^+$, $x^+$ is the attracting fixed point, so the direction of translation is the same as for $S$ and $T$.

For the second part, let us first assume that $S$ and $T$ have the repelling fixed point in common, i.e. $s^- = t^-$. Then $s^-$ is the repelling fixed point of $S \circ T$ and using Lemma 2.10 we have

$$\begin{aligned}
\tau(S \circ T) &= \log\big((S \circ T)'(s^-)\big) \\
&= \log\big(S'(T(s^-)) \cdot T'(s^-)\big) \\
&= \log\big(S'(s^-)\big) + \log\big(T'(t^-)\big) \\
&= \tau(S) + \tau(T). \qquad (5.2)
\end{aligned}$$

If $S$ and $T$ have the attracting fixed point in common, i.e. $s^+ = t^+$ then the inverses $S^{-1}$ and $T^{-1}$ share the repelling fixed point $s^+$. Since $\tau(\varphi) = \tau(\varphi^{-1})$ for any hyperbolic transformation $\varphi$ it follows with (5.2) that

$$\tau(S \circ T) = \tau(T^{-1} \circ S^{-1}) = \tau(T^{-1}) + \tau(S^{-1}) = \tau(S) + \tau(T).$$

Let now $\ell_S$ and $\ell_T$ not have any endpoint in common. Then there exists a unique geodesics $\ell$ that is orthogonal to both $\ell_S$ and $\ell_T$ (see [Bea83, Ch. 7.22]). Set $\ell_1 := T^{-1}(\ell)$ and $\ell_2 := S(\ell)$. Then $S \circ T$ maps $\ell_1$ to $\ell_2$.

*Fact.* $\ell_1$ and the axis $\ell^*$ of $S \circ T$ intersect.

*Proof of the fact.* Assume $\ell_1$ and $\ell^*$ do not intersect. Then $\ell_1$ and $\ell_2 = S \circ T(\ell_1)$ lie on the same side of $\ell^*$. By construction, $\ell_1$ intersects $\ell_T$ and $\ell_2$ intersects $\ell_S$. As neither $\ell_T$ nor $\ell_S$ intersect $\ell^*$, they all have to lie on the same side of $\ell^*$, contradicting the fact that $\ell^*$ weakly separates $\ell_T$ and $\ell_S$. $\square$

Let $z_1$ be the intersection point of $\ell_1$ and $\ell^*$ and let $z_2 := S \circ T(z_1) \in \ell_2$. Then, as the translation length is realized in every point on the axis, we have

$$\tau(S \circ T) = \inf_{z \in \mathbb{H}^2} \mathrm{d}(z, S \circ T(z)) = \mathrm{d}(z_1, z_2) \geq \inf_{z \in \ell_1,\ w \in \ell_2} \mathrm{d}(z, w) =: \mathrm{d}(\ell_1, \ell_2), \tag{5.3}$$

Let $\ell_3$ be the common perpendicular to $\ell_1$ and $\ell_2$. Denote the intersection of $\ell_3$ with $\ell_1, \ell, \ell_2$ by $x_1, x, x_2$ respectively. Then

$$\mathrm{d}(x_1, x_2) = \mathrm{d}(x_1, x) + \mathrm{d}(x, x_2) \tag{5.4}$$

where we have equality since the points all lie on the geodesic $\ell_3$. Further, by definition of $x_1$ and $x_2$ we have

$$\begin{aligned} \mathrm{d}(x_1, x_2) &= \mathrm{d}(\ell_1, \ell_2), \\ \mathrm{d}(x_1, x) &\geq \mathrm{d}(\ell_1, \ell) = \tau(T), \\ \mathrm{d}(x, x_2) &\geq \mathrm{d}(\ell, \ell_2) = \tau(S). \end{aligned}$$

The last two equalities on the right hold since the distance between two geodesics is realized along the unique geodesic orthogonal to both (see [Bea83, Ch. 7.23]). In the case of $\ell$ and $\ell_1$ this is $\ell_T$, so $\mathrm{d}(\ell_1, \ell)$ is the length of the geodesic segment of $\ell_T$ between $\ell$ and $\ell_1$ and this is just $\tau(T)$. The same argument works for $\tau(S)$. Together with (5.3) and (5.4) we obtain the inequality (5.1). $\square$

**Lemma 5.7.** *If $\lambda$ is a finite lamination, a map $E\colon \mathbb{H}^2 \to \mathbb{H}^2$ that is an isometry on every stratum of $\lambda$ is a left earthquake map if and only if the comparison maps for adjacent strata $A$ and $B$ are hyperbolic transformations with axis the leaf weakly separating $A$ and $B$, translating to the left as viewed from $A$.*

*Proof.* One implication is immediate. For the other, let $A$ be a fixed stratum of $\lambda$. For any other stratum $B$ consider a path $c$ from $A$ to $B$. As $\lambda$ is finite, $c$ intersects only finitely many strata $A_0 = A, A_1, \dots, A_n = B$, numbered in the order they intersect $c$. Then

$$\begin{aligned}\operatorname{cmp}(A,B) &= \operatorname{cmp}(A,A_1) \circ \operatorname{cmp}(A_1,B)\\ &= \operatorname{cmp}(A,A_1) \circ \operatorname{cmp}(A_1,A_2) \circ \dots \circ \operatorname{cmp}(A_{n-1},B)\ .\end{aligned}$$

Those are all hyperbolic with non-intersecting axes, translating in the same direction. By Lemma 5.6 it follows that the axis of $\operatorname{cmp}(A,B)$ weakly separates $A$ and $B$ and translates to the left as viewed from $A$.
For injectivity, let $x, y \in \mathbb{H}^2$ with $E(x) = E(y)$. Then $x$ and $y$ cannot lie in different strata $A, B$, since then $(E|A)(x) = E(x) = E(y) = (E|B)(y)$ implies

$$\operatorname{cmp}(A,B)(y) = (E|A)^{-1} \circ (E|B)(y) = x.$$

But we have just proved that $\operatorname{cmp}(A,B)$ is a hyperbolic transformation with axis $\ell$ weakly separating $A$ from $B$. Then $x$ and $y$ either lie on different sides of $\ell$ or one lies on $\ell$ and the other does not. Either way, $\operatorname{cmp}(A,B)$ cannot map $y$ to $x$. So $x$ and $y$ have to lie in the same stratum $A$ and $(E|A)(x) = E(x) = E(y) = (E|A)(y)$. By injectivity of $(E|A)$ it follows that $x = y$, so also $E$ has to be injective.
It remains to show that $E$ is surjective. By looking at the comparison isometries, the boundary $\partial E(A)$ for a gap $A$ is just $E(\partial A)$. In particular, if $A$ has boundary component $\ell$, then $E(A)$ has boundary component $E(\ell)$. As the lamination is finite, it follows that $\operatorname{Im}(E)$ is closed. If now $y \notin \operatorname{Im}(E)$, then there must be a maximal open ball $U_\varepsilon(y)$ that is disjoint from $\operatorname{Im}(E)$. By maximality of $U_\varepsilon(y)$, there is some $\overline{y} \in \overline{U_\varepsilon(x)}$ that lies in the image of $E$. As $\overline{y}$ has to lie in the boundary of the image, it follows that there is some leaf $\ell$ of the lamination with $\overline{y} \in E(\ell)$. In $\lambda$, $\ell$ is adjacent to two gaps $A$ and $B$, so $E(\ell)$ is adjacent to $E(A)$ and $E(B)$ and hence $U_\varepsilon(x)$ either intersects $E(A)$ or $E(B)$, contradicting the fact that it is disjoint from $\operatorname{Im}(E)$. So indeed, $E$ is surjective. $\square$

*Remark* 5.8. The proof of injectivity does not use the fact that the lamination is finite.

Hence, a map that is an isometry on every stratum of an arbitrary lamination $\lambda$ and satisfies the property of the comparison maps for an earthquake is already injective.

The statement of the following Lemma is taken from [Š06].

**Lemma 5.9.** *Let $E$ be a map that satisfies all properties of an earthquake except possibly surjectivity. Then $E$ is surjective if and only if the following holds:*
*For any sequence $(\ell_n)_{n\in\mathbb{N}}$ of leaves and a sequence $(H_n)_{n\in\mathbb{N}}$ half-planes with boundary $\ell_n$ having the property that $H_{n+1} \subseteq H_n$ and $\bigcap_{n\in\mathbb{N}} H_n$ consists of a unique point in $S^1_\infty$, the sequence $(E(\ell_n))_{n\in\mathbb{N}}$ satisfies the same property.*

*Proof.* First, from looking at the comparison isometries, we make some observations:

i) As $E$ is an isometry on every stratum, $E(\ell)$ is a geodesic for a leaf $\ell$ and $E(A)$ is open for a gap $A$. The boundary components of $E(A)$ are geodesics. For a gap $A$ it holds that $\partial E(A) = E(\partial A)$.

ii) If $\ell$ is a leaf and $(\ell_n)_{n\in\mathbb{N}}$ is a sequence of leaves accumulating to it from one side, then also the geodesics $(E(\ell_n))_{n\in\mathbb{N}}$ accumulate to $E(\ell)$ from one side.

Now assume that $y \notin \mathrm{Im}(E)$. Then without loss of generality we can assume that $y \in \overline{\mathrm{Im}(E)}$, because else we expand a ball around $y$ until it hits $\partial\mathrm{Im}(E)$ and call the point of tangency $\overline{y}$. Then $\overline{y}$ is not in $\mathrm{Im}(E)$, because otherwise, by i), it would lie in $E(\ell)$ for some leaf $\ell$. In the source lamination $\lambda$ either $\ell$ is adjacent to gaps on both sides or there is a sequence of strata accumulating to it. In both cases, by i) and ii), the same holds for $E(\ell)$, so $E(\ell)$ cannot lie in the boundary of $\mathrm{Im}(E)$ and hence $\overline{y} \notin \mathrm{Im}(E)$. It follows that there must be a sequence $(\ell_n)_{n\in\mathbb{N}}$ of leaves such that $y \in \lim_{n\to\infty} E(\ell_n)$. The only situation in which this can happen is when we have a sequence of strata $(\ell_n)_{n\in\mathbb{N}}$ as in the statement for which the sequence $(E(\ell_n))_{n\in\mathbb{N}}$ does not satisfy the stated properties.
Vice versa, let $E$ be surjective. We have to show that the limit of the image leaves $E(\ell_n)$ consists of one single point of the boundary. Assume by contradiction that there is more than one point in the limit of the image leaves, i.e. $\lim_{n\to\infty}(\mathrm{diam}\ E(\ell_n)) \neq 0$, where we consider the diameter with respect to the Euclidean metric on the closed disk $\mathbb{H}^2 \cup S^1_\infty$. Let $x_n$ and $y_n$ be the endpoints of the geodesics $E(\ell_n)$. Up to taking a subsequence, those converge to points $x$ respectively $y$ in $S^1_\infty$. As the diameter does not go to zero, we have $x \neq y$. Let $\ell$ be the geodesic connecting $x$ and $y$ and let $p \in \ell$. Then $E(\ell_n)$ gets arbitrarily close to $p$. It follows that there are $p_n \in \ell_n$ and $p \in \mathbb{H}^2$ with $\lim_{n\to\infty} E(p_n) = p$. Since $E$ is surjective, $p$ lies in the image of some stratum $A$, i.e. there is some $q \in A$ such that $p = E(q)$. Then $A$ is a leaf, since otherwise, there is some neighbourhood of $p$

that lies completely in $E(A)$. In particular, for $n$ large enough, $E(p_n)$ lies in $E(A)$ and hence $\ell_n = A$. This contradicts the fact that all $\ell_n$ are leaves. Passing to a subsequence if necessary, we can further assume that $\ell_n \neq A$ for all $n \in \mathbb{N}$ and that all $\ell_n$ lie on the same side of $A$.
Let $a_n$ be the axis of the comparison isometry $\mathrm{cmp}(A, \ell_n)$ and $b_n$ be the axis of $\mathrm{cmp}(\ell_n, \ell_{n+1})$. We have

$$\mathrm{cmp}(A, \ell_{n+1}) = (E|A)^{-1} \circ (E|\ell_n) \circ (E|\ell_n)^{-1} \circ (E|\ell_{n+1}) = \mathrm{cmp}(A, \ell_n) \circ \mathrm{cmp}(\ell_n, \ell_{n+1}).$$

From Lemma 5.6 it follows that $a_{n+1}$ separates $a_n$ and $b_n$. As a result, all $a_n$ lie on the same side of $a_1$. The hyperbolic transformation $\mathrm{cmp}(A, \ell_n)$ cannot move any point on $\ell_n$ to a point on the other side of its axis $a_n$, and $a_n$ weakly separates $\ell_n$ and $A$ by construction. Now all image points $\mathrm{cmp}(A, \ell_n)(p_n)$ lie on the same side of $a_n$ as $\ell_n$, so they all lie on the same side of $a_1$ as $\ell_1$. In particular, $a_1$ separates the points $\mathrm{cmp}(A, \ell_n)(p_n)$ from $A$. But we have

$$\mathrm{cmp}(A, \ell_n)(p_n) = (E|A)^{-1} \circ (E|\ell_n)(p_n) \underset{n\to\infty}{\longrightarrow} (E|A)^{-1}(E(q)) = q \in A,$$

giving a contradiction. This shows that indeed, $\lim_{n\to\infty} \mathrm{diam} E(\ell_n) = 0$, so the limit consist of one single point in $S^1_\infty$. □

## 5.2. Examples

We now look at some examples for earthquakes.

*Example* 5.10. We start with the simplest non-trivial example of an earthquake. Consider the half-plane model $H^2$. Let the lamination $\lambda$ be given by one single geodesic $\ell$. Such an earthquake is called elementary. Let $\ell$ be the geodesic connecting 0 and $\infty$. Then we have two gaps: $A := \{z \in H^2 \mid \mathrm{Re}(z) > 0\}$ and $B := \{z \in H^2 \mid \mathrm{Re}(z) < 0\}$. In total, $\lambda$ consists of the three strata $\ell$, $A$ and $B$. Let $\varphi \in \mathrm{Isom}^+(H^2)$ be defined by $\varphi(z) = e^{-2}z$. It is a hyperbolic transformation with axis $\ell$ shifting towards 0 and with translation distance

$$\tau(\varphi) = \inf_{z\in\mathbb{H}^2} \mathrm{d}(\varphi(z), z) = \mathrm{d}(i, e^{-2}i) = |\log(e^{-2})| = 2.$$

We set $(E|A) := id$ and $(E|B) := (E|\ell) := \varphi$. Then $\mathrm{cmp}(A, B) = \varphi$ is a hyperbolic transformation with axis $\ell$ weakly separating $A$ and $B$ shifting towards 0, so to the left as viewed from $A$. It follows immediately that $\mathrm{cmp}(B, A) = \mathrm{cmp}(A, B)^{-1}$ shifts to the left as viewed from $B$. Further, as $(E|\ell) = (E|B)$, we do not have to check the comparison isometries for $A$ and $\ell$ separately, so $E$ is indeed an earthquake map. Note that defining an earthquake $E'$ by setting $(E'|B) := (E'\ell) = id$ and $(E'|A) = \varphi$ does not give us a left earthquake, but a right earthquake since the corresponding comparison isometry $\mathrm{cmp}'(A, B) = \varphi^{-1}$ translates to the right as viewed from $A$. However, we can define a left earthquake that is the identity on $B \cup \ell$ by setting $(E'|A) := \varphi^{-1}$. In this case, $E$ and $E'$ differ by an isometry, namely $E = \varphi \circ E'$.

*Example* 5.11. In Example 5.10, $E$ maps every stratum to itself and hence the source lamination $\lambda_s$ and the target lamination $\lambda_t$ agree. However, in general this does not have to be the case. We modify Example 5.10 slightly to obtain a left earthquake that changes the lamination. Let $\lambda$ and $\varphi$ be given as in Example 5.10. Set $\psi(z) = z + 1$ and $(E|A) := \psi$, $(E|B) = (E|\ell) := \psi \circ \varphi$. Now $E$ maps $\ell$ to the geodesic $E(\ell)$ connecting 1 and $\infty$. For the comparison isometry, we have as before

$$\mathrm{cmp}(A, B) = \psi^{-1} \circ (\psi \circ \varphi) = \varphi,$$

so $E$ is an earthquake map.

*Example* 5.12. We now give an example that shears along two geodesics that do not have a common endpoint in $S^1_\infty$. An earthquake whose lamination has finitely many leaves is called simple. Let $\lambda$ be given by $\ell_1$ connecting 0 and $\infty$ and by $\ell_2$ connecting 1 and 3. The map

$$\psi(z) = \frac{3z + 1}{z + 1}$$

sends $\ell_1$ to $\ell_2$. If we let $\varphi$ be as in Example 5.10, then the conjugated map $\psi \circ \varphi \circ \psi^{-1}$ is a hyperbolic transformation with axis $\ell_2$ and translation distance 2. Let $A$ be the half-plane bounded by $\ell_2$ that does not contain $\ell_1$, $C$ be the half-plane bounded by $\ell_1$ that does not contain $\ell_2$ and let $B$ be the remaining gap. Now $\lambda$ consists of two leaves $\ell_1$, $\ell_2$ and three gaps $A, B, C$. Set

$$\begin{aligned}
&(E|A) = id,\\
&(E|B) = (E|\ell_2) = \psi \circ \varphi \circ \psi^{-1} \text{ and}\\
&(E|C) = (E|\ell_1) = (\psi \circ \varphi \circ \psi^{-1}) \circ \varphi.
\end{aligned}$$

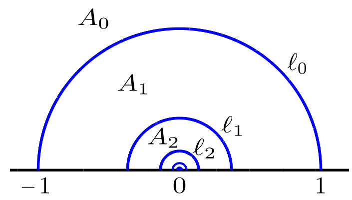



Figure 5.2: A lamination can also consist of infinitely many strata. In this example, there is a sequence of strata with endpoints $-e^{-n}$ and $e^{-n}$ that accumulate to the point 0 in the boundary.

Now $\mathrm{cmp}(A, B) = \psi \circ \varphi \circ \psi^{-1}$ is hyperbolic with axis $\ell_2$ shifting from 3 to 1, so to the left as viewed from $A$. Further

$$\mathrm{cmp}(B, C) = (\psi \circ \varphi \circ \psi^{-1})^{-1} \circ (\psi \circ \varphi \circ \psi^{-1}) \circ \varphi = \varphi.$$

Thus, all comparison isometries behave as they should, so $E$ is indeed a left earthquake. Note that given a finite lamination $\lambda$ and for every leaf a translation distance, one can proceed analogously to construct a left earthquake map realizing the given translation distance assigned to the leaf $\ell$ as translation distance of the comparison isometry of the two strata adjacent to $\ell$.

*Example* 5.13. Our last example is an earthquake with infinitely many leaves. For $n \in \mathbb{N} \cup \{0\}$ let $\ell_n$ be the geodesic connecting $-e^{-n}$ and $e^{-n}$. Let $\lambda$ be the lamination given by the geodesics $\ell_n$, $A_n$ the gap between $\ell_{n-1}$ and $\ell_n$ and $A_0$ the remaining gap (see Figure (5.2)). We construct a left earthquake map that is the identity on $A_0$ and shears along every leaf by distance 2. For all $n \in \mathbb{N} \cup \{0\}$, let $\psi_n$ be the isometry sending the imaginary axis to $\ell_n$. It is given by

$$\psi_n(z) = \frac{e^{-n}z - e^{-n}}{z + 1}.$$

Set $\varphi(z) = e^2 z$. It is a hyperbolic translation with axis the imaginary axis, translation distance 2 and 0 as repelling fixed point. Now for all $n$, $\psi_n \circ \varphi \circ \psi_n^{-1}$ is a hyperbolic transformation with axis $\ell_n$ and translation distance 2. Set $(E|A_0) = id$ and define inductively $(E|A_n) = (E|A_{n-1}) \circ (\psi_n \circ \varphi \circ \psi_n^{-1})$. On the leaves, we let $E$ act as on one of the adjacent gaps, i.e. we set $(E|\ell_n) = (E|A_n)$. On every stratum, $E$ agrees with a concatenation of hyperbolic translations, so it is an isometry. For the comparison isometries of adjacent gaps $A_n$ and $A_{n+1}$ we have

$$\operatorname{cmp}(A_n, A_{n+1}) = (E|A_n)^{-1} \circ (E|A_n) \circ (\psi_n \circ \varphi \circ \psi_n^{-1}) = \psi_n \circ \varphi \circ \psi_n^{-1},$$

so this is a hyperbolic translation with axis $\ell_n$ weakly separating $A_n$ and $A_{n+1}$. Since $\varphi$ has 0 as repelling fixed point, the repelling fixed point of $\operatorname{cmp}(A_n, A_{n+1})$ is $\psi(0) = -e^{-n}$, so it translates to the left as viewed from $A_n$. If now $A_m$ and $A_n$ are arbitrary gaps with $m > n$, then

$$\operatorname{cmp}(A_n, A_m) = \operatorname{cmp}(A_n, A_{n+1}) \circ \operatorname{cmp}(A_{n+1}, A_{n+2}) \circ \cdots \circ \operatorname{cmp}(A_{m-1}, A_m).$$

As the maps on the left are all hyperbolic transformations with disjoint axes translating in the same direction as seen from $A_n$, it follows with Lemma 5.6 that $\operatorname{cmp}(A_n, A_m)$ is hyperbolic with axis weakly separating $A_n$ and $A_m$ and translating to the left as viewed from $A_n$.

## 5.3. Extending earthquakes to the boundary

Even though a non-trivial earthquake is not continuous, it can be extended to the boundary such that the resulting map is continuous at every boundary point.

**Proposition 5.14.** *Let $E$ be an earthquake map. Then there exists a unique map $E_\infty \colon S^1_\infty \to S^1_\infty$ such that $E$ together with $E_\infty$ form a map $\hat{E}$,*

$$\hat{E} \colon \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2} \cup S^1_\infty, \qquad \hat{E}(x) = \begin{cases} E(x) & \text{for } x \in \mathbb{H}^2, \\ E_\infty(x) & \text{for } x \in S^1_\infty, \end{cases}$$

*that is continuous at any point $x \in S^1_\infty$. Further, $E_\infty \colon S^1_\infty \to S^1_\infty$ is an orientation-preserving homeomorphism.*

*Proof.* We want to define $E_\infty$ as extension of $E$ on $S^1_\infty$. For $x \in S^1_\infty$, we distinguish two cases.

Let us first assume that $x$ lies in the closure of some stratum $A$. We set $E_\infty(x) := (E|A)(x)$. Clearly, $E_\infty(x) \in S^1_\infty$, as $(E|A)$ preserves $S^1_\infty$. Further, $E_\infty(x)$ is well-defined: Suppose $x$ lies in the intersection of the closures of two strata $A$ and $B$. In that case, those strata have to meet at infinity, precisely at the point $x$. Consider the comparison isometry $\operatorname{cmp}(A, B) = (E|A)^{-1} \circ (E|B)$. It is a hyperbolic transformation whose axis weakly separates $A$ and $B$ and hence its closure has to contain $x$. Therefore, $\operatorname{cmp}(A, B)$ fixes $x$,

so $(E|A)(x) = (E|B)(x)$ and $E_\infty(x)$ is well-defined.
If $x$ does not lie in the closure of any stratum, the lamination has to consist of infinitely many geodesics (see for instance Example 5.13 and Figure 5.2 with $x = 0$) and $x$ has a neighbourhood basis in $\mathbb{H}^2 \cup S^1_\infty$ consisting of neighbourhoods bounded by leaves $\ell_n$, $n \in \mathbb{N}$, of $\lambda$. By Lemma 5.9, the limit of the image leaves $E(\ell_n)$ consists of one single point in $S^1_\infty$ that we define to be $E_\infty(x)$.
We have to show that $E_\infty$ is bijective and continuous. The inverse of $E$ is a right earthquake map $F$ (Lemma 5.4). In the same way as $E$, $F$ can be extended to a map $F_\infty$ at $S^1_\infty$. It then holds that $F_\infty = E_\infty^{-1}$: If $x \in S^1_\infty$ is in the closure of some stratum $A$, then

$$F_\infty\left(E_\infty(x)\right) = \left(F|(E(A))\right)\left((E|A)(x)\right) = x.$$

If $x$ does not lie in the closure of any stratum, then $E_\infty$ was defined to be the unique point in the limit of the $E(\ell_n)$, so $F_\infty(E_\infty(x))$ is the unique point in the limit of the $F(E(\ell_n)) = \ell_n$, which equals $x$. In the same way, $E_\infty \circ F_\infty = id_{S^1_\infty}$, so $F_\infty$ is the inverse of $E_\infty$ and $E_\infty$ is bijective.
We now show that $E_\infty$ is orientation-preserving. For that, we can show that $E_\infty$ preserves the cyclic order by Lemma 3.6. Let $(x, y, z) \in C$ be an ordered triple, where $C$ denotes the cyclic order on $\mathbb{S}^1$ (see Section 3). Assume first that there is some stratum $A$ such that $x, y, z \in \overline{A}$. As $(E|A)$ preserves orientation, we have that

$$((E|A)(x), (E|A)(y), (E|A)(z)) = (E_\infty(x), E_\infty(y), E_\infty(z)) \in C\ .$$

If there are two strata $A \neq B$ with $x, y \in \overline{A}$ and $z \in \overline{B}$, then the axis $\ell$ of $\mathrm{cmp}(A, B)$ separates $x$ and $y$ from $z$, hence $\ell$ separates $x$ and $y$ from $\mathrm{cmp}(A, B)(z)$. Thus, $((x, y, \mathrm{cmp}(A, B)(z))$ is ordered. By applying the orientation-preserving isometry $(E|A)$ it follows that

$$((E|A)(x), (E|A)(y), (E|B)(z)) = (E_\infty(x), E_\infty(y), E_\infty(z)) \in C\ .$$

The cases $x, z \in \overline{A},\ y \in \overline{B}$ and $y, z \in \overline{A},\ x \in \overline{B}$ can be treated analogously. Now assume that $x, y, z$ lie in the closures of different strata $x \in \overline{A}, y \in \overline{B}, z \in \overline{C}$. Assume by contradiction that $(E_\infty(x), E_\infty(y), E_\infty(z)) \notin C$. Then $(E_\infty(y), E_\infty(x), E_\infty(z)) \in C$. Without loss of generality let $(E|C) = id$. Hence $\mathrm{cmp}(C, A) = (E|A)$ and $\mathrm{cmp}(C, B) = (E|B)$ are hyperbolic with axes separating $z$ from $x$ respectively $y$, both shifting to the left as viewed from $z \in \overline{C}$. We denote the fixed points of $(E|A)$ (respectively, $(E|B)$) by $A^+, A^-$

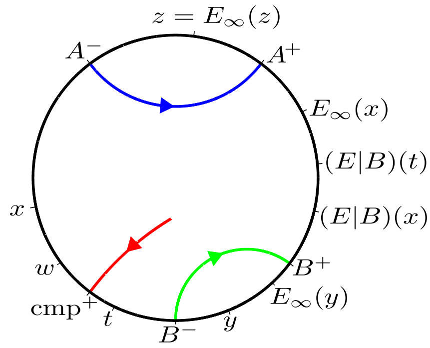



Figure 5.3: To show that $E_\infty$ preserves orientation, we assume that the image of some ordered triple of points $(x, y, z)$ is not ordered. Looking at the axes of comparison isometries, we find that $\mathrm{cmp}(A, B)$ moves $B^-$ towards $\mathrm{cmp}^+$ - contradicting the fact that it translates to the left as viewed from $x \in A$.

(respectively, $B^+, B^-$) where $+$ indicates that the fixed point is attracting. We derive a contradiction through several steps. The situation is shown in Figure 5.3.

*Step 1:* $(x, y, A^+)$ is ordered, since otherwise, $(x, A^+, y)$ would be ordered. Considering where the axis of $(E|A)$ lies and that $E_\infty(x) = (E|A)(x)$, this would imply that $(E_\infty(x), E_\infty(y), E_\infty(z))$ is ordered, contradicting our assumption.

*Step 2:* $\mathrm{cmp}(B, A) = (E|B)^{-1} \circ (E|A)$ has axis weakly separating $x$ and $y$ , translating to the left as viewed from $y$. We denote the fixed points by $\mathrm{cmp}^+$ and $\mathrm{cmp}^-$. Then $(x, \mathrm{cmp}^+, y)$ is ordered. Set

$$w := \mathrm{cmp}(B, A)(x) \qquad \text{and} \qquad t := \mathrm{cmp}(B, A)(y).$$

Then the tuple $(x, w, \mathrm{cmp}^+, t, y)$ is ordered.

*Step 3:* As $(x, w, y) \in C$, also

$$((E|B)(x), (E|B)(w), (E|B)(y)) = ((E|B)(x), E_\infty(x), E_\infty(y)) \in C \ .$$

*Step 4:* Applying $(E|A)$ to the ordered triple $(x, y, A^+)$ we deduce that

$$\big((E|A)(x), (E|A)(y), (E|A)(A^+)\big) = \big(E_\infty(x), (E|B)(t), A^+\big) \in C \ .$$

*Step 5:* $(y, B^+, x) \in C$ since otherwise, we would have $(y, x, B^+) \in C$. Together with the fact that $(y, z, x) \in C$ it would follow that $(y, z, B^+) \in C$, contradicting the fact that the axis of $(E|B)$ separates $y$ from $z$ and that the direction of translation is to the left as viewed from $z$. We deduce that

$$\big((E|B)(y), (E|B)(B^+), (E|B)(x)\big) = \big(E_\infty(y), B^+, (E|B)(x)\big) \in C \ .$$

*Step 6:* As $(E|B)(t)$ and $(E|B)(x)$ lie on the same side of the axis of $(E|B)$, also $x$ and $t$ have to lie on this side of the axis. Hence, $(t, B^-, y) \in C$ and thus also $(x, \mathrm{cmp}^+, B^-) \in C$.
*Step 7:* Now

$$\mathrm{cmp}(A, B)(B^-) = (E|A)^{-1} \circ (E|B)(B^-) = (E|A)(B^-),$$

so $\mathrm{cmp}(A, B)$ moves $B^-$ away from $A^+$, i.e. towards $\mathrm{cmp}^+$. But we also know that $\mathrm{cmp}(A, B) = \mathrm{cmp}(B, A)^{-1}$ has repelling fixed point $\mathrm{cmp}^+$, so moves $B^-$ away from $\mathrm{cmp}^+$ - a contradiction. This shows that $(E_\infty(x), E_\infty(y), E_\infty(z))$ has to be ordered.
We are left with the cases where at least one of $x, y, z$ does not lie in the closure of any stratum. Without loss of generality let $y \notin \overline{A}$ for all strata $A$, but let $x, z$ be contained in closures of strata. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence converging to $y$ and such that for all $n$, $y_n \in \ell_n$ for leaves $\ell_n$. Then for $n$ large enough, $(x, y_n, z)$ is ordered, so also $((E_\infty(x), E_\infty(y_n), E_\infty(z))$ is ordered. Since $(E_\infty(y_n))_{n\in\mathbb{N}}$ converges to $E_\infty(y)$ and we have $E_\infty(z) \neq E_\infty(y) \neq E_\infty(x)$ if follows that $((E_\infty(x), E_\infty(y), E_\infty(z))$ is ordered. Now let $x, y \notin \overline{A}$ for all strata $A$ and let $z \in \overline{A}$ for some $A$. Let $w$ be chosen such that $(x, w, y)$ is ordered. Then also $(x, w, z)$ and $(w, y, z)$ are ordered. It follows that $((E_\infty(x), E_\infty(w), E_\infty(z))$ and $((E_\infty(w), E_\infty(y), E_\infty(z))$ are ordered, hence also $((E_\infty(x), E_\infty(y), E_\infty(z))$ is ordered. The case when no point $x, y, z$ lies in the closure of any stratum follows analogously. So, whenever $(x, y, z)$ is ordered, also $((E_\infty(x), E_\infty(y), E_\infty(z))$ is ordered, so $E_\infty$ preserves orientation.
It follows that $E_\infty$ is continuous: The open intervals $((x, y))$ for $x, y \in \mathbb{S}^1$ form a basis of the topology of $\mathbb{S}^1$. Hence it suffices to show that $E_\infty^{-1}\left(((x, y))\right)$ is open for all $x, y \in \mathbb{S}^1$. Let $x, y, \in \mathbb{S}^1$. As $E_\infty$, also $E_\infty^{-1}$ preserves the circular order, so with bijectivity of $E_\infty$ we have

$$\begin{aligned}
E_\infty^{-1}\left(((x, y))\right) &= \{E_\infty^{-1}(z) \mid (x, z, y) \text{ is ordered}\} \\
&= \{E_\infty^{-1}(z) \mid (E_\infty^{-1}(x), E_\infty^{-1}(z), E_\infty^{-1}(y)) \text{ is ordered}\} \\
&= \{w \in \mathbb{S}^1 \mid (E_\infty^{-1}(x), w, E_\infty^{-1}(y)) \text{ is ordered}\} \\
&= ((E_\infty^{-1}(x), E_\infty^{-1}(y))) \ .
\end{aligned}$$

It follows that $E_\infty$ is continuous. Since $\mathbb{S}^1$ is a compact Hausdorff space, $E_\infty$ is a

homeomorphism.
It remains to check that the map $\hat{E}$ is continuous at every $x \in \mathbb{S}^1$, i.e. that we do not only have continuity on the boundary, but also when approaching $x$ from inside the disk. When approaching a boundary point from within one single stratum $A$, continuity is clear by construction. However, there could be a sequence $(x_n)_{n \in \mathbb{N}}$ in $D^2$ converging to a boundary point $x$ such that all $x_n$ lie in different strata. Also in this case, we have to make sure that $E(x_n)$ converges to $E_\infty(x)$. Let $x \in \mathbb{S}^1$ and $\varepsilon > 0$ be fixed. We do a case distinction.
*Case 1:* $x$ does not lie in $\overline{A}$ for all strata $A$. Then there exists a basis of neighbourhoods $U_n$ of $x$ bounded by leaves $\ell_n$ with endpoints $x_n$ and $y_n$ for all $n \in \mathbb{N}$. Since $E_\infty$ is continuous, there exists $\tilde{\delta} > 0$ such that for all $t \in \mathbb{S}^1$

$$\|x - t\| < \tilde{\delta} \;\Rightarrow\; \|E_\infty(x) - E_\infty(t)\| < \varepsilon \ .$$

Let $N \in \mathbb{N}$ be such that $x_n, y_n \in U_{\tilde{\delta}}(x)$ for all $n \geq N$. Set $\delta := \min\{\|x - x_N\|,\ \|x - y_N\|\}$. Then by construction $U_\delta(x) \subseteq U_N$. As $\hat{E}(U_N)$ is bounded by $\hat{E}(\ell_N)$ we have by choice of $\delta$ that $\hat{E}(U_\delta(x)) \subseteq U_\varepsilon\left(\hat{E}(x)\right)$, so

$$\|x - y\| < \delta \;\Rightarrow\; \|\hat{E}(x) - \hat{E}(y)\| < \varepsilon.$$

*Case 2:* $x$ lies in $\overline{A}$ for some stratum $A$. Without los of generality let $(E|A) = id$. Then for every stratum $B \neq A$ we have $\mathrm{cmp}(A, B) = (E|B)$. Let $\tilde{\delta} > 0$ be such that for all $t \in \mathbb{S}^1$

$$\|x - t\| < \tilde{\delta} \;\Rightarrow\; \|E_\infty(x) - E_\infty(t)\| < \varepsilon \ .$$

Let $B$ be some stratum intersecting $U_{\tilde{\delta}}(x)$ and let $\ell_B$ be the leaf in the boundary of $B$ separating $A$ from $B$. We distinguish two cases. If both endpoints of $\ell_B$ lie in $U_{\tilde{\delta}}(x)$, then it can easily be seen that $B \subseteq U_{\tilde{\delta}}(x)$. Hence for $y \in U_{\tilde{\delta}}(x) \cap B$ we have $\hat{E}(y) \in \hat{E}(B) \subseteq U_\varepsilon(\hat{E}(x))$. If only one endpoint of $\ell_B$ lies in $U_{\tilde{\delta}}(x)$, then let $\delta_B > 0$ such that for all $t, w \in \overline{\mathbb{H}^2}$

$$\|t - w\| < \delta_B \;\Rightarrow\; \|(E|B)(t) - (E|B)(w)\| < \varepsilon \ .$$

Set $\mathcal{B} := \{B \text{ stratum of } \lambda \mid \ell_B \text{ has exactly one endpoint in } U_{\tilde{\delta}}(x)\}$.
*Fact.* $\inf\, \{\delta_B \mid B \in \mathcal{B}\} > 0$.

*Proof of the Fact.* When $\mathcal{B}$ is finite, then the claim is trivial. For the general case, let $B \in \mathcal{B}$. As $(E|B) = \mathrm{cmp}(A, B)$ is hyperbolic, $\delta_B$ can be characterized by the translation length: Consider the half-plane model $H^2$ and assume that $(E|B)$ has axis connecting 0 and $\infty$. Then $(E|B)(z) = e^{\lambda} z$ for some $\lambda \neq 0$. Now

$$\|(E|B)(t) - (E|B)(w)\| = e^{\lambda}\|t - w\|.$$

For $\lambda < 0$, 0 is the attracting fixed point and we can choose $\delta_B = \varepsilon > 0$. If 0 is repelling, then

$$e^{\lambda}\|t - w\| < \varepsilon \;\Leftrightarrow\; \|t - w\| < \frac{\varepsilon}{e^{\lambda}} \;,$$

so we can choose $\delta_B = \frac{\varepsilon}{e^{\lambda}}$ with $\lambda = \tau\left((E|B)\right)$. Now $\inf \{\delta_B \mid B \in \mathcal{B}\} > 0$ is equivalent to $\sup \{\tau\left((E|B)\right) \mid B \in \mathcal{B}\} < \infty$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of strata in $\mathcal{B}$. Up to passing to a smaller neighbourhood of $x$ and up to passing to a subsequence, we can assume that $\ell_n := \ell_{B_n}$ weakly separates $A$ from $\ell_{n+1} = \ell_{B_{n+1}}$, so the $\ell_n$ get closer to $A$. Now the axes of $\mathrm{cmp}(A, B_{n+1}) = (E|B_{n+1})$ and $\mathrm{cmp}(B_{n+1}, B_n)$ cannot intersect and both are hyperbolic transformations translating in the same direction. By Lemma 5.6 we have

$$\begin{aligned}\tau\left((E|B_n)\right) &= \tau\left(\mathrm{cmp}(A, B_{n+1}) \circ \mathrm{cmp}(B_{n+1}, B_n)\right)\\ &\geq \tau\left(\mathrm{cmp}(A, B_{n+1})\right) + \tau\left(\mathrm{cmp}(B_{n+1}, B_n)\right)\\ &\geq \tau\left((E|B_{n+1})\right) \;. \end{aligned} \tag{5.5}$$

Hence, the sequence $\left(\tau\left((E|B_n)\right)\right)_{n\in\mathbb{N}}$ is non-increasing. This shows that the supremum of the translation distances is bounded from above, so $\inf \{\delta_B \mid B \in \mathcal{B}\} > 0$. □

Set $\delta := \min\{\tilde{\delta}, \inf\{\delta_B \mid B \in \mathbb{B}\}\}$. Then for $y \in U_\delta(x)$, we have $y \in \overline{B}$ for some $B$ intersecting $U_\delta(x) \subseteq U_{\tilde{\delta}}(x)$. Now for $B$, we either have $\ell_B \subseteq U_\delta$ or only one endpoint of $\ell_B$ lies in $U_\delta(x)$. In both cases, we have $\hat{E}(y) \in U_\varepsilon(\hat{E}(x))$, so $\hat{E}$ is continuous in $x$. Summing up, we have shown that $E$ can be continuously extended to an orientation-preserving homeomorphism $E_\infty$ of $S^1_\infty$. □

# 6. The earthquake theorem in the hyperbolic plane

The main goal of this thesis is to establish a correspondence between continuous relative hyperbolic structures and earthquakes. More precisely, we will show that any two continuous relative hyperbolic structures differ by an earthquake - where we have to make clear what this means exactly (see Corollary 6.16). Since we have already seen that continuous relative hyperbolic structures correspond to orientation-preserving homeomorphisms of $\mathbb{S}^1$ up to post-composition with elements in $\mathrm{PSL}(2,\mathbb{R})$ (Theorem 4.2), we first consider a correspondence between earthquakes and homeomorphisms of the circle. As seen in Proposition 5.14, every earthquake map $E$ gives an orientation-preserving homeomorphism $E_\infty$ of $\mathbb{S}^1$. We now show the converse of this statement.

**Theorem 6.1** (Thurston's earthquake theorem [Thu06])**.** *Every orientation-preserving homeomorphism $f$ of $S^1_\infty$ to itself arises as the limiting value $E_\infty$ of a left earthquake map $E$. The underlying lamination $\lambda$ of $E$ is uniquely determined by $f$ and $E$ is uniquely determined on all gaps. For any leaf $\ell$ two possible isometries for $(E|\ell)$ differ by a hyperbolic isometry with axis $\ell$ and translation length between $0$ and the infimum of the translation lengths of the comparison maps for $E$ on the two sides of $\ell$.*

We devote most of the chapter to the proof of this result.

## 6.1. Extreme left homeomorphisms

Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism of $\mathbb{S}^1$. Fix an extension of $f$ to the disk $\mathbb{D}^2$ which we denote by $f$ as well. We want to construct an earthquake map, a map that piecewise is a hyperbolic isometry. Therefore, we consider the left coset $\mathcal{C} := \mathrm{PSL}(2,\mathbb{R}) \circ f$ of $f$ in $)\backslash\mathrm{Homeo}^+(\mathbb{S}^1\ \mathrm{PSL}(2,\mathbb{R})$. When acting on the circle, some elements of $\mathcal{C}$ have fixed points, others do not. For example, if $f = id$, then $\mathcal{C} = \mathrm{PSL}(2,\mathbb{R})$ and hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$ have two fixed points, while elliptic elements have none. If a homeomorphism $h$ of $\mathbb{S}^1$ has at least one fixed point, then there is a unique lift $H$ of $h$ to $\mathbb{R}$ that also has fixed points (Lemma 3.7).

**Definition 6.2.** *A homeomorphism $h$ of $\mathbb{S}^1$ is an* extreme left homeomorphism *if*

*i) $h$ has at least one fixed point and*

*ii) the unique lift $H$ of $h$ that has fixed points satisfies $H(x) \geq x$ for all $x \in \mathbb{R}$.*

*We set* $XL := \{h \in \mathcal{C} \mid h \text{ is an extreme left homeomorphism}\}$.

Geometrically, an extreme left homeomorphism moves all points counterclockwise, except for those that it fixes. The elements in $XL$ will be crucial for the construction of the earthquake map. However, we do not know a lot about $XL$ yet. It turns out that topologically, $XL$ is a plane. To establish this, we find a homeomorphism between $XL$ and $\mathbb{H}^2$.

**Proposition 6.3.** *The set $XL$ is homeomorphic to the hyperbolic plane.*

*Proof.* Since $\mathrm{PSL}(2,\mathbb{R})$ is a topological space, so is $\mathcal{C} = \mathrm{PSL}(2,\mathbb{R}) \circ f$ - just identify $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ with $\varphi \circ f \in \mathcal{C}$. We fix a base point $x_0 \in \mathbb{H}^2$, for instance $x_0 = 0$ in the disk model. We define

$$\begin{aligned} p : \qquad \mathcal{C} &\longrightarrow \mathbb{H}^2 \\ \varphi \circ f &\longmapsto \varphi \circ f(x_0), \end{aligned}$$

so $p$ is the evaluation at $x_0$. As seen in Lemma 2.13, $p$ is continuous and surjective. We claim that the restriction of $p$ to $XL$ is a homeomorphism. For the proof, we use the disk model $D^2$. We first show that $p|_{XL}$ is a bijection. Fix $y \in \mathbb{H}^2$. We want to prove that there exists exactly one element in $XL \cap p^{-1}(\{y\}) \subseteq \mathcal{C}$. Since $p$ is surjective, there exists some element $h_0 \in C$ with $p(h_0) = y$. Any other element $h$ in $p^{-1}(\{y\})$ is of the form $\varphi \circ h_0$ for some isometry $\varphi$. We have

$$\varphi(y) = \varphi(h_0(x_0)) = h(x_0) = y,$$

i.e. the isometry $\varphi$ fixes $y$. Let us first assume that $y = 0$. Then the isometries fixing $y$ act as rotations of the circle. In total, we have

$$p^{-1}(\{0\}) = \{r_\alpha \circ h_0 \mid \alpha \in [0, 2\pi)\},$$

where $r_\alpha$ denotes the rotation by angle $\alpha$ and center 0, so the fibre of $p$ over 0 is a circle. The same will be true for all other points $y \in \mathbb{H}^2$. To show that $p^{-1}(\{0\})$ contains exactly one extreme left homeomorphism, we work with lifts of circle maps (see Section 3). Let $H_0$ be any lift of $h_0$ to $\mathbb{R}$, i.e.

$$H_0 : \mathbb{R} \to \mathbb{R}, \qquad e^{iH_0(y)} = h_0\left(e^{iy}\right) \;\; \forall y \in \mathbb{R}.$$

If now $h = r_\alpha \circ h_0 \in p^{-1}(\{0\})$ and $H$ is a lift of $h$, then

$$e^{iH(y)} = h\left(e^{iy}\right) = r_\alpha \circ h_0\left(e^{iy}\right) = r_\alpha \circ e^{iH_0(y)} = e^{i(\alpha + H_0(y))} \ .$$

Hence, $H$ has to be of the form $H_0 + \alpha + 2\pi k$ for some $k \in \mathbb{Z}$. As a result, all lifts of elements in $p^{-1}(\{0\})$ are of the form $H_0 + T$ for some real constant $T$. Modulo $2\pi$, $T$ only depends on $\alpha$. Remember that we are searching for an extreme left homeomorphism and that an element $h$ of $\mathcal{C}$ is an extreme left homeomorphism if it has some lift $H$ satisfying $H(y) \geqq y$, or equivalently $H(y) - y \geqq 0$, for all $y \in \mathbb{R}$, where we use the symbol $\geqq$ to indicate that the inequality holds with equality at least once. Since $h_0$ is orientation-preserving it holds that

$$H_0(y + 2\pi) = H_0(y) + 2\pi \qquad \forall y \in \mathbb{R},$$

so the function $g(y) := H_0(y) - y$ satisfies for all $y \in \mathbb{R}$

$$g(y + 2\pi) = H_0(y + 2\pi) - (y + 2\pi) = H_0(y) + 2\pi - y - 2\pi = H_0(y) - y = g(y) \ ,$$

i.e. $g$ is $2\pi$-periodic and attains its minimum on the compact interval $[0, 2\pi]$. Set $T := -\min_{y\in[0,2\pi]}(H_0(y) - y)$. Then for all $y \in \mathbb{R}$ we have

$$(H_0(y) + T) - y = (H_0(y) - y) + T \geqq -T + T = 0 \ .$$

It follows that the element $h \in \mathbb{C}$ with lift $H_0 + T$ is an extreme left homeomorphism. Since $T$ is uniquely determined by $H_0$, so is $h$. All other elements in $p^{-1}(\{0\})$ correspond to different $T$, so cannot satisfy the inequality with equality at least once. We showed that $\#(p^{-1}(\{0\}) \cap XL) = 1$. Now we have to adapt the proof for an arbitrary $y \in \mathbb{H}^2$ by changing coordinates.
Let $\psi$ be an isometry satisfying $\psi(y) = 0$. We introduce y-coordinates by $z_y = \psi(z)$, i.e. a point $z$ is identified with its image under $\psi$. In particular, $y$ corresponds to 0 in y-coordinates, so $y_y = 0$. If $h\colon \mathbb{H}^2 \to \mathbb{H}^2$ is a map given in standard coordinates, then the corresponding map in y-coordinates is $h_y = \psi \circ h \circ \psi^{-1}$, since for any $z_y \in \mathbb{H}^2$ given in y-coordinates

$$h_y(z_y) = \psi \circ h \circ \psi^{-1}(\psi(z)) = \psi \circ h(z) = (h(z))_y.$$

If now $\varphi \in \mathrm{PSL}(2, \mathbb{R})$ fixes $y$ in standard coordinates, then $\varphi_y$ in y-coordinates acts as rotation of the circle, since

$$\varphi_y(0) = \psi \circ \varphi \circ \psi^{-1}(0) = \psi \circ \varphi(y) = \psi(y) = 0.$$

We have seen before that all elements $h$ in $p^{-1}(\{y\})$ are of the form $h = \varphi \circ h_0$ for some fixed $h_0 \in p^{-1}(\{y\})$ and some $\varphi \in \mathrm{PSL}(2, \mathbb{R})$ with $\varphi(y) = y$. If we now change to y-coordinates, we have

$$h_y = \psi \circ (\varphi \circ h_0) \circ \psi^{-1} = (\psi \circ \varphi \circ \psi^{-1}) \circ (\psi \circ h_0 \circ \psi^{-1}) = \varphi_y \circ (h_0)_y,$$

where $\varphi_y$ acts as rotation of the circle. So, in y-coordinates, we have exactly the same situation as before and know that there exists a unique extreme left homeomorphism in $p^{-1}(\{y\})$. It remains to check that being an extreme left homeomorphism is independent of coordinates. Let $h$ be an extreme left homeomorphism and $H$ a lift satisfying $H(y) \geqq y$ for all $y \in \mathbb{R}$. Let $\tilde{\psi}, \tilde{\psi}^{-1}$ be lifts of $\psi, \psi^{-1}$, chosen such that they are inverse to each other (Lemma 3.8). Since $\psi, \psi^{-1}$ are orientation-preserving, $\tilde{\psi}, \tilde{\psi}^{-1}$ are monotonically increasing. Then $\tilde{\psi} \circ H \circ \tilde{\psi}^{-1}$ is a lift of $h_y$ since

$$\begin{aligned} e^{i(\tilde{\psi} \circ H \circ \tilde{\psi}^{-1}(y))} &= \psi \circ e^{i(H \circ \tilde{\psi}^{-1}(y))} \\ &= \psi \circ h \circ e^{i(\tilde{\psi}^{-1}(y))} \\ &= \psi \circ h \circ \psi^{-1}\left(e^{iy}\right) \\ &= h_y\left(e^{iy}\right). \end{aligned}$$

As $\tilde{\psi}$ is monotonically increasing and $\tilde{\psi}^{-1}$ is bijective, $H(y) \geqq y$ for all $y \in \mathbb{R}$ implies $\tilde{\psi} \circ H \circ \tilde{\psi}^{-1}(y) \geqq y$ for all $y \in \mathbb{R}$. So indeed, $h_y$ is an extreme left homeomorphism. In the same way one can show that if $h_y$ is an extreme left homeomorphism, so is $h$, i.e. being an extreme left homeomorphism is independent of coordinates. In total, we know that for every $y \in \mathbb{H}^2$ there is exactly one extreme left homeomorphism in $p^{-1}(\{y\})$, so $p|_{XL}\colon XL \to \mathbb{H}^2$ is a bijection.

It remains to show that $p|_{XL}$ is a homeomorphism. Since $p$ is continuous on $\mathcal{C}$ (Lemma 2.13), also $p|_{XL}$ is continuous. Recall the steps we followed for finding the unique $h \in XL$ with $p(h) = y \in \mathbb{H}^2$:

i) For $y \in \mathbb{H}^2$ we chose $h_0 \in XL$ with $h_0(x_0) = y$.

ii) We fixed a lift $H_0$ of $h$ and found $T = -\inf_{x \in \mathbb{R}}(H_0(x) - x)$.

iii) The element of $p^{-1}(\{y\})$ with lift $H_0(x) + T$ is the extreme left homeomorphism we searched for.

To show that $(p|_{XL})^{-1}$ is continuous, we show that all steps can be done in a continuous fashion, i.e.

i) $h_0$ can be chosen such that it depends continuously on $y$,

ii) $H_0$ can be chosen such that it depends continuously on $y$, and $T$ depends continuously on $H$ and

iii) a circle homeomorphism h depends continuously on its lift $H$.

Here, the spaces $\mathbb{R}$, $H^2$ and $D^2$ are equipped with the standard topology induced by the Euclidean metric and the function spaces are endowed with the uniform topology with respect to the Euclidean metric for functions on $\mathbb{R}$ and with respect to the chordal metric for functions on $\overline{H^2} \subseteq \mathbb{C} \cup \{\infty\}$ (see (2.4)). We start with showing i). Any $h \in \mathcal{C}$ is of the form $h = \varphi \circ f$ for some $\varphi \in \mathrm{PSL}(2, \mathbb{R})$, so finding $h_0$ with $h_0(x_0) = y$ is equivalent to finding $\varphi$ with $\varphi(f(x_0)) = y$. It suffices to show that we can find an isometry $\alpha$ of the hyperbolic disc with $\alpha(0) = y$ depending continuously on $y$. If we found such an $\alpha$, we set $\varphi := \alpha \circ \psi$ for a fixed $\psi$ satisfying $\psi(f(x_0)) = 0$. Then $h_0 := \varphi \circ f \in \mathcal{C}$ satisfies $h_0(x_0) = \alpha \circ \psi(f(x_0)) = \alpha(0) = y$. Since $f$ and $\psi$ are independent of $y$ and $\alpha$ depends continuously on $y$, also $h_0$ depends continuously on $y$.
Consider the disk model $D^2$ and let $\alpha$ be the isometry defined by

$$\alpha(z) = \frac{z - y}{1 - \overline{y}z} \qquad \forall z \in \mathbb{D}^2.$$

$\alpha$ corresponds to the matrix

$$A = \frac{1}{1 - \|y\|^2} \begin{pmatrix} 1 & -y \\ -\overline{y} & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

This indeed defines an isometry of the hyperbolic plane in the disk model (see Subsection 2.2). Clearly, $A$ depends continuously on $y$. Further, the map $\Phi\colon \mathrm{SL}(2, \mathbb{C}) \to \mathcal{M}$ sending a matrix to the corresponding Möbius transformation is continuous (by Theorem 2.12). We have the following conjunction of continuous mappings:

$$\mathbb{R} \longrightarrow \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathcal{M}, \quad y \longmapsto A \longmapsto \alpha,$$

so the assignment $y \mapsto \alpha$ is continuous. This shows that we can choose $h_0 \in \mathcal{C}$ with $h_0(x_0) = y$ in a continuous fashion. To prove ii), we first want a lift of $h_0$ that depends

continuously on $h_0$.
In the following, $x$ and $y$ will always be points in $\mathbb{R}$ and $t$ and $w$ will be on $\mathbb{S}^1$. Further, we use $|\cdot|$ for the Euclidean norm on $\mathbb{R}$ and $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^2$ and on subsets, for instance on $\mathbb{S}^1$.
One would like to make an assignment $\mathrm{Homeo}^+(\mathbb{S}^1) \to \mathrm{Homeo}^+(\mathbb{R})$ sending a map $h$ to some lift $H$ by requiring that for instance, $H(y) \in [0, 2\pi)$ for some fixed $y \in \mathbb{R}$. However, this map would not be continuous, as we could have $h(e^{iy})$ and $g(e^{iy})$ close in $\mathbb{S}^1$ with the lifts satisfying $H(y)$ close to 0 and $G(y)$ close to $2\pi$, getting farther away from each other when $h(e^{iy})$ and $g(e^{iy})$ are getting closer. We can fix this issue by assuming that we already have some lift $H$ of a map $h$ and a map $g$ that is $\delta$-close to $h$. All functions considered are either on $\mathbb{R}$ or on $\mathbb{S}^1$, so in the following, all distances are the Euclidean distances. We choose a lift $G$ of $g$ satisfying

$$|H(0) - G(0)| \leq \pi.$$

We can do so since all lifts of $g$ differ by an additive constant that is a multiple of $2\pi$.
Let $\delta < 2$. As $h$ and $g$ are $\delta$-close, we have

$$\sup_{y \in \mathbb{R}} \|h\left(e^{iy}\right) - g\left(e^{iy}\right)\| < \delta < 2,$$

so for no $y \in \mathbb{R}$, $h(e^{iy})$ and $g(e^{iy})$ are antipodal. Since $|H(0) - G(0)| = \pi$ would imply that $h(e^{iy})$ and $g(e^{iy})$ are antipodal, we conclude $|H(0) - G(0)| < \pi$. If $|H(y) - G(y)| > \pi$ for some $y$, then by the intermediate value theorem there would be some $x$ between 0 and $y$ with $|H(x) - G(x)| = \pi$, so $h(e^{ix})$ and $g(e^{ix})$ are antipodal - a contradiction. Hence, $\delta < 2$ and $\sup_{t \in \mathbb{S}^1} \|h(t) - g(t)\| < \delta$ together imply $\sup_{y \in \mathbb{R}} |H(y) - G(y)| < \pi$. Note that we cannot have equality as $H(y) - G(y)$ is $2\pi$-periodic, so the supremum is in fact a maximum.
Since

$$\pi \colon \mathbb{R} \to \mathbb{S}^1 \ , \ \pi(x) = e^{ix}$$

is a covering map, for any point in $\mathbb{S}^1$ there is a neighbourhood $U$ and open sets $V_k \subseteq \mathbb{R}$ for $k \in \mathbb{Z}$ such that

$$\pi^{-1}(U) = \bigcup_{k \in \mathbb{Z}} V_k, \qquad \pi|_{V_k} \colon V_k \to U \text{ is a homeomorphism } \forall k \in \mathbb{Z} \ . \tag{6.1}$$

One says that $U$ is an *evenly covered neighbourhood*. If we fix one open set $V_0 \subseteq \mathbb{R}$, all others are of the form $V_k = V_0 + 2\pi k,\ k \in \mathbb{Z}$. If $U$ is chosen small enough, then $x, y \in \pi^{-1}(U)$ and $|x - y| < \pi$ together imply that $x, y$ lie in the same open set $V_k$. Now for any $k$, $(\pi|_{V_k})^{-1}$ is continuous, so

$$\forall \varepsilon > 0\ \exists\ \delta_\varepsilon > 0:\ \|t - w\| < \delta_\varepsilon\ \Rightarrow |(\pi|_{V_k})^{-1}(t) - (\pi|_{V_k})^{-1}(w)| < \varepsilon\ . \tag{6.2}$$

Note that $\delta_\varepsilon$ can be chosen independently of $k$. For a fixed $t \in \mathbb{S}^1$, evenly covered by open sets $V_k \subseteq \mathbb{R}$, and for $x \in \mathbb{R}$ such that $t = \pi(x) = e^{ix}$ and $H(x) \in V_k$ for some $k$, we have

$$\pi|_{V_k}^{-1}(h(t)) = \pi|_{V_k}^{-1}\left(h\left(\pi|_{V_k}(x)\right)\right) = \pi|_{V_k}^{-1}\left(\pi|_{V_k}(H(x))\right) = H(x),$$

since $\pi|_{V_k}$ is a homeomorphism. Now $\|h(t) - g(t)\| < \delta_\varepsilon$ implies that $H(x)$ and $G(x)$ lie in the same $V_k$ and with (6.2) it follows that $\|H(x) - G(x)\| < \varepsilon$.
Fix $\varepsilon > 0$. As $\mathbb{S}^1$ is compact, there are finitely many open balls $U_i$, $i = 1, \ldots, n$, of radius at most $\delta_\varepsilon$ that cover $\mathbb{S}^1$ and that are all evenly covered. Set $\delta < \min\{\delta_\varepsilon, 2\}$. For all $U_i$ and all $t \in U_i$ with $x \in \mathbb{R}$ satisfying $t = e^{ix}$, we have shown that $|h(t) - g(t)| < \delta$ implies $|H(x) - G(x)| < \varepsilon$. It follows that

$$\sup_{t\in\mathbb{S}^1} \|h(t) - g(t)\| < \delta \quad \Rightarrow \quad \sup_{x\in\mathbb{R}} |H(x) - G(x)| < \varepsilon.$$

Hence, if we have a homeomorphism $h$ and a lift $H$ and vary $h$ just a bit to a homeomorphism $g$, then there is a lift $G$ of $g$ that is close to $H$.
We now show that the infimum of a function depends continuously on the function, i.e. for $h, g \in \mathrm{Homeo}^+(\mathbb{R})$

$$\forall \varepsilon > 0\ \exists \delta > 0:\ \sup_{x\in\mathbb{R}} |h(x) - g(x)| < \delta\ \Rightarrow |\inf_{x\in\mathbb{R}} h(x) - \inf_{x\in\mathbb{R}} g(x)| < \varepsilon\ .$$

Let $\varepsilon > 0$, $\delta$ such that $0 < \delta < \varepsilon$ and let $\sup_{x\in\mathbb{R}} |h(x) - g(x)| < \delta$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} g(x_n) = \inf_{x\in\mathbb{R}} g(x)$. Then $g(x_n) > h(x_n) - \delta$ for all $n \in \mathbb{N}$ and

$$\inf_{x\in\mathbb{R}} g(x) = \lim_{n\to\infty} g(x_n) \geq \liminf_{n\to\infty} h(x_n) - \delta \geq \inf_{x\in\mathbb{R}} h(x) - \delta.$$

Analogously, one shows that $\inf_{x\in\mathbb{R}} h(x) \geq \inf_{x\in\mathbb{R}} g(x) - \delta$, so in total

$$|\inf_{x\in\mathbb{R}} h(x) - \inf_{x\in\mathbb{R}} g(x)| \leq \delta < \varepsilon.$$

This shows continuity in step ii). For iii), we only have to show that a function on $\mathbb{S}^1$ depends continuously on its lift. Note that $\pi\colon \mathbb{R} \to \mathbb{S}^1$ is continuous, so

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; |x - y| < \delta \;\Rightarrow\; \|\pi(x) - \pi(y)\| < \varepsilon \; .$$

Let $\varepsilon > 0$ and let $\delta$ be as above. Let $F, G \in \mathrm{Homeo}^+(\mathbb{R})$ be lifts of $f, g \in \mathrm{Homeo}^+(\mathbb{S}^1)$ and let $F, G$ be $\delta$-close, i.e. $\sup_{x\in\mathbb{R}} |F(x) - G(x)| < \delta$. Then by surjectivity and continuity of $\pi$ it follows that

$$\begin{aligned} \sup_{t\in\mathbb{S}^1} |f(t) - g(t)| &= \sup_{y\in\mathbb{R}} |f(\pi(y)) - g(\pi(y))| \\ &= \sup_{y\in\mathbb{R}} |\pi(F(y)) - \pi(G(y))| < \varepsilon, \end{aligned}$$

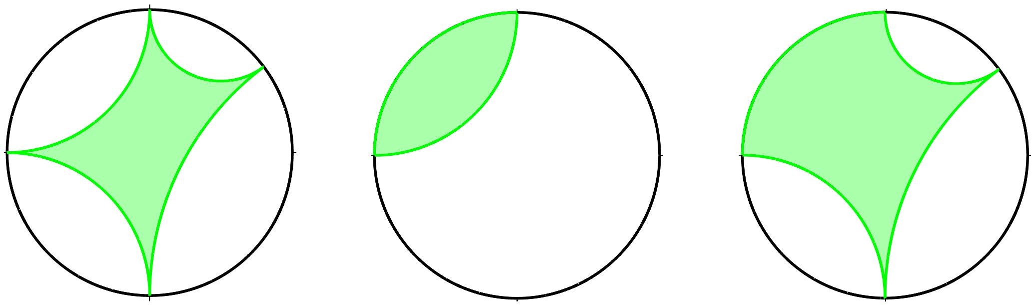
so the map assigning to a lift $F$ the map $f$ is continuous. In total, we have proven that the construction of the inverse of $p$ is continuous, so indeed $p|_{XL}\colon XL \to \mathbb{H}^2$ is a homeomorphism. $\square$

## 6.2. Convex hulls

From the set $XL$, we now want to construct a lamination. Given $g \in XL$, let $\mathrm{fix}(g)$ be the set of fixed points of $g$ on $S^1_\infty$ and let $H(g) \subseteq \mathbb{D}^2 \cong \mathbb{H}^2 \cup S^1_\infty$ be the convex hull of $\mathrm{fix}(g)$, that is the smallest convex set containing $\mathrm{fix}(g)$ (see Figure 6.1). Here, we consider the convex hull in the hyperbolic sense: A set $C$ is convex if for any $x, y \in C$, the hyperbolic geodesic arc connecting $x$ and $y$ is contained in $C$. For instance

- if $\mathrm{fix}(g)$ consists of one single point, then $H(g) = \mathrm{fix}(g)$,
- if $\mathrm{fix}(g)$ consists of two points, then $H(g)$ is the hyperbolic geodesic connecting those points,
- if $\mathrm{fix}(g)$ consists of $n$ points, then $H(g)$ is the ideal hyperbolic $n$-gon with vertices the points in $\mathrm{fix}(g)$.

If $H(g)$ is two-dimensional, it is bounded by geodesics that connect points in $\mathrm{fix}(g)$. Since the leaves of a lamination do not cross and we want to use the convex hulls $H(g)$ for



(a) (b) (c)

Figure 6.1: The convex hulls $H(g)$ are defined using fix$(g)$. If fix$(g)$ is finite, then $H(g)$ is an ideal hyperbolic polygon (6.1a). In general fix$(g)$ can contain an interval of $S^1_\infty$ (6.1b and 6.1c).

$g \in XL$ to construct a lamination, we would like to have a similar property for the $H(g)$. Indeed, the following holds:

**Lemma 6.4.** *The sets $H(g)$ for $g \in XL$ do not cross each other. More precisely, if $g_1, g_2 \in XL$ satisfy $\#fix(g_i) > 1$ for $i = 1, 2$ and $H(g_1) \neq H(g_2)$, then $H(g_2)$ is contained in the closure of $\mathbb{D}^2 \setminus H(g_1)$.*

*Proof.* Let $g_1, g_2 \in XL$ be as in the statement and let $\varphi_1, \varphi_2 \in \mathrm{PSL}(2, \mathbb{R})$ be such that $g_i = \varphi_i \circ f$ for $i = 1, 2$. We have

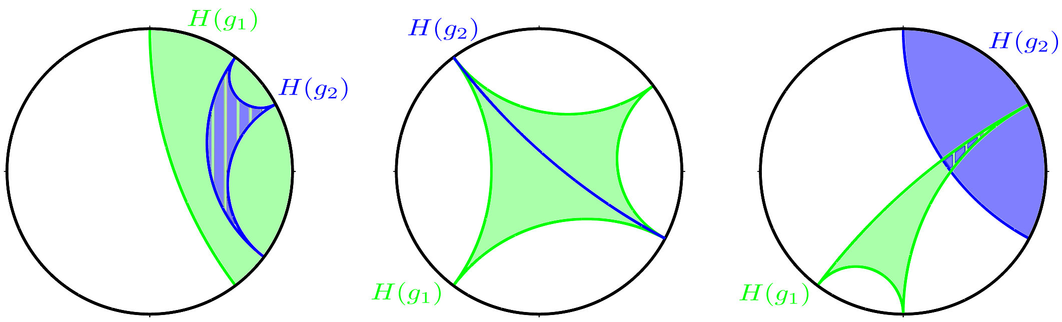
$$g_1 \circ g_2^{-1} = (\varphi_1 \circ f) \circ (\varphi_2 \circ f)^{-1} = \varphi_1 \circ f \circ f^{-1} \circ \varphi_2^{-1} = \varphi_1 \circ \varphi_2^{-1} \in \mathrm{PSL}(2, \mathbb{R}),$$

so $g_1 \circ g_2^{-1}$ can be extended to a unique isometry of $\mathbb{H}^2$. In particular, $g_1 \circ g_2^{-1}$ preserves angles and maps geodesics to geodesics. Suppose by contradiction that $H(g_2)$ does not lie in the closure of $\mathbb{D}^2 \setminus H(g_1)$. Then either

i) $H(g_2) \subseteq H(g_1)$ or

ii) some geodesics $\ell_1$ and $\ell_2$ bounding $H(g_1)$ and $H(g_2)$ have to meet in the interior of $\mathbb{H}^2$

(see Figure 6.2). In Case ii), for $i = 1, 2$, let $x_i$ and $y_i$ by the endpoints of $\ell_i$ in fix$(g_i)$. Since $H(g_1) \neq H(g_2)$ we can assume that $\ell_1 \neq \ell_2$, so at least three of the points $x_1, y_1, x_2, y_2$ are pairwise distinct. Assume that $\ell_1$ and $\ell_2$ meet at a point in $\mathbb{H}^2$. In this case, as $\ell_1 \neq \ell_2$, the geodesics have no common endpoint on $S^1_\infty$. We choose one of the hyperbolic angles between $\ell_1$ and $\ell_2$ and denote it with $\alpha$. In Figure 6.3 this is the angle corresponding




(a) (b) (c)

Figure 6.2: If $H(g_2)$ is not contained in the closure of $\mathbb{D}^2 \setminus H(g_1)$, it is either contained in $H(g_1)$ (6.2a and 6.2b) or some bounding geodesics of $H(g_1)$ and $H(g_2)$ intersect (6.2c).

to the segment of $S^1_\infty$ between $x_1$ and $y_2$. We know that $\alpha$ equals the angle between $g_1 \circ g_2^{-1}(\ell_1)$ and $g_1 \circ g_2^{-1}(\ell_2)$, as $g_1 \circ g_2^{-1}$ is an isometry. Let us have a look at what $g_1 \circ g_2^{-1}$ does to the $\ell_i$. Since $g_1 \circ g_2^{-1}$ maps geodesics to geodesics, it suffices to consider the images of the endpoints of the $\ell_i$. As $g_2^{-1}$ fixes $x_2$ and $y_2$ and $g_1$ is an extreme left homeomorphism, $g_1 \circ g_2^{-1}$ either fixes $x_2$ and $y_2$ or moves them counterclockwise. Note that those points are only fixed by $g_1 \circ g_2^{-1}$ if they are also fixed points of $g_1$. Similarly, $x_1$ and $y_1$ are either fixed by $g_1 \circ g_2^{-1}$ or moved clockwise (see Figure 6.3). Considering the angle between the resulting geodesics, we have

$$\measuredangle(g_1 \circ g_2^{-1}(\ell_1), g_1 \circ g_2^{-1}(\ell_2)) \geq \alpha.$$

Since we already know that we have equality of the angles, it follows that $g_1 \circ g_2^{-1}$ fixes the geodesics $\ell_1$ and $\ell_2$, so in particular, fixes their endpoints $x_1, y_1, x_2, y_2$. As any element in $\mathrm{PSL}(2, \mathbb{R})$ fixing at least three points is the identity, we hae $g_1 \circ g_2^{-1} = id$, hence $g_1 = g_2$.

In Case i), we have $\mathrm{fix}(g_2) \subseteq \mathrm{fix}(g_1)$. If $\#\mathrm{fix}(g_2) > 2$, then $g_1$ and $g_2$ have at least three common fixed points $x_i$, $i = 1, 2, 3$. It follows that $g_1 \circ g_2^{-1}(x_i) = x_i$ for $i = 1, 2, 3$. Since $g_1 \circ g_2^{-1}$ is an isometry, this implies $g_1 \circ g_2^{-1} = id$, so $g_1 = g_2$. If $\#\mathrm{fix}(g_2) = 2$, then $H(g_2)$ is a geodesic $\ell$ with endpoints $x_2, y_2$ the fixed points of $g_2$. As above, those are also fixed points of $g_1$, so $g_1 \circ g_2^{-1}$ has two fixed points and hence is hyperbolic wih axis $\ell$. Note that $H(g_2) \subseteq \overline{\mathbb{D}^2 \setminus H(g_1)}$ implies that $g_1$ has fixed points $x_1$ and $y_1$ on different sides of $\ell$ (Figure 6.2b). Now $g_1 \circ g_2^{-1}$ moves both points clockwise, as $g_1, g_2$ are extreme left homeomorphisms and $g_1$ fixes $x_1$ and $y_1$. But this contradicts the fact that $g_1 \circ g_2^{-1}$ is a hyperbolic transformation and $x_1$ and $y_1$ lie on different sides of the translation axis $\ell$.

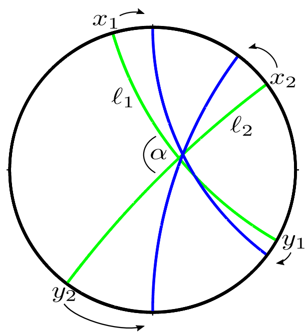



Figure 6.3: To show that the convex hulls for distinct $g_1, g_2 \in XL$ do not cross, one first considers geodesics $\ell_1 \subseteq H(g_1)$ and $\ell_2 \subseteq H(g_2)$. If they have an intersection in $\mathbb{H}^2$, then applying the isometry $g_1 \circ g_2^{-1}$ increases the angle $\alpha$ - a contradiction. Hence $g_1 = g_2$.

Thus, for $g_1 \neq g_2$ with $H(g_1) \neq H(g_2)$ the interiors of the convex hulls are disjoint. □

We want to use the geodesics bounding the convex hulls $H(g)$ for $g \in XL$ to build a lamination. For that, we need to show that the union of these geodesics is closed (Lemma 6.7). To define an the earthquake map, we also need a strong property for the convex hulls, namely, that they cover all of the hyperbolic plane (Proposition 6.8). Working towards these statements, we first show a continuity property for the convex hulls. For that, it is easiest to use the Klein model $K^2$. If $\text{fix}(g)$ is finite, the convex hull $H(g)$ then becomes a Euclidean convex polygon with vertices on the circle.

**Proposition 6.5.** *For any $h \in XL$ and any open neighbourhood $U$ of $H(h)$ in $\mathbb{D}^2$, there is a neighbourhood $V$ of $h$ in $XL$ such that*

$$g \in V \quad \Rightarrow \quad H(g) \subseteq U.$$

In other words, the map $H\colon XL \to \mathcal{P}(\mathbb{D}^2)$ mapping $h$ to $H(h)$ is continuous in the sense that if $h$ changes a little bit, all that $H(h)$ can do is change a little bit.

*Proof.* We start with showing that $U$ contains a convex open neighbourhood $U_0$ of $H(h)$. The set $H(h)$ is closed and since $U$ is open, also $\mathbb{D}^2 \setminus U$ is closed. As $H(h) \cap (\mathbb{D}^2 \setminus U) = \emptyset$, those sets have positive euclidean distance: $\text{dist}(H(h), \mathbb{D}^2 \setminus U) = \varepsilon > 0$. Set

$$U_0 := U_{\frac{\varepsilon}{2}}(H(h)) = \{x \in \mathbb{D}^2 \mid \text{dist}(x, H(h)) < \frac{\varepsilon}{2}\} \subseteq U\ .$$

We show that $U_0$ is convex: Let $x, y$ in $U_0$ and $x', y' \in H(h)$ such that $\|x - x'\| < \frac{\varepsilon}{2}$ and $\|y - y'\| < \frac{\varepsilon}{2}$. For $t \in [0, 1]$ let $z := tx + (1-t)y$. We show that $z$ lies in $U_0$: Set

$z' := tx' + (1-t)y'$. By convexity of $H(h)$, $z'$ lies in $H(h)$ and we have

$$\begin{aligned}\|z - z'\| &= \|\ (tx + (1-t)y) - (tx' + (1-t)y')\ \| \\ &= \|\ t(x - x') + (1-t)(y - y')\ \| \\ &\leq t \cdot \|x - x'\| + (1-t) \cdot \|y - y'\| \\ &< t \cdot \frac{\varepsilon}{2} + (1-t) \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\ .\end{aligned}$$

Hence $z \in U_0$, so indeed, we found a convex neighbourhood $U_0$ of $H(h)$ with $U_0 \subseteq U$ and can therefore assume that $U$ is convex.
We claim that there is a lower bound to the Euclidean distance that $h$ moves points in $S^1_\infty \setminus U$. We identify $S^1_\infty$ with $\mathbb{S}^1$. Assume there was no lower bound. Then there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^1 \setminus U$ with $\lim_{n\to\infty} \|t_n - h(t_n)\| = 0$. Since $\mathbb{S}^1$ is compact, after passing to a subsequence, $(t_n)_{n\in\mathbb{N}}$ converges to some $t \in \mathbb{S}^1$ and hence $(h(t_n))_{n\in\mathbb{N}}$ converges to $h(t)$ as $h$ is continuous. If follows that $h(t) = t$, so $t \in \operatorname{fix}(h) \subseteq H(h) \subseteq U$. But we also know that $\mathbb{S}^1 \setminus U$ is closed, so the limit of the sequence $(t_n)_{n\in\mathbb{N}}$ has to lie in $\mathbb{S}^1 \setminus U$. This gives us $t \in (\mathbb{S}^1 \setminus U) \cap \operatorname{fix}(h) \subseteq (\mathbb{S}^1 \setminus U) \cap U$ - a contradiction. So there has to be some lower bound $m > 0$ with $m < \|t - h(t)\|$ for all $t \in \mathbb{S}^1 \setminus U$. Let now $0 < \varepsilon < m$ and let $V$ be the $\varepsilon$-neighbourhood of $h$ in $XL$:

$$V := \{g \in XL \mid \sup_{t\in\mathbb{S}^1} \|g(t) - h(t)\| < \varepsilon\}\ .$$

We show that the fixed points of all elements in $V$ lie in $U \cap S^1_\infty$: Let $g \in V$ and $t_0$ be a fixed point of $g$. Assume that $t_0 \in S^1_\infty \setminus U$. Then $h$ moves $t_0$ at least by distance $m$, so $\varepsilon > \|g(t_0) - h(t_0)\| = \|t_0 - h(t_0)\| > m$, a contradiction. Hence for all $g \in V$, $\operatorname{fix}(g) \subseteq U \cap \mathbb{S}^1$ and since $U$ is convex and $H(g) = \operatorname{conv}(\operatorname{fix}(g))$, it follows that $H(g) \subseteq U$. $\square$

We showed in Proposition 6.3 that $XL$ is homeomorphic to $\mathbb{H}^2$ via the evaluation $p$ at $x_0$. $\mathbb{H}^2$ has as compactification the closed disk $\mathbb{D}^2$(see Section 2.1). By identifying $XL$ with $\mathbb{H}^2$, we obtain a compactification $\overline{XL}$ of $XL$ and we define $\partial XL := \overline{XL} \setminus XL$. The elements in $\partial XL$ cannot be interpreted as functions like the elements in $XL$ but correspond to points in $S^1_\infty$. We define

$$\overline{H} \colon \overline{XL} \to \mathcal{P}(\mathbb{D}^2), \quad \overline{H}(g) := \begin{cases} H(g) & \text{for } g \in XL, \\ \{g\} & \text{for } g \in \partial XL. \end{cases}$$

Thus, $\overline{H}$ extends $H$, mapping the boundary points to singleton sets. Also $\overline{H}$ satisfies a continuity property.

**Proposition 6.6.** *For all $h \in \overline{XL}$ and any open neighbourhood $U$ of $\overline{H}(h)$ in $\mathbb{D}^2$, there is some open neighbourhood $V$ of $h$ in $\overline{XL}$ such that*

$$g \in V \quad \Rightarrow \quad \overline{H}(g) \subseteq U. \tag{6.3}$$

*Proof.* If $h \in XL$ and $U$ is a neighbourhood of $\overline{H}(h) = H(h)$ in $\overline{XL}$, then by Proposition 6.5, there is a neighbourhood $V$ of $h$ in $XL$ such that $g \in V$ implies $\overline{H}(g) = H(g) \subseteq U$. As $V$ is also a neighbourhood of $h$ in $\overline{XL}$, this proves the statement.
So let $h \in \partial XL \cong \mathbb{S}^1$ and let $U$ be a neighbourhood of $\overline{H}(h) = \{h\}$ in $\mathbb{D}^2$. Remember that we identified $g \in XL$ with the point $g(x_0) \in \mathbb{H}^2$ for some fixed base point $x_0$. We can assume that $U$ is given as $U_\eta(h)$ for some $\eta > 0$, where

$$U_\eta(h) = \{g \in \mathbb{S}^1 \mid \|g - h\| < \eta\} \cup \{g \in XL \mid \|g(x_0) - h\| < \eta\}$$

is the $\eta$-neighbourhood of $h$ in $\overline{XL}$. Let $0 < \varepsilon < \eta$ and set $V := U_\varepsilon(h)$. Later we will make precise how $\varepsilon$ has to be chosen. For $g \in V \cap \mathbb{S}^1$ it holds that $\overline{H}(g) = \{g\} \subseteq V = U_\varepsilon(h) \subseteq U_\eta(h) = U$ and we have nothing to show. Now let $g \in V \cap XL$. As in the proof of Proposition 6.5 we can assume that $U$ is convex. It then suffices to show that $\mathrm{fix}(g) \subseteq U$. Assume by contradiction that $g$ has a fixed point $y$ that does not lie in $U$, so $\|y - h\| \geq \eta$. Let $y_1 \in U_{\eta-\varepsilon}(x)$ be a point close to $y$ in clockwise direction. Note that $y_1$ cannot lie in $V$ as

$$\|y_1 - h\| \geq \|y - h\| - \|y_1 - y\| > \eta - (\eta - \varepsilon) = \varepsilon.$$

Before going on, we prove the following:
*Fact.* Fix $y_0 \in \mathbb{H}^2$. Any $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ can be written as $\varphi = t \circ \rho$, where $\rho$ is an elliptic element of $\mathrm{PSL}(2,\mathbb{R})$ fixing $y_0$ and $t$ is a hyperbolic transformation sending $y_0$ to $\varphi(y_0)$.

*Proof of the Fact.* Let $t$ be the hyperbolic transformation with axis the geodesic through $y_0$ and $\varphi(y_0)$, sending $y_0$ to $\varphi(y_0)$. Then $t^{-1} \circ \varphi(y_0) = y_0$, so $\rho := t^{-1} \circ \varphi$ is an elliptic element fixing $y_0 \in \mathbb{H}^2$. Then $\varphi = t \circ t^{-1} \circ \varphi = t \circ \rho$. □

We write $g = \varphi \circ f$ for some $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ and $\varphi = t \circ \rho$, where $\rho$ is an elliptic transformation fixing $f(x_0)$ and $t$ is a hyperbolic transformation sending $f(x_0)$ to $\varphi(f(x_0)) = g(x_0)$, where $x_0 \in \mathbb{H}^2$ is the point we used to identify $XL$ with $\mathbb{H}^2$. Without loss of generality we can assume that $f(x_0)$ is the centre of the disk - else, we use a change of coordinates as in the proof of Proposition 6.3. We can do so since changing coordinates preserves continuity properties. Now $t$ maps $f(x_0) = 0$ to $g(x_0)$. The translation distance $\tau(t)$ of $t$ is attained in any point on the axis, so

$$\tau(t) = \inf_{z \in D^2} \mathrm{d}(z, t(z)) = \mathrm{d}(0, g(x_0)) = \log\left(\frac{1 + ||g(x_0)||}{1 - ||g(x_0)||}\right).$$

(see Proposition 2.3). As $g \in U_\varepsilon(h)$, we have $||g(x_0)|| > 1 - \varepsilon$, so $1 - ||g(x_0)|| < \varepsilon$. For $\varepsilon \ll 1$ this gives us by monotonicity of the logarithm

$$\tau(t) = \log\left(\frac{1 + ||g(x_0)||}{1 - ||g(x_0)||}\right) > \log\left(\frac{2 - \varepsilon}{\varepsilon}\right) > \log\left(\frac{1}{\varepsilon}\right). \tag{6.4}$$

This shows that by decreasing $\varepsilon$ we can obtain arbitrary high translation distances for $t$. As $g(x_0)$ lies inside $U_\varepsilon(h)$ and the translation axis $\ell$ of $t$ goes through 0 and $g(x_0)$, also the attracting endpoint of $t$ has to lie in $U_\varepsilon(h)$. It follows that neither $y$ nor $y_1$ agree with this fixed point, since they both lie in $\mathbb{S}^1 \setminus U_\varepsilon(h)$. As continuous function on the compact set $\mathbb{S}^1$, $f$ is uniformly continuous. So we have $\delta_\eta > 0$ depending on $\eta$ such that $\|z - w\| < \eta$ implies $\|f(z) - f(w)\| < \delta_\eta$ for all $z, w$ on $\mathbb{S}^1$. Set $\delta := \|f(y) - f(y_1)\|$; note that $\delta < \delta_\eta$ as $\|y - y_1\| < \eta$. As $\rho$ is a rotation, it preserves the Euclidean distance between points on the boundary, so $\|\rho \circ f(y) - \rho \circ f(y_1)\| = \delta$. Now $y$ is a fixed point of $g = t \circ \rho \circ f$, so $t^{-1}(y) = t^{-1}(g(y)) = \rho \circ f(y)$ and hence $\|t^{-1}(y) - \rho \circ f(y_1)\| = \delta$. Further, all maps considered are orientation-preserving, so $\rho \circ f(y_1)$ is clockwise from $\rho \circ f(y)$. Applying $t$ to those points will give us $g(y_1)$ and $y$, where again, $g(y_1)$ is clockwise from $y$. Also $y_1$ is clockwise from $y$. If we can show that $\|y - g(y_1)\| > \|y - y_1\|$, then $g$ moves $y_1$ clockwise, contradiction the fact that it is an extreme left homeomorphism. Note that for large translation distance, $t^{-1}(y)$ is close to the repelling fixed point. Intuitively, $t$ expands the interval between two points that are close to the repelling fixed point by a huge factor - the larger the translation distance, the larger that factor. If we make sure that the translation distance is large enough, then indeed we obtain $\|y - g(y_1)\| > \|y - y_1\|$. To show this more precisely, we consider the half-plane model, where we can assume that the translation axis is the imaginary axis and the repelling fixed point is 0. The fact that $y_1$ is clockwise from $y$ translates to $y_1 < y$. In the half-plane model, $t$ is of the form $z \mapsto e^\lambda z$ for some $\lambda > 0$ where the translation distance is $\lambda$. Now for all $z, w \in \mathbb{R}$ we have

$$\|t(z) - t(w)\| = \|e^\lambda z - e^\lambda w\| = e^\lambda \|z - w\|.$$

Note that $e^\lambda$ can be seen as expansion factor for any interval $[z, w]$. In particular, we have

$$\|y - g(y_1)\| = \|t(t^{-1}(y)) - t(\rho \circ f(y_1))\| = e^\lambda \|t^{-1}(y) - \rho \circ f(y_1)\| = e^\lambda \delta.$$

Remember that $\log(\lambda^2) = \tau(t)$ and from (6.4)we know that we can obtain arbitrarily high $\tau(t)$ by decreasing $\varepsilon$. In particular, for $\varepsilon$ small enough, we have $\|y - g(y)\| = e^\lambda \tilde{\delta} > \|y - y_1\|$. Note that strictly speaking, we have cross dependencies in this inequality: $\tilde{\delta}$ depends on $y_1$ and this depends on $\varepsilon$, as we required that $\|y - y_1\| < \eta - \varepsilon$. So decreasing $\varepsilon$ could possibly change $y_1$, increase the distance between $y$ and $y_1$ and hence $\delta$. But when $\varepsilon$ decreases, $\delta$ could only increase and it is bounded by $\delta_\eta$. Also, the right-hand side could increase as we decrease $\varepsilon$, but it is bounded by $\eta$, whereas the left-hand side can grow beyond all limits. Hence, it holds that if $\varepsilon$ is chosen small enough, we have $\|y - g(y_1)\| > \|y - y_1\|$, where both $y_1$ and $g(y_1)$ are clockwise from $y$ (or right in the half-plane model), contradicting the fact that $g \in XL$. In total, $g$ cannot have a fixed point outside $U$, so $\overline{H}(g) \subseteq U$ for all $g \in V = U_\varepsilon(h)$. □

We now define the lamination for the earthquake using the geodesics bounding the convex hulls for the elements in $XL$. For any $g \in XL$, $H(g)$ is bounded by geodesics $\ell_g^i$, where $i$ ranges in some index set $I_g$. Let now

$$\lambda := \{\ell_g^i \mid g \in XL,\ i \in I_g\}$$

be the set of all geodesics bounding the convex hulls $H(g)$ for $g \in XL$. As seen in Lemma 6.4, those geodesics do not intersect. In order to show that they form a lamination, we have to prove the following:

**Lemma 6.7.** *The union of all geodesics in $\lambda$ is closed.*

*Proof.* For the proof, we again consider the situation in the Klein model. Let

$$L := \bigcup_{\ell \in \lambda} \ell$$

be the union of all geodesics in $\lambda$. Consider a sequence $(p_n)_{n \in \mathbb{N}}$ in $L$ that converges to some $p \in \mathbb{H}^2$. We want to prove that $p \in L$. For any $n \in \mathbb{N}$, let $\ell_n \in \lambda$ be the geodesic with $p_n \in \ell_n$. It is unique as the convex hulls do not cross (Lemma 6.4). Let $x_n, y_n \in S^1_\infty \cong \mathbb{S}^1$ be the endpoints of $\ell_n$. After passing to a subsequence if necessary, we can assume that there are $x, y \in \mathbb{S}^1$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ as $\mathbb{S}^1$ is compact. We have $x \neq y$ since otherwise, the geodesics $\ell_n$ would accumulate to a point on the boundary and this would imply $p \notin \mathbb{H}^2$. Let $\ell$ be the geodesic connecting $x$ and $y$. By definition of $\lambda$, for any $n \in \mathbb{N}$ there exists $g_n \in XL$ such that $\ell_n$ is a boundary component of $H(g_n)$ and hence $x_n, y_n \in \operatorname{fix}(g_n)$. Since $\overline{XL}$ is compact, there has to be a subsequence of $(g_n)_{n\in\mathbb{N}}$ converging to some $g \in \overline{XL}$. Without loss of generality let this already be $(g_n)_{n\in\mathbb{N}}$. If $g \in \partial XL$, then $\overline{H}(g) = \{g\}$ and as $\overline{H}$ is continuous (Proposition 6.6) for $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $\operatorname{fix}(g_n) \subseteq \overline{H}(g_n) \subseteq U_\varepsilon(\overline{H}(g)) = U_\varepsilon(g)$ for all $n \geq N$. Letting $\varepsilon$ go to zero, it follows that $x = \lim_{n\to\infty} x_n = g = \lim_{n\to\infty} y_n = y$, contradicting the fact that $x \neq y$. Hence this case cannot occur and we have $g \in XL$. We show that $g$ has to have $x$ and $y$ as fixed points. Consider

$$\begin{aligned} \|x - g(x)\| &= \lim_{n\to\infty} \|x_n - g(x)\| \\ &= \lim_{n\to\infty} \|g_n(x_n) - g(x)\| \\ &\leq \lim_{n\to\infty} \|g_n(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| \ . \end{aligned} \tag{6.5}$$

Let $\varepsilon > 0$. As $(g_n)_{n\in\mathbb{N}}$ converges to $g$ in the topology of uniform convergence, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\sup_{y\in\mathbb{S}^1} \|g_n(y) - g(y)\| < \frac{\varepsilon}{2}.$$

Further, as $g$ is continuous, there is $\delta > 0$ such that $\|z - w\| < \delta$ implies $\|g(z) - g(x)\| < \frac{\varepsilon}{2}$ for all $z, w \in \mathbb{S}^1$. As $(x_n)_{n\in\mathbb{N}}$ converges to $x$, there is some $N_2 \in \mathbb{N}$ such that $\|x_n - x\| < \delta$ for all $n \geq N_2$. Setting $N := \max\{N_1, N_2\}$, we obtain for all $n \geq N$

$$\|x_n - g(x)\| \leq \|g_n(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

Letting $\varepsilon$ go to zero, we obtain with (6.5) $\|x - g(x)\| = 0$, and hence $g(x) = x$. The same argument shows that also $y$ is a fixed point of $g$, so we have $\ell \subseteq H(g)$. Because the geodesics $\ell_n$ are getting arbitrary close to $\ell$ it is not possible that $\ell$ lies in the interior of $H(g)$, so it has to be a boundary component, so $\ell \subseteq L$. We claim that $p \in \ell$. As $\ell$ is the

Euclidean line segment between $x$ and $y$ we have the equivalence

$$p \in \ell \quad \Leftrightarrow \quad \|x - p\| + \|p - y\| = \|x - y\| \ . \tag{6.6}$$

From the triangle inequality we already know that

$$\|x - p\| + \|p - y\| \geq \|x - y\| \ . \tag{6.7}$$

As distance is continuous, we further have

$$\begin{aligned} \|x - p\| + \|p - y\| &= \lim_{n\to\infty} (\|x - p_n\| + \|p_n - y\|) \\ &\leq \lim_{n\to\infty} (\|x - x_n\| + \|x_n - p_n\| + \|p_n - y_n\| + \|y_n - y\|) \\ &= \lim_{n\to\infty} (\|x - x_n\| + \|x_n - y_n\| + \|y_n - y\|) \\ &= \|x - y\| \ , \end{aligned} \tag{6.8}$$

where we used $p_n \in \ell_n$ for all $n \in \mathbb{N}$ and (6.6). Combining (6.6), (6.7) and (6.8) we obtain $p \in \ell \subseteq L$. So indeed, $L$ has to be closed. □

## 6.3. Covering property of convex hulls

We are now almost ready to define the $\lambda$-left earthquake $E$. For a point $x \in \mathbb{H}^2$, we will define $E(x)$ using the extreme left homeomorphism $g \in XL$ with $x \in H(g)$. In order to do so, we first have to make sure that such a $g$ always exist, i.e. that the convex hulls $H(g)$ for $g \in XL$ cover the plane $\mathbb{H}^2$.

**Proposition 6.8.** *The convex hulls $H(g)$ for $g \in XL$ cover the hyperbolic plane, i.e.*

$$\bigcup_{g\in XL} H(g) \supseteq \mathbb{H}^2.$$

Unfortunately, the proof of the proposition is not constructive. The idea is to replace $H$ by a continuous function $\overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^2$ using an averaging technique.

*Proof.* As before we consider the situation in the Klein model $K^2$. We break down the proof in several steps.
*Step 1: Construction of the convolution $\beta * h$.* For $g \in \overline{XL}$ let $h(g) \in \mathbb{D}^2$ be the centroid of $\overline{H}(g)$. This is the point in the mean position of all points in $H(g)$ with respect to all coordinate directions. It can be calculated as follows: If $\overline{H}(g) = \{x\}$ is just one point,

then $h(g) := x$. If $g \in XL$ and $\#\mathrm{fix}(g) = 2$, then $\overline{H}(g)$ is a Euclidean straight line and we set $h(g)$ to be its Euclidean midpoint. If $\#\mathrm{fix}(g) > 2$, let $\mu$ be the Lebesgue-measure on $H(g)$ and let $x_1, x_2$ be the cartesian coordinates on $\mathbb{R}^2$. We let $h(g)$ be given by its coordinates

$$h(g)_i = \frac{1}{\mathrm{vol}(H(g))} \int x_i \mathrm{d}\mu \qquad \text{for } i = 1, 2,$$

where

$$\mathrm{vol}(H(g)) := \int_{H(g)} 1 d\mu$$

is the area of $H(g)$. We will not need to determine the point $h(g)$ explicitly. What matters to us is that it can be interpreted as midpoint of $\overline{H}(g)$ and that it always lies in $\overline{H}(g)$ as $\overline{H}(g)$ is convex.

The next ingredient in the proof is a bump function $\beta$. Let $\varepsilon > 0$ be fixed and let $\beta$ be a smooth non-negative function on $\mathbb{R}^2$ with $\mathrm{supp}(\beta) \subseteq U_\varepsilon(0)$, so $\beta \equiv 0$ on $\mathbb{R}^2 \setminus U_\varepsilon(0)$. By normalizing we can require that $\int_{\mathbb{R}^2} \beta(x, y) d(x, y) = 1$. For the existence of such a function we refer to [Hör83]. Also the details of the following definition as well as the properties used can be found there. We consider the convolution

$$\beta * h(x) := \int_{\mathbb{D}^2} \beta(x - y) h(y) \mathrm{d}y \qquad \forall x \in \mathbb{D}^2. \tag{6.9}$$

As the convolution is commutative and $\mathrm{supp}(\beta) \subseteq U_\varepsilon(0)$, we have for $x \in \mathbb{D}^2$

$$\beta * h(x) = h * \beta(x) = \int_{U_\varepsilon(0)} h(x - y)\beta(y)\mathrm{d}y = \int_{U_\varepsilon(x)} h(y)\beta(x - y)\mathrm{d}y\ .$$

The convolution $\beta * h$ can be seen as mean value of $h$ in a neighbourhood of $x$ with weighting governed by $\beta$. Note that $\beta * h$ is a function from $\mathbb{D}^2$ to $\mathbb{D}^2$. As $h$ is bounded with $\mathrm{supp}(h) \subseteq \mathbb{D}^2$, it is integrable. Since $\beta$ is smooth, it follows that $\beta * h$ is continuous.

*Step 2: $\beta * h$ is close to the identity near the boundary.* We first have to show the following fact:

*Fact.* For all $g_0 \in XL$, $\beta * h(g_0)$ is contained in the convex hull of the union of the $\overline{H}(g)$ where $g$ ranges over the $\varepsilon-$neighbourhood of $g_0$.

*Proof of the Fact.* Let $x_0 \in \mathbb{D}^2$ be the element corresponding to $g_0$ in the identification from Proposition 6.3. Then by definition

$$\beta * h(g_0) = \beta * h(x_0) = \int_{U_\varepsilon(x_0)} \beta(x_0 - y)h(y)\mathrm{d}y.$$

The centroid $h(y)$ lies in $\overline{H}(y)$. Weighting with $\beta$ and integrating over all values in $U_\varepsilon(x_0)$ will give some convex combination of the $h(y)$ that might not necessarily lie in $\bigcup_{y \in U_\varepsilon(x_0)} \overline{H}(y)$, but will lie in the convex hull. $\square$

By continuity of $\overline{H}$ (Proposition 6.6) it follows that $\beta * h(g_0)$ is contained in some small neighbourhood $U_\delta(H(g_0))$ of $H(g_0)$ where $\delta$ depends on $\varepsilon$. To show that $\beta * h$ is close to the identity near the boundary, let $x \in \mathbb{S}^1$ be fixed and set $V := U_{2\varepsilon}(x)$. As $\overline{H}$ satisfies a continuity property (Proposition 6.6), there is some $\eta > 0$ depending on $\varepsilon$ with $\overline{H}(y) \subseteq U_\eta(x)$ for all $y \in V$. Let $y \in U_\varepsilon(x)$. Then $U_\varepsilon(y) \subseteq U_{2\varepsilon}(x) = V$, so for all $\tilde{y} \in U_\varepsilon(y)$, we have $\overline{H}(\tilde{y}) \subseteq U_\eta(x)$. Since $U_\eta(x)$ is convex and contains all $\overline{H}(\tilde{y})$ for $\tilde{y}$ in the $\varepsilon-$neighbourhood of $y$ , it follows with the fact we have shown above that $\beta * h(y) \in U_\eta(x)$, so for all $y \in U_\varepsilon(x)$

$$\|\beta * h(y) - id(y)\| = \|\beta * h(y) - y\| \leq \|\beta * h(y) - x\| + \|x - y\| < \eta + \varepsilon \ .$$

As $\mathbb{S}^1$ is compact it can be covered by finitely many balls $U_{\frac{\varepsilon}{2}}(x_i)$ for $i = 1, \dots, n$ with $x_i \in \mathbb{S}^1$. Let $\eta_1, \dots, \eta_n$ be as above such that $\|\beta * h(y) - y\| < \eta_i + \varepsilon$ for all $y \in U_\varepsilon(x_i)$ for $i = 1, \dots, n$. Set $\eta := \max_{i=1,\dots,n} \eta_i + \varepsilon$. Let

$$N_{\frac{\varepsilon}{2}}(\mathbb{S}^1) := \{y \in \mathbb{D}^2 \mid \|y\| \geq 1 - \frac{\varepsilon}{2}\}$$

be the closed $\frac{\varepsilon}{2}-$neighbourhood of $\mathbb{S}^1$ in $\mathbb{D}^2$. For all $y \in N_{\frac{\varepsilon}{2}}(\mathbb{S}^1)$, there is some $z \in \mathbb{S}^1$ with $y \in \overline{U_{\frac{\varepsilon}{2}}(z)}$ and there is some $i \in \{1, \dots, n\}$ with $z \in \overline{U_{\frac{\varepsilon}{2}}(x_i)}$. In total, this gives us $y \in \overline{U_\varepsilon(x_i)}$, so

$$\|\beta * h(y) - y\| \leq \eta \qquad \forall y \in N_{\frac{\varepsilon}{2}}(\mathbb{S}^1).$$

In particular, $\sup_{y \in N_{\frac{\varepsilon}{2}}(\mathbb{S}^1)} \|\beta * h(y) - y\| \leq \eta$, so on $N_{\frac{\varepsilon}{2}}(\mathbb{S}^1)$, $\beta * h$ is $\eta-$close to the identity. Note that $\eta$ depends on $\varepsilon$ and gets smaller as $\varepsilon$ does.
*Step 3: $\beta * h$ is surjective onto all except possibly a small neighbourhood of* $\mathbb{S}^1$. The proof uses homology theory and is similar to the proof that a map $g\colon \mathbb{S}^1 \to \mathbb{S}^2$ with non-zero degree is surjective (Lemma A.9). The definitions and results we need from homology theory are collected in the Appendix A. We know by Step 2 that $\beta * h(\mathbb{S}^1) \subseteq N_\eta(\mathbb{S}^1)$. Let

$\iota$ denote the inclusion of $\mathbb{S}^1$ in $N_\eta(\mathbb{S}^1)$. The two maps

$$(\beta * h)|_{\mathbb{S}^1} \colon \mathbb{S}^1 \to N_\eta(\mathbb{S}^1) \quad \text{and} \quad \iota \colon \mathbb{S}^1 \to N_\eta(\mathbb{S}^1)$$

are homotopic by linear interpolation. Explicitly, the homotopy is given by

$$H \colon \mathbb{S}^1 \times [0,1] \to N_\eta(\mathbb{S}^1), \quad H(x,t) := t \cdot \iota(x) + (1-t) \cdot (\beta * h)(x).$$

We define the degree of $\beta * h$ and $\iota$ just as for a map $\mathbb{S}^1 \to \mathbb{S}^1$ (Definition A.8). This is possible as $\mathbb{S}^1$ and $N_\eta(\mathbb{S}^1)$ are homotopy equivalent: The map

$$g \colon N_\eta(\mathbb{S}^1) \to \mathbb{S}^1, \; g(x) := \frac{x}{\|x\|}$$

is a homotopy equivalence since $\iota \circ g \simeq id$ by linear interpolation as above and $g \circ \iota = id$. Hence the homology groups of $N_\eta(\mathbb{S}^1)$ and $\mathbb{S}^1$ are isomorphic (Proposition A.3).
Now the induced maps $(\beta * h)_*, \; \iota_* \colon H_1(\mathbb{S}^1) \to H_1(N_\eta(\mathbb{S}^1))$ can be interpreted as maps from $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ to itself, so it does make sense to talk about $\deg(\beta * h)$ and $\deg(\iota)$. As both maps are homotopic we have $\deg(\beta * h) = \deg(\iota)$ and clearly, $\deg(\iota) = 1$.
Assume now that $\beta * h$ misses a point $x \in \mathbb{D}^2 \setminus N_\eta(\mathbb{S}^1)$. Then $N_\eta(\mathbb{S}^1)$ is a subset of $\mathbb{D}^2 \setminus \{x\}$. As $\beta * h$ is continuous it induces maps between the (relative) homology groups of the pairs $(\mathbb{D}^2, \mathbb{S}^1)$ and $(\mathbb{D}^2 \setminus \{x\}, N_\eta(\mathbb{S}^1))$. By Proposition A.5 we have the following diagram coming from long exact sequences of the pairs:

$$\begin{tikzcd}
\dots \arrow[r] & H_2(\mathbb{D}^2, \mathbb{S}^1) \arrow[r, "\partial"] \arrow[d, "(\beta * h)_*"] & H_1(\mathbb{S}^1) \arrow[r] \arrow[d, "(\beta * h)_*"] & \dots \\
\dots \arrow[r] & H_2(\mathbb{D}^2 \setminus \{x\}, N_\eta(\mathbb{S}^1)) \arrow[r, "d"] & H_1(N_\eta(\mathbb{S}^1)) \arrow[r] & \dots
\end{tikzcd}$$

We make the following observations:

- As $\mathbb{D}^2 \setminus \{x\}$, $N_\eta(\mathbb{S}^1)$ and $\mathbb{S}^1$ are homotopy equivalent, we have $H_1(N_\eta(\mathbb{S}^1)) = H_1(\mathbb{S}^1) = \mathbb{Z}$.

- Since for any space $X$, the relative homology group $H_n(X, X)$ is trivial by construction, it follows that $H_2(\mathbb{D}^2 \setminus \{x\}, N_\eta(\mathbb{S}^1)) \cong H_2(\mathbb{S}^1, \mathbb{S}^1) = 0$.

It follows that $d$ is trivial. Since $(\beta * h)_*$ is a chain map, we have $(\beta * h)_* \circ \partial = d \circ (\beta * h)_* = 0$. But $\partial$ is an isomorphism and $H_1(\mathbb{S}^1) \cong \mathbb{Z}$ is non-trivial, so $(\beta * h)_* \colon H_1(\mathbb{S}^1) \to H_1(N_\eta(\mathbb{S}^1))$ has to be trivial and hence $\deg(\beta * h) = 0$ - a contradiction. In total, $\beta * h$ has to be

surjective, except possibly onto the small neighbourhood $N_\eta(\mathbb{S}^1)$ of $\mathbb{S}^1$.
*Step 4: $\mathbb{H}^2$ is covered by the convex hulls.* The idea of the proof is to consider a sequence of bump functions whose support goes to zero. The annulus $N_\eta(\mathbb{S}^1)$ where the convolution with $h$ is not surjective then gets smaller. Let $x \in \mathbb{H}^2$ be arbitrary. We want to show that $x \in H(f_0)$ for some $f_0 \in XL$. If $\eta$ is small enough, then $x \in \mathbb{D}^2 \setminus N_\eta(\mathbb{S}^1)$. For all $n$, let $\beta_n$ be a bump function with $\mathrm{supp}(\beta_n) \subseteq U_{\frac{1}{n}}(0)$. Then $\beta_n * h$ is surjective onto $\mathbb{D}^2 \setminus N_{\eta_n}(\mathbb{S}^1)$ where $\eta_n$ depends on $\frac{1}{n}$ as in Step 2. As $\eta_n$ converges to 0 there is some $N \in \mathbb{N}$ such hat $x \in \mathbb{D}^2 \setminus N_{\eta_n}(\mathbb{S}^1)$ for all $n \geq N$. Up to taking a subsequence, we can therefore assume that $x \in \mathbb{D}^2 \setminus N_{\eta_n}(\mathbb{S}^1)$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ there exists $f_n \in \overline{XL}$ with $\beta_n * h(f_n) = x$ by Step 3. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{XL}$. Up to passing to a subsequence, we can assume that $(f_n)_{n\in\mathbb{N}}$ converges to some $f_0 \in \overline{XL}$ as $\overline{XL}$ is compact. We claim that $x \in \overline{H}(f_0)$ and that $f_0 \in XL$. Let $\delta_1 > 0$. As $\overline{H}$ is continuous by Proposition 6.6, there is some $\delta_2 > 0$ such that $\overline{H}(g) \subseteq U_{\delta_1}(\overline{H}(f_0))$ for all $g \in U_{\delta_2}(f_0)$. As $(f_n)_{n\in\mathbb{N}}$ converges to $f_0$, there exists $N_1$ depending on $\delta_2$ with $f_n \in U_{\delta_2}(f_0)$ for all $n \geq N_1$. It follows that

$$\overline{H}(f_n) \subseteq U_{\delta_1}(\overline{H}(f_0)) \qquad \forall n \geq N_1. \tag{6.10}$$

By continuity of $\overline{H}$, there is some $\delta(n) > 0$ for all $n$ such that $g \in U_{\frac{1}{n}}(f_n)$ implies $\overline{H}(g) \subseteq U_{\delta(n)}(\overline{H}(f_n))$. By the fact we have proven in Step 2, it follows that

$$\beta * h(f_n) \in \operatorname{conv}\left(\bigcup_{g\in U_{\frac{1}{n}}(f_n)} \overline{H}(g)\right) \subseteq \operatorname{conv}\left(\bigcup_{g\in U_{\frac{1}{n}}(f_n)} U_{\delta(n)}(\overline{H}(f_n))\right) = U_{\delta(n)}(\overline{H}(f_n))$$

as $U_{\delta(n)}(\overline{H}(f_n))$ is convex. As $\delta(n)$ decreases with increasing $n$ there is some $N_2 \in \mathbb{N}$ such that $\delta(n) \leq \delta_1$ for all $n \geq N_2$ and hence

$$x \in U_{\delta_1}(\overline{H}(f_n)) \qquad \forall n \geq N_2. \tag{6.11}$$

Set $N := \max\{N_1, N_2\}$. Then (6.10) and (6.11) give us

$$x \in U_{\delta_1}(\overline{H}(f_n)) \subseteq U_{\delta_1}(U_{\delta_1}(\overline{H}(f_0)) \subseteq U_{2\delta_1}(\overline{H}(f_0)).$$

If we let $\delta_1$ go to zero it follows that $x \in \overline{H}(f_0)$. This also gives us that $f_0$ lies in $XL$ since for $f_0 \in \partial XL$ we would have $x \in \overline{H}(f_0) = \{f_0\}$, so $x = f_0$, contradicting the fact that $x \in \mathbb{H}^2$. In total, we have shown that the convex hulls $H(g)$ for $g \in XL$ cover $\mathbb{H}^2$. $\square$

## 6.4. Construction of an earthquake map

We are now ready to finish the proof of Theorem 6.1, i.e. to construct an earthquake map $E$ with $E_\infty = f$. We have seen that the union of all geodesic boundary components of $H(g)$ for $g \in XL$ forms a lamination $\lambda$ (Lemma 6.4 and Lemma 6.7). We now construct a $\lambda$-left earthquake.

Let $A$ be a stratum of $\lambda$. We choose $f_A \in XL$ with $A \subseteq H(f_A)$:

If $A$ is a gap, there is only one possible choice for $f_A$. Assume $A \subseteq H(f_A)$ and $A \subseteq H(g_A)$ for $f_A, g_A \in XL$. Then, since $A$ is open, $A \subseteq \mathrm{int}(H(f_A)) \cap \mathrm{int}(H(g_A))$, where for a set $B$ $\mathrm{int}(B)$ denotes the interior of $B$. This can only be true for $f_A = g_A$, as by Lemma 6.4 the interiors of $H(f_A)$ and $H(g_A)$ do not intersect if $f_A$ and $g_A$ are distinct.

If $A$ is a leaf with endpoints $x$ and $y$, there may be various choices for $f_A$, but they all share the two fixed points $x$ and $y$. For now, we fix one choice. Later, we will shortly examine what choices are possible for $f_A$. Since $f_A \in XL \subseteq \mathcal{C} = \mathrm{PSL}(2, \mathbb{R}) \circ f$, there is some $\varphi_A \in \mathrm{PSL}(2, \mathbb{R})$ with $f_A = \varphi_A^{-1} \circ f$, or equivalently, $f = \varphi_A \circ f_A$. We set $E|_A := (\varphi_A)|_A$ - so on $A$, $E$ agrees with $\varphi_A$. We claim that $E$ is an earthquake extending the homeomorphism $f$. We have to show the following:

i) For any stratum $A$, $E|_A$ agrees with the restriction of an isometry $\mathbb{H}^2$.

ii) For two strata $A \neq B$, the comparison isometry $\mathrm{cmp}(A, B)$ is a hyperbolic transformation whose axis weakly separates $A$ and $B$ and translates to the left as viewed from $A$.

iii) $E$ is injective.

iv) $E$ is surjective.

v) $E_\infty$ coincides with $f$.

As a last step, we will show uniqueness up to translation on leaves as formulated in Theorem 6.1.

Property i) holds by construction of $E$. To show ii), observe that for two strata $A \neq B$

$$\mathrm{cmp}(A, B) = \varphi_A^{-1} \circ \varphi_B = (\varphi_A^{-1} \circ f) \circ (f^{-1} \circ \varphi_B) = f_A \circ f_B^{-1}, \tag{6.12}$$

where $\varphi_A$ and $\varphi_B$ are as above. The proof is now similar to the proof of Lemma 5.6. Let $\ell_A$ be a geodesic in $\partial A$ weakly separating $A$ and $B$ and with endpoints $x_A$ and $y_A$ on $S^1_\infty$. Analogously, let $\ell_B$ a geodesic in $\partial B$ weakly separating $A$ and $B$ and with endpoints $x_B$

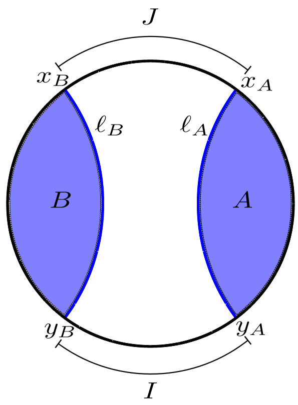



Figure 6.4: To show that for two strata $A$ and $B$ the comparison isometry is hyperbolic with axis weakly separating $A$ and $B$ and translates to the left as viewed from $A$, one has a look at what happens when applying $\mathrm{cmp}(A,B)$ to the intervals $I$ and $J$.

and $y_B$ on $S^1_\infty$. In the case that $A$ (respectively, $B$) is a leaf, we set $\ell_A := A$ (respectively, $\ell_B := B$). We can assume that the tuple $(x_A, x_B, y_B, y_A)$ is ordered. Let $J := [[x_A, x_B]]$ and $I := [[y_B, y_A]]$ (see Figure 6.4). We also allow that one of the intervals is degenerate, i.e. a point. In this case, $\ell_A$ and $\ell_B$ share an endpoint. The case that $A$ and $B$ are adjacent, i.e. both of the intervalls $I$ and $J$ are degenerate, requires some extra thought and will be considered later. As $f_A$ is an extreme left homeomorphism fixing $y_A$, we have $f_A(I) \subseteq I$ . Moreover, $f_B^{-1}$ moves all points at most clockwise and fixes $y_B$, so $f_B^{-1}(I) \subseteq I$. Using (6.12), we obtain that $\mathrm{cmp}(A,B)$ maps $I$ into itself. By Brouwer's fixed point theorem (Theorem 5.5), $\mathrm{cmp}(A,B)$ has a fixed point $y$ in $I$. Analogously, $(\mathrm{cmp}(A,B))^{-1} = f_B \circ f_A^{-1}$ has a fixed point $x$ in $J$, so also $\mathrm{cmp}(A,B)$ has $x$ as a fixed point. Hence $\mathrm{cmp}(A,B)$ is an isometry with at least two fixed points on $S^1_\infty$. As the fixed point sets of $f_A$ and $f_B$ do not coincide, $\mathrm{cmp}(A,B)$ is not the identity. If follows that it is a hyperbolic transformation with axis connecting $x$ and $y$, separating $A$ from $B$. Since $\mathrm{cmp}(A,B)$ maps $I$ into itself, the translation has to be from $x$ to $y$, i.e. to the left as viewed from $A$. If now both intervals are degenerate, $I = \{x\}$ and $J = \{y\}$, then $A$ and $B$ are adjacent and we obtain again, that $\mathrm{cmp}(A,B)$ is a hyperbolic transformation with axis weakly separating $A$ and $B$. In this case, the direction of translation cannot be deduced from the fact that $I = \{x\}$ is mapped into itself. But as $A \neq B$, one of the homeomorphisms $f_A$ and $f_B$ has a fixed point $w$ that is not a fixed point of the other. Without loss of generality let $w \in \mathrm{fix}(f_B) \setminus \mathrm{fix}(f_A)$ and let $A$ and $B$ be such that translation to the left as viewed from $A$ is translating from $x$ to $y$. Now the image point

$$\mathrm{cmp}(A,B)(w) = f_A \circ f_B^{-1}(w) = f_A(w)$$

is counterclockwise from $w$. So indeed, $\mathrm{cmp}(A,B)$ translates to the left as viewed from $A$.

Now that we have proven i) and ii), the injectivity iii) immediately follows with Remark 5.8.

To show surjectivity, we use Lemma 5.9. Let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence of leaves and $(H_n)_{n\in\mathbb{N}}$ be a sequence of half-planes bounded by $\ell_n$ such that $H_{n+1} \subseteq H_n$ and $\bigcap_{n\in\mathbb{N}} H_n$ contains a unique point $x$ in $S^1_\infty$. Let $x_n$ and $y_n$ be the endpoints of $\ell_n$. We know that $\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} y_n$. Now for any $n$, $\ell_n$ corresponds to an extreme left homeomorphism $\varphi_n \circ f \in XL$ and by definition of $E$ we have $(E|\ell_n) = \varphi_n^{-1}$. To show surjectivity, is suffices to show that the geodesics $E(\ell_n) = \varphi_n^{-1}(\ell_n)$ accumulate to a single point at the boundary, so that $\lim_{n\to\infty} \varphi_n^{-1}(x_n) = \lim_{n\to\infty} \varphi_n^{-1}(y_n)$. By construction, $\varphi_n \circ f$ fixes $x_n$ and $y_n$, so we have for all $n$

$$\varphi_n^{-1}(x_n) = \varphi_n^{-1} \circ \varphi_n \circ f(x_n) = f(x_n)$$

and analogous $\varphi_n^{-1}(y_n) = f(y_n)$. By continuity of $f$ it follows that $\lim_{n\to\infty} \varphi_n^{-1}(x_n) = \lim_{n\to\infty} \varphi_n^{-1}(y_n)$, so $E$ is surjective.

It remains to check property v): $E_\infty$ coincides with $f$. Let $x \in S^1_\infty$ and suppose that $x$ lies in the closure of some stratum $A$. Then $x$ is a fixed point of $f_A$ with $f_A = \varphi_A^{-1} \circ f$ and hence

$$E_\infty(x) = (E|A)(x) = \varphi_A(x) = \varphi_A(f_A(x)) = f(x).$$

If $x$ is not contained in the closure of any stratum, then $x$ has some neighbourhood basis bounded by leaves as seen in the proof of Propostion 5.14. In particular there is some sequence $(x_n)_{n\in\mathbb{N}}$ in $S^1_\infty$ converging to $x$ such that for all $n$, $x_n$ lies in the closure of a leaf $\ell_n$. Since both $E_\infty$ and $f$ are continuous it follows that

$$E_\infty(x) = \lim_{n\to\infty} E(x_n) = \lim_{n\to\infty} f(x_n) = f(x).$$

So indeed the extension $E_\infty$ of the earthquake $E$ coincides with $f$ on $S^1_\infty$. This completes the proof of existence of an earthquake $E$ extending $f$.

Before proving uniqueness, we have a closer look at what different possibilities we have for $E$ on a leaf $\ell$ of the lamination $\lambda$ constructed above. Let $\ell$ be a leaf with endpoints $x$

and $y$ and let $f_1, f_2$ be two different extreme left homeomorphisms both having $x$ and $y$ as fixed points, i.e. $\ell \subseteq H(f_1) \cap H(f_2)$. Then there are $\varphi_1, \varphi_2 \in \mathrm{PSL}(2,\mathbb{R})$ such that $f_1 = \varphi_1^{-1} \circ f$ and $f_2 = \varphi_2^{-1} \circ f$. Two choices for $E$ on $\ell$ then are $\varphi_1$ and $\varphi_2$. We claim that they differ by a hyperbolic isometry with translation distance between 0 and the infimum of the translation distance for strata on different sides of $\ell$. We have

$$\varphi_1^{-1} \circ \varphi_2 = f_1 \circ f_2^{-1} \tag{6.13}$$

as seen in (6.12). Since both $x$ and $y$ are fixed points of $f_1 \circ f_2^{-1}$, this is hyperbolic with axis $\ell$. Without loss of generality let $x$ be the repelling fixed point. Let $A, B$ be two strata on opposite sides of $\ell$ with notation chosen such that $\mathrm{cmp}(A,B)$ translates in the same direction as $\varphi_1^{-1} \circ \varphi_2$ and let $\varphi_A, \varphi_B \in \mathrm{PSL}(2,\mathbb{R})$ such that $(E|A) = \varphi_A$ and $(E|B) = \varphi_B$ as above. We consider the following hyperbolic transformations:

$$\varphi_A^{-1} \circ \varphi_1 = f_A \circ f_1^{-1} \tag{6.14}$$
$$\varphi_1^{-1} \circ \varphi_B = f_1 \circ f_B^{-1} \tag{6.15}$$
$$\varphi_2^{-1} \circ \varphi_B = f_2 \circ f_B^{-1} \tag{6.16}$$

As seen in (6.12), those are all hyperbolic transformations translating in the same direction as $\varphi_1^{-1} \circ \varphi_2$. Assume first that all axes, except possibly those of (6.15) and (6.16), are pairwise non-intersecting and distinct. We can apply Lemma 5.6 to obtain

$$\begin{aligned}
\tau(\mathrm{cmp}(A,B)) &= \tau(\varphi_A^{-1} \circ \varphi_B) \\
&= \tau\left((\varphi_A^{-1} \circ \varphi_1) \circ (\varphi_1^{-1} \circ \varphi_B)\right) \\
&\geq \tau(\varphi_A^{-1} \circ \varphi_1) + \tau(\varphi_1^{-1} \circ \varphi_B) \\
&= \tau(\varphi_A^{-1} \circ \varphi_1) + \tau\left((\varphi_1^{-1} \circ \varphi_2) \circ (\varphi_2^{-1} \circ \varphi_B)\right) \\
&\geq \tau(\varphi_A^{-1} \circ \varphi_1) + \tau(\varphi_1^{-1} \circ \varphi_2) + \tau(\varphi_2^{-1} \circ \varphi_B) \\
&\geq \tau(\varphi_1^{-1} \circ \varphi_2)\ .
\end{aligned} \tag{6.17}$$

If some of the axes of (6.14), (6.15) and (6.16) coincide, then the proof of Lemma 5.6 shows tat for hyperbolic transformations $S, T$ with the same axis translating in the same direction, we have

$$\tau(S \circ T) = \tau(S) + \tau(T).$$

Hence, the inequality remains valid if some of the axes coincide. In total, we have shown that the translation distance of $\varphi_1^{-1} \circ \varphi_2$ is bounded from above by $\tau(\mathrm{cmp}(A, B))$, where $A$ and $B$ are strata on opposite sides of $\ell$. In particular,

$$\tau(\varphi_1^{-1} \circ \varphi_2) \leq \inf\{\tau(\mathrm{cmp}(A, B) \mid A, B \text{ strata on opposite sides of } \ell\}.$$

*Remark* 6.9. In the special case that $\ell$ is adjacent to two gaps $A$ and $B$, then we do not have to consider an infimum and have

$$\tau(\varphi_1^{-1} \circ \varphi_2) \leq \tau(\mathrm{cmp}(A, B)) \; .$$

This can be seen from (6.17): If $\tilde{A}$ is a gap on the same side of $\ell$ as $A$ and $\tilde{B}$ is a gap on the same side of $\ell$ as $B$, both not adjacent to $\ell$, then we have

$$\begin{aligned}\tau(\mathrm{cmp}(\tilde{A}, \tilde{B})) &\geq \tau(\varphi_{\tilde{A}}^{-1} \circ \varphi_A) + \tau(\mathrm{cmp}(A, B)) + \tau(\varphi_B^{-1} \circ \varphi_{\tilde{B}}) \\ &\geq \tau(\mathrm{cmp}(A, B))\end{aligned}$$

and the claim follows.

To finish the proof of the earthquake theorem, the only thing that is left to show is uniqueness. Suppose $E'$ is any left earthquake satisfying $E'_\infty = f = E_\infty$ and let $\lambda'$ be the underlying lamination of $E'$. Let $A$ be a stratum of $\lambda'$. We want to show that $A$ is also a stratum of $\lambda$, the underlying lamination of $E$. The idea is to find an extreme left homeomorphism $f_A$ with $A \subseteq H(f_A)$. As $E'$ is an earthquake we have $(E'|A) = \varphi_A$ for some isometry $\varphi_A$. Set $h := \varphi_A^{-1} \circ E'$ on $\mathbb{H}^2$. Then $h$ extends to $S^1_\infty$ as $h_\infty = \varphi_A^{-1} \circ E'_\infty = \varphi_A^{-1} \circ f$. By construction, $h$ acts on $A$ as the identity:

$$h|_A = \varphi_A^{-1} \circ E|_A = \varphi_A^{-1} \circ (\varphi_A)|_A = id|_A.$$

We claim that $h_\infty$ is an extreme left homeomorphism with at least two fixed points. Since $A$ is either a leaf or a gap, we have $\#(\overline{A} \cap S^1_\infty) \geq 2$ and as $h|_A = id$, $h_\infty$ fixes all points in $\overline{A} \cap S^1_\infty$. For any $x \in S^1_\infty \setminus \overline{A}$, there are two cases. If $x$ lies in the closure of some other stratum $B$ of $\lambda'$, $B \neq A$, with $\varphi_B := (E'|B)$, then

$$h_\infty(x) = \varphi_A^{-1} \circ E'_\infty(x) = \varphi_A^{-1} \circ \varphi_B(x) = \mathrm{cmp}'(A, B)(x),$$

where we consider the comparison isometry $\mathrm{cmp}'$ with respect to $E'$. As $\mathrm{cmp}'(A, B)$ moves left as viewed from $A$ and $x$ lies in the closure of $B$, the translation axis of

$\mathrm{cmp}'(A, B)$ weakly separates $x$ from $A$ and $x$ is moved by $h_\infty$ to the left as viewed from $A$. If $x$ does not lie in the closure of any stratum, then again there is some sequence $(x_n)_{n\in\mathbb{N}}$ in $S^1_\infty$ converging to $x$ with $x_n \in \ell_n$ for a leaf $\ell_n$ of $\lambda'$ for all $n \in \mathbb{N}$. By continuity of $h_\infty$ we have $h_\infty(x) = \lim_{n\to\infty} h_\infty(x_n)$. Since all $x_n$ are moved counterclockwise by $h_\infty$, also $x$ is moved at most counterclockwise. This shows that $h_\infty$ is an extreme left homeomorphism, i.e. $h_\infty \in XL$. It follows that $A \subseteq H(h_\infty)$, so in particular, $A$ is contained in a stratum $A_E := H(h_\infty)$ of the lamination $\lambda$ of $E$. If $A$ is a gap, then also $A_E$ has to be a gap and $f_{A_E}$ is uniquely determined and has to agree with $h_\infty = \varphi_A^{-1} \circ f$. By construction of $E$ we have $(E|(A_E)) = \varphi_A$ and thus

$$E|_A = (E|A_E)|_A = (\varphi_A)|_A = E'|_A, \tag{6.18}$$

so $E$ and $E'$ agree on any gap $A$ of $\lambda'$. Until now, we only know $A \subseteq A_E$. To show that both laminations coincide, we need to have equality. If $A$ and $B$ are two gaps of $\lambda'$ contained in the same gap $A_E$ of $\lambda$, it follows using (6.18) that

$$(E'|A) = (E|A_E) = (E'|B),$$

so $\mathrm{cmp}'(A, B) = id$. This can only occur if one the strata is contained in the closure of the other. As both $A$ and $B$ are gaps, it follows that $A = B$. The fact that the strata of $\lambda'$ cover all of $\mathbb{H}^2$ then gives us $A = A_E$, so any gap $A$ of $\lambda'$ is a gap of $\lambda$. If $A$ is a leaf, so is $A_E$ and we again have $A = A_E$. In total, the laminations $\lambda'$ and $\lambda$ agree.

As seen above, $E'|_A = E|_A$ for any gap $A$. For a leaf $\ell$ we have $\ell \subseteq H(h_\infty) = H(\varphi_\ell^{-1} \circ f)$. On $\ell$, there are various possibilities for $E|_\ell$, one of them being $(E'|\ell) = \varphi_\ell$. In particular, as seen above, $(E'|\ell) = \varphi_\ell$ and $(E|\ell)$ differ by a hyperbolic transformation with translation axis $A$ and translation length between 0 and the infimum of the comparison isometries from both sides of $\ell$.

*Remark* 6.10. The proof of uniqueness also shows that any earthquake map $E$ can be obtained from the construction using convex hulls of extreme left homeomorphisms: If $E$ is an earthquake map and $f := E_\infty$, then for any stratum $A$, $(E|A)^{-1} \circ f$ is an extreme left homeomorphism, just as we have seen above for $h_\infty$. Using exactly those elements in $XL$ of the form $(E|A)^{-1} \circ f$, in particular for the choice on leaves, to define an earthquake $E'$, this will give us $E = E'$.

The earthquake theorem in the hyperbolic plane also holds for right earthquakes. The proof is analogous with small modifications. One uses extreme right instead of extreme left homeomorphisms, defined in the obvious way. Now in some steps of the proof one has to replace *minimum* by *maximum* or *clockwise* by *counterclockwise*, but the ideas of the proofs are the same. However, in general, the right and left earthquake map having the same effect on the boundary are very different (see Example 6.14).

## 6.5. Examples

In general circumstances, the proof of the main theorem is not constructive. However, for very well-behaved circle maps $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ it gives us a recipe to construct an earthquake realizing $f$ on the boundary.

*Example* 6.11. We start with the trivial example when $f = id$. The resulting earthquake should be trivial as well. We have $\mathcal{C} = \mathrm{PSL}(2,\mathbb{R}) \circ id = \mathrm{PSL}(2,\mathbb{R})$. What are the extreme left homeomorphisms? Clearly, $id \in XL$. Let $\varphi \in \mathrm{PSL}(2,\mathbb{R}) \setminus \{id\}$. If $\varphi$ is elliptic then it has no fixed points on the boundary, so it is not in $XL$. If $\varphi$ is hyperbolic, then it has two fixed points on the boundary, but it moves some points clockwise, others counterclockwise. Hence, hyperbolic elements cannot lie in $XL$. Thus we are left with parabolic elements. All parabolic elements have exactly one fixed point on $\partial\mathbb{H}^2$ and move all other points in one direction. So $XL$ consists of the identity and all parabolic elements that move the non-fixed points counterclockwise. Now that we found $XL$, we can construct the lamination $\lambda$. For all elements $\varphi \in XL$, we have either $\mathrm{fix}(\varphi) = \overline{\mathbb{H}^2}$ in case that $\varphi = id$ or $\mathrm{fix}(\varphi) = \{a\}$ if $\varphi$ is parabolic with fixed point $a \in S^1_\infty$. Thus $H(id) = \overline{\mathbb{H}^2}$ and $H(\varphi) = \{a\}$ for all other $\varphi \in XL$. The latter do not contribute to the lamination. It follows that there are no leaves and the only gap is $A = \mathbb{H}^2$ corresponding to $f_A = id \in XL$. We have $E|_A = id$. So indeed, the earthquake giving rise to the identity at the boundary is trivial.

*Remark* 6.12. In this special case, one can also explicitly compute the element in $XL$ corresponding to $i \in \mathbb{H}^2$ using the construction from Proposition 6.3 - it is the identity. However, as soon as we pick another element, the computation gets more complicated.

*Example* 6.13. Let

$$f\colon \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}, \qquad f(z) = \begin{cases} z & \text{for } z > 0, \\ e^{-2}z & \text{for } z \leq 0, \\ \infty & \text{for } z = \infty. \end{cases}$$

$f$ is an orientation-preserving homeomorphism. Note that $f$ agrees with the boundary homeomorphism induced by the elementary earthquake from Example 5.10. So, by uniqueness in the main theorem, the earthquake we construct should agree with the one in Example 5.10 - up to possibly on leaves. What are the elements in $XL \subseteq \mathrm{PSL}(2,\mathbb{R}) \circ f$? Recall that in the half-plane model, being in $XL$ means having at least one fixed point and moving all points not fixed to the right. We first note that $f$ itself is an extreme left homeomorphism, hence $id \circ f \in XL$ with fixed point set $\mathrm{fix}(id \circ f) = \mathbb{R}_{\geq 0} \cup \{\infty\}$. Let now $\lambda \in \mathbb{R} \setminus \{0\}$ and $\varphi_\lambda(z) = e^\lambda z$ be a hyperbolic transformation with axis $\ell$ connecting 0 and $\infty$. Then

$$\varphi_\lambda \circ f(z) = \begin{cases} e^\lambda z & \text{for } z > 0, \\ e^{\lambda - 2} z & \text{for } z \leq 0, \\ \infty & \text{for } z = \infty. \end{cases}$$

We have a look what happens for different values of $\lambda$:

- $\lambda < 0 \;\Rightarrow\; \varphi_\lambda \circ f$ moves all points $z > 0$ to the left $\Rightarrow\; \varphi_\lambda \circ f \notin XL$.
- $\lambda > 2 \;\Rightarrow\; \varphi_\lambda \circ f$ moves all points $z < 0$ to the left $\Rightarrow\; \varphi_\lambda \circ f \notin XL$.
- $\lambda = 2 \;\Rightarrow\; \varphi_2 \circ f$ fixes all points $z \leq 0$ and $z = \infty$ and moves all points $z > 0$ to the right $\Rightarrow\; \varphi_\lambda \circ f \in XL$ with $\mathrm{fix}(\varphi_2 \circ f) = \mathbb{R}_{\leq 0} \cup \{\infty\}$.
- $\lambda \in (0,2) \;\Rightarrow\; \varphi_\lambda \circ f$ fixes $z = 0$ and $z = \infty$ and moves all other points to the right $\Rightarrow\; \varphi_\lambda \circ f \in XL$ with $\mathrm{fix}(\varphi_\lambda \circ f) = \{0, \infty\}$.

Note that we can also allow $\lambda = 0$. Then $\varphi_0 = id$ and $\varphi_0 \circ f \in XL$ as seen above. Until now, we have the following convex hulls for elements in $XL$, drawn in Figure 6.5:

- $H(id \circ f) = \mathrm{conv}(\mathbb{R}_{\geq 0} \cup \{\infty\}) = \{z \in \mathbb{H}^2 \mid \mathrm{Re}(z) \geq 0\ \} \cup \mathbb{R}_{\geq 0} \cup \{\infty\}$
- $H(\varphi_2 \circ f) = \mathrm{conv}(\mathbb{R}_{\leq 0} \cup \{\infty\}) = \{z \in \mathbb{H}^2 \mid \mathrm{Re}(z) \leq 0\ \} \cup \mathbb{R}_{\leq 0} \cup \{\infty\}$
- $H(\varphi_\lambda \circ f) = \mathrm{conv}(\{0, \infty\}) = \{z \in \mathbb{H}^2 \mid \mathrm{Re}(z) = 0\ \} \cup \{0, \infty\}$ for $\lambda \in (0,2)$

These convex hulls already cover $\mathbb{H}^2$. Since we know that the convex hulls do not intersect, the lamination $\lambda$ is already determined by these elements in $XL$. All other elements in $XL$ have to have singleton fixed point sets and thus do not contribute to the lamination. Now our lamination consists of one geodesic $\ell$ connecting 0 and $\infty$, the gap $A := \{z \in \mathbb{H}^2 \mid \mathrm{Re}(z) > 0\}$ and the gap $B := \{z \in \mathbb{H}^2 \mid \mathrm{Re}(z) < 0\}$. The unique $f_A \in XL$ with $A \subseteq H(f_A)$ is $f_A = id \circ f$, so we set $(E|A) := id^{-1} = id$. The unique $f_B \in XL$ with $B \subseteq H(f_B)$ is $f_B = \varphi_2 \circ f$, so we set $(E|B) := \varphi_2^{-1} = (z \mapsto e^{-2}z)$. As

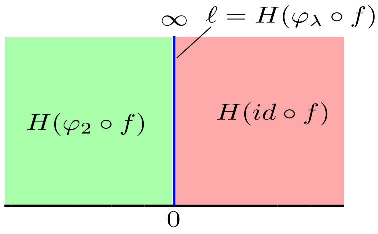



Figure 6.5: The convex hulls for the extreme left homeomorphisms $id \circ f$, $\varphi_2 \circ f$ and $\varphi_\lambda \circ f$ for some $\lambda \in [0, 2]$ already cover the hyperbolic plane.

seen in the proof of the main theorem, there is no unique $f_\ell \in XL$ with $\ell \subseteq H(f_\ell)$. In fact, we have $\ell \subseteq H(\varphi_\lambda \circ f)$ for all $\lambda \in [0, 2]$. So we pick one $\lambda \in [0, 2]$ and set $(E|\ell) := \varphi_\lambda^{-1} = (z \mapsto e^{-\lambda} z)$. For two different choices $\lambda_1, \lambda_2 \in [0, 2]$ we have

$$\varphi_{\lambda_1}^{-1} \circ \varphi_{\lambda_2}(z) = e^{\lambda_1 - \lambda_2} z.$$

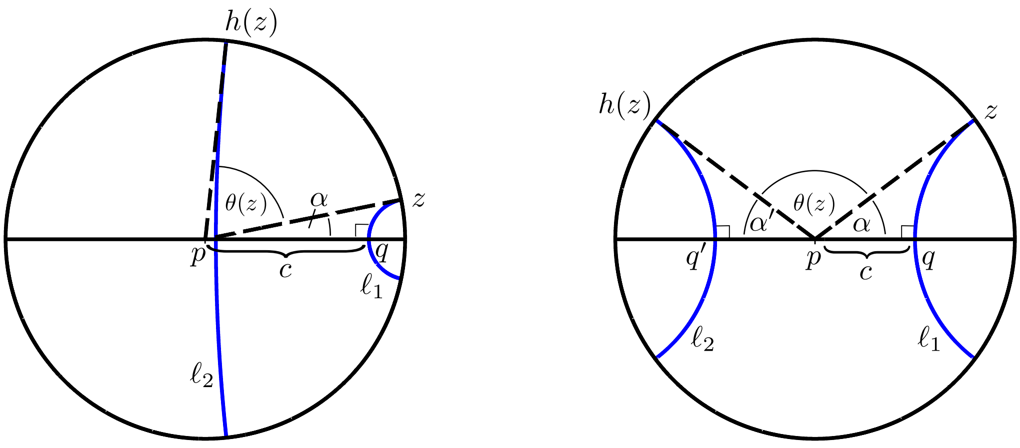
The translation distance of this map is $|\log(e^{\lambda_1 - \lambda_2})| = |\lambda_1 - \lambda_2| \in [0, 2]$. So here we see again that the choices for $E|_\ell$ differ by a hyperbolic translation with translation distance between 0 and $\tau(\mathrm{cmp}(A, B)) = 2$.

In these two examples we could construct the earthquake giving rise to a given homeomorphism. However, if the homeomorphism $f$ is more complicated, for instance if it is not piecewise an isometry, there is no easy way to determine $XL$ and hence the earthquake map. We now show that the left and right earthquakes with the same boundary homeomorphism are in general very different.

*Example* 6.14. As mentioned before, the earthquake theorem holds as well for right earthquakes. We now want to find the right earthquake map having the same effect on the boundary as a given elementary left earthquake. Let $\ell, A$ and $B$ be as constructed in Example 6.13, let $d > 0$ and let $(E|A)(z) := e^{\frac{d}{2}} z$ and $(E|\ell)(z) = (E|B)(z) = e^{-\frac{d}{2}} z$. Note that for $d = 2$ this coincides with the earthquake from the previous example composed with the hyperbolic transformation $\varphi(z) = e^{-1} z$. The corresponding boundary map $h := E_\infty$ is

$$h(z) = \begin{cases} e^{\frac{d}{2}} z & \text{for } z > 0, \\ e^{-\frac{d}{2}} z & \text{for } z \leq 0, \\ \infty & \text{for } z = \infty. \end{cases}$$

To find the right earthquake $F$ with $F_\infty = h$, we have to find the extreme right homeo-




(a) Case 1: $c > \frac{d}{2}$.    (b) Case 2: $c < \frac{d}{2}$.

Figure 6.6: The angle $\theta(z)$ between $z$ and $h(z)$ at $p$ can be calculated from the angles $\alpha$ and $\alpha'$. The explicit formula depends on the position of $q$ and in particular on the length of the side $c$ of the triangle with vertices $p, q$ and $z$.

morphisms in $\mathrm{PSL}(2,\mathbb{R}) \circ h$. We switch to the disk model. Using the Cayley transform, we find that for the upper half of $\mathbb{S}^1$, i.e. $\mathbb{S}^1_+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subseteq \partial B$, respectively, for the lower half $\mathbb{S}^1_- := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\} \subseteq \partial A$, $h$ has as matrix representation

$$\begin{pmatrix} \cosh\left(\frac{d}{4}\right) & \sinh\left(\frac{d}{4}\right) \\ \sinh\left(\frac{d}{4}\right) & \cosh\left(\frac{d}{4}\right) \end{pmatrix} \quad \text{respectively,} \begin{pmatrix} \cosh\left(\frac{d}{4}\right) & \sinh\left(-\frac{d}{4}\right) \\ \sinh\left(-\frac{d}{4}\right) & \cosh\left(\frac{d}{4}\right) \end{pmatrix} .$$

The leaf $\ell$ then is the geodesic between $-1$ and $1$ in $D^2$. One easily computes that $h(-z) = -h(z)$ for all $z \in \mathbb{S}^1$, i.e. $h$ is symmetric with respect to the origin. Thus, we can focus on the upper half $\mathbb{S}^1_+$ of $\mathbb{S}^1$. The idea for finding an extreme right homeomorphism is the following: Take a point $z_0 \in \mathbb{S}^1_+$ and use an elliptic element $\rho$ rotating in the clockwise direction that maps $h(z_0)$ to $z_0$. Then $\rho \circ h$ fixes $z_0$. We want to find $\rho$ such that all other points are either fixed as well or moved clockwise.

Let $p = 0 \in \ell$ be the origin. For $z \in \mathbb{S}^1_+$ let $\theta(z)$ be the interior angle at $p$ of the triangle with vertices $p, z$ and $h(z)$. The function $\theta$ is continuous as $h$ is continuous. We want to find a maximum of $\theta$ in $\mathbb{S}^1_+$. To compute $\theta(z)$ for a fixed $z \in \mathbb{S}^1_+$, let $\ell_1$ be the geodesic orthogonal to $\ell$ ending at $z$ and let $\ell_2$ be the geodesic orthogonal to $\ell$ ending at $h(z)$. Then $(E|B)(\ell_1) = \ell_2$ and the distance $\mathrm{d}(\ell_1, \ell_2) = \frac{d}{2}$ is realized along the common orthogonal $\ell$. Denote the intersection of $\ell_1$ and $\ell$ by $q$ and let $c := \mathrm{d}(p, \ell_1) = \mathrm{d}(p, q)$ (see Figure 6.6). Consider the hyperbolic triangle with vertices $p, q$ and $z$ where $z$ is an ideal vertex. Let $\alpha$ be the angle at $p$. The angle at $q$ is $\frac{\pi}{2}$. The interior angle at $z$ is 0. By a hyperbolic triangle formula (see [Bea83, Ch. 7.10]) it holds that

$$\cosh(c) = \frac{1+\cos(\alpha)\cos(\frac{\pi}{2})}{\sin(\alpha)\sin(\frac{\pi}{2})} = \frac{1}{\sin(\alpha)} \ ,$$

so $\alpha = \arcsin\left(\frac{1}{\cosh(c)}\right)$. Analogously, this holds for the triangle with vertices $p, h(z)$ and $\tilde{q}$ where $\tilde{q}$ denotes the intersection of $\ell$ and $\ell_2$. We denote the angle at $p$ by $\tilde{\alpha}$. The point $q$ lies on $\ell$ which can as be identified with the open interval $(-1,1)$. We distinguish several cases:

*Case 1:* If $q$ lies in $(p,1)$ and $c > \frac{d}{2}$, then $\mathrm{d}(p,\ell_2) = c - \frac{d}{2}$ (see Figure 6.6a). It follows that

$$\theta(z) = \tilde{\alpha} - \alpha = \arcsin\left(\frac{1}{\cosh(c-\frac{d}{2})}\right) - \arcsin\left(\frac{1}{\cosh(c)}\right) \ .$$

We can view $\theta$ as a function in $c$. We observe that $\theta$ decreases as $c$ increases and that $\theta$ tends to $\frac{\pi}{2} - \arcsin\left(\frac{1}{\cosh(\frac{d}{2})}\right)$ as $c$ tends to $\frac{d}{2}$. If $c = \frac{d}{2}$, then $p = \tilde{q}$ and the triangle with vertices $p, h(z), \tilde{q}$ is degenerate. However, we can still compute

$$\theta(z) = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cosh\left(\frac{d}{2}\right)}\right) \ .$$

Thus, for $q \in (p,1)$ and $c \in [\frac{d}{2}, \infty)$, $\theta$ achieves its maximum at $c = \frac{d}{2}$.

*Case 2:* If $q$ lies in $(p,1)$ and $c \in (0, \frac{d}{2})$, then $\mathrm{d}(p,\ell_2) = \frac{d}{2} - c$ (see Figure 6.6b). For the angle $\theta$, we have

$$\theta(z) = \pi - \tilde{\alpha} - \alpha = \pi - \arcsin\left(\frac{1}{\cosh\left(\frac{d}{2}-c\right)}\right) - \arcsin\left(\frac{1}{\cosh(c)}\right) \ .$$

Viewing $\theta$ again as function in $c$, it is differentiable on $(0, \frac{d}{2})$. By explicit computation we find

$$\frac{\mathrm{d}}{\mathrm{d}c}\theta(c) = -\frac{1}{\cosh\left(\frac{d}{2}-c\right)} + \frac{1}{\cosh\left(c\right)} \ .$$

As $c$ and $\frac{d}{2} - c$ are both positive, this is 0 exactly when $c = \frac{d}{2} - c$, so $c = \frac{d}{4}$. The second derivative is negative at $c = \frac{d}{4}$, so we have a maximum at $\frac{d}{4}$. As this is the only extremal point in $(0, \frac{d}{2})$, it follows that $\theta(\frac{d}{4}) > \theta(\frac{d}{2})$. We denote the point on $\mathbb{S}^1$ corresponding to

$c = \frac{d}{2}$ in this case with $z_0$.
*Case 3:* If $q = p$, then $c = 0$ and we have $\theta(0) = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cosh\left(\frac{d}{2}\right)}\right)$ which equals $\theta\left(\frac{d}{2}\right)$.
*Case 4:* If $q \in (-1, p)$, then $\mathrm{d}(p, \ell_2) = c + \frac{d}{2}$ and

$$\theta(z) = \alpha - \tilde{\alpha} = \arcsin\left(\frac{1}{\cosh(c)}\right) - \arcsin\left(\frac{1}{\cosh\left(c + \frac{d}{2}\right)}\right) .$$

As in Case 1, this is smaller than the limit for $c$ going to 0, which is $\frac{\pi}{2} - \arcsin\left(\frac{1}{\cosh\left(\frac{d}{2}\right)}\right)$. This is equal to $\theta(\frac{d}{2})$ from Case 1 and we already know that this is smaller than $\theta(z_0)$.

In total, it follows that $\theta$ attains its maximum at the point $z_0$, corresponding to $c = \frac{d}{4}$. Let $\theta^* := \theta\left(\frac{d}{4}\right)$ and let $\rho_p$ be the clockwise rotation around $p$ by angle $\theta^*$. Then $\rho \circ h$ fixes $z_0$. By symmetry of $\rho_p$ and $h$ with respect to $p$ it follows that $\rho_p \circ h$ also fixes $-z_0$. Further, all other points on $\mathbb{S}^1$ are moved by $h$ by an angle less than $\theta^*$, so $\rho_p \circ h$ moves them clockwise. In total, this shows that $\rho_p \circ h$ is an extreme right homeomorphism and $H(\rho_p \circ h)$ is the geodesic connecting $z_0$ and $-z_0$, where $z_0$ is an endpoint of the geodesic that is orthogonal to $\ell$ at distance $\frac{d}{4}$ from $p$. Note that almost all considerations from above work for an arbitrary $p \in \ell$, not only for the origin. The only thing that changes is that $\rho_p \circ h$, fixing $z_0 \in \mathbb{S}^1_+$, does not fix $-z_0$, but the endpoint of $\ell_2$ that lies in $\mathbb{S}^1_-$. If $p$ is the origin, then $H(\rho_p \circ h)$ is a diameter of the circle, so it is immediate that $p \in H(\rho_p \circ h)$. For other $p$, we can see this using a change of coordinates mapping $p$ to the origin. Further, the angle $\alpha$ between $H(\rho_p \circ h)$ and $\ell$ is the same for every $p \in \ell$.

It follows that the lamination $\lambda$ of $F$ consists of all geodesics making constant angle $\alpha$ to $\ell$. On $\ell$, $F$ acts as the identity. On a leaf $\ell_p = H(\rho_p \circ h)$ intersecting $\ell$ at $p$, $(E|\ell_p) = \rho_p^{-1}$ is a counterclockwise rotation by angle $\theta^* = \pi - 2\alpha$ around $p$ and $E$ maps $\ell_p$ to a geodesic intersecting $\ell$ at $p$ at complementary angle $\pi - \alpha$ (see Figure 6.7).

## 6.6. Relation to relative hyperbolic structures

We constructed an earthquake $E$ having a given $f \in \mathrm{Homeo}^+(\mathbb{S}^1)$ as boundary homeomorphism. If we change $f$ by pre- or post-composing it with an isometry $\varphi \in \mathrm{PSL}(2, \mathbb{R})$, then also the corresponding earthquake maps can be obtained by pre- or post-composing $E$ with $\varphi$.

**Lemma 6.15.** *Let $f \in \mathrm{Homeo}^+(\mathbb{S}^1)$ and let $E$ be a left earthquake map such that $E_\infty = f$ as constructed in the proof of Theorem 6.1. Let $\varphi \in \mathrm{Isom}(\mathbb{H}^2)$. Then*

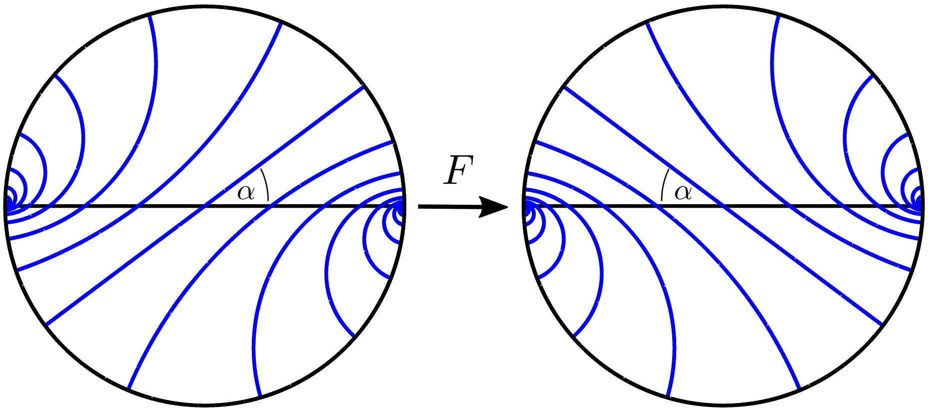



Figure 6.7: The underlying lamination of the right earthquake $F$ consists of geodesics making constant angle $\alpha$ to the geodesic $\ell$ connecting $-1$ and 1. Every such geodesic is mapped by $F$ to a geodesic meeting $\ell$ at the same point, but with complementary angle $\pi - \alpha$.

*i)* $E_1 := \varphi \circ E$ *is a left earthquake map satisfying* $(E_1)_\infty = \varphi \circ f$.

*ii)* $E_2 := E \circ \varphi$ *is a left earthquake map satisfying* $(E_2)_\infty = f \circ \varphi$.

*Proof.* We have to show that $E_1$ and $E_2$ are indeed earthquake maps. The fact that they extend to $\varphi \circ f$ respectively $f \circ \varphi$ is then clear by construction and continuity of the extension. Bijectivity of $E_1$ and $E_2$ immediately follows from bijectivity of $E$ and $\varphi$. Let $A$ be a stratum of the underlying lamination $\lambda$ of $E$. Then

$$(E_1)|_A = (\varphi \circ E)|_A = (\varphi \circ (E|A))|_A$$

with $\varphi \circ (E|A) \in \mathrm{Isom}^+(\mathbb{H}^2)$. Thus on $A$, $E_1$ agrees with an isometry. For two strata $A$ and $B$, we have

$$\begin{aligned}\mathrm{cmp}_1(A, B) &= (E_1|A)^{-1} \circ (E_1|B) = (\varphi \circ (E|A))^{-1} \circ (\varphi \circ (E|B)) \\ &= (E|A)^{-1} \circ (E|B) = \mathrm{cmp}(A, B),\end{aligned}$$

where $\mathrm{cmp}_1$ is the comparison isometry with respect to the map $E_1$. As $E$ is an earthquake map, it follows that also $E_1$ is an earthquake map with the same underlying lamination. Note that this fact is also reflected in the proof of Theorem 6.1: The lamination of $E$ depends on $XL \subseteq \mathrm{PSL}(2, \mathbb{R}) \circ f$ and is invariant under post-composition of $f$ with elements in $\mathrm{PSL}(2, \mathbb{R})$.

From $\lambda$, we obtain a lamination $\lambda_2$ as follows: If $\lambda$ is given by geodesics $\{\ell_i\}_{i \in I}$, then $\lambda_2$

is given by $\{\varphi^{-1}(\ell_i)\}_{i\in I}$. Note that this is indeed a lamination since the fact that $\varphi$ is a homeomorphism guarantees that the locus of $\lambda_2$ is closed and disjoint leaves do not intersect. Now for any stratum $A$ of $\lambda$ there is a stratum $\varphi^{-1}(A)$ of $\lambda_2$ and

$$(E_2)|_{\varphi^{-1}(A)} = (E \circ \varphi)|_{\varphi^{-1}(A)} = ((E|A) \circ \varphi)|_{\varphi^{-1}(A)}$$

with $(E|A) \circ \varphi \in \mathrm{Isom}^+(\mathbb{H}^2)$. For the comparison isometries of strata $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ we have

$$\mathrm{cmp}_2(\varphi^{-1}(A), \varphi^{-1}(B)) = ((E|A) \circ \varphi)^{-1} \circ ((E|B) \circ \varphi) = \varphi^{-1} \circ \mathrm{cmp}(A, B) \circ \varphi.$$

Let $\ell$ be the axis of $\mathrm{cmp}(A, B)$. Then $\mathrm{cmp}_2(\varphi^{-1}(A), \varphi^{-1}(B))$ fixes the endpoints of $\varphi^{-1}(\ell)$ and hence is a hyperbolic transformation with axis weakly separating $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ and shifting to the left as viewed from $\varphi^{-1}(A)$ since $\varphi$ and $\varphi^{-1}$ preserve orientation. So indeed, $E_2$ is an earthquake map. $\square$

We now want to relate two different continuous relative hyperbolic structures on $\mathbb{H}^2$ by earthquakes.

**Corollary 6.16.** *For two continuous relative hyperbolic structures $[h], [h']$ on $\mathbb{H}^2$ there exists an earthquake map sending one to the other.*

The Corollary has to be understood in the following sense: Identify $[h], [h']$ with their images $[f_h], [f_{h'}]$ under $\mathcal{B}$ in the right cosets of $\mathrm{Homeo}^+(\mathbb{S}^1)$ up to $\mathrm{PSL}(2,\mathbb{R})$ (see Theorem 4.2). Pick representatives $f \in [f_h]$ and $g \in [f_{h'}]$. Then there exists an earthquake map $E$ satisfying $E_\infty \circ f = g$. $E$ is unique up to pre- and post-composition with elements in $\mathrm{PSL}(2,\mathbb{R})$ and up to translation on leaves as specified in Theorem 6.1.

*Proof.* As described above, pick $f \in [f_h] = \mathcal{B}([h])$ and $g \in [f_{h'}] = \mathcal{B}([h'])$. Then $g \circ f^{-1} \in \mathrm{Homeo}^+(\mathbb{S}^1)$, so by the earthquake theorem 6.1 there exists a left earthquake map $E$ with $E_\infty = g \circ f^{-1}$, unique except on leaves. If we choose different representatives $\varphi \circ f \in [f_h]$ and $\psi \circ g \in [f_{h'}]$, then the map becomes $\psi \circ g \circ f^{-1} \circ \varphi^{-1}$. As seen in Lemma 6.15, the corresponding earthquake map is given by $\psi \circ E \circ \varphi^{-1}$. Hence, $E$ is uniquely determined except on leaves and up to pre- and post-composition with elements in $\mathrm{PSL}(2,\mathbb{R})$. $\square$

We now have proven Thurston's earthquake theorem in two versions: First, we showed that every orientiation-preserving homeomorphism of the circle arises as boundary homeomorphism of an earthquake map (Theorem 6.1). Then we deduced from this that any two relative hyperbolic structures can be related by an earthquake map that is essentially unique (Corollary 6.16). In his paper [Thu06], Thurston defines relative hyperbolic structures not only on the hyperbolic plane, but also on hyperbolic surfaces, and uses the plane version of the earthquake theorem to show an analogous result for hyperbolic surfaces. The understanding of earthquakes gained in this thesis gives the possibility to better understand the proof of the earthquake theorem on hyperbolic surface and maybe also to apply it to other settings.

# A. Basics in homology theory

For one step in the proof of the earthquake theorem we need homology theory. In this section, we recall the definitions and results needed. Further explanation and proofs can be found in Chapter 2 of Hatcher's introductory book "Algebraic topology" [Hat02].

**Definition A.1.** *Let $X$ be a topological space. A* singular n-simplex *is a continuous map $\sigma\colon \Delta^n \to X$, where $\Delta^n$ denotes the standard n-simplex*

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\ \forall i\ \}.$$

*With $C_n(X)$ we denote the free abelian group with basis the set of singular n-simplices.*

If $v_0, \dots, v_n$ are the vertices of $\Delta^n$, we also denote the n-simplex by $[v_0, \dots, v_n]$. The elements of $C_n(X)$ are abstract finite linear combinations

$$\sigma = \sum_i n_i \sigma_i$$

where $n_i \in \mathbb{Z}$ and the $\sigma_i$ are singular n-simplices. For all $n \in \mathbb{N}$ there is a boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ defined by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}$$

where $\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}$ is the map $\sigma$ restricted to the $(n-1)$-dimensional face of $\Delta^n$ that does not contain $v_i$. It holds that $\partial_{n-1} \circ \partial_n = 0$.

**Definition A.2.** *The* n-th homology group *is*

$$H_n(X) := \ker(\partial_n) \big/ \mathrm{Im}(\partial_{n+1}) \quad .$$

Let $X$ and $Y$ be topological spaces and $f\colon X \to Y$ a continuous map. Let $\sigma$ be a singular n-simplex for $X$. Then $f_\#(\sigma) := f \circ \sigma\colon \Delta^n \to Y$ is a singular n-simplex for $Y$. Extending $f_\#$ linearly by

$$f_\# \left( \sum_i n_i \sigma_i \right) = \sum_i n_i f_\#(\sigma_i)$$

gives a homomorphism $f_\#\colon C_n(X) \to C_n(Y)$ satisfying $f_\# \circ \partial_n = \partial_n \circ f_\#$ for all $n$. Therefore $f_\#$ induces a homomorphism $f_*\colon H_n(X) \to H_n(Y)$.

**Proposition A.3.** *If $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for all $n$.*

*Proof.* See [Hat02, Cor. 2.11]. □

We will need to work with relative homology groups. They are defined as follows: Let $A \subset X$ be a subspace. Set

$$C_n(X, A) := {C_n(X)}\big/{C_n(A)} .$$

The boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, hence it induces a boundary map $\partial_n\colon C_n(X, A) \to C_{n-1}(X, A)$.

**Definition A.4.** *The* n-th relative homology group *is*

$$H_n(X, A) := {\ker(\partial_n)}\big/{\mathrm{Im}(\partial_{n+1})} .$$

Clearly, $C_n(X, \emptyset) = C_n(X)$ and $H_n(X, \emptyset) = H_n(X)$.

**Proposition A.5.** *Let $\iota\colon A \to X$ be the inclusion map and $j\colon X \to {X}/{A}$ the quotient map, where $A \subset X$ is a non-empty closed subspace that is a deformation retract of some neighbourhood in $X$. Then there is a long exact sequence*

$$\ldots \to \tilde{H}_n(A) \xrightarrow{\iota_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X, A) \xrightarrow{\partial_n} \tilde{H}_{n-1}(A) \to \ldots \to \tilde{H}_0(X, A) \to 0 . \tag{A.1}$$

*Proof.* See [Hat02, Th.2.13]. □

Here, $\tilde{H}_n(X)$ denote the so-called reduced homology groups. For us, it is sufficient to know that for $n > 0$ it holds that $\tilde{H}_n(X) = H_n(X)$, as we will only need the part of the long exact sequence (A.1) for $n = 1$ and $n = 2$.
We now take a look at the concrete spaces that we need in the proof of the earthquake theorem.

*Example* A.6. Let $X = \mathbb{D}^2$ and $A = \mathbb{S}^1$. $\mathbb{D}^2$ is homotopy equivalent to a point $\{x_0\}$, $x_0 \in \mathbb{D}^2$, i.e. contractible. Thus

$$H_n(\mathbb{D}^2) \cong H_n(\{x_0\}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

For $\mathbb{S}^1$, we have

$$H_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{for } n > 1. \end{cases}$$

This can be shown using the so-called simplicial homology together with the fact that $\mathbb{S}^1$ is a $\Delta$-complex and on $\Delta$-complexes, simplicial and singular homology agree (see [Hat02, Th.2.27 and Ex.2.2]). In view of Proposition A.5 we obtain an exact sequence

$$\begin{tikzcd} & 0 \arrow[d, equal] & & \mathbb{Z} \arrow[d, equal] & 0 \arrow[d, equal] & \\ \dots \arrow[r] & H_2(\mathbb{D}^2) \arrow[r] & H_2(\mathbb{D}^2, \mathbb{S}^1) \arrow[r] & H_1(\mathbb{S}^1) \arrow[r] & H_1(\mathbb{D}^2) \arrow[r] & \dots \end{tikzcd}$$

As the sequence is exact, it follows that $H_2(\mathbb{D}^2, \mathbb{S}^1) \to H_1(\mathbb{S}^1)$ is an isomorphism, so $H_2(\mathbb{D}^2, \mathbb{S}^1) \cong \mathbb{Z}$.

*Example* A.7. Let $x_0 \in \mathbb{D}^2$ and $X = \mathbb{D}^2 \setminus \{x_0\}$. Without loss of generality we can assume $x_0 = 0$. Else, we use some $\varphi \in \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(D^2)$ with $\varphi(x_0) = 0$. As $\varphi$ is a homotopy equivalence from $\mathbb{D}^2 \setminus \{x_0\}$ to $\mathbb{D}^2 \setminus \{0\}$ with inverse $\varphi^{-1}$ we have $H_n(\mathbb{D}^2 \setminus \{x_0\}) \cong H_n(\mathbb{D}^2 \setminus \{0\})$ for all $n$. Let

$$N_\varepsilon(\mathbb{S}^1) := \{x \in \mathbb{D}^2 \mid |x| \geq 1 - \varepsilon \}$$

be the closed $\varepsilon$-neighbourhood of $\mathbb{S}^1$ in $\mathbb{D}^2$. Let $r \colon \mathbb{D}^2 \setminus \{0\} \to \mathbb{S}^1$ be the retraction of $\mathbb{D}^2 \setminus \{0\}$ to $\mathbb{S}^1$, given by

$$r(x) = \frac{x}{|x|} .$$

We denote by $\iota \colon \mathbb{S}^1 \to \mathbb{D}^2$ the standard inclusion. Then we have $r \circ \iota = id_{\mathbb{S}^1}$ and $\iota \circ r$ is homotopic to $id_{\mathbb{D}^2}$. Thus, $\mathbb{D}^2 \setminus \{0\}$ and $\mathbb{S}^1$ are homotopy equivalent. The same holds for $N_\varepsilon(\mathbb{S}^1)$. Hence, $H_n(\mathbb{D}^2 \setminus \{0\}) \cong H_n(\mathbb{S}^1) \cong H_n(N_\varepsilon(\mathbb{S}^1))$. Since for any topological space $X$, $C_n(X, X)$ is trivial, we have

$$H_2(\mathbb{D}^2 \setminus \{0\}, N_\varepsilon(\mathbb{S}^1)) \cong H_2(\mathbb{S}^1, \mathbb{S}^1) = 0.$$

and

$$H_n(\mathbb{D}^2 \setminus \{0\}) \cong H_n(N_\varepsilon(\mathbb{S}^1)) \cong H_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{for } n > 1. \end{cases}$$

The resulting long exact sequence then is

$$\begin{tikzcd} & 0 \arrow[d, equal] & 0 \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] & \\ \dots \arrow[r] & H_2(\mathbb{D}^2 \setminus \{0\}) \arrow[r] & H_2(\mathbb{D}^2 \setminus \{0\}, N_\varepsilon(\mathbb{S}^1)) \arrow[r] & H_1(N_\varepsilon(\mathbb{S}^1)) \arrow[r] & H_1(\mathbb{D}^2 \setminus \{0\}) \arrow[r] & \dots \end{tikzcd}$$

A further notion we need is the degree of a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$.

**Definition A.8.** *Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be continuous and $f_*\colon H_1(\mathbb{S}^1) \to H_1(\mathbb{S}^1)$ be the induced map in homology. Let $\alpha \in H_1(\mathbb{S}^1)$ be a generator of $H_1(\mathbb{S}^1) \cong \mathbb{Z}$. The* degree of $f$, *denoted by* $\deg f$ *is the unique integer $d \in \mathbb{Z}$ such that $f_*(\alpha) = d \cdot \alpha$.*

Note that this definition is independent on the choice of the generator since $f_*$ is a group homomorphism.

**Lemma A.9.** *Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be continuous. The degree has the following properties:*

*i)* *If $f$ and $g$ are homotopic, then* $\deg f = \deg g$.

*ii)* $\deg id_{\mathbb{S}^1} = 1$.

*iii)* $\deg fg = \deg f \cdot \deg g$.

*iv)* *If $f$ is not surjective, then* $\deg f = 0$.

*Proof.* We only prove iv). Assume $x_0 \in \mathbb{S}^1 \setminus f(\mathbb{S}^1)$. Then $\mathbb{S}^1 \setminus \{x_0\}$ is contractible, so $H_1(\mathbb{S}^1 \setminus \{x_0\}) = 0$. Now $f$ can be written as composition $f = \iota \circ f_0$ where $\iota\colon \mathbb{S}^1 \setminus \{x_0\} \to \mathbb{S}^1$ denotes the inclusion and $f_0\colon \mathbb{S}^1 \to \mathbb{S}^1 \setminus \{x_0\}$ is given by $f_0(x) = f(x)$ for all $x \in \mathbb{S}^1$. Now $f_* = \iota_* \circ (f_0)_* = 0$, since $(f_0)_*(H_1(\mathbb{S}^1)) \subseteq H_1(\mathbb{S}^1 \setminus \{x_0\}) = 0$. □

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